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On the global bifurcation diagram for the one-dimensional Ginzburg–Landau model of superconductivity

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Some new global results are given about solutions to the boundary value problem for the Euler–Lagrange equations for the Ginzburg–Landau model of a one-dimensional superconductor. The main advance is a proof that in some parameter range there is a branch of asymmetric solutions connecting the branch of symmetric solutions to the normal state. Also, simplified proofs are presented for some local bifurcation results of Bolley and Helffer. These proofs require no detailed asymptotics for solution of the linear equations. Finally, an error in Odeh's work on this problem is discussed.

1 Introduction

In 1950 Ginzburg & Landau [16] proposed a model for the electromagnetic properties of a film of superconducting material of width $2d$ subjected to a tangential external magnetic field. Under the assumption that all quantities are functions only of the transverse coordinate, they proposed that the electromagnetic properties of the superconducting material are described by a pair (ϕ, a) which minimizes the free energy functional

$$G = \frac{1}{2d}\int_{-d}^{d}(\phi^2(\phi^2-2) + \frac{2(\phi')^2}{\kappa^2} + 2\phi^2a^2 + 2(a'-h)^2)dx.$$

The functional G is now known as the Ginzburg–Landau energy, and provides a measure of the difference between normal and superconducting states of the material. The variable ϕ is the 'order parameter', which measures the density of superconducting electrons, and a is the magnetic field potential. Also, h is the external magnetic field, and κ is the dimensionless constant distinguishing different superconductors. So-called 'type I' superconductors have $0 < \kappa < \frac{1}{\sqrt{2}}$, with $\kappa > \frac{1}{\sqrt{2}}$ for type II superconductors. (However, this is really only valid for large d; see Aftalion & Troy [2].)

The existence of minimizers for the functional G is proved in a standard way, and such minimizers satisfy the following Euler–Lagrange boundary value problem:

$$\phi'' = \kappa^2\phi(\phi^2 + a^2 - 1), \tag{1.1}$$

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$$a'' = \phi^2 a, \tag{1.2}$$

with boundary conditions

$$\phi'(\pm d) = 0, \quad a'(\pm d) = h. \tag{1.3}$$

It is not hard to show that the solutions of physical interest are such that $\phi > 0$ on $[-d, d]$, and this is the only kind of solution we will consider in this paper. Also, $h > 0$. Our goal is to determine for what values of h, d and κ the problem has solutions, and how many solutions there are in various parameter ranges.

There are two kinds of solutions of interest, so-called *symmetric solutions*, where $\phi(x)$ is an even function of x while $a(x)$ is odd, and asymmetric solutions, where these conditions are not satisfied. There is a family of trivial solutions, called 'normal states', of the form

$$\phi(x) = 0, \quad a(x) = h(x + c),$$

which are obviously symmetric when $c = 0$ and asymmetric otherwise. In an early paper, Odeh [20] studied when non-trivial solutions may bifurcate off these normal solutions. He concluded that symmetric solutions did bifurcate from the branch of normal solutions, but as we shall see just before Lemma 2, his argument had a flaw. He also considered whether asymmetric solutions could bifurcate off the normal state, but reached no definitive conclusion.

Subsequently, Bolley & Helffer wrote a series of papers on the problem [8–12] (see also references cited in [11]). They gave a quite thorough treatment of the local bifurcations which can occur from the normal state [11, 12]. Among many results, they gave the correct formulation of when and how symmetric solutions bifurcate from the normal state, and they did not make the error made by Odeh, though they appear not to have noticed the discrepancy with his assertions. However, some of their proofs are complicated, so we will give some simplifications. The proofs below are self-contained, and in particular, we note that at least for the results considered below, it is not necessary to use detailed asymptotics for the parabolic cylinder functions which solve the relevant bifurcation equation.

A problem of particular physical interest is whether, as the strength of the magnetic field is lowered, asymmetric solutions bifurcate from the normal state before the symmetric solutions. This problem is discussed by Boeck & Chapman [7] and Aftalion & Troy [2]. They relate this question to the formation of vortices in the medium, a phenomenon that cannot be seen in the one-dimensional model. According to these authors, if d is neither too small nor too large, and if the asymmetric solutions bifurcate first, then interference between the two symmetrically placed solutions at either edge of the slab can produce a row of vortices down the centre of the slab. The only rigorous result on this problem is by Bolley and Helffer, who show that when d is sufficiently large, it is indeed the case that $h_a > h_s$. This is one of the results for which we give a simpler proof below. A related result in two dimensions with radial symmetry appears in Bauman *et al.* [6].

After formulating our results, we received a new paper by Aftalion & Troy [2], who did a thorough numerical study of how the bifurcation curves change with κ and d. Based on these computations, they make a variety of conjectures, some of which are related to our work. They conjecture in particular that $h_a > h_s$ for any (κ, d) such that asymmetric solutions exist. (This has not been proved, though it is well accepted by physicists [2].)

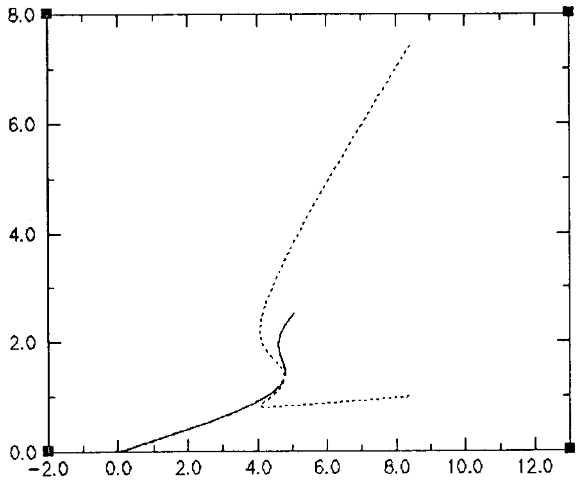

FIGURE 1. The horizontal axis is h and the vertical axis is $a(d)$. The solid curve is the branch of symmetric solutions while the dotted curve is the branch of asymmetric solutions. The end points of these curves, other than $(0, 0)$, are bifurcations from a normal state.

Subsequently, Aftalion and Chapman used methods of matched asymptotic expansions to study some of the phenomena found by Aftalion & Troy [4, 5].

The first rigorous study of the global bifurcation diagram for symmetric solutions was by Kwong [19]. He proved that for any (κ, d) there is a unique curve of symmetric solutions which can be given in the form $h = h(\phi(0))$ for $0 < \phi(0) < 1$. This curve is smooth, and $h(1) = 0$, $h(0) = h_s$. Hastings, Kwong and Troy studied the nature of this curve for large d, showing that it has at least one minimum, followed by at least one maximum, if $\kappa > \sqrt{1/2}$. This implies that, for some values of h, there will be at least three solutions of the boundary value problem in this range of κ and d. They also showed that for any fixed $\kappa \in (0, \frac{1}{\sqrt{2}})$, if d is sufficiently large then for some range of h there will be at least two solutions. More recently, Aftalion & Troy [3] proved that for sufficiently small κd, there is only one symmetric solution, and there are no asymmetric solutions. (Numerically, it appears that asymmetric solutions begin when κd reaches approximately 0.905 [7, 2].)

Up to now, very little has been done concerning the global structure of asymmetric solutions (in the parameters κ, d, h) or of bifurcations away from the normal states. Some initial conjectures were made by Aftalion [1]. However, a numerical study by Seydel [23] shows that the picture can be quite complicated. He considers only a single configuration, namely $d = 2.5$, $\kappa = 1$, and presents essentially the graph in Figure 1, in which h is plotted against the value of a at the right-hand end of the interval $[-d, d]$. (Seydel uses $a(-d)$ instead of $a(d)$. There are a number of possible 'bifurcation curves' which one can draw for this problem. For example, we could plot h vs. $\phi(0)$, as was done for symmetric solutions by Hastings *et al.* [18]. We elect here to follow Seydel and plot $a(d)$ vs. h. Either kind of curve gives the important information of how many solutions there are for a given h.)

Among the features we see here are the existence of up to seven solutions for a given h, and the bifurcation of asymmetric solutions from the symmetric branch. It must be remembered, though, that asymmetric solutions occur in pairs, and modulo a symmetric reflection, Seydel finds up to two asymmetric solutions and three symmetric solutions for fixed h.

There are two obvious questions to ask concerning the Seydel result. How does the picture change as d and κ vary, and what are the stability and minimization properties of these solutions?

As stated above, the first of these questions was studied thoroughly by Aftalion & Troy [2]. From their bifurcation diagrams, and results in Bolley & Helffer [12], one can infer results about local stability near bifurcation points. Global minimization was studied by Hastings & Troy [17]. In addition to demonstrating the existence of asymmetric solutions for large d, they showed that in some parameter range there are asymmetric solutions, but no non-trivial symmetric solutions. They also showed that the energy of the asymmetric solutions can be negative, so that a global minimizer of the Ginzburg–Landau functional G must be asymmetric.

We have done some numerical work to consider the robustness of an asymmetric global minimizer as we move into the region where both symmetric and asymmetric solutions exist. More precisely, choosing the 'Seydel' values $\kappa = 1, d = 2.5$, so that asymmetric solutions exist, we started with $h = h_a$, the asymmetric bifurcation point. As h is lowered, initially there are only asymmetric solutions, and at points along the branch of asymmetric solutions we evaluated the Ginzburg–Landau functional G and found it to be negative, so that the solutions must be global minimizers. (This is in accord with the result of Hastings and Troy.) Lowering h further, we reached the region where there are both symmetric and asymmetric solutions. At a decreasing sequence of h values, we evaluated G at each of the solutions existing for these values of h; up to five distinct solutions. We found for a considerable distance down the original curve of asymmetric solutions that G takes its minimum on this curve. Thus, the minimization property of this asymmetric branch appears to be very robust.

However, this was a relatively crude examination, and by no means a thorough study of the (κ, d) parameter space. In this paper, we consider only the existence of solutions of (1.1)–(1.3), and not the stability properties of these solutions.

We used the program *Auto* [15] to produce many bifurcation diagrams for different parameter pairs (κ, d). (Aftalion and Troy have a much more extensive study, also using Auto.) Figure 2 gives some samples.

Referring to the graphs of $a(d)$ vs. h, there are at least the following possibilities, not all of which are shown above:

(1) A single-valued curve of symmetric solutions in the $(h, a(d))$ plane, and no asymmetric solutions.

(2) A single-valued curve of symmetric solutions, from which bifurcates a C-shaped curve of asymmetric solutions.

(3) A C-shaped curve of symmetric solutions, no asymmetric solutions.

(4) A C-shaped curve of symmetric solutions and a C-shaped curve of asymmetric solutions.

(5) A C-shaped curve of symmetric solutions, from which there bifurcates a W-shaped curve of asymmetric solutions.

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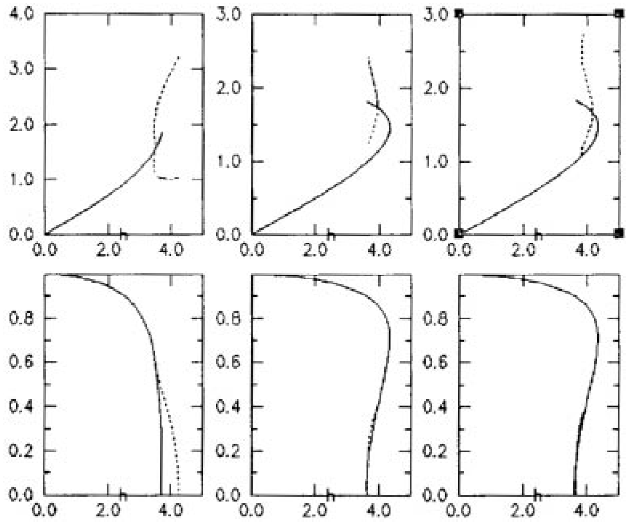

FIGURE 2. Sample bifurcation diagrams, showing $a(d)$ vs. h (top row) and $\max\phi(x)$ vs. h (bottom row), for the parameter pairs $(\kappa, d) = (1.25, 1), (2, 0.5)$ and $(2, 0.55)$, left to right. Solid curves are symmetric branches and dashed curves are asymmetric branches. Between the second and third examples there is a transition from a 'C-shaped symmetric branch, C-shaped asymmetric branch' to a 'C-shaped symmetric branch, W shaped asymmetric branch', referring to the plots of $a(d)$ vs. h. In the top row the upper and lower corresponding parts of the asymmetric branches refer to pairs of solutions $(\phi(x), a(x))$ and $(\phi(-x), -a(-x))$. These show as one curve in the bottom row. The lower diagrams show how both branches end at a bifurcation from a normal state, since $\max\phi$ tends to zero.

(6) An S-shaped curve of symmetric solutions from which there bifurcates a W-shaped curve of asymmetric solutions. (This is the type found by Seydel.)

We also see that the asymmetric curves can bifurcate from various parts of the C- or S-shaped symmetric curves. Some of these features, and others, are discussed in more detail by Aftalion & Troy [2].

2 Statement of results

First we consider bifurcations from the normal state. Since the normal state has $\phi = 0$, we rescale by letting

$$\phi = \alpha\psi,$$

where $\alpha = \phi(-d)$ and $\psi(-d) = 1$. Then,

$$\psi'' = \kappa^2(\alpha^2\psi^2 + a^2 - 1), \tag{2.1}$$

$$a'' = \alpha^2\psi^2 a, \tag{2.2}$$

$$\psi(-d) = 1, \psi'(\pm d) = 0, \quad a'(\pm d) = h. \tag{2.3}$$

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For $\alpha = 0$ there is a family of solutions $(\psi_0, h(x+c))$ where $h = h(c)$ is chosen so that the linear problem

$$\psi_0'' = \kappa^2(h^2(x+c)^2 - 1)\psi_0 \tag{2.4}$$

$$\psi_0(-d) = 1, \ \ \psi_0'(\pm d) = 0 \tag{2.5}$$

has a unique positive solution ψ_0. From standard linear theory [13], we have

Lemma 1 *For each c there is a unique $h = h(c) > 0$ such that* (2.4)–(2.5) *has a positive solution. When it exists, this solution is unique.*

Thus (2.1)–(2.3) is degenerate at $\alpha = 0$, in the sense that there is a continuum of solutions, because c is arbitrary.

To remove this degeneracy we reformulate the problem. Consider equations (2.1)–(2.2) with initial conditions $\psi(-d) = 1, \psi'(-d) = 0, a(-d) = h(c-d)$, and $a'(-d) = h$. Then consider $\psi'(d)$ and $a'(d)$ as functions of (α, c, h). We wish to solve the equations

$$\psi'(d) = 0, a'(d) - h = 0 \tag{2.6}$$

for (c, h) as functions of α. Since

$$a'(d) - h = \int_{-d}^{d} \alpha^2 \psi(x)^2 a(x) dx$$

we replace (2.6) with

$$\psi'(d) = 0, \int_{-d}^{d} \psi(x)^2 a(x) dx = 0. \tag{2.7}$$

We get a solution at $\alpha = 0$ by setting $h = h(c)$ as given in Lemma 1 and looking for values of c such that

$$I(c) := \int_{-d}^{d} (x+c)\psi_0^2 dx = 0. \tag{2.8}$$

Suppose that (2.8) is satisfied for some $c = c_1$. This gives a solution to (2.7) for $\alpha = 0$, and this solution can be extended to $\alpha > 0$ provided that at $(\alpha, c, h) = (0, c_1, h(c_1)$ the determinant

$$\det \begin{pmatrix} \dfrac{\partial \psi'(d)}{\partial h} & \dfrac{\partial \psi'(d)}{\partial c} \\ \dfrac{\partial I}{\partial h} & I'(c) \end{pmatrix}$$

is nonzero. We will see later (equation (3.6)) that $I(c) = 0$ implies that $(\partial \psi'(d))/(\partial c) = 0$, and standard theory implies that $(\partial \psi'(d))/(\partial h) \neq 0$. Hence, a unique branch of solutions bifurcates from $(0, c_1, h(c_1))$ provided that $I'(c_1) \neq 0$.

For $c = 0$, $a = hx$ is an odd function, and this implies that ψ_0 is even. Thus, it is automatic that $I(0) = 0$. Further, for $\alpha > 0$ we can consider instead of (2.1)–(2.3) the problem for symmetric solutions. This is (2.1)–(2.2) on $[0, d]$ with $\psi'(0) = \psi'(d) = 0$, $a(0) = 0$, $a'(d) = h$. Kwong's result [19] shows that this has a unique solution for each

$\alpha \in (0,1)$, defining h as a function of α. If $I'(0) \neq 0$, then this is also the unique solution to (2.1)–(2.3). Hence, a unique solution bifurcates from $(0,0,h(0))$ when $I'(0) \neq 0$. However, this condition depends upon d. When it is not satisfied we expect a more complicated picture.

Odeh [20] claimed that $I'(0)$ is always positive (for any d). He may have thought this was so because he thought that $(\partial\psi_0)/(\partial c)$ was an even function of x when $c = 0$, but in fact, this function is neither even nor odd. We have the following result, also proved by Bolley [9], but with a much longer proof:

Lemma 2 *Suppose that for each positive κ, and d, and each c, h is chosen as in Lemma 1. Then for sufficiently small κd, $I'(0) > 0$, while for sufficiently large κd, $I'(0) < 0$.*

The (relatively short) proof will be given in § 3. As a consequence, we have

Theorem 3 *For any (κ, d) there is a bifurcation of symmetric solutions from the normal state. For sufficiently large κd and for sufficiently small κd, a unique curve of symmetric solutions bifurcates from the normal state. In other words, for sufficiently small α, and for some $\delta > 0$, (2.1)–(2.3) has a unique solution with $|h - h(0)| < \delta$, and this solution is symmetric.*

Further, suppose that (κ_1, d_1) and (κ_2, d_2) are such that $I'(0) < 0$ if $(\kappa, d) = (\kappa_1, d_1)$ and $I'(0) > 0$ if $(\kappa, d) = (\kappa_2, d_2)$, and assume that $(\kappa(t), d(t))$ is a real analytic curve C joining (κ_1, d_1) and (κ_2, d_2), with $\kappa(0) = \kappa_1$, $d(0) = d_1$ and $\kappa(1) = \kappa_2$ and $d(1) = d_2$. Then there exists a $t_0 \in (0,1)$ and asymmetric solutions (which are nearly symmetric) arbitrarily close to $(0, \tilde{h}_0 x)$ with h near $\tilde{h}_0$, κ near $\kappa(t_0)$ and d near $d(t_0)$. Here $\tilde{h}_0$ is the eigenvalue found in Lemma 1 for $(\kappa, d) = (\kappa(t_0), d(t_0))$ and $c = 0$.

Remark 4 *Note that, unlike Bolley & Helffer* [9]*, we do not need a transversality assumption, and with care we could avoid assuming that the curve is analytic. Note also that Lemma 2 ensures that suitable points (κ_1, d_1) and (κ_2, d_2) exist.*

The asymmetric solutions obtained in Theorem 3 are, at least initially, nearly symmetric, since they start from $c = 0$. A different sort of bifurcation of asymmetric solutions was obtained by Bolley & Helffer [11], and independently, with a different proof, by Hastings & Troy [17]. In this case we consider a fixed large κd, and vary c, looking for other values of c where $I(c) = 0$. The symmetry in the problem means we only have to consider positive c. It is obvious that for $c \geqslant d$, $I(c) > 0$, so using Lemma 2 we have:

Corollary 5 *For sufficiently large κd, there is at least one $c > 0$ where $I(c) = 0$.*

In fact, there is only one such c and asymmetric bifurcation occurs at this point. Thus, at this positive c where $I(c) = 0$, we have $h = h_a$. The uniqueness of this positive c was initially shown by Bolley & Helffer [11], but here we will give a simpler, self-contained, proof.

Theorem 6 *For sufficiently large κd there is exactly one $c = c_1 > 0$ such that $I(c_1) = 0$. (By symmetry there is also one negative c with this property.) Further, $h(c_1) > h(0)$.*

Finally, it is not hard to show that bifurcation does not occur for small κd [11].

Now we turn to more global results. The goal now is to show that bifurcation of asymmetric solutions can occur from the interior of the symmetric branch, rather than just from normal states, and show that the resulting branch of asymmetric solutions can be continued in the $(h, a(-d))$ plane (d large and fixed) to the asymmetric bifurcation point which was found in Theorem 6. To state this result, we must first recall a result of Kwong [19]. This result is about symmetric solutions, and concerns the global bifurcation curve of symmetric solutions, for any fixed d and κ. Since we are considering only symmetric solutions, we consider (1) with the following *initial* conditions:

$$\phi(0) = \alpha, \;\; \phi'(0) = 0, \;\; a(0) = 0, \;\; a'(0) = \delta, \tag{2.9}$$

where $\alpha \in (0, 1)$ and $\delta > 0$ are to be chosen such that $\phi > 0$ on $[0, d]$ and $\phi'(d) = 0$. By continuing the resulting solution with ϕ even and a odd to the entire interval $[-d, d]$, we get a symmetric solution to (1.1, 1.2). Kwong's result is:

Lemma 7 *For each $\alpha \in (0, 1]$ there is a unique $\delta > 0$ such that the solution of* (1.1, 2.8) *is positive and satisfies $\phi'(d) = 0$.*

Hence, for each α we obtain a unique $h = a'(d)$ and a unique $a(d)$. Plotting $a(d)$ vs h gives the global bifurcation curve for symmetric solutions in the form referred to above. An alternative form as used by Hastings *et al.* [18] is to plot h vs. α. The main new result of this paper is:

Theorem 8 *If the product κd is sufficiently large, then bifurcation of asymmetric solutions occurs somewhere along the curve of symmetric solutions. For any fixed κ, if d is sufficiently large there is a continuum of asymmetric solutions which connects the curve of symmetric solutions to an asymmetric normal state.*

Remark 9 *Here we are identifying points in the $(a(d), h)$ bifurcation diagram corresponding to asymmetric pairs of solutions. The asymmetric normal state referred to in this theorem must be that discussed in Corollary 5 and Theorem 6, since to within a reflection there is only one asymmetric bifurcation point from the normal state. By real analyticity the continuum in this theorem contains a curve which is parametrized (in some sense) by h.*

Remark 10 *Also, it is expected that for $\kappa > 1/\sqrt{2}$, if d is sufficiently large then the curve of symmetric solutions is S-shaped, so that for some values of h there are three solutions. In Hastings et al.* [18] *it was shown that there are* at least *three solutions, which is consistent with this conjecture. Theorem 8 shows that bifurcation must occur somewhere along this curve, but we are not able to prove anything about the location of the bifurcation point on this curve. Similarly, for $\kappa < 1/\sqrt{2}$ there are at least two solutions for large d, and the bifurcation seems to occur on either of the two branches.*

Remark 11 *A problem which we have not been able to solve is to determine the direction of the bifurcation. As a result, we have not been able to prove that there are some values of the parameters (d, κ, h) where there are five distinct solutions, as was seen in Seydel's original* *numerical result. In Bolley & Helffer* [11] *there is a discussion of the stability of solutions bifurcating from the normal states. This involves consideration of the energy functional G, and we have not studied this topic here.*

3 Proofs

3.1 Local results

In this section we denote ψ_0 by ψ, since we are only considering the linear equations. Thus, ψ is assumed to satisfy (7)–(8).

Proof of Lemma 2

In (2.4)–(2.5) we can make the change of variables $x \to \kappa x$, which removes the term κ^2 from the differential equation, while replacing d with $D = \kappa d$, and also introducing a new h and c. The latter changes are immaterial in this and subsequent results about bifurcation from the normal state, so without loss of generality we will simply assume in (2.4)–(2.5) that $\kappa = 1$, and to remind us of this, replace d with D. Multiply (2.4) by ψ' and integrate by parts to get

$$2h^2 I(c) = \left(h^2(D+c)^2 - 1\right)\psi(D)^2 - \left(h^2(-D+c)^2 - 1\right)\psi(-D)^2. \tag{3.1}$$

Consider (2.4) with the initial conditions $\psi(-D) = 1, \psi'(-D) = 0$, and denote the solution by $\psi(x,h,c)$. Let $p = \partial\psi/\partial h$ and $q = \partial\psi/\partial c$. Then

$$p'' = \left(h^2(x+c)^2 - 1\right)p + 2h(x+c)^2\psi, \quad p(-D) = p'(-D) = 0, \tag{3.2}$$

and

$$q'' = \left(h^2(x+c)^2 - 1\right)q + 2h^2(x+c)\psi, \quad q(-D) = q'(-D) = 0. \tag{3.3}$$

Lemma 1 tells us that (2.4)–(2.5) define h as a function of c. We can also see this locally by applying the implicit function theorem, solving the equation

$$\psi'(D,h,c) = 0$$

for h as a function of c. We can do this if $p'(D) \neq 0$. Multiply (3.2) by ψ and (2.4) by p and integrate from $-D$ to D. With the boundary and initial conditions for ψ and q, this gives

$$\psi(D)p'(D) = 2h\int_{-D}^{D}(x+c)^2\psi(x)dx > 0. \tag{3.4}$$

Hence h is a smooth function of c. Further,

$$\frac{dh}{dc} = -\frac{q'(D)}{p'(D)}. \tag{3.5}$$

Now multiply (3.3) by ψ, (2.4) by q, subtract and integrate, and use the boundary conditions again to get

$$\psi(D)q'(D) = 2h^2\int_{-D}^{D}(x+c)\psi(x)^2dx = 0. \tag{3.6}$$

This shows that $dh/dc = 0$ whenever $I = 0$. In particular, this is true at $c = 0$.

Now differentiate (3.1) with respect to c and then set $c = 0$. Since then $\psi(\pm D) = 1$, we get

$$h^2 I'(0) = 2h^2 D + (h^2D^2 - 1)q(D). \tag{3.7}$$

From (2.4) and (3.2) the equation obtained at $c = 0$ is

$$\left(\frac{q}{\psi}\right)' = \frac{\int_{-D}^{x} 2h^2 s\psi(s)^2 ds}{\psi(x)^2}. \tag{3.8}$$

Since ψ is an even function at $c = 0$, the right side of (3.8) is zero when $x = D$. Further, the integrand is negative for $s < 0$ and positive for $s > 0$, and this implies that the integral is strictly negative for $-D < x < D$.

Now return to (2.4). We see that $\psi'' \geqslant -\psi$, and since $\psi'(0) = 0$, this leads to $\psi(x) \geqslant \psi(0)\cos x$ on $[0, D]$ for small D. It can be seen that as $D \to 0$, $\max_{x\in[0,D]} |\psi(x) - 1| \to 0$. Integrating the right side of the differential equation for ψ then shows that $\lim_{D\to 0} hD = \sqrt{3}$, so $h \to \infty$. From (3.8), it follows that $\left(\frac{q}{\psi}\right)' = O(h)$ as $D \to 0$, so $q(D) = O(hD) = O(1)$. Hence from (3.7), we see that for sufficiently small D, $I'(0) > 0$.

To complete the proof of Lemma 2, there remains to show that $I'(0) < 0$ when D is sufficiently large. We first need to show that h is bounded as $D \to \infty$. In fact, it is well known [17] that $h \to 1$ as $D \to \infty$, but the following technique quickly shows that h is at least bounded: Let $\rho = \psi'/\psi$. Then

$$\rho' = h^2x^2 - 1 - \rho^2, \rho(0) = \rho(D) = 0, \tag{3.9}$$

with $\rho < 0$ in $(0, D)$. (Remember that we are only considering $c = 0$ here.) It is easy to see from (3.9) that if $h \to \infty$, then $D \to 0$. Hence, as $D \to \infty$, h must remain bounded.

It is also well known that $h > 1$, but for completeness here is a quick proof: Compare ρ from (3.9) with the solution $\sigma = -x$ of

$$\sigma' = x^2 - 1 - \sigma^2, \quad \sigma(0) = 0.$$

An easy comparison shows that if $h \leqslant 1$ then $\rho(0) = \sigma(0) = 0$ implies that $\rho \leqslant \sigma$ for all $x \geqslant 0$, which contradicts $\rho(D) = 0$.

Lemma 2 now follows from (3.7) if we can show that $q(D)$ does not tend to zero as D tends to infinity. To do this we again use (1.2) and (2.4). Multiply (2.4) by ψ' and integrate from $-D$ to x. This, with (2.5) and (3.8), leads to

$$\left(\frac{q}{\psi}\right)' \leqslant (h^2x^2 - 1).$$

Since h is bounded for large D, there is some interval around $x = 0$ of fixed length $\mu > 0$ in which $\left(\frac{q}{\psi}\right)' \leqslant -\frac{1}{2}$. In addition, q/ψ is decreasing on the entire interval $[-D, D]$, so $(q(D))/(\psi(D)) \leqslant -\frac{1}{2}\mu$. But $\psi(D) = 1$ when $c = 0$. Then (3.7) shows that $I'(0) < 0$ for large D. This proves Lemma 2.

Proof of Theorem 3

Note that $I'(0)$ is a function of κ and d. (We are no longer assuming that $\kappa = 1$.) In fact, it is a real analytic function of κ and d, which is seen by observing that one can use the implicit function theorem to prove that ψ is a real analytic function of (κ, d) [14]. Thus,

$I'(0)$ is a real analytic function of t along the curve C, since C is real analytic. Since $I'(0)$ does not vanish identically, being nonzero at the endpoints of C, its zeros are isolated. Therefore, there is a $t_0 \in (0, 1)$ such that $I'(0)$ has a strict change of sign as we cross $(\kappa(t_0), d(t_0))$ on C.

We now look for solutions $\phi = \alpha(\psi_0 + w)$, $a = h(x + c) + \rho$, where ψ_0 is the eigenfunction at $t = t_0$, h is close to h_0 and the numbers α and c and functions w and ρ are small, and where w is orthogonal to ψ_0 and ρ is orthogonal to 1 over $(-d, d)$. Equation (1.1) can be written as

$$-(\psi_0 + w)'' = \kappa^2(\psi_0 + w)(1 - a^2 - \alpha^2(\psi_0 + w)^2). \tag{3.10}$$

We now use Fredholm alternative theory and the implicit function theorem in a standard way [22, Chap. 9] to show that for every (α, c, h) with α, c, and $(h - h_0)$ sufficiently small, there is a unique pair (β, γ) so that the equations

$$-(\psi_0 + w)'' = \kappa^2(\psi_0 + w)(1 - (h(x + c) + \rho)^2 - \alpha^2(\psi_0 + w)^2) + \beta\psi_0 \tag{3.11}$$

$$\rho'' = \alpha^2(\psi_0 + w)^2(h(x + c) + \rho) + \gamma \tag{3.12}$$

have a unique solution (w, ρ) such that $w'(\pm d) = \rho'(\pm d) = 0$. In other words, we subtract suitable multiples $\beta\psi_0$ of ψ_0 and $\gamma \cdot 1$ of 1 in equations (3.11)–(3.12), respectively, to find solutions (w, ρ) in the space

$$\{w \in C^1[-d, d] : w \text{ is orthogonal to } \psi_0, w'(\pm d) = 0\} \times \{\rho \in C^1[-d, d] \text{ and } \rho \text{ is orthogonal to } 1, \rho'(\pm d) = 0\} \tag{3.13}$$

We obtain w, ρ, β, γ as smooth functions of α, c, h. Here we are really solving a projected equation. This is a Lyapunov–Schmidt reduction. We want to find solutions of (3.11)–(3.12) with $\beta = \gamma = 0$.

Let $\hat{\rho}$ denote the solution of

$$\hat{\rho}'' = \psi_0^2 x, \quad \hat{\rho}'(\pm d) = 0 \tag{3.14}$$

with mean value zero. Using (1.2), it is easily shown that

$$\rho(\alpha, h, c) = h\alpha^2(\hat{\rho} + o(1))$$

as $(\alpha, h, c) \to (0, h_0, 0)$. Note here that $s\psi_0^2(s)$ has a mean value zero. Hence,

$$a^2 = (h(x + c) + \alpha^2(\hat{\rho} + o(1)))^2 = h^2(x + c)^2 + 2\alpha^2 h(x + c)\hat{\rho} + \text{terms of higher order.}$$

(That is, higher order in α.) Note that as $(\alpha, h, c) \to (0, h_0, 0)$, w and ρ tend to zero uniformly in $[-d, d]$.

Now multiply (3.11) by ψ_0 and integrate, using the orthogonality of w and ψ_0, to get

$$(h^2 - h_0^2)\left(\int_{-d}^{d} s^2\psi_0(s)^2 ds + o(1)\right) ds \;+\; \int_{-d}^{d} o(1) ds = \beta \int_{-d}^{d} \psi_0(s)^2 ds, \tag{3.15}$$

where the $o(1)$ terms are terms in w and ρ which are smooth and tend to zero as $(\alpha, h, c) \to (0, h_0, 0)$. (Also, the derivative with respect to h of the second $o(1)$ term is small if α is small, h is near h_0 and c is small.) To obtain a solution of (3.10), we set $\beta = 0$ in (3,15) and use the implicit function theorem to solve this equation for h as a function of (α, c) near $h = h_0, \alpha = c = 0$. The coefficient of $(h^2 - h_0^2)$ in the first term on the left of (3.15) is of order 1. For each small $\alpha \neq 0$ and each small c, there will be a unique $h = h(\alpha, c)$ near to h_0 for which (3.15) implies that $\beta = 0$.

Now integrate (1.2) over $[-d, d]$ and use the boundary conditions to obtain, upon dividing by $\alpha^2 h$ the equation

$$R(\alpha, h, c) \equiv \frac{1}{\alpha^2 h} \int_{-d}^{d} \phi^2 a \, dx = \int_{-d}^{d} ((x+c)\psi_0^2(x) + o(1))dx = 0,$$

where the small $o(1)$ term is a smooth function of α, h, c and is zero if $\alpha = c = h - h_0 = 0$. Note that $R(\alpha, h, c) = 0$ is equivalent to $\gamma = 0$. Also, $R(0, h_0, c) = I(c)$. Letting $R_3'(\alpha, h, c) = \partial R/\partial c$, we have for large κd, by Lemma 2, that $R_3'(0, h_0, 0) < 0$, so $R_3'(\alpha, h, c) < 0$ if α and c are small and h is close to h_0. Also, $R(\alpha, h, 0) = 0$, because then both w and ψ_0 are even functions of x and a is an odd function of x. Hence, if $I'(0) < 0$, then $R(\alpha, h, c) < 0$ if α is small, c is positive and small and h is near to h_0. Similarly, if $I'(0) > 0$, then $R(\alpha, h, c) > 0$ if α is small, c is small and positive, and h is near to h_0.

We now solve (3.15) for h as a function of c and α, for small positive α and c. (We do not know the sign of $h - h_0$.) We do this for (κ, d) on the curve C with (κ, d) close to $(\kappa(t_0), d(t_0))$. On this curve, on one side of $(\kappa(t_0), d(t_0))$, $I'(0) < 0$, and hence $R(\alpha, h(\alpha, c), c) < 0$ if α and c are small and $c > 0$. How small α and c must be depends upon the particular point on the curve C. For a given $t_1 < t_0$ but close to t_0, find positive α_1 and c_1 such that $R(\alpha, h(\alpha, c), c) < 0$ if $0 < \alpha \leqslant \alpha_1$ and $0 < c \leqslant c_1$. Choose a fixed $t_2 > t_0$, but close to t_0, and then lower α_1 and c_1 if necessary so that $R(\alpha, h(\alpha, c), c) > 0$ if $0 < \alpha \leqslant \alpha_1$ and $0 < c \leqslant c_1$. Then somewhere between t_1 and t_2, as we keep α and c nonzero and fixed in $(0, \alpha_1]$ and $(0, c_1]$ and move along C, R must equal 0, which gives the required solution. (Note that h_0 varies continuously with (κ, d), but this does not affect the argument, since everything varies continuously along C.) As we can choose c arbitrarily small, the solutions will be nearly symmetric.

Proof of Theorem 6

As previously for the linearized problem (2.4)–(2.8), we can rescale to eliminate κ, so we will again assume that $\kappa = 1$ and replace d with D. We saw in the proof of Lemma 2 that for large D, $I(0) = 0$, $I'(0) < 0$, and that consequently Corollary 5 holds. Further, the definition of $I(c)$ shows that $I(c) > 0$ for $c \geqslant D$.

From (2.4)–(2.5) with $\psi > 0$, we see that $\tau(x) = h(x+c)^2 - 1$ must change sign in $(-D, D)$, and from (13) it follows that if $I(c) = 0$, then $\tau(-D)$ and $\tau(D)$ must have the same sign, so $\tau()$ has exactly two zeros in $(-D, D)$, with $\psi''(x)$ changing from positive to negative and back to positive as x increases from $-D$ to D. Therefore, ψ has a local maximum at some $x_0 \in (-D, D)$, with $\psi' > 0$ on $(-D, x_0)$ and $\psi' < 0$ on (x_0, D).

Lemma 12 *For any* $c \in (0, D)$, $h > 1$.

Proof Let $\rho(x) = (\psi'(x))/(\psi(x))$. Then $\rho(-D) = \rho(x_0) = \rho(D) = 0$. Also,

$$\rho' = h^2(x+c)^2 - 1 - \rho^2. \tag{3.16}$$

Further, let $\sigma(x) = -x - c$. Then

$$\sigma' = (x+c)^2 - 1 - \sigma^2, \quad \sigma(-c) = 0. \tag{3.17}$$

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However, $\rho(-D) < \sigma(-D)$, and an easy comparison of (3.16) and (3.17) shows that if $h \leqslant 1$, then $\rho < \sigma$ on $[-D, D]$, so $\rho(D) < 0$, a contradiction. This proves Lemma 12. □

Lemma 13 *For sufficiently large D, (2.4)–(2.8) has no solution with $\psi > 0$ and $0 < c \leqslant D/2$.*

Proof Recall that $I(0) = 0$, $I'(0) < 0$. Let $c_1 = \inf\{c > 0 | I(c) = 0\}$. Then $I'(c_1) \geqslant 0$. Assume that $c_1 \leqslant D/2$. We will obtain a contradiction by differentiating (3.1) with respect to c and setting $c = c_1$.

Recall that $dh/dc = 0$ whenever $I(c) = 0$. Therefore, from (3.1) we get

$$\begin{aligned} 2h^2 I'(c_1) &= 2h^2(D + c_1)\psi(D)^2 \\ &\quad + 2(h^2(D + c_1)^2 - 1)\psi(D)q(D) + 2h^2(D - c_1)\psi(-D)^2 \end{aligned} \tag{3.18}$$

where $q = (\partial\psi)/(\partial c)$, so q satisfies (3.3). From (3.3) and (2.4)–(2.5), we obtain (3.8) with s replaced by $(s + c_1)$, and as in the proof of Lemma 2, we then see that $\left(\frac{q}{\psi}\right)' < 0$ in $(-D, D)$ and $\left(\frac{q}{\psi}\right)' \leqslant h^2(x + c_1)^2 - 1$. Also, as in Lemma 2, it is seen that h is bounded as $D \to \infty$, and this leads to a negative upper bound of the form

$$\left(\frac{q(D)}{\psi(D)}\right) \leqslant -\eta < 0, \tag{3.19}$$

where η is independent of D and $c_1 \in [0, D/2]$.

From $I(c_1) = 0$ and (13), we obtain

$$\psi(D)^2 = \frac{h^2(D - c_1)^2 - 1}{h^2(D + c_1)^2 - 1}\psi(-D)^2 \geqslant \frac{h^2(\frac{D}{2})^2 - 1}{2h^2D^2 - 1}\psi(-D)^2.$$

Hence, $(\psi(-D)^2/(\psi(D)^2)$ is bounded as $D \to \infty$. Then (3.19) and (3.18) show that $I'(c_1) < 0$ for sufficiently large D. This contradiction proves Lemma 13. □

Continuing with the proof of Theorem 6, we now assume that $c \geqslant D/2$. The result will follow if we can show that $I'(c_1) > 0$ for any solution of (2.4)–(2.5) in this range of c. As before, we assume that (c_1, h, ψ) is a solution, and that ψ has its maximum over $[-D, D]$ at $x_0 \in (-D, D)$. Also, as before, we know that $|x_0 + c_1| < 1$, so $x_0 \leqslant -D/2 + 1$. We translate the origin to x_0, letting $\psi(x) = \chi(x - x_0) = \chi(y)$, so that

$$\chi''(y) = (h^2(y + x_0 + c_1)^2 - 1)\chi(y).$$

Now let $\rho(y) = (\chi'(y))/(\chi(y))$ (a shift from the previous ρ), and let $\omega(y) = -\rho(-y)$. Then

$$\rho' = h^2(y + x_0 + c_1)^2 - 1 - \rho^2, \quad \rho(0) = \rho(D - x_0) = 0 \tag{3.20}$$

and

$$\omega' = h^2(y - x_0 - c_1)^2 - 1 - \omega^2, \quad w(0) = w(D + x_0) = 0. \tag{3.21}$$

It is important to recall that $x_0 < -D/2 + 1$.

We now need estimates on ρ and ω, which we obtain from the following result:

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Lemma 14 *Suppose that for some constants δ and $\varDelta$, with $h|\delta| \leqslant 1$ and $\varDelta$ large, $\eta(\cdot)$ solves the boundary value problem*

$$\eta'(y) = h^2(y+\delta)^2 - 1 - \eta^2, \ \eta(0) = \eta(\varDelta) = 0, \ \eta < 0 \text{ in } (0, \varDelta). \tag{3.22}$$

Then,

$$\eta(y) > -hy - \beta \tag{3.23}$$

on $[0, \varDelta]$, *where* $\beta = \sqrt{h - 1 + h^2\delta^2}$, *and*

$$\eta(y) < -h(y+\delta) + 1 \tag{3.24}$$

on $[0, \varDelta - 1]$.

Proof Inequality (3.23) follows by assuming equality at some $y_1 \in (0, \varDelta)$ and using (3.22) to show that $\eta'(y_1) < -h$. This would imply that $\eta(y) < -hy - \beta$ for $y > y_1$, so that η could not vanish at $\varDelta$. Next, observe that (3.24) holds as long as $h(y+\delta) \leqslant 1$, since $\eta < 0$ on $(0, \varDelta)$. If equality holds at some $y_1 > 0$, then $h(y_1+\delta) > 1$ and $\eta'(y_1) = -2 + 2h(y_1+\delta) > 0$. Also, $\eta < 0$ and $\eta' > 0$ imply that $\eta'' > 0$. Hence, $\eta' > -2 + 2h(y_1+\delta) > 0$ as long as $\eta < 0$, and if $\eta < 0$ on $[y_1, y_1+1]$, then

$$\eta(y_1+1) \geqslant \eta(y_1) - 2 + 2h(y_1+\delta) = -1 + h(y_1+\delta) > 0.$$

This contradiction shows that $\eta = 0$ somewhere in $[y_1, y_1+1]$. If $y_1 \leqslant \varDelta - 1$, then $\eta = 0$ before $y = \varDelta$, a contradiction which proves Lemma 14. □

Applying (3.24) to ρ with $\delta = x_0 + c_1$ and $\varDelta = D - x_0$ shows that

$$\psi(D) = \chi(D - x_0) \leqslant \psi(x_0) e^{-\frac{h}{2}(D+c_1-1)^2 + \frac{h}{2}(x_0+c_1)^2 + D - x_0 - 1},$$

while applying (3.23) to ω with $\delta = -x_0 - c_1$ gives

$$\psi(-D) \geqslant \psi(x_0) e^{-\frac{h}{2}(D+x_0)^2 - \beta(D+x_0)}.$$

Combining these and noticing that $2Dx_0 < -2x_0^2$ and $\beta < h$, we find that for $c_1 \geqslant D/2$,

$$\frac{\psi(D)}{\psi(-D)} \leqslant e^{-rhD^2} \tag{3.25}$$

for some r which is independent of c_1, h and D.

We now return to (3.18). Since $\psi''(-D) \geqslant 0$, we have $h^2(D - c_1) \geqslant h$. Further, the proof (following equation (3.8)) that when $c = 0$, h is bounded as $D \to \infty$ easily extends to $c \geqslant 0$, since $x_0 \leqslant -c + 1$. From (3.25), it follows that a bound of the form

$$q(D) \geqslant -Lh^m D^n \psi(D) \tag{3.26}$$

for some $L > 0$ independent of c_1, h or D will imply that $I'(c_1) > 0$ for large D. There are two cases to consider, namely, $-c_1 - 1 \leqslant x_0 \leqslant -c_1$ and $-c_1 \leqslant x_0 \leqslant -c_1 + 1$. We consider the first, the two cases being similar. Repeating the derivation of (3.8) for $c_1 > 0$, we obtain that

$$\frac{q(D)}{\psi(D)} = 2h^2 \left(\int_{-D}^{x_0} + \int_{x_0}^{-c_1} + \int_{-c}^{D} \frac{1}{\psi(x)^2} \int_{-D}^{x} (s + c_1)\psi(s)^2 ds dx \right). \tag{3.27}$$

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In the first of the three integrals with respect to x, $-D \leqslant s \leqslant x \leqslant x_0$, so $\psi(s) \leqslant \psi(x)$, and this term contributes less than $O(D^3)$ to $(q(D))/(\psi(D))$ as $D \to \infty$. In the third of the three integrals, we use the fact that $I(c_1) = 0$ to write

$$\int_{-c_1}^{D} \frac{1}{\psi(x)^2} \int_{-D}^{x} (s + c_1)\psi(s)^2 ds dx = -\int_{-c_1}^{D} \frac{1}{\psi(x)^2} \int_{x}^{D} (s + c_1)\psi(s)^2 ds dx, \tag{3.28}$$

and because $-c_1 \geqslant x_0$, we again have $\psi(s) \leqslant \psi(x)$ and get a contribution $O(D^3)$. The second term in (3.27) is bounded by

$$\int_{x_0}^{x_0+1} \frac{\psi(x_0)^2}{\psi(x_0+1)^2} \int_{-D}^{x} (s + c_1) ds dx.$$

Again we let $\rho = (\psi'(x - x_0))/(\psi(x - x_0))$, and note that $\rho' \geqslant -1 - \rho^2$, $\rho(0) = 0$. This gives a bound $(\psi(x_0))/(\psi(x_0 + 1)) \leqslant e^{\delta}$ for some δ independent of D, h or c_1, and so the second term in (3.27) is $O(D^2)$ as $D \to \infty$.

This proves the desired bound (3.26). Hence, (3.25) shows that the dominant term in (3.18) is the last one, proving that $I'(c_1) > 0$ for large D, if D is sufficiently large and $I(c_1) = 0$. Thus, there is a unique $c_1 > 0$ with $I(c_1) = 0$. It follows from (3.4)–(3.6) that $h(c_1) > h(0)$, since $I(c) < 0$ for $0 < c < c_1$. This completes the proof of Theorem 6.

4 Proofs of global results

We now turn to the global results on bifurcation from the curve of symmetric solutions. We consider the full problem (1.1)–(1.3). The symmetric problem can be studied on the interval $[0, d]$. Let $(\phi(x, \alpha, \delta), a(x, \alpha, \delta))$ denote the solution of (1.1), (1.2) which satisfies the initial conditions $\phi(0) = \alpha$, $\phi'(0) = 0$, $a(0) = 0$, $a'(0) = \delta$. Recall that Kwong [19] proved that for each $\alpha \in (0, 1]$ there is a unique $\delta = \delta_0(\alpha)$, such that ϕ is positive on $[0, d]$ and $\phi'(d) = 0$. Now let

$$s(x) = \frac{\partial \phi(x)}{\partial \delta} \quad \text{and} \quad z(x) = \frac{\partial a(x)}{\partial \delta}.$$

Let $\delta = \delta_0(\alpha)$ and let (ϕ, a) be the corresponding solution on $[0, d]$.

Lemma 15 $s'(d) > 0$.

Proof The pair (s, z) satisfies the system

$$\begin{aligned} \theta'' &= \kappa^2[(\phi^2 + a^2 - 1)\theta + 2a\phi\mu + 2\phi^2\theta] \\ \mu'' &= \phi^2\mu + 2a\phi\theta \end{aligned} \tag{4.1}$$

and

$$s(0) = s'(0) = 0, \; z(0) = 0, \; z'(0) = 1.$$

It is clear that z, z', z'' are all positive on $(0, d]$ as long as $s > 0$, since $a > 0$, $\phi > 0$. Also, $s''(0) = s'''(0) = 0$, while $s''''(0) > 0$, so initially s is positive. Suppose that $s(x_0) = 0$ at some first $s_0 > 0$ in $(0, d]$, with $s > 0$ on $(0, x_0)$. Multiply (4.1) by ϕ and (1) by s, subtract

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and integrate. We conclude that

$$\phi s' - s\phi'|_0^{x_0} = \int_0^{x_0} (2a\phi^2 z + 2\phi^3 s)dx > 0, \tag{4.2}$$

and applying the boundary conditions shows that $s'(x_0) > 0$, a contradiction. Hence $s > 0$ on $(0, d]$ and then from (4.2) we find that $s'(d) > 0$, as desired. □

To continue our study of bifurcation from the symmetric branch, now consider (1.1) with initial conditions which are possibly asymmetric, namely

$$\phi(0) = \alpha, \phi'(0) = \beta, a(0) = \gamma, a'(0) = \delta. \tag{4.3}$$

Let

$$p(x) = \frac{\partial\phi}{\partial\alpha},\ q(x) = \frac{\partial\phi}{\partial\beta},\ r(x) = \frac{\partial\phi}{\partial\gamma},\ s(x) = \frac{\partial\phi}{\partial\delta}$$
$$u(x) = \frac{\partial a}{\partial\alpha},\ v(x) = \frac{\partial a}{\partial\beta}, w(x) = \frac{\partial a}{\partial\gamma},\ z(x) = \frac{\partial a}{\partial\delta}.$$

Then the pairs (p, u), (q, v) , (r, w) and (s, z) all satisfy the system (4.1). The initial conditions are

$$\begin{aligned} (p, p', u, u') &= (1, 0, 0, 0) \\ (q, q', v, v') &= (0, 1, 0, 0) \\ (r, r', w, w') &= (0, 0, 1, 0) \\ (s, s', z, z') &= (0, 0, 0, 1) \end{aligned} \tag{4.4}$$

at $x = 0$. Thus, these four pairs form a fundamental solution for (4.1).

We wish to solve the three equations

$$F(\alpha, \beta, \gamma, \delta) = G(\alpha, \beta, \gamma, \delta) = H(\alpha, \beta, \gamma, \delta) = 0, \tag{4.5}$$

where

$$\begin{aligned} F(\alpha, \beta, \gamma, \delta) &= \phi'(d) \\ G(\alpha, \beta, \gamma, \delta) &= \phi'(-d) \\ H(\alpha, \beta, \gamma, \delta) &= a'(d) - a'(-d). \end{aligned} \tag{4.6}$$

For each $\alpha \in (0, 1]$, $(\alpha, 0, 0, \delta_0(\alpha))$ is a solution. Starting at $\alpha = 1$, where $\delta = 0$, this solution continues uniquely as the smooth curve of symmetric solutions as α decreases, so long as $J \neq 0$, where

$$J = \det \begin{bmatrix} q'(d) & r'(d) & s'(d) \\ q'(-d) & r'(-d) & s'(-d) \\ v'(d) - v'(-d) & w'(d) - w'(-d) & z'(d) - z'(-d) \end{bmatrix}$$

where in (41), (ϕ, a) is the solution at $(\alpha, 0, 0, \delta_0(\alpha))$.

Since ϕ is even and a is odd, the initial conditions (4.4) imply that p, v, w, and s are even functions of x, while u, q, r and z are odd functions of x. As a result, J simplifies to

$$J = 4s'(d)(q'(d)w'(d) - r'(d)v'(d)).$$

Lemma 16 *For sufficiently large κd, J changes sign between $\alpha = 0$ and $\alpha = 1$.*

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Proof Lemma 15 showed that $s'(d) > 0$. Therefore, to obtain a bifurcation point on the curve of symmetric solutions, we must show that $M(\alpha) = q'(d)w'(d) - r'(d)v'(d)$ changes sign along this curve.

For $\alpha = 1$ we have $\phi \equiv 1$, $a \equiv 0$. Equations (4.1) and (4.4) can then be solved explicitly to show that $q'(d) > 0$, $v'(d) = 0$, $r'(d) = 0$, and $w'(d) > 0$ so that $M(1) > 0$.

We now analyze $M(\alpha)$ for small α. First, with (q, v) substituted for (θ, μ) and $\alpha\psi$ for ϕ in (4.1), we multiply the equation for q'' by ψ, (4) by q, subtract, integrate, and use (2.3) and (4.4) to obtain

$$\psi(d)q'(d) = 1 + 2\kappa^2\alpha \int_0^d (a\psi^2 v + \psi^3 \alpha q)dx.$$

As $\alpha \to 0$, $\psi \to \psi_0$, where

$$\psi_0'' = \kappa^2(h_0^2x^2 - 1)\psi_0, \;\; \psi_0(0) = 1, \psi_0'(\pm d) = 0 \tag{4.7}$$

and h_0 is the unique positive number such that (4.7) has a positive solution. (Earlier we referred to the solution ψ_0 of (4.7) as ψ, but now we must distinguish ψ_0 from $\psi = \frac{\phi}{\alpha}$ for $\alpha \neq 0$.) Since

$$v'' = \alpha^2\psi^2 v + 2a\alpha\psi q,$$

(4.4) implies that $v \to 0$ on $[0, d]$ as $\alpha \to 0$, so we consider, instead, v/α, and see that as $\alpha \to 0$,

$$\frac{v'(d)}{\alpha} \to \int_0^d 2h_0 x\psi_0 q_0 dx,$$

where

$$q_0'' = \kappa^2(h_0^2x^2 - 1)q_0, \;\; q_0(0) = 0, \;\; q_0'(0) = 1. \tag{4.8}$$

We proceed in the same way with (r, w) to obtain, finally, that

$$\lim_{\alpha\to 0} \frac{M(\alpha)}{\alpha^2} = \frac{1}{\psi_0(d)} \left(\int_0^d \psi_0^2 + 2h_0 x\psi_0 R_0 dx - 4\kappa^2 h_0^2 \int_0^d x\psi_0^2 dx \cdot \int_0^d x\psi_0 q_0 dx \right). \tag{4.9}$$

In this expression, the only term we have not defined is R_0, which is $\lim_{\alpha\to 0} \frac{r}{\alpha}$ and satisfies

$$R_0'' = \kappa^2(h_0^2x^2 - 1)R_0 + 2\kappa^2 h_0 x\psi_0, \;\; R_0(0) = R_0'(0) = 0. \tag{4.10}$$

To prove that J changes sign, it is again convenient to rescale, letting $\psi_0(x) = g(\kappa x)$, so that

$$g'' = (\lambda^2y^2 - 1)g, \;\; g(0) = 1, \;\; g'(0) = 0, \;\; g'(D) = 0, \tag{4.11}$$

where $D = \kappa d$ and $\lambda = h_0/\kappa$. It was shown in the proof of Lemma 2 that $\lambda > 1$.

Making the same change of variables in (4.9), we must show that for large $D = \kappa d$,

$$\frac{1}{\kappa} \int_0^D (g(y)^2 + 2\frac{h_0}{\kappa} y g(y)P(y))dy < 4\frac{h_0^2}{\kappa^2} \int_0^D y g(y)^2 dy \cdot \int_0^D y g(y)Q(y)dy, \tag{4.12}$$

where $R_0(x) = P(\kappa x)$, and $q_0(x) = Q(\kappa x)$. Hence, we have

$$Q'' = (\lambda^2y^2 - 1)Q, \quad Q(0) = 0, \quad Q'(0) = \frac{1}{\kappa}, \tag{4.13}$$

and

$$P'' = (\lambda^2 y^2 - 1)P + 2\lambda y g(y), \quad P(0) = P'(0) = 0 \tag{4.14}$$

as well as (4.11). Multiplying (4.13) by g and (4.11) by Q and subtracting and integrating gives

$$\left(\frac{Q}{g}\right)' = \frac{1}{\kappa g(y)^2},$$

and similarly from (4.14), we obtain

$$\left(\frac{P}{g}\right)' = 2\lambda \frac{1}{g(y)^2} \int_0^y s g(s)^2 ds.$$

From these we find that

$$P(y) = g(y) \int_0^y 2 \frac{h_0 \int_0^x s g(s)^2 ds}{\kappa g(x)^2} dx, \tag{4.15}$$

and

$$Q(y) = \frac{g(y)}{\kappa} \int_0^y \frac{1}{g(x)^2} dx.$$

With these substitutions, (4.12) becomes

$$\int_0^D g(y)^2 dy < 4\lambda^2 \int_0^D y g(y)^2 \int_0^y \frac{1}{g(s)^2} \int_s^D t g(t)^2 dt\, ds\, dy. \tag{4.16}$$

It is tempting to approach this result by studying the asymptotic behaviour of $g(y)$ as $D \to \infty$. In fact, one can show that $g(y) \to \mathrm{e}^{-(y^2/2)}$ point-wise, and further effort can refine this result. It turns out to be a mistake, however, to study the result of substituting $e^{-(y^2/2)}$) for g in (4.16), because this function does not satisfy the boundary conditions, and this turns out to make the required estimates much more difficult, or indeed, impossible.

Instead, we proceed directly, and this is possible primarily because using (4.11) we can evaluate integrals of the form $\int y g(y)^2 dy$. In particular, multiplying (4.11) by g' and integrating by parts, we find that

$$\lambda^2 \int y g(y)^2 dy = (\lambda^2 y^2 - 1) \frac{g(y)^2}{2} - \frac{g'(y)^2}{2}.$$

Substituting this in (4.16) and using the boundary conditions gives

$$\lambda^2 \int_0^y \frac{1}{g(s)^2} \int_s^D t g(t)^2 dt ds = \frac{\lambda^2 D^2 - 1}{2} g(D)^2 \int_0^y \frac{1}{g(s)^2} ds - \frac{1}{2} \frac{g'(y)}{g(y)}. \tag{4.17}$$

Substituting this in the right side of (4.16) gives

$$\begin{aligned} &\lambda^2 \int_0^D y g(y)^2 \int_0^y \frac{1}{g(s)^2} \int_s^D t g(t)^2 dt\, ds\, dy = \\ &\frac{\lambda^2 D^2 - 1}{2} g(D)^2 \int_0^D y g(y)^2 \int_0^y \frac{1}{g(s)^2} ds\, dy + \tfrac{1}{4} \int_0^D g(y)^2 dy - \frac{D g(D)^2}{4}. \end{aligned} \tag{4.18}$$

Using this in (4.16) implies that the following inequality is sufficient for our result:

$$\frac{\lambda^2 D^2 - 1}{2} g(D)^2 \int_0^D y g(y)^2 \int_0^y \frac{1}{g(s)^2} ds\, dy - \frac{D g(D)^2}{4} > 0 \tag{4.19}$$

(for large D).

We need only one additional fact about $g(y)$; namely, that $1 \geqslant g(y) \geqslant e^{-(y^2/2)}$. We already know that $g' \leqslant 0$ giving the first inequality, and for the second, differentiate

$$\left(\frac{g}{e^{-\frac{y^2}{2}}}\right)$$

twice and use (4.11) to see that this expression, which is 1 at $y = 0$, increases on $[0, D]$.

Using this information, it can be seen that the double integral on the left of (4.19) does not tend to zero with D, and this proves (4.19) for large D. This implies (4.12), and completes the proof of Lemma 16. □

We also need a global bound on the solutions, for fixed κ, d.

Lemma 17 *For given κ and d, there is an M such that if (ϕ, a, h) is a solution of* (1.1)–(1.3) *with $\phi > 0$ on $[-d, d]$, then $|\phi| + |\phi'| + |a| + |a'| \leqslant M$ on $[-d, d]$.*

Proof Letting

$$\psi = \frac{\phi}{\max_{-d \leqslant x \leqslant d} \phi(x)},$$

we see that $\psi'(\pm d) = 0$ and $\psi'' \geqslant -\kappa$. This implies that ψ' is bounded independent of which solution is being considered. Further, for any solution there is an $x_0 \in (-d, d)$ with $a(x_0) = 0$. If $a'(x_0)$ is bounded then we are done, so we can assume that there are solutions with $a'(x_0)$ arbitrarily large. Since a' is a minimum at x_0, this implies that for any A, and ε, there is a Λ such that if $a'(x_0) \geqslant \Lambda$, then the length of the interval in which $|a| \leqslant A$ is less than ε.

Since $\psi'(\pm d) = 0$ we must have

$$\int_{-d}^{d} \left(\psi - \psi^3\right) dx = \int_{-d}^{d} \psi a^2 dx. \tag{4.20}$$

Since $\max_{[-d,d]} \psi(x) = 1$, and $\psi' \geqslant -\kappa$, there must be an $\varepsilon > 0$ such that for any solution of (1.1)–(1.3), $\psi \geqslant \frac{1}{2}$ in some interval Ω of length ε. For solutions with $a'(x_0)$ sufficiently large, we must have $|a|$ large in at least half of Ω, which means that the right side of (4.20) can be arbitrarily large, while the left side is bounded by $2d$. This contradiction proves Lemma 17. □

We note that Kwong [19] gave a proof using Sturmian methods.

Completion of proof of Theorem 8

With M as in Lemma 17, truncate (1.1)–(1.2) by letting

$$g(x) = \min(x, M^2)$$

and considering

$$\phi'' = \kappa^2(g(a^2 + \phi^2) - 1)\phi \tag{4.21}$$

and

$$a'' = g(\phi)^2 a$$

with boundary conditions (1.3). With the notation of (4.6), let $\tilde{F} = (F, G, H) = \tilde{F}(\alpha, \tilde{\beta})$, where $\tilde{\beta} = (\beta, \gamma, \delta)$. We use a truncation to ensure that whatever the initial condition at $x = 0$, the solutions exist up to $x = d$.

Since $J > 0$ when α is close to 1, (when $\tilde{F}(\alpha, (0, 0, \delta_0(\alpha))) = 0$), this solution $t(\alpha) = (0, 0, \delta_0(\alpha))$ is non-degenerate and isolated, and has Brouwer degree sgn $J = 1$ (for the map $\tilde{F}$ with fixed α). Similarly, if α is small and positive, then $J < 0$ and $t(\alpha)$ has degree -1. Hence, there is a change of degree along the branch of symmetric solutions between α small and $\alpha = 1$. Thus, by a slight variant of Theorem 1.16 of Rabinowitz [21], there is a connected set of solutions of $\tilde{F}(\alpha, \tilde{\beta}) = 0$ branching off the symmetric solution $(\alpha, t(\alpha))$ at a point $\alpha \in (0, 1)$ where $J = 0$, and this branch will either be unbounded in R^4, or return to (0,0,0,0), or meet the symmetric branch again.

Note that there are no solutions with $h = 0$, since then a would have to be of constant sign, and we could not have $\int_{-d}^{d} \phi^2 a\, dx = 0$. Also, solutions cannot bifurcate from the symmetric branch near $\alpha = 0$ or $\alpha = 1$, because $J \neq 0$ there.

The solutions of interest are positive. (That is, ϕ is positive.) Solutions on the bifurcating branch of asymmetric solutions start off positive as the branch leaves the symmetric branch, from continuity. From (1.1), it follows that solutions can only fail to be positive along this branch if $\phi \to 0$ (uniformly on $[-d, d]$). Also, Lemma 17 shows the solutions must remain bounded, and in fact, remain within the truncated region where (1.1)–(1.2) apply. If $\phi \to 0$, then $a \to h(x + c)$.

It is conceivable that there are several points α where $J = 0$. However, the change of degree ensures that one branch does not return to the branch of symmetric solutions, and so by Rabinowitz's global result, must tend to $(0, h(x+c))$ for some $c \neq 0$, i.e. to a normal solution. In fact, by the symmetry of the problem, there will be two branches bifurcating from the same point, one tending to the unique bifurcation point from the normal solution with $c > 0$, and the other to the symmetric reflection of this solution around 0. (The uniqueness of this bifurcation point follows from Theorem 6.) This completes the proof of Theorem 8.

5 Conclusion

The initial motivation for this paper was Seydel's bifurcation diagram (Figure 1). Our goal was to prove that in some parameter range the problem could have as many as seven solutions (five essentially distinct). Unfortunately we have not achieved this goal. There are at least two features of Seydel's curve that seem important in obtaining such a proof. We would like to determine where on the symmetric branch the bifurcation to asymmetric solutions does exist, and we want to know the direction of bifurcation at this point. These remain challenges for future work. However, we have verified that for large κd the desired bifurcation from the symmetric branch occurs, and furthermore, there is a curve of asymmetric solutions going from the symmetric branch to the normal state. We have also shown how earlier results on bifurcation from the normal state can be obtained without the use of detailed asymptotics for the linear problem.

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