How to prove it (third edition) by Daniel J. Velleman, pp 458, £29.99 (paper), ISBN 978-1-10843-953-4, Cambridge University Press (2019). (Also available as hardback, and as an e-book ISBN 978-1-10833-745-8.)

It is a truth universally acknowledged that a textbook which reaches its third edition must have something in it. Or, to put the matter more formally, $\forall x\ (x \in \{\text{books}\} \wedge x \in \{\text{third editions}\} \rightarrow \neg(x = \phi))$. The book under review has a great deal in it, and aims to take the reader from the most basic ideas of logic to sophisticated proofs in elementary number theory and infinite sets including least and greatest bounds, but stopping short of limits except for one optional example. The chapter on number theory is new to this edition, and the author has included many more examples and also the idea of closure of a subset $C \subseteq A$ under a function $f : A \rightarrow A$, the existence of closures being proved with the logical apparatus developed earlier. The examples and exercises are a very strong point of this book, many examples being preceded by a commentary or discussion of tactics, and a rough outline ('scratch work') of how a solution might be found. Putative theorems are presented, sometimes with bogus proofs which the reader is invited to dissect. Hints and solutions are given for some of the exercises in a 54-page Appendix.

Here is a brief overview of the contents, with some commentary. The Introduction shows what proofs look like, mostly taking examples from elementary number theory. Chapter 1 covers deductive reasoning rules, truth tables, variables and implication. Is $2 \in \{w \mid 6 \in \{x \mid x \text{ is divisible by } w\}\}$ true or false? Chapter 2 is about quantifiers and sets, and several exercises are of the form 'turn a statement about sets into a logical statement', for example $x \in \cup_{i \in I}(A_i \backslash B_i)$ is to be re-expressed using logical symbols $\in$, $\notin$, $=$. $\neq$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\forall$ and $\exists$. Chapter 3 introduces methods of proof, including contrapositives, proofs of negative statements, uniqueness proofs and the use of contradiction. Statements being proved are mostly from set theory or number theory. The chapter is full of good advice as well as formal proof-writing: Leave a proof out for a day or two so you can admire it; set out your givens and goals before starting a proof; proofs should contain words as well as symbols; when introducing variables be explicit about what they mean; scratch work should indicate your thought processes but the final writeup avoids the use of logical symbols. Chapter 4 on relations (subsets of a cartesian product of sets) affords plenty of scope for false or almost-true results which need careful thought: $A \times B = B \times A \Leftrightarrow A = B$? (not quite); $A \times B \subseteq C \times D \Rightarrow A \subseteq C$ and $B \subseteq D$? (no, despite a 'proof' being given). Equivalence relations and partitions conclude this chapter. Chapter 5 on functions doesn't strictly need all of the preceding work but covers one-to-one (injective), onto (surjective) functions, domain and range, inverses, images and inverse images. Chapter 6 on induction has many interesting and tricky examples, and introduces the arithmetic-geometric mean inequality (with general proof as a guided exercise), the rearrangement inequality, and examples such as $n \geqslant 2k^2 \Rightarrow n! \geqslant k^n$ where the base case $n = 2k^2$ needs its own induction. Strong induction is used for basic theorems of number theory such as division and prime factorization; Fibonacci numbers and well-ordering are explored. How about $a, b > 0 \Rightarrow a^2 + 2b^2, 2a^2 + b^2$ can't both be squares, using simple properties of divisibility by 3?

A danger of having a chapter (7) on number theory starting at page 324 is that it might suggest all the previous material is needed to understand this very concrete and attractive topic. Also, although equivalence classes are an important concept, needed for example with quotient groups, I have never seen the merit in introducing them for the purpose of understanding modular arithmetic. Thus for example $12x \equiv 7 \pmod{25}$ becomes $[12]_{25}[x]_{25} = [7]_{25}$ which takes longer to write and

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manipulate. The set of equivalence classes of integers mod m is denoted, logically but rather messily, by $Z/\equiv_m$. That said, the chapter has the expected ingredients, including public key cryptography, and contains many interesting and challenging exercises (for example showing in several steps that if a is a number of at most 100 decimal digits then gcd (a, b) by the Euclidean algorithm takes at most 479 divisions). Chapter 8 on infinite sets is quite ambitions and covers the inclusion-exclusion formula and the uncountability of the power set of the positive integers, leading to the same for the real numbers, and the Cantor-Schröder-Bernstein theorem.

Altogether this is an ambitious and largely very successful introduction to the writing of good proofs, laced with many good examples and exercises, and with a pleasantly informal style to make the material attractive and less daunting than the length of the book might suggest. I particularly liked the many discussions of fallacious or incomplete proofs, and the associated challenges to readers to untangle the errors in proofs and to decide for themselves whether a result is true.

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Probability and random processes (4th edition) by Geoffrey R. Grimmett and David R. Stirzaker, pp. 669, £40 (paper), ISBN 978-019884759, Oxford University Press (2020)

One thousand exercises in probability (3rd edition) by Geoffrey R. Grimmett and David R. Stirzaker, pp. 580, £29.99 (paper), ISBN 978-0198847618, Oxford University Press (2020)

This pair of books, first published in 1982, has become a classic, well worthy of comparison with those by Feller [1] and Ross [2]. The third edition of *Probability and random processes* (*PRP*) and the then new *One thousand exercises in probability* were reviewed by David Applebaum in March 2002; for those who don't have ready access to the earlier review I shall summarise David's comments here. *PRP* offers a full account of virtually every aspect of the subject that is likely to feature in an undergraduate course, and plenty more. It goes from axioms (including an early discussion of sigma algebras that provides motivation missing in some comparable texts) to generating functions and the key limit theorems, with 'a healthy minimum' of measure theoretic technicalities. There follow chapters on Markov chains (in both discrete and continuous time), convergence and martingales, random processes, queuing theory, martingales and diffusion processes. The authors tell us that for the new edition the section on Markov chains in continuous time has been revised extensively, and sections have been added on coupling from the past, Lévy processes, self-similarity and stability, and time changes. Unlike Feller, *PRP* covers both discrete and continuous distributions in a single volume.

Features of *PRP* include brief but helpful motivational introductions to each subsection, and copious references to historical applications. To aid navigation, definitions, theorems and other key results are highlighted, using three different colours. The tone throughout is rigorous but the touch is human; for instance, one question concerns the lengths of tails of a troupe of chimeras, while the introduction to queuing theory, which '[draws] strongly from our intuitions' adds as a footnote 'and frustrations, including listening to the wrong types of music on a telephone'. There are helpful teaching points, such as, concerning the integral for the characteristic function of the exponential distribution, 'Do not fall into the trap of