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Homogenization of Dirichlet parabolic systems with variable monotone operators in general perforated domains

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We consider the homogenization of parabolic systems with Dirichlet boundary conditions when the operators and the domains in which the problems are posed vary simultaneously. We assume the operators do not depend on t. Then we show that the corrector obtained in a previous paper for the elliptic problem still gives a corrector for the parabolic one. From this result, we easily obtain the limit problem in the parabolic case.

1. Introduction

In a previous paper [1], we studied the asymptotic behaviour of the solutions of the nonlinear Dirichlet system (see also [10] for the linear case and $M = 1$),

$$\left.\begin{aligned} -\operatorname{div} a_n(x, Du_n) = f \quad \text{in } \mathcal{D}'(\Omega_n),\\ u_n \in W_0^{1,p}(\Omega_n)^M. \end{aligned}\right\} \tag{1.1}$$

Here, Ω_n is an arbitrary sequence of open sets contained in a fixed open bounded set $\Omega \subset \mathbb{R}^N$ and a_n is a sequence of Carathéodory functions that define monotone operators in $W_0^{1,p}(\Omega)^M$ and are uniformly bounded and elliptic (see the exact hypotheses on a_n in definition 2.1). We proved in [1] that, taking a as the homogenized limit of a_n, there exist μ in the set $\mathcal{M}_0^p(\Omega)$ of non-negative Borel measures that vanish on the sets of C_p-capacity zero and a μ-Carathéodory function $F : \Omega \times \mathbb{R}^M \to \mathbb{R}^M$ that satisfies similar properties to a_n, such that, for every $f \in W^{-1,p'}(\Omega)^M$, the solutions of (1.1) (extended by zero outside Ω_n) converge weakly in $W_0^{1,p}(\Omega)^M$ to the solution u of

$$\left.\begin{aligned} u \in W_0^{1,p}(\Omega)^M,\\ \int_\Omega a(x, Du) : Dv \,\mathrm{d}x + \int_\Omega F(x, u)v \,\mathrm{d}\mu = \langle f, v\rangle,\\ \forall v \in W_0^{1,p}(\Omega)^M. \end{aligned}\right\} \tag{1.2}$$

When μ is Radon, this problem can be written in the distributional sense as

$$-\operatorname{div} a(x, Du) + F(x, u)\mu = f \quad \text{in } \mathcal{D}'(\Omega)^M,$$
$$u \in W^{1,p}(\Omega)^M.$$

The new term is the 'strange term' of Cianorescu and Murat [5], which usually appears in the homogenization of Dirichlet problems in perforated domains. A corrector of Du_n, i.e. an approximation in the strong topology of $L^p(\Omega, \mathcal{M}_{M\times N})$, it is also obtained in [1].

The goal of the present paper is to show how these results can be used to solve the parabolic homogenization problem

$$\left.\begin{aligned} \partial_t u_n - \operatorname{div} a_n(x, Du_n) = f \quad \text{in } \mathcal{D}'(\Omega_n \times (0,T))^M,\\ u_n \in L^p(0,T; W_0^{1,p}(\Omega_n)^M),\\ u_n(x,0) = 0 \quad \text{in } \Omega_n. \end{aligned}\right\} \tag{1.3}$$

For this purpose, we show that the corrector for the elliptic problem is still a corrector for the parabolic one. This will imply that the limit problem of (1.3) can be written as

$$\left.\begin{aligned} u \in L^p(0,T; W_0^{1,p}(\Omega)^M \cap L^p_\mu(\Omega)^M), u(x,0) = 0 \qquad \text{in } \Omega,\\ \langle \partial_t u, v\rangle + \int_\Omega a(x, Du) : Dv \,\mathrm{d}x + \int_\Omega F(x,u) v \,\mathrm{d}\mu = \langle f, v\rangle \quad \text{in } \mathcal{D}'(0,T),\\ \forall v \in W_0^{1,p}(\Omega)^M \cap L^p_\mu(\Omega)^M, \end{aligned}\right\} \tag{1.4}$$

where a, F and μ are the same as those that appear in the elliptic case. Indeed, as in [1, 10], we consider a more general problem than (1.3). For this, we remark that if, following Dal Maso and Mosco [8], we consider the sequence of measures $\mu_n \in \mathcal{M}_0^p(\Omega)$ defined by

$$\mu_n(B) = \begin{cases} +\infty & \text{if } C_p(B \cap (\Omega \setminus \Omega_n)) > 0,\\ 0 & \text{if } C_p(B \cap (\Omega \setminus \Omega_n)) = 0 \ \forall B \subset \Omega \text{ Borel}, \end{cases}$$

then (1.3) is equivalent to (1.4) with $a = a_n$, $\mu = \mu_n$ and $F(x, s) = |s|^{p-2}s$. So, instead of problem (1.3), we consider the homogenization problem

$$\left.\begin{aligned} u_n \in L^p(0,T; W_0^{1,p}(\Omega)^M \cap L^p_{\mu_n}(\Omega)^M), u_n(x,0) = 0 \quad \text{in } \Omega,\\ \langle \partial_t u_n, v\rangle + \int_\Omega a_n(x, Du_n) : Dv \,\mathrm{d}x + \int_\Omega F_n(x,u_n) v \,\mathrm{d}\mu = \langle f_n, v\rangle\\ \text{in } \mathcal{D}'(0,T),\\ \forall v \in W_0^{1,p}(\Omega)^M \cap L^p_{\mu_n}(\Omega)^M, \end{aligned}\right\} \tag{1.5}$$

and we prove that the limit is still (1.4), i.e. in this form, the structure of the problem does not vary by homogenization.

We finish this introduction with some bibliographical references.

To the homogenization of the elliptic case, we refer to [11, 14, 15] when Ω_n is fixed, and to [2–9, 18] when a_n does not vary. As we mentioned above, the cases

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where Ω_n and a_n vary simultaneously are studied in [10] for the linear problem and [1] for the monotone one.

For the parabolic problem, we refer to [15] and the references in it when the domains do not vary and to [19] when the operators are fixed. The case where the operators and the domains vary simultaneously has been studied in [16, 17], assuming that the variations hold in a periodic way.

2. Notation

For $M, N \in \mathbb{N}$, we denote by $\mathcal{M}_{M\times N}$ the space of $M \times N$ real matrices. The scalar product of two matrices $A, B \in \mathcal{M}_{M\times N}$ will be denoted by $A : B$.

We represent by $\Omega \subset \mathbb{R}^N$ a bounded open set and by Q_R, $R > 0$, the cylinder $Q_R = \Omega \times [0, R]$.

For a measure μ in Ω, we denote by $L^p_\mu(\Omega, \mathbb{R}^M)$, $1 \leqslant p \leqslant +\infty$, the usual Lebesgue spaces relatives to the measure μ. If μ is the Lebesgue measure, we write $L^p(\Omega, \mathbb{R}^M)$.

For a normed space X, $x \in X$, $x' \in X'$ (the dual space of X), we denote by $\langle x', x\rangle$ the duality product between x' and x.

For every $A \subset \Omega$ and $p \in (1, +\infty)$, we denote by $C_p(A, \Omega)$ the C_p-capacity of A (in Ω), which is defined as the infimum of

$$\int_\Omega |\nabla u|^p \, \mathrm{d}x$$

over the set of functions $u \in W^{1,p}_0(\Omega)$ such that $u \geqslant 1$ a.e. in a neighbourhood of A.

We say that a property $\mathcal{P}(x)$ holds C_p-quasi everywhere (abbreviated as q.e.) in a set E if there exists $N \subset E$ with $C_p(N, \Omega) = 0$ such that $\mathcal{P}(x)$ holds for all $x \in E \setminus N$.

A function $u : \Omega \to \mathbb{R}^M$ is said to be C_p-quasi continuous if, for every $\varepsilon > 0$, there exists $N \subset \Omega$, with $C_p(N, \Omega) < \varepsilon$, such that the restriction of u to $\Omega \setminus N$ is continuous. It is well known that every $u \in W^{1,p}(\Omega)^M$ has a C_p-quasi continuous representative (see [12, 13, 20], etc.). We always identify u with its C_p-quasi continuous representative.

We denote by $\mathcal{M}^p_0(\Omega)$ the class of all non-negative Borel measures that vanish on the sets of C_p-capacity zero and satisfy

$$\mu(B) = \inf\{\mu(A) : A\ C_p\text{-quasi open},\ B \subseteq A \subseteq \Omega\},$$

for every Borel set $B \subseteq \Omega$.

Definition 2.1. We denote by $a_n : \Omega \times \mathcal{M}_{M\times N} \to \mathcal{M}_{M\times N}$ a sequence of Carathéodory functions and we define

$$\hat{a}_n : \Omega \times \mathcal{M}_{M\times N} \to \mathcal{M}_{M\times N} \quad \text{and} \quad \check{a}_n : \Omega \times \mathcal{M}_{M\times N} \times \mathcal{M}_{M\times N} \to \mathcal{M}_{M\times N}$$

by

$$\hat{a}_n(x, \xi) = a_n(x, \xi) : \xi \quad \forall \xi \in \mathcal{M}_{M\times N} \quad \text{a.e. } x \in \Omega$$

and

$$\check{a}_n(x, \xi_1, \xi_2) = (a_n(x, \xi_1) - a_n(x, \xi_2)) : (\xi_1 - \xi_2) \quad \forall \xi_1, \xi_2 \in \mathcal{M}_{M\times N} \quad \text{a.e. } x \in \Omega.$$

The sequence a_n is supposed to satisfy the following properties.

There exists $p \geqslant 2$, such that

(i) $a_n(x, 0) = 0 \ \forall n \in \mathbb{N}$ a.e. $x \in \Omega$;

(ii) there exists $\alpha > 0$ such that

$$\check{a}_n(x, \xi_1, \xi_2) \geqslant \alpha|\xi_1 - \xi_2|^p \quad \forall n \in \mathbb{N} \quad \forall \xi_1, \xi_2 \in \mathcal{M}_{M\times N} \quad \text{a.e. } x \in \Omega; \tag{2.1}$$

(iii) there exist $\gamma > 0$, $\sigma \in (0, 1]$ and $r \in L^1(\Omega)$ such that

$$\begin{aligned} &|a_n(x, \xi_1) - a_n(x, \xi_2)| \\ &\quad \leqslant \gamma(r(x) + \hat{a}_n(x, \xi_1) + \hat{a}_n(x, \xi_2))^{(p-1-\sigma)/p}\check{a}_n(x, \xi_1, \xi_2)^{\sigma/p} \\ &\qquad\qquad \forall n \in \mathbb{N} \quad \forall \xi_1, \xi_2 \in \mathcal{M}_{M\times N} \quad \text{a.e. } x \in \Omega. \end{aligned} \tag{2.2}$$

Remark 2.2. Hypotheses (i), (ii) and (iii) imply the following.

(iii′) There exist $\gamma' > 0$ and $r' \in L^p(\Omega)$ such that

$$\begin{aligned} &|a_n(x, \xi_1) - a_n(x, \xi_2)| \\ &\quad \leqslant \gamma'(r'(x) + |\xi_1| + |\xi_2|)^{p(p-1-\sigma)/(p-\sigma)}|\xi_1 - \xi_2|^{\sigma/(p-\sigma)} \\ &\qquad\qquad \forall n \in \mathbb{N} \quad \forall \xi_1, \xi_2 \in \mathcal{M}_{M\times N} \quad \text{a.e. } x \in \Omega. \end{aligned} \tag{2.3}$$

In particular, we have the following.

(iv′) There exist $\beta > 0$ and $h \in L^{p'}(\Omega)$ such that

$$|a_n(x, \xi)| \leqslant h(x) + \beta|\xi|^{p-1} \quad \forall n \in \mathbb{N} \quad \forall \xi \in \mathcal{M}_{M\times N} \quad \text{a.e. } x \in \Omega. \tag{2.4}$$

Reciprocally, if we assume (i), (ii) and (iii′), then a_n satisfy (iii) with constants $\tilde{\gamma}$, $\tilde{\sigma}$ and a function $\tilde{r}$. We remark that $\tilde{\sigma} = \sigma/(p - \sigma)$ only coincides with σ for $p = 2$ and $\sigma = 1$.

Remark 2.3. Hypothesis (i) can be replaced by $a_n(\cdot, 0)$ belongs to $L^{p'}(\Omega)$. In this case, it is enough in the following to replace a_n by $\bar{a}_n$, defined by

$$\bar{a}_n(x, \xi) = a_n(x, \xi) - a_n(x, 0) \quad \forall n \in \mathbb{N} \quad \forall \xi \in \mathcal{M}_{M\times N} \quad \text{a.e. } x \in \Omega.$$

Definition 2.4. We consider a sequence of measures $\mu_n \in \mathcal{M}_0^p(\Omega)$ and a sequence of functions $F_n : \Omega \times \mathbb{R}^M \to \mathbb{R}^M$ such that $F_n(\cdot, s)$ is μ_n-measurable for every $s \in \mathbb{R}^M$. Analogously to a_n, we define

$$\hat{F}_n : \Omega \times \mathbb{R}^M \to \mathbb{R}^M \quad \text{and} \quad \check{F}_n : \Omega \times \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}^M$$

by

$$\hat{F}_n(x, s) = F_n(x, s)s \quad \forall n \in \mathbb{N} \quad \forall s \in \mathbb{R}^M \quad \mu_n\text{-a.e. } x \in \Omega$$

and

$$\begin{aligned} &\check{F}_n(x, s_1, s_2) = (F_n(x, s_1) - F_n(x, s_2))(s_1 - s_2) \\ &\qquad\qquad \forall n \in \mathbb{N} \quad \forall s_1, s_2 \in \mathbb{R}^M \quad \mu_n\text{-a.e. } x \in \Omega. \end{aligned}$$

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The sequence F_n is assumed to satisfy

$$F_n(x,0)=0 \qquad \forall n\in\mathbb{N} \quad \mu_n\text{-a.e. } x\in\Omega, \tag{A}$$

$$\check{F}_n(x,s_1,s_2)\geqslant \alpha|s_1-s_2|^p \quad \forall n\in\mathbb{N} \quad \forall s_1,s_2\in\mathbb{R}^M \quad \mu_n\text{-a.e. } x\in\Omega \tag{B}$$

and

$$|F_n(x,s_1)-F_n(x,s_2)|\leqslant \gamma[\hat{F}_n(x,s_1)+\hat{F}_n(x,s_2)]^{(p-1-\sigma)/p}|\check{F}_n(x,s_1,s_2)|^{\sigma/p} \\ \forall n\in\mathbb{N} \quad \forall s_1,s_2\in\mathbb{R}^M \quad \mu_n\text{-a.e. } x\in\Omega. \tag{C}$$

Remark 2.5. Analogously to a_n, hypotheses (A), (B), (C) imply that there exists $\gamma'>0$ such that

$$|F_n(x,s_1)-F_n(x,s_2)|\leqslant \gamma'(|s_1|+|s_2|)^{p(p-1-\sigma)/(p-\sigma)}|s_1-s_2|^{\sigma/(p-\sigma)} \\ \forall s_1,s_2\in\mathbb{R}^M \quad \mu_n\text{-a.e. } x\in\Omega \quad \forall n\in\mathbb{N} \tag{C$'$}$$

and

$$\text{there exists } \beta>0 \text{ such that } |F_n(x,s)|\leqslant \beta|s|^{p-1} \\ \forall s\in\mathbb{R}^M \quad \mu_n\text{-a.e. } x\in\Omega \quad \forall n\in\mathbb{N}. \tag{D}$$

Remark 2.6. Our results can easily be extended to $p\in(1,2)$, but we prefer to remain with the case $p\geqslant 2$ to simplify the exposition. In this case, hypotheses (ii) and (B) must be replaced by

$$\check{a}_n(x,\xi_1,\xi_2)\geqslant \alpha\frac{|\xi_1-\xi_2|^p}{|\xi_1|^{2-p}+|\xi_2|^{2-p}} \quad \forall n\in\mathbb{N} \quad \forall \xi_1,\xi_2\in\mathcal{M}_{M\times N} \quad \text{a.e. } x\in\Omega$$

and

$$\check{F}_n(x,s_1,s_2)\geqslant \alpha\frac{|s_1-s_2|^p}{|s_1|^{2-p}+|s_2|^{2-p}} \quad \forall n\in\mathbb{N} \quad \forall s_1,s_2\in\mathbb{R}^M \quad \mu_n\text{-a.e. } x\in\Omega.$$

respectively.

In order to write shorter expressions, we do not specify the dependence in x of a_n and F_n. For example, we write $a_n(Du)$ to mean $a_n(x,Du(x))$ and $F_n(u)$ to mean $F_n(x,u(x))$.

We denote by C a generic constant that only depends on p, N, γ and β and can change from one line to another one.

3. Preliminary results

We start this section by recalling some results for the stationary homogenization problem. The following definition was introduced in [7,10].

Definition 3.1. We define w_n as the solution of the problem

$$\left.\begin{aligned} &w_n\in W_0^{1,p}(\Omega)\cap L^p_{\mu_n}(\Omega),\\ &\int_\Omega |\nabla w_n|^{p-2}\nabla w_n\nabla v\,\mathrm{d}x+\int_\Omega |w_n|^{p-2}w_n v\,\mathrm{d}\mu_n=\int_\Omega v\,\mathrm{d}x,\\ &\forall v\in W_0^{1,p}(\Omega)\cap L^p_{\mu}(\Omega). \end{aligned}\right\} \tag{3.1}$$

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The main properties we need about w_n are given by the following result (see [4, 7, 9]).

Proposition 3.2. *The sequence w_n is non-negative C_p-q.e. in Ω and its norm in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)\cap L^p_{\mu_n}(\Omega)$ is bounded. So, extracting a subsequence if necessary, there exists a non-negative function $w\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ such that w_n converges weakly to w in $W_0^{1,p}(\Omega)$ and weakly-$*$ in $L^\infty(\Omega)$. The convergence is also strong in $W_0^{1,q}(\Omega)$, $1\leqslant q<p$. Moreover, there exists a measure $\mu\in\mathcal{M}_0^p(\Omega)$ such that, analogously to w_n, w satisfies*

$$\left.\begin{aligned} &w\in W_0^{1,p}(\Omega)\cap L^p_\mu(\Omega),\\ &\int_\Omega |\nabla w|^{p-2}\nabla w\nabla v\,\mathrm{d}x+\int_\Omega |w|^{p-2}wv\,\mathrm{d}\mu=\int_\Omega v\,\mathrm{d}x,\\ &\forall v\in W_0^{1,p}(\Omega)\cap L^p_\mu(\Omega). \end{aligned}\right\} \tag{3.2}$$

The sequences w_n, μ_n, the function w and the measure μ satisfy the following.

(a) *The space $\{w\varphi:\varphi\in\mathcal{D}(\Omega)\}$ is dense in $W_0^{1,p}(\Omega)\cap L^p_\mu(\Omega)$.*

(b) *For every $\psi\in\mathcal{D}(\Omega)$, we have*

$$\lim_{n\to\infty}\left(\int_\Omega|\nabla(w_n\psi)|^p\,\mathrm{d}x+\int_\Omega|w_n\psi|^p\,\mathrm{d}\mu_n\right)=\int_\Omega|\nabla(w\psi)|^p\,\mathrm{d}x+\int_\Omega|w\psi|^p\,\mathrm{d}\mu.$$

(c) *For every sequence $u_n\in W_0^{1,p}(\Omega)\cap L^p_{\mu_n}(\Omega)$ that converges weakly to a function u in $W_0^{1,p}(\Omega)$, and such that $\|u_n\|_{L^p_{\mu_n}(\Omega)}$ is bounded, we get that u belongs to $W_0^{1,p}(\Omega)\cap L^p_\mu(\Omega)$ and*

$$\int_\Omega|\nabla u|^p\,\mathrm{d}x+\int_\Omega|u|^p\,\mathrm{d}\mu\leqslant\liminf_{n\to\infty}\left(\int_\Omega|\nabla u_n|^p\,\mathrm{d}x+\int_\Omega|u_n|^p\,\mathrm{d}\mu_n\right).$$

From proposition 3.2 (a), it is easy to prove the following result.

Corollary 3.3. *Assume that w_n defined by (3.1) converges weakly to w and define μ by proposition 3.2. For every $u\in L^p(0,T;W_0^{1,p}(\Omega)^M\cap L^p_\mu(\Omega)^M)$ such that $\partial_t u\in L^{p'}(0,T;(W^{1,p}(\Omega)^M\cap L^p_\mu(\Omega)^M)')$, there exists $\psi_m\in\mathcal{D}((0,T]\times\Omega)$ such that $w\psi_m$ and $w\partial_t\psi_m$ converge to u and $\partial_t u$ in $L^p(0,T;W_0^{1,p}(\Omega)^M\cap L^p_\mu(\Omega)^M)$ and $L^{p'}(0,T;(W^{1,p}(\Omega)^M\cap L^p_\mu(\Omega)^M)')$, respectively.*

With respect to the homogenization problem

$$\left.\begin{aligned} &u_n\in W_0^{1,p}(\Omega)^M\cap L^p_{\mu_n}(\Omega)^M,\\ &\int_\Omega a_n(Du_n):Dv\,\mathrm{d}x+\int_\Omega F_n(u_n)v\,\mathrm{d}\mu_n=\langle f_n,v\rangle,\\ &\forall v\in W_0^{1,p}(\Omega)^M\cap L^p_{\mu_n}(\Omega)^M, \end{aligned}\right\} \tag{3.3}$$

the following result is given in [1] (see also [10] for the linear problem).

THEOREM 3.4. *Assume that w_n defined by (3.1) converges weakly in $W_0^{1,p}(\Omega)$ to w, and define μ as the measure given by proposition 3.2. Then there exits a subsequence of n, still denoted by n, a Carathéodory function $a:\Omega\times\mathcal{M}_{M\times N}\to\mathcal{M}_{M\times N}$ and a function $F:\Omega\times\mathbb{R}^M\to\mathbb{R}^M$ such that, for every sequence f_n that converges strongly in $W^{-1,p'}(\Omega)^M$ to a distribution f, the solution u_n of (3.3) converges weakly in $W_0^{1,p}(\Omega)^M$ to the unique solution u of*

$$\left.\begin{aligned} &u\in W_0^{1,p}(\Omega)^M\cap L^p_\mu(\Omega)^M,\\ &\int_\Omega a(Du):Dv\,\mathrm{d}x+\int_\Omega F(u)v\,\mathrm{d}\mu=\langle f,v\rangle,\\ &\forall v\in W_0^{1,p}(\Omega)^M\cap L^p_\mu(\Omega)^M.\end{aligned}\right\}\tag{3.4}$$

Moreover, the functions a and F, and the measure μ, respectively, satisfy (i), (ii), (iii) and (A), (B), (C) of the previous section with the same constants α, γ and σ and the same function r. The function a does not depend on μ_n or F_n. In particular, it coincides with the function that appears in the homogenization of (3.3) when μ_n is zero.

The following result, which will be used later, was also obtained in [1].

PROPOSITION 3.5. *Consider the subsequence of n given by theorem 3.4. Then there exists a constant $C>0$ such that, for every sequence f_n^1, f_n^2 that converge to two distributions f^1, f^2, respectively, in $W^{-1,p'}(\Omega)^M$, the solutions u_n^1, u_n^2 of (3.3) with f_n replaced by f_n^1 and f_n^2, respectively, satisfy*

$$\begin{aligned}\limsup_{n\to\infty}\int_\Omega |D(u_n^1-u_n^2-\bar u_n^2+\bar u_n^2)|^p\,\mathrm{d}x\qquad&\\ \leqslant\left(\int_\Omega(|u^1|+|u^2|)^p\,\mathrm{d}x\right)^{(p-1-\sigma)/(p-\sigma)}&\left(\int_\Omega|u^1-u^2|^p\,\mathrm{d}\mu\right)^{1/(p-\sigma)},\end{aligned}\tag{3.5}$$

where u^1, u^2 are the solutions of (3.4) with f replaced by f^1 and f^2, respectively, and $\bar u_n^1$, $\bar u_n^2$ the solutions of

$$\begin{gathered}\bar u_n^i\in W_0^{1,p}(\Omega)^M,\\ \int_\Omega a(D\bar u_n^i):Dv\,\mathrm{d}x=\int_\Omega a(Du^i):Dv\,\mathrm{d}x,\quad i=1,2,\\ \forall v\in W_0^{1,p}(\Omega)^M.\end{gathered}$$

4. Homogenization

In this section we use the results stated in § 3 to realize the homogenization of (1.5). The main result of the present paper is next theorem.

THEOREM 4.1. *Let n be the subsequence of n given by theorem 3.4 and consider the measure μ and the functions a and F that appear in this theorem. Then, for every sequence f_n that converges strongly to a distribution f in $L^{p'}(0,T;W^{-1,p'}(\Omega)^M)$, the solution u_n of (1.5) converges weakly in $L^p(0,T;W_0^{1,p}(\Omega)^M\cap L^p_\mu(\Omega)^M)$ to the*

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unique solution u of the problem

$$\left.\begin{array}{ll} u \in L^p(0,T;W_0^{1,p}(\Omega)^M \cap L^p_\mu(\Omega)^M), u(x,0)=0 & a.e.\ x\in\Omega, \\ \langle \partial_t u, v\rangle + \displaystyle\int_\Omega a(Du):Dv\,\mathrm{d}x + \int_\Omega F(u)v\,\mathrm{d}\mu = \langle f,z\rangle & in\ \mathcal{D}'(0,T) \\ \quad\forall v\in W_0^{1,p}(\Omega)^M\cap L^p_\mu(\Omega)^M, & \end{array}\right\} \tag{4.1}$$

Proof. Let us show the result in several steps.

STEP 1. We consider f_n, f and u_n as in the statement of theorem 4.1. Using u_n as a test function in (1.5), we deduce

$$\int_\Omega |u_n(T)|^2\,\mathrm{d}x + \int_{Q_T}|Du_n(x,t)|^p\,\mathrm{d}x\mathrm{d}t + \int_{Q_T}|u_n(t)|^p\,\mathrm{d}\mu_n\mathrm{d}t \leqslant C\int_0^T \|f_n(t)\|^{p'}_{W^{-1,p'}(\Omega)^M}\,\mathrm{d}t.$$

So, extracting a subsequence if necessary and taking into account proposition 3.2 (c), we deduce that there exists $u\in L^p(0,T;W_0^{1,p}(\Omega)^M\cap L^p_\mu(\Omega)^M)$ such that u_n converges weakly to u in $L^p(0,T;W_0^{1,p}(\Omega)^M)$.

When we prove that u is the solution of (4.1), we will deduce by uniqueness, so it is not necessary to extract any subsequence.

STEP 2. We prove that $\partial_t u$ belongs to $L^{p'}(0,T;(W^{1,p}(\Omega)^M\cap L^p_\mu(\Omega)^M)')$. For this purpose, we consider $\varphi\in\mathcal{D}(Q_T)^M$. Then we take $w_n\varphi$ as a test function in (1.5). Since the norms of $a_n(Du_n)$ and $F_n(u_n)$ are bounded in $L^{p'}(0,T;L^{p'}(\Omega_n)^M)$ and $L^{p'}(0,T;L^{p'}_{\mu_n}(\Omega)^M)$, respectively, we get

$$\begin{aligned}\int_{Q_T} u_n\partial_t(w_n\varphi)\,\mathrm{d}x\mathrm{d}t &= \int_{Q_T} a_n(Du_n):D(w_n\varphi)\,\mathrm{d}x\mathrm{d}t \\ &\quad + \int_{Q_T}F_n(u_n)w_n\varphi\,\mathrm{d}\mu_n\mathrm{d}t - \int_{Q_T}\langle f_n, w_n\varphi\rangle\,\mathrm{d}t \\ &\leqslant M\|w_n\varphi\|_{L^p(0,T;W_0^{1,p}(\Omega_n)^M\cap L^p_{\mu_n}(\Omega)^M)},\end{aligned} \tag{4.2}$$

where M is a positive constant that does not depend on n. Using the fact that u_n converges weakly to u in $L^p(Q_T)^M$ and

$$\int_{Q_T}|\nabla(w_n\varphi)|^p\,\mathrm{d}x\mathrm{d}t + \int_{Q_T}|w_n\varphi|^p\,\mathrm{d}\mu_n\mathrm{d}t \to \int_{Q_T}|\nabla(w\varphi)|^p\,\mathrm{d}x\mathrm{d}t + \int_{Q_T}|w\varphi|^p\,\mathrm{d}\mu\mathrm{d}t, \tag{4.3}$$

which is an easy consequence of proposition 3.2 (b), and the Lebesgue dominated convergence theorem, we can pass to the limit in (4.2) to deduce that

$$\int_{Q_T} u\partial_t(w\varphi)\,\mathrm{d}x\mathrm{d}t \leqslant M\|w\varphi\|_{L^p(0,T;W_0^{1,p}(\Omega)^M\cap L^p_\mu(\Omega)^M)} \quad \forall\varphi\in\mathcal{D}(Q_T)^M.$$

Since $\{w\varphi:\varphi\in\mathcal{D}(Q_T)^M\}$ is dense in $L^p(0,T,W_0^{1,p}(\Omega)^M\cap L^p_\mu(\Omega)^M)$, we conclude that $\partial_t u$ belongs to $L^{p'}(0,T;(W^{1,p}(\Omega)^M\cap L^p_\mu(\Omega)^M)')$.

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STEP 3. We consider a sequence $\psi_m \in \mathcal{D}(Q_T)^M$ such that $w\psi_m$ and $\partial_t(w\psi_m)$ converge to u and $\partial_t u$ in

$$L^p(0,T;W_0^{1,p}(\Omega)^M \cap L^p_\mu(\Omega)^M) \quad \text{and} \quad L^{p'}(0,T;(W^{1,p}(\Omega)^M \cap L^p_\mu(\Omega)^M)'),$$

respectively. Such sequences exists by corollary 3.3. Then, for $n, m \in \mathbb{N}$, we define $\tilde{u}_{m,n} \in L^p(0,T;W_0^{1,p}(\Omega)^M \cap L^p_\mu(\Omega)^M)$ as the solution of

$$\left.\begin{aligned} &\tilde{u}_{m,n}(t) \in W_0^{1,p}(\Omega)^M \cap L^p_{\mu_n}(\Omega)^M, \\ &\int_\Omega a_n(D\tilde{u}_{m,n}(t)) : Dv\,\mathrm{d}x + \int_\Omega F_n(\tilde{u}_{m,n}(t))v\,\mathrm{d}\mu_n \\ &\qquad\qquad + m\int_\Omega [\tilde{u}_{m,n}(t) - w_n\psi_m(t)]v\,\mathrm{d}x = 0, \\ &\forall v \in W_0^{1,p}(\Omega)^M \cap L^p_{\mu_n}(\Omega)^M \quad \text{a.e. } t \in (0,T). \end{aligned}\right\} \tag{4.4}$$

STEP 4. Let us prove some properties of $\tilde{u}_{m,n}$. Taking $\tilde{u}_{m,n} - w_n\psi_m$ as a test function in (4.4), the properties of a_n and F_n easily imply that

$$\begin{aligned} &\int_\Omega |D\tilde{u}_{m,n}(t)|^p\,\mathrm{d}x + \int_\Omega |\tilde{u}_{m,n}(t)|^p\,\mathrm{d}\mu_n \\ &\quad + m\int_\Omega |\tilde{u}_{m,n}(t) - w_n\psi_m(t)|^2\,\mathrm{d}x \\ &\qquad \leqslant C\biggl(\int_\Omega |h(x)|^{p'}\,\mathrm{d}x + \int_\Omega |\nabla(w_n\psi_m(t))|^p\,\mathrm{d}x\mathrm{d}t + \int_\Omega |w_n\psi_m(t)|^p\,\mathrm{d}\mu_n\biggr) \\ &\qquad\qquad \text{a.e. } t \in (0,T). \end{aligned} \tag{4.5}$$

Thus, for a.e. $t \in (0,T)$ and every $m \in \mathbb{N}$, there exists a subsequence of n (which depends on t and m) such that $\tilde{u}_{m,n}(t)$ converges weakly in $W_0^{1,p}(\Omega)^M$. But, using theorem 3.4, we conclude that it is not necessary to extract such a subsequence and that, defining $\tilde{u}_m \in L^p(0,T;W_0^{1,p}(\Omega)^M \cap L^p_\mu(\Omega)^M)$ as the solution of

$$\left.\begin{aligned} &\tilde{u}_m(t) \in W_0^{1,p}(\Omega)^M \cap L^p_\mu(\Omega)^M, \\ &\int_\Omega a(D\tilde{u}_m(t)) : Dv\,\mathrm{d}x + \int_\Omega F(\tilde{u}_m(t))v\,\mathrm{d}\mu \\ &\qquad\qquad + m\int_\Omega [\tilde{u}_m(t) - w\psi_m(t)]v\,\mathrm{d}x = 0, \\ &\forall v \in W_0^{1,p}(\Omega)^M \cap L^p_\mu(\Omega,\mathbb{R}^M) \quad \text{a.e. } t \in (0,T), \end{aligned}\right\} \tag{4.6}$$

we have

$$\tilde{u}_{m,n}(t) \rightharpoonup \tilde{u}_m(t) \quad \text{in } W_0^{1,p}(\Omega)^M \tag{4.7}$$

for every $m \in \mathbb{N}$ and a.e. $t \in (0,T)$. In particular, by the Rellich–Kondrachov compactness theorem,

$$\tilde{u}_{m,n}(t) \to \tilde{u}_m(t) \quad \text{in } L^p(\Omega)^M \quad \forall m \in \mathbb{N} \quad \text{a.e. } t \in (0,T), \tag{4.8}$$

and thus

$$\int_\Omega |\tilde{u}_{m,n}(t)|^p\,\mathrm{d}x \to \int_\Omega |\tilde{u}_m(t)|^p\,\mathrm{d}x \quad \forall m \in \mathbb{N} \quad \text{a.e. } t \in (0,T).$$

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By (4.5) and the Poincaré inequality, we also have

$$\int_\Omega |\tilde{u}_{m,n}(t)|^p \,\mathrm{d}x \leqslant C\biggl(\int_\Omega |h(x)|^{p'}\,\mathrm{d}x + \int_\Omega |\nabla(w_n\psi_m(t))|^p\,\mathrm{d}x + \int_\Omega |w_n\psi_m(t)|^p\,\mathrm{d}\mu_n\biggr).$$

Since $h(x) \in L^{p'}(\Omega)$, the norm of w_n in $W_0^{1,p}(\Omega)\cap L^p_{\mu_n}(\Omega)$ is bounded and ψ_m belongs to $\mathcal{D}(\Omega)$, we deduce that, for every $m \in \mathbb{N}$, the second term of this inequality is bounded independently of n and t. So we can apply the Lebesgue convergence theorem to deduce that

$$\int_{Q_T} |\tilde{u}_{m,n}|^p \,\mathrm{d}x\mathrm{d}t \to \int_{Q_T} |\tilde{u}_m|^p\,\mathrm{d}x\mathrm{d}t,$$

which implies that

$$\tilde{u}_{m,n} \to \tilde{u}_m \quad \text{in } L^p(Q_T) \quad \forall m \in \mathbb{N}. \tag{4.9}$$

Using (4.7) and the Lebesgue convergence theorem as above, we have

$$\tilde{u}_{m,n} \rightharpoonup \tilde{u}_m \quad \text{in } L^p(0,T;W_0^{1,p}(\Omega)^M). \tag{4.10}$$

Using the properties of a and F, it is also easy to check that

$$\tilde{u}_m \to u \quad \text{in } L^p(0,T;W_0^{1,p}(\Omega)^M \cap L^p_\mu(\Omega)^M). \tag{4.11}$$

On the other hand, defining $\bar{u}_{m,n} \in L^p(0,T;W^{1,p}(\Omega)^M)$ as the solution of

$$\left.\begin{gathered}\bar{u}_{m,n}(t) \in W_0^{1,p}(\Omega)^M,\\ \int_\Omega a_n(D\bar{u}_{m,n}(t)) : Dv\,\mathrm{d}x = \int_\Omega a(D\tilde{u}_m(t)) : Dv\,\mathrm{d}x,\\ \forall v \in W_0^{1,p}(\Omega)^M \quad \text{a.e. } t\in(0,T),\end{gathered}\right\} \tag{4.12}$$

we deduce by (3.5) that

$$\begin{aligned}\limsup_{n\to\infty}&\int_\Omega |D(\tilde{u}_{m,n}(t) - \tilde{u}_{m,n}(r) - \bar{u}_{m,n}(t) + \bar{u}_{m,n}(r))|^p\,\mathrm{d}x\\ &\leqslant C\biggl(\int_\Omega (|\tilde{u}_m(t)| + |\tilde{u}_m(r)|)^p d\mu\biggr)^{(p-1-\sigma)/(p-\sigma)}\biggl(\int_\Omega |\tilde{u}_m(t) - \tilde{u}_m(r)|^p\,\mathrm{d}\mu\biggr)^{1/(p-\sigma)}\end{aligned} \tag{4.13}$$

for every $m \in \mathbb{N}$ a.e. $t, r \in (0,T)$.

Taking $\bar{u}_{m,n}(t) - \bar{u}_{m,n}(r)$ as a test function in the difference of the problems satisfied by $\bar{u}_{m,n}(t)$ and $\bar{u}_{m,n}(r)$, passing to the limit in n and using the properties of a_n and a, we also deduce that there exits $l \in L^p(\Omega)$ such that

$$\begin{aligned}\limsup_{n\to\infty}&\biggl(\int_\Omega |D(\bar{u}_{m,n}(t) - \bar{u}_{m,n}(r))|^p\,\mathrm{d}x\biggr)\\ &\leqslant C\biggl(\int_\Omega (l(x) + |D\tilde{u}_m(t)| + |D\tilde{u}_m(r)|)^p\,\mathrm{d}x\biggr)^{(p-1-\sigma)/(p-\sigma)}\\ &\qquad\times\biggl(\int_\Omega |D(\tilde{u}_m(t) - \tilde{u}_m(r))|^p\,\mathrm{d}x\biggr)^{1/(p-\sigma)}\end{aligned} \tag{4.14}$$

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for every $m \in \mathbb{N}$ a.e. $t, r \in (0, T)$. From (4.9), (4.13) and (4.14), we conclude that

$$\begin{aligned}\limsup_{n\to\infty}\biggl(\int_\Omega |D(\tilde u_{m,n}(t)-\tilde u_{m,n}(r))|^p\,\mathrm{d}x+\int_\Omega |\tilde u_{m,n}(t)-\tilde u_{m,n}(r)|^p\,\mathrm{d}\mu_n\biggr)&\\ \leqslant C\biggl(\int_\Omega (l(x)+|D\tilde u_m(t)|+|D\tilde u_m(r)|)^p\,\mathrm{d}x\biggr)^{(p-1-\sigma)/(p-\sigma)}&\\ \times\biggl(\int_\Omega |D(\tilde u_m(t)-\tilde u_m(r))|^p\,\mathrm{d}x\biggr)^{1/(p-\sigma)}&\\ +C\biggl(\int_\Omega (|\tilde u_m(t)|+|\tilde u_m(r)|)^p\,\mathrm{d}\mu\biggr)^{(p-1-\sigma)/(p-\sigma)}&\\ \times\biggl(\int_\Omega |\tilde u_m(t)-\tilde u_m(r)|^p\,\mathrm{d}\mu\biggr)^{1/(p-\sigma)}&\end{aligned}\tag{4.15}$$

for every $m \in \mathbb{N}$ a.e. $t, r \in (0, T)$.

Step 5. Let us now prove that, for every $R < T$, we have

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\biggl(\int_{Q_R}|D(u_n(t)-\tilde u_{m,n}(t))|^p\,\mathrm{d}x\mathrm{d}t+\int_{Q_R}|u_n(t)-\tilde u_{m,n}(t)|^p\,\mathrm{d}\mu_n\mathrm{d}t\biggr)=0. \tag{4.16}$$

We consider $\zeta \in C^1[0, T]$ such that $\zeta = 1$ in $[0, R]$, $\zeta = 0$ in $[\frac{1}{2}(R + T), T]$, ζ decreasing. For $0 < t < \frac{1}{2}(R + T)$, $0 < s < \frac{1}{2}(T - R)$, we take $(u_n(t) - \tilde u_{m,n}(t + s))\zeta(t)$ as a test function in the difference of (1.5) and (4.4), and get

$$\begin{aligned}\int_0^T \langle\partial_t u_n(t), u_n(t)-\tilde u_{m,n}(t+s)\rangle\zeta(t)\,\mathrm{d}x\mathrm{d}t&\\ +\int_{Q_T}\breve a_n(Du_n(t), D\tilde u_{m,n}(t+s))\zeta(t)\,\mathrm{d}x\mathrm{d}t&\\ +\int_{Q_T}\breve F_n(u_n(t),\tilde u_{m,n}(t+s))\zeta(t)\,\mathrm{d}\mu_n\mathrm{d}t&\\ -m\int_{Q_T}[\tilde u_{m,n}(t+s)-w_n\psi_m(t+s)]&\\ \times(u_n(t)-\tilde u_{m,n}(t+s))\zeta(t)\,\mathrm{d}x\mathrm{d}t&\\ =\int_0^T\langle f_n(t), u_n(t)-\tilde u_{m,n}(t+s)\rangle\zeta(t)\,\mathrm{d}t.&\end{aligned}\tag{4.17}$$

Integrating with respect to s between zero and $1/m$, multiplying by m and denoting

$$\tilde{\tilde u}_{m,n}(t)=m\int_0^{1/m}\tilde u_{m,n}(t+s)\,\mathrm{d}s,$$

the above inequality gives

$$\begin{aligned}\int_0^T\langle\partial_t u_n(t), u_n(t)-\tilde{\tilde u}_{m,n}(t)\rangle\zeta(t)\,\mathrm{d}x\mathrm{d}t&\\ +\alpha m\int_0^{1/m}\int_{Q_T}|D(u_n(t)-\tilde u_{m,n}(t+s))|^p\zeta(t)\,\mathrm{d}x\mathrm{d}t\mathrm{d}s&\end{aligned}$$

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$$
\begin{aligned}
&+ \alpha m \int_0^{1/m} \int_{Q_T} |u_n(t) - \tilde{u}_{m,n}(t+s)|^p \zeta(t)\, \mathrm{d}\mu_n \mathrm{d}t\mathrm{d}s \\
&\quad - m^2 \int_0^{1/m} \int_{Q_T} [\tilde{u}_{m,n}(t+s) - w_n \psi_m(t+s)] \\
&\qquad\qquad \times (u_n(t) - \tilde{u}_{m,n}(t+s))\zeta(t)\, \mathrm{d}x\mathrm{d}t\mathrm{d}s \\
&\qquad\qquad\qquad \leqslant \int_0^T \langle f_n(t), u_n(t) - \tilde{\tilde{u}}_{m,n}(t)\rangle \zeta(t)\, \mathrm{d}t.
\end{aligned} \tag{4.18}
$$

Let us now pass to the limit in (4.18), first in n and then in m.

For the first term of (4.18), we use

$$
\begin{aligned}
\int_0^T &\langle \partial_t u_n(t), u_n(t) - \tilde{\tilde{u}}_{m,n}(t)\rangle \zeta(t)\, \mathrm{d}t \\
&= -\frac{1}{2} \int_{Q_T} |u_n(t)|^2 \zeta'(t)\, \mathrm{d}x\mathrm{d}t \\
&\quad + m \int_{Q_T} u_n(t) \left(\tilde{u}_{m,n}\left(t + \frac{1}{m}\right) - \tilde{u}_{m,n}(t) \right) \zeta(t)\, \mathrm{d}x\mathrm{d}t \\
&\qquad + \int_{Q_T} u_n(t) \tilde{\tilde{u}}_{m,n}(t) \zeta'(t)\, \mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{4.19}
$$

Taking into account the weak lower semicontinuity for the weak convergence in $L^2(Q_T)$ of the function

$$
v \to -\int_{Q_T} |v|^2 \zeta'(t)\, \mathrm{d}x\mathrm{d}t
$$

and (4.9), we easily get from this equality that

$$
\begin{aligned}
\liminf_{n\to\infty} \int_0^T &\langle \partial_t u_n(t), u_n(t) - \tilde{\tilde{u}}_{m,n}(t)\rangle \zeta(t)\, \mathrm{d}x\mathrm{d}t \\
&\geqslant -\frac{1}{2} \int_{Q_T} |u(t)|^2 \zeta'(t)\, \mathrm{d}x\mathrm{d}t \\
&\quad + m \int_{Q_T} u(t) \left(\tilde{u}_m\left(t + \frac{1}{m}\right) - \tilde{u}_m(t) \right) \zeta(t)\, \mathrm{d}x\mathrm{d}t \\
&\qquad + \int_{Q_T} u(t) \tilde{\tilde{u}}_m(t) \zeta'(t)\, \mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{4.20}
$$

where we have denoted

$$
\tilde{\tilde{u}}_m(t) = m \int_0^{1/m} \tilde{u}_m(t+s)\, \mathrm{d}s.
$$

For the second and third terms of the left-hand side of (4.18), we use

$$
\begin{aligned}
\int_{Q_R} &|D(u_n(t) - \tilde{u}_{m,n}(t))|^p\, \mathrm{d}x\mathrm{d}t + \int_{Q_R} |u_n(t) - \tilde{u}_{m,n}(t)|^p\, \mathrm{d}\mu_n \mathrm{d}t \\
&\leqslant Cm \int_0^{1/m} \int_{Q_T} |D(u_n(t) - \tilde{u}_{m,n}(t+s))|^p \zeta(t)\, \mathrm{d}x\mathrm{d}t\mathrm{d}s
\end{aligned}
$$

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$$+ Cm \int_0^{1/m} \int_{Q_T} |D(\tilde{u}_{m,n}(t+s) - \tilde{u}_{m,n}(t))|^p \zeta(t)\, \mathrm{d}x\mathrm{d}t\mathrm{d}s$$
$$+ Cm \int_0^{1/m} \int_{Q_T} |u_n(t) - \tilde{u}_{m,n}(t+s)|^p \zeta(t)\, \mathrm{d}\mu_n \mathrm{d}t\mathrm{d}s$$
$$+ Cm \int_0^{1/m} \int_{Q_T} |\tilde{u}_{m,n}(t+s) - \tilde{u}_{m,n}(t)|^p \zeta(t)\, \mathrm{d}\mu_n \mathrm{d}t\mathrm{d}s. \tag{4.21}$$

Taking into account (4.15) and (4.5), we can apply the Lebesgue dominated convergence theorem to deduce

$$\begin{aligned}
\limsup_{n\to\infty} \Big(\int_0^{1/m} \int_{Q_T} & |D(\tilde{u}_{m,n}(t+s) - \tilde{u}_{m,n}(t))|^p \zeta(t)\, \mathrm{d}x\mathrm{d}t\mathrm{d}s \\
& + \int_0^{1/m} \int_{Q_T} |\tilde{u}_{m,n}(t+s) - \tilde{u}_{m,n}(t)|^p \, \mathrm{d}\mu_n \zeta(t) \mathrm{d}t\mathrm{d}s \Big) \\
\leqslant C \int_0^{1/m} \int_0^T & \Big(\int_\Omega (|D\tilde{u}_m(t+s)| + |D\tilde{u}_m(t)|)^p \, \mathrm{d}x \Big)^{(p-1-\sigma)/(p-\sigma)} \\
& \times \Big(\int_\Omega |D(\tilde{u}_m(t+s) - \tilde{u}_m(t))|^p \, \mathrm{d}x \Big)^{1/(p-\sigma)} \mathrm{d}t\mathrm{d}s \\
+ C \int_0^{1/m} \int_0^T & \Big(\int_\Omega (|\tilde{u}_m(t+s)| + |\tilde{u}_m(t)|)^p \, \mathrm{d}x \Big)^{(p-1-\sigma)/(p-\sigma)} \\
& \times \Big(\int_\Omega |\tilde{u}_m(t+s) - \tilde{u}_m(t)|^p \, \mathrm{d}\mu \Big)^{1/(p-\sigma)} \mathrm{d}t\mathrm{d}s
\end{aligned} \tag{4.22}$$

for every $m \in \mathbb{N}$.

For the fourth term of (4.18), we use (4.9), which implies

$$\begin{aligned}
\lim_{n\to\infty} \int_0^{1/m} \int_{Q_T} & [\tilde{u}_{m,n}(t+s) - w_n \psi_m(t+s)](u_n(t) - \tilde{u}_{m,n}(t+s))\zeta(t)\, \mathrm{d}x\mathrm{d}t\mathrm{d}s \\
& = \int_0^{1/m} \int_{Q_T} [\tilde{u}_m(t+s) - w\psi_m(t+s)](u(t) - \tilde{u}_m(t+s))\zeta(t)\, \mathrm{d}x\mathrm{d}t\mathrm{d}s
\end{aligned} \tag{4.23}$$

for every $m \in \mathbb{N}$ and every $\varepsilon > 0$.

Using (4.20), (4.21), (4.22), and (4.23), we can pass to the limit in (4.18) in n to obtain

$$- \frac{1}{2} \int_{Q_T} |u(t)|^2 \zeta'(t)\, \mathrm{d}x\mathrm{d}t$$
$$+ m \int_{Q_T} u(t) \Big(\tilde{u}_m \Big(t + \frac{1}{m} \Big) - \tilde{u}_m(t) \Big) \zeta(t)\, \mathrm{d}x\mathrm{d}t + \int_{Q_T} u(t) \tilde{\tilde{u}}_m(t) \zeta'(t)\, \mathrm{d}x\mathrm{d}t$$

$$
\begin{aligned}
&+\limsup_{n\to\infty}\biggl(\int_{Q_R}|D(u_n(t)-\tilde{u}_{m,n}(t))|^p\,\mathrm{d}x\mathrm{d}t\\
&\qquad\qquad+\int_{Q_R}|u_n(t)-\tilde{u}_{m,n}(t)|^p\,\mathrm{d}\mu_n\mathrm{d}rt\biggr)\\
&-m^2\int_0^{1/m}\int_{Q_T}[\tilde{u}_m(t+s)-w\psi_m(t+s)](u(t)-\tilde{u}_m(t+s))\zeta(t)\,\mathrm{d}x\mathrm{d}t\\
\leqslant{}& Cm\int_0^{1/m}\int_0^T\biggl(\int_\Omega(|D\tilde{u}_m(t+s)|+|D\tilde{u}_m(t)|)^p\,\mathrm{d}x\biggr)^{(p-1-\sigma)/(p-\sigma)}\\
&\qquad\times\biggl(\int_\Omega|D(\tilde{u}_m(t+s)-\tilde{u}_m(t))|^p\,\mathrm{d}x\biggr)^{1/(p-\sigma)}\mathrm{d}t\mathrm{d}s\\
&+Cm\int_0^{1/m}\int_0^T\biggl(\int_\Omega(|\tilde{u}_m(t+s)|+|\tilde{u}_m(t)|)^p\,\mathrm{d}\mu\biggr)^{(p-1-\sigma)/(p-\sigma)}\\
&\qquad\times\biggl(\int_\Omega|\tilde{u}_m(t+s)-\tilde{u}_m(t)|^p\,\mathrm{d}\mu\biggr)^{1/(p-\sigma)}\mathrm{d}t\mathrm{d}s\\
&+\int_0^T\langle f,u(t)-\tilde{\tilde{u}}_m(t)\zeta(t)\rangle\,\mathrm{d}t
\end{aligned}\tag{4.24}
$$

for every $m\in\mathbb{N}$. Let us now pass to the limit in (4.24) when m tends to infinity.

First, we remark that equation (4.11) implies that $\tilde{\tilde{u}}_m$ converges to u in $L^p(0,T;W_0^{1,p}(\Omega)^M\cap L^p_\mu(\Omega)^M)$, and then it is easy to check that the third term in the left-hand side of (4.24) tends to $\int_{Q_T}u(t)^2\zeta'(t)\,\mathrm{d}x\mathrm{d}t$ and that the right-hand side of (4.24) tends to zero. For the fifth term on the left-hand side of (4.24), we use that $\tilde{u}_m$ satisfies (4.6) and (4.11), then we get

$$
\begin{aligned}
&m^2\int_0^{1/m}\int_{Q_T}[\tilde{u}_m(t+s)-w\psi_m(t+s)](u(t)-\tilde{u}_m(t+s))\zeta(t)\,\mathrm{d}x\mathrm{d}t\mathrm{d}s\\
&\quad=-\int_0^1\int_{Q_T}a\biggl(D\tilde{u}_m\biggl(t+\frac{r}{m}\biggr)\biggr):D\biggl(u(t)-\tilde{u}_m\biggl(t+\frac{r}{m}\biggr)\biggr)\zeta(t)\,\mathrm{d}x\mathrm{d}t\mathrm{d}r\\
&\qquad-\int_0^1\int_{Q_T}F\biggl(\tilde{u}_m\biggl(t+\frac{r}{m}\biggr)\biggr)\biggl(u(t)-\tilde{u}_m\biggl(t+\frac{r}{m}\biggr)\biggr)\zeta(t)\,\mathrm{d}\mu\mathrm{d}t\mathrm{d}r\\
&\quad\to 0.
\end{aligned}
$$

It remains to pass the limit in the second term of (4.24). We use

$$
\begin{aligned}
&m\int_{Q_T}u(t)\biggl(\tilde{u}_m\biggl(t+\frac{1}{m}\biggr)-\tilde{u}_m(t)\biggr)\zeta(t)\,\mathrm{d}x\mathrm{d}t\\
&\qquad=m\int_{Q_T}u(t)\biggl(\tilde{u}_m\biggl(t+\frac{1}{m}\biggr)-w\psi_m\biggl(t+\frac{1}{m}\biggr)\biggr)\zeta(t)\,\mathrm{d}x\mathrm{d}t\\
&\qquad\quad+m\int_{Q_T}u(t)\biggl(w\psi_m\biggl(t+\frac{1}{m}\biggr)-w\psi_m(t)\biggr)\zeta(t)\,\mathrm{d}x\mathrm{d}t\\
&\qquad\quad+m\int_{Q_T}u(t)(w\psi_m(t)-\tilde{u}_m(t))\zeta(t)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{4.25}
$$

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Using (4.6) as above, the first and third terms of the right-hand side of (4.25) tend to zero. For the second term, we have

$$m\int_{Q_T} u(t)\Big(w\psi_m\Big(t+\frac{1}{m}\Big)-w\psi_m(t)\Big)\zeta(t)\,\mathrm{d}x\mathrm{d}t = \int_0^1\int_{Q_T} w\partial_t\psi_m\Big(t+\frac{r}{m}\Big)\zeta(t)u(t)\,\mathrm{d}x\mathrm{d}t\mathrm{d}r. \tag{4.26}$$

So, since $w\partial_t\psi_m$ converges to $\partial_t u$ in $L^{p'}(0,T;(W^{1,p}(\Omega)^M\cap L^p_\mu(\Omega)^M)')$, we get

$$m\int_{Q_T} u(t)\Big(\tilde{u}_m\Big(t+\frac{1}{m}\Big)-\tilde{u}_m(t)\Big)\zeta(t)\,\mathrm{d}x\mathrm{d}t \to \int_0^T\langle u',u\rangle\zeta(t)\,\mathrm{d}t = -\frac{1}{2}\int_{Q_T}|u(t)|^2\zeta'(t)\,\mathrm{d}x\mathrm{d}t.$$

Thus, from (4.24), we conclude (4.16).

Step 6. To finish the proof of theorem 4.1, all that is left to show is that u satisfies (4.1). For $\varphi\in\mathcal{D}(Q_T)^M$, we take $w_n\varphi$ as a test function in (1.5). Then we get

$$-\int_{Q_T} u_n w_n\partial_t\varphi\,\mathrm{d}x\mathrm{d}t+\int_{Q_T} a_n(Du_n):D(w_n\varphi)\,\mathrm{d}x\mathrm{d}t + \int_{Q_T} F_n(u_n)w_n\varphi\,\mathrm{d}\mu_n\mathrm{d}t = \int_0^T\langle f_n,w_n\varphi\rangle\,\mathrm{d}x\mathrm{d}t. \tag{4.27}$$

Since w_n converges weakly in $W^{1,p}_0(\Omega)$ to w, we have

$$-\int_{Q_T} u_n w_n\partial_t\varphi\,\mathrm{d}x\mathrm{d}t \to -\int_{Q_T} u\partial_t(w\varphi)\,\mathrm{d}x\mathrm{d}t$$

and

$$\int_0^T\langle f_n,w_n\varphi\rangle\,\mathrm{d}x \to \int_0^T\langle f,w\varphi\rangle\,\mathrm{d}x.$$

From step 5 and the properties of a_n and F_n, we have

$$\lim_{n\to\infty}\Big(\int_{Q_T} a_n(Du_n):D(w_n\varphi)\,\mathrm{d}x\mathrm{d}t+\int_{Q_T} F_n(u_n)w_n\varphi\,\mathrm{d}\mu_n\mathrm{d}t\Big) = \lim_{m\to\infty}\lim_{n\to\infty}\Big(\int_{Q_T} a_n(D\tilde{u}_{m,n}):D(w_n\varphi)\,\mathrm{d}x\mathrm{d}t+\int_{Q_T} F_n(\tilde{u}_{m,n})w_n\varphi\,\mathrm{d}\mu_n\mathrm{d}t\Big),$$

but taking $w_n\varphi$ as a test function in (4.4), and using (4.10), (4.11) and (4.6), we get

$$\lim_{m\to\infty}\lim_{n\to\infty}\Big(\int_{Q_T} a_n(D\tilde{u}_{m,n}):D(w_n\varphi)\,\mathrm{d}x\mathrm{d}t+\int_{Q_T} F_n(\tilde{u}_{m,n})w_n\varphi\,\mathrm{d}\mu_n\mathrm{d}t\Big) = \int_{Q_T} a(Du):D(w\varphi)\,\mathrm{d}x\mathrm{d}t+\int_{Q_T} F(u)w\varphi\,\mathrm{d}\mu\mathrm{d}t.$$

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Thus we deduce that

$$\int_{Q_T} \langle \partial_t u, w\varphi \rangle \, dt + \int_{Q_T} a(Du) : D(w\varphi) \, dx dt + \int_{Q_T} F(u) w\varphi \, d\mu dt = \int_0^T \langle f, w\varphi \rangle$$

for every $\varphi \in \mathcal{D}(Q_T)^M$. In order to show that u is the solution of (4.1), since the space $\{w\varphi : \varphi \in \mathcal{D}(Q_T)^M\}$ is dense in $L^p(0,T,W_0^{1,p}(\Omega)^M \cap L^p_\mu(\Omega)^M)$, it only remains to prove the condition $u(x,0) = 0$ in Ω. To this end, we take u_n as a test function in (1.5) in Q_s for $s > 0$. We get

$$\frac{1}{2} \int_\Omega |u_n(s)|^2 \, dx = -\int_0^s \int_\Omega a(Du_n(t)) : Du_n(t) \, dx dt - \int_0^s \int_\Omega F(u_n(t)) u_n(t) \, d\mu_n dt + \int_0^s \langle f_n(t), u_n(t) \rangle \, dt.$$

Integrating this inequality in s between 0 and ε, we get

$$\begin{aligned} \frac{1}{2} \int_0^\varepsilon \int_\Omega |u_n(s)|^2 \, dx ds = &-\int_0^\varepsilon \int_\Omega a(Du_n(t)) : Du_n(t)(\varepsilon - t) \, dx dt \\ &- \int_0^\varepsilon \int_\Omega F(u_n(t)) u_n(t)(\varepsilon - t) \, d\mu_n dt \\ &+ \int_0^\varepsilon \langle f_n(t), u_n(t) \rangle (\varepsilon - t) \, dt. \end{aligned} \tag{4.28}$$

By step 5, it is easy to check that

$$\begin{aligned} \lim_{n\to\infty} \bigg(\int_0^\varepsilon \int_\Omega a_n\Big(Du_n(t)\Big) &: Du_n(t)(\varepsilon - t) \, dx dt \\ &+ \int_0^\varepsilon \int_\Omega F_n(u_n(t)) u_n(t)(\varepsilon - t) \, d\mu_n dt \bigg) \\ = \lim_{m\to\infty} \lim_{n\to\infty} \bigg(\int_0^\varepsilon \int_\Omega a_n\Big(D\tilde{u}_{m,n}(t)\Big) &: D\tilde{u}_{m,n}(t)(\varepsilon - t) \, dx dt \\ &+ \int_0^\varepsilon \int_\Omega F_n(\tilde{u}_{m,n}(t)) \tilde{u}_{m,n}(t)(\varepsilon - t) \, d\mu_n dt \bigg) \\ = \int_0^\varepsilon \int_\Omega a(Du(t)) : Du(t)(\varepsilon - t) \, dx dt &+ \int_0^\varepsilon \int_\Omega F(u(t)) u(t)(\varepsilon - t) \, d\mu dt. \end{aligned}$$

So we can pass to the limit in (4.28) to conclude that

$$\begin{aligned} \frac{1}{\varepsilon} \int_0^\varepsilon \int_\Omega |u(s)|^2 \, dx ds \leqslant &\liminf_{n\to\infty} \int_0^\varepsilon \int_\Omega |u_n(s)|^2 \, dx ds \\ &= -\frac{1}{\varepsilon} \int_0^\varepsilon \int_\Omega a(Du(t)) : Du(t)(\varepsilon - t) \, dx dt \\ &\quad - \frac{1}{\varepsilon} \int_0^\varepsilon \int_\Omega F(u(t)) u(t)(\varepsilon - t) \, d\mu dt \\ &\quad + \frac{1}{\varepsilon} \int_0^\varepsilon (\varepsilon - t) \langle f(t), u(t) \rangle \, dt, \end{aligned}$$

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which implies that

$$\int_{\Omega} |u(x,0)|^2 \, \mathrm{d}x = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^{\varepsilon} \int_{\Omega} |u(x,s)|^2 \, \mathrm{d}x \mathrm{d}s = 0.$$

This completes the proof of theorem 4.1. □

REMARK 4.2. We have also proved in theorem 4.1 that

$$Du_n \sim D\tilde{u}_{m,n} \quad \text{in } L^p(0,T;W_0^{1,p}(\Omega)^M).$$

This means that we can obtain a corrector for the parabolic problem from the corrector given in [1] for the elliptic one.

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