J. Appl. Prob. **53,** 279–292 (2016)
*doi:*10.1017/jpr.2015.24

MODERATE DEVIATION PRINCIPLE FOR A CLASS OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

PARISA FATHEDDIN,* *University of Tennessee, Knoxville*
JIE XIONG,** *University of Macau*

Abstract

We establish the moderate deviation principle for the solutions of a class of stochastic partial differential equations with non-Lipschitz continuous coefficients. As an application, we derive the moderate deviation principle for two important population models: super-Brownian motion and the Fleming–Viot process.

Keywords: Moderate deviation principle; Fleming–Viot process; stochastic partial differential equation; super-Brownian motion

2010 Mathematics Subject Classification: Primary 60F10
Secondary 60H15; 60J68

1. Introduction

Many problems in the field of applications can be modeled by measure-valued processes. Among them are two of the most commonly studied population models, namely, super-Brownian motion (SBM) and the Fleming–Viot process (FVP). These population models have been the focus of numerous recent publications, among which is [6] in which the authors established the central limit theorem for the two models. One of the interesting problems for these models is to set the branching rate for SBM and a resampling rate for the FVP to tend to 0 and to study the rate at which the population's measure converges to a deterministic limit. This rate of convergence is best given by the large deviation principle (LDP). In [7], we achieved the LDP for SBM and FVP as the above mentioned rates go to 0 and obtained an explicit form of the rate of convergence for each model. However, the topology introduced there is not a natural one. Namely, we used the double quotient space due to the nonuniqueness of the controlled partial differential equation (PDE) in the definition of the rate function. Here we achieve the moderate deviation principle (MDP), which provides the convergence rate of the models as the branching/resampling rate tends to 0 at a speed slower than that considered for the LDP. The topology we will use is the standard one, and there is no need to introduce the quotient space.

The MDP for SBM has also been established by Schied in [13]. There the author used the space $\mathcal{C}([0, 1]; M(\mathbb{R}^d))$ equipped with compact open topology, where $M(\mathbb{R}^d)$ is the space of finite signed measures on $\mathbb{R}^d$ with the coarsest topology in which $\mu \mapsto \langle \mu, f\rangle$ are continuous for every bounded Lipschitz function on $\mathbb{R}^d$. The main tool the author applied was the Gärtner–Ellis theorem; see [3, Theorem 4.6.1]. Here we have used a similar space and have obtained the same result; however, with a different approach. Other authors including those of [9], [10], [15], and [16] have investigated the MDP for processes related to SBM. These processes include

Received 9 September 2014; revision received 22 January 2015.
* Current address: Department of Mathematics and Statistics, Air Force Institute of Technology, 2950 Hobson Way, Wright-Patterson AFB, OH 45433, USA. Email address: graduate24@gmail.com
** Postal address: Department of Mathematics, FST, University of Macau, PO Box 3001, Macau, China.

SBM with super-Brownian immigration and SBM with immigration governed by the Lebesgue measure. Here we have also derived the MDP for the FVP, which, to the best of the authors' knowledge, has not yet been shown in the literature.

In this paper we study the SBM and FVP based on their characterization by solutions to certain stochastic partial differential equations (SPDEs). We formulate a general class of SPDEs by observing the similarities between the two SPDEs and in Section 3 derive the MDP for this class by applying [1, Theorem 6]. In Section 4 we then establish the MDP for the two population models with the help of the contraction principle; see [4, Theorem 4.2.1]. We note that since the formulation of SBM and FVP by SPDEs offered by [14] was given only for dimension 1 then our result on the MDP is limited to this dimension. For higher dimensions further investigation is required.

2. Notations and main results

Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\{\mathcal{F}_t\}$ is a family of nondecreasing right-continuous sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F}_0$ contains all $\mathbb{P}$-null subsets of Ω. We denote $\mathcal{C}_{\mathrm{b}}(\mathbb{R})$ to be the space of continuous bounded functions on $\mathbb{R}$, and $\mathcal{C}_{\mathrm{c}}(\mathbb{R})$ be the set of continuous functions in $\mathbb{R}$ with compact support. In addition, for $0 < \beta \in \mathbb{R}$, we let $\mathcal{M}_\beta(\mathbb{R})$ denote the set of σ-finite measures μ on $\mathbb{R}$ such that

$$\int \mathrm{e}^{-\beta|x|}\,\mathrm{d}\mu(x) < \infty. \tag{1}$$

We endow this space with the topology defined by a modification of the usual weak topology: $\mu^n \to \mu$ in $\mathcal{M}_\beta(\mathbb{R})$ if and only if for every $f \in \mathcal{C}_{\mathrm{b}}(\mathbb{R})$,

$$\int_{\mathbb{R}} f(x)\mathrm{e}^{-\beta|x|}\mu^n(\mathrm{d}x) \to \int_{\mathbb{R}} f(x)\mathrm{e}^{-\beta|x|}\mu(\mathrm{d}x).$$

This topology is given by the following modified Wasserstein distance:

$$\rho_\beta(\mu,\nu) := \inf\left\{\left|\int_{\mathbb{R}} f(x)\mathrm{e}^{-\beta|x|}(\mu(\mathrm{d}x)-\nu(\mathrm{d}x))\right| : f \in \mathcal{C}_{\mathrm{b}}^1(\mathbb{R}),\ \|f\|_\infty \vee \|f'\|_\infty \le 1\right\}.$$

We denote the probability measures on $\mathbb{R}$ with the above topology by $\mathcal{P}_\beta(\mathbb{R})$. Let $(S, \mathcal{S})$ be the measurable space defined as $(S, \mathcal{S}) := (\mathcal{C}([0,1];\mathbb{R}^\infty), \mathbb{B}\mathcal{C}([0,1];\mathbb{R}^\infty)))$, where $\mathbb{R}^\infty$ is the Polish space with the metric given as

$$d(\{x_i\},\{y_i\}) := \sum_{i=1}^{\infty} 2^{-i}(|x_i - y_i| \wedge 1).$$

Throughout this paper, we assume that $\beta_0 \in (0,\beta)$ and K is a constant which may take different values in different lines. Also the notation Δ stands for the second derivative in the spatial variable x. This notation will be used when both spatial and time variables are involved, or when the dual operator will be needed. Otherwise, we will use the simpler notation f''. The same convention is used for ∇, the first derivative in the spatial variable. For $\alpha \in (0,1)$, we consider the space $\mathbb{B}_{\alpha,\beta}$ composed of all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for every $m \in \mathbb{N}$ there exist constants $K > 0$ with the following conditions:

$$\begin{aligned} |f(y_1) - f(y_2)| &\le K\mathrm{e}^{\beta m}|y_1 - y_2|^\alpha \quad \text{for all } |y_1|,\ |y_2| \le m, \\ |f(y)| &\le K\mathrm{e}^{\beta|y|} \quad \text{for all } y \in \mathbb{R} \end{aligned} \tag{2}$$

https://doi.org/10.1017/jpr.2015.24 Published online by Cambridge University Press

and with the metric

$$d_{\alpha,\beta}(u,v)=\sum_{m=1}^{\infty}2^{-m}(\|u-v\|_{m,\alpha,\beta}\wedge 1),\qquad u,v\in\mathbb{B}_{\alpha,\beta},$$

where

$$\|u\|_{m,\alpha,\beta}=\sup_{x\in\mathbb{R}}\mathrm{e}^{-\beta|x|}|u(x)|+\sup_{y_1\neq y_2|y_1|,|y_2|\leq m}\frac{|u(y_1)-u(y_2)|}{|y_1-y_2|^{\alpha}}\mathrm{e}^{-\beta m}.$$

Note that the collection of continuous functions on $\mathbb{R}$ satisfying (2), referred to as $\mathbb{B}_\beta$, is a separable Banach space with norm

$$\|f\|_\beta=\sup_{x\in\mathbb{R}}\mathrm{e}^{-\beta|x|}|f(x)|.$$

For the convenience of the reader, we now offer a quick introduction to the two population models considered. In the SBM model each individual has an exponentially distributed lifetime and the population evolves as a 'cloud'. It is studied by taking a scaled limit of a branching process with an associated branching rate. SBM with branching rate ε, denoted by μ_t^ε, is a measure-valued Markov process that can be characterized by one of the following.

(i) For a fixed constant $p>d$, let $\phi_p(x):=(1+|x|^2)^{-p/2}$ for $x\in\mathbb{R}^d$ and

$$C_p(\mathbb{R}^d):=\{f\in C_b(\mathbb{R}^d)\colon |f(x)|\leq K\phi_p(x)\}$$

then SBM, $(\mu_t^\varepsilon)_t$, is a measure-valued Markov process with transition probabilities given by

$$\mathbb{E}\exp(-\langle\mu_t^\varepsilon,f\rangle)=\exp(-<\mu_0^\varepsilon,n(t,\cdot)>)\quad\text{for } f\in C_p^{+}(\mathbb{R}^d),$$

where $n(\cdot,\cdot)$ is the unique mild solution of the evolution equation

$$\dot n(t)=\Delta n(t)-n^2(t),\qquad n(0)=f.$$

(ii) The process (μ_t^ε) as the unique solution to a martingale problem. For all $f\in\mathcal{C}_{\mathrm{b}}^2(\mathbb{R})$,

$$M_t(f):=\langle\mu_t^\varepsilon,f\rangle-\langle\mu_0^\varepsilon,f\rangle-\int_0^t\Big\langle\mu_s^\varepsilon,\frac{1}{2}\Delta f\Big\rangle\,\mathrm{d}s$$

is a square-integrable martingale with quadratic variation

$$\langle M(f)\rangle_t=\varepsilon\int_0^t\langle\mu_s^\varepsilon,f^2\rangle\,\mathrm{d}s.$$

(iii) In [14] SBM was studied by its 'distribution' function-valued process u_t^ε defined as

$$u_t^\varepsilon(y)=\int_0^y\mu_t^\varepsilon(\mathrm{d}x)\quad\text{for all } y\in\mathbb{R},\tag{3}$$

and using (3), SBM was characterized by the following SPDE:

$$u_t^\varepsilon(y)=F(y)+\sqrt{\varepsilon}\int_0^t\int_0^{u_s^\varepsilon(y)}W(\mathrm{d}s\,\mathrm{d}a)+\int_0^t\frac{1}{2}\Delta u_s^\varepsilon(y)\,\mathrm{d}s,\tag{4}$$

where $F(y)=\int_0^y\mu_0(\mathrm{d}x)$ is the 'distribution' function of μ_0 and W is an $\mathcal{F}_t$-adapted space-time white noise random measure on $\mathbb{R}^+\times\mathbb{R}$ with intensity measure $\mathrm{d}s\,\mathrm{d}a$.

On the other hand, FVP is a population model with its evolution based on the genetic types of the individuals. It is a probability measure-valued diffusion process with mutation rate ε, studied as a scaled limit of a step-wise mutation model in which the population size is assumed to stay constant throughout time and individuals move in $\mathbb{Z}^d$ according to a continuous-time simple random walk. As in the case for SBM, this population model denoted as μ_t^ε is a Markov process and can be characterized by one of the following.

(i) Let $M_1([0, 1])$ be the space of all probability measures on $[0, 1]$ with weak topology and Prohorov metric ρ. Furthermore, let A be the generator of a Markov process on E with domain $D(A)$. Then FVP is a Markov process with generator

$$\mathcal{L}g(\mu_t^\varepsilon) = f'(\langle \mu_t^\varepsilon, \phi\rangle)\langle \mu_t^\varepsilon, A\phi\rangle + \frac{\varepsilon}{2}\int int\phi(x)\phi(y)Q(\mu_t^\varepsilon, \mathrm{d}x, \mathrm{d}y) \tag{5}$$

and domain

$$\mathcal{D} = \{g\colon g(\mu_t^\varepsilon) = f(\langle \mu_t^\varepsilon, \phi\rangle),\ f \in C_{\mathrm{b}}^\infty(\mathbb{R}),\ \phi \in D(A),\ \mu_t^\varepsilon \in M_1([0, 1])\},$$

where $Q(\mu_t^\varepsilon;\ \mathrm{d}x,\ \mathrm{d}y) = \mu_t^\varepsilon(\mathrm{d}x)\delta_x(\mathrm{d}y) - \mu_t^\varepsilon(\mathrm{d}x)\mu_t^\varepsilon(\mathrm{d}y)$ with δ_x being the Dirac measure at $x \in E$. Here E is the type space, A is the mutation operator, and the second term in (5) describes the continuous sampling. If the mutation operator has the form $Af(x) = (\varepsilon/2)\int(f(y) - f(x))\nu_0(\mathrm{d}y)$ with $\nu_0 \in M_1(E)$, then the FVP is said to have neutral mutation. For more information on this characterization; see [2] and [8].

(ii) The process (μ_t^ε) as a unique solution to the following martingale problem. For $f \in \mathcal{C}_{\mathrm{c}}^2(\mathbb{R})$,

$$M_t(f) = \langle \mu_t^\varepsilon, f\rangle - \langle \mu_0^\varepsilon, f\rangle - \int_0^t \left\langle \mu_s^\varepsilon, \frac{1}{2}\Delta f\right\rangle \mathrm{d}s$$

is a continuous square-integrable martingale with quadratic variation

$$\langle M_t(f)\rangle = \varepsilon\int_0^t (\langle \mu_s^\varepsilon, f^2\rangle - \langle \mu_s^\varepsilon, f\rangle^2)\,\mathrm{d}s.$$

(iii) An alternative formulation of FVP was also made in [14]. There, by using $u_t^\varepsilon(y) = \mu_t^\varepsilon((-\infty, y])$, FVP was proved to be given by the solution to the following SPDE:

$$u_t^\varepsilon(y) = F(y) + \sqrt{\varepsilon}\int_0^t\int_0^1 (\mathbf{1}_{\{a \le u_s^\varepsilon(y)\}} - u_s^\varepsilon(y))W(\mathrm{d}s\,\mathrm{d}a) + \int_0^t \frac{1}{2}\Delta u_s^\varepsilon(y)\,\mathrm{d}s. \tag{6}$$

Based on context, $\varepsilon > 0$ represents the branching rate for SBM and the resampling rate for the FVP. Note that the main difference between (4) and (6) is in the second term; therefore, in [14] a general SPDE with small noise term of the form

$$u_t^\varepsilon(y) = F(y) + \sqrt{\varepsilon}\int_0^t\int_U G(a, y, u_s^\varepsilon(y))W(\mathrm{d}s\,\mathrm{d}a) + \int_0^t \frac{1}{2}\Delta u_s^\varepsilon(y)\,\mathrm{d}s \tag{7}$$

was considered with conditions

$$\int_U |G(a, y, u_1) - G(a, y, u_2)|^2\lambda(\mathrm{d}a) \le K|u_1 - u_2|, \tag{8}$$

$$\int_U |G(a, y, u)|^2\lambda(\mathrm{d}a) \le K(1 + |u|^2),$$

where $(U, \mathcal{U}, \lambda)$ is a measure space such that $L^2(U, \mathcal{U}, \lambda)$ is separable. Furthermore, $u_1, u_2, u, y \in \mathbb{R}$, F is a function on $\mathbb{R}$, and $G\colon U \times \mathbb{R}^2 \to \mathbb{R}$. Here we prove the MDP for $\{u_t^\varepsilon\}$ by considering the LDP for $\{v_t^\varepsilon\}$ given by

$$v_t^\varepsilon(y) := \frac{a(\varepsilon)}{\sqrt{\varepsilon}}(u_t^\varepsilon(y) - u_t^0(y)). \tag{9}$$

Hence, we have

$$v_t^\varepsilon(y) = a(\varepsilon)\int_0^t \int_U G_s^\varepsilon(a, y, v_s^\varepsilon(y)) W(\mathrm{d}s\,\mathrm{d}a) + \frac{1}{2}\int_0^t \Delta v_s^\varepsilon(y)\,\mathrm{d}s, \tag{10}$$

where $G_s^\varepsilon(a, y, v) := G(a, y, (\sqrt{\varepsilon}/a(\varepsilon))v + u_s^0(y))$, $a(\varepsilon)$ satisfies $0 \le a(\varepsilon) \to 0$, and $a(\varepsilon)/\sqrt{\varepsilon} \to \infty$ as $\varepsilon \to 0$. To form the controlled PDE of (10), we replace the noise by $h \in L^2([0,1] \times U,\ \mathrm{d}s\lambda(\mathrm{d}a))$ and obtain

$$v_t(y) = \int_0^t \int_U G(a, y, u_s^0(y)) h(s, a)\lambda(\mathrm{d}a)\,\mathrm{d}s + \frac{1}{2}\int_0^t \Delta v_s(y)\,\mathrm{d}s. \tag{11}$$

Note that for every $h \in L^2([0,1] \times U,\ \mathrm{d}s\lambda(\mathrm{d}a))$, (11) has a unique solution, which we denote as $\gamma(h)$ for a map $\gamma\colon L^2([0,1] \times U,\ \mathrm{d}s\lambda(\mathrm{d}a)) \to \mathcal{C}([0,1]; \mathbb{B}_\beta)$. We are now ready to state the first result of this paper.

Theorem 1. *If $F \in \mathbb{B}_{\alpha,\beta_0}$ for $\alpha \in (0, \frac{1}{2})$ then the family $\{v_\cdot^\varepsilon\}$ given by (10) satisfies the LDP in $\mathcal{C}([0,1]; \mathbb{B}_\beta)$ with speed $a(\varepsilon)$ and rate function*

$$I(v) = \frac{1}{2}\inf\bigg\{\int_0^1 \int_U |h_s(a)|^2 \lambda(\mathrm{d}a)\,\mathrm{d}s\colon v = \gamma(h)\bigg\}, \tag{12}$$

which implies that the family $\{u_t^\varepsilon\}$ obeys the MDP.

Similar to [5] we consider the Cameron–Martin space which is defined as follows. Let $\mathcal{M}_\beta^{\mathrm{S}}(\mathbb{R})$ be the space of signed measures $\mu = \mu_+ - \mu_-$ with $\mu_\pm \in \mathcal{M}_\beta(\mathbb{R})$. Let $\mathcal{D}$ be the Schwartz space of test functions with compact support in $\mathbb{R}$ and continuous derivatives of all orders. Denote the dual space of real distributions on $\mathbb{R}$ by $\mathcal{D}^*$. We say that the generalized function $\nu \in \mathcal{D}^*$ is absolutely continuous with respect to the measure $\mu \in \mathcal{M}_\beta^{\mathrm{S}}$ if there exists a function $g \ge 0$ which is locally μ-integrable and satisfies $\langle \nu, \phi\rangle = \langle \mu, g\phi\rangle$, $\phi \in \mathcal{D}$. Then we write $g = \mathrm{d}\nu/\mathrm{d}\mu$ and call g the Radon–Nikodym derivative of ν with respect to μ. The Cameron–Martin space, H, is composed of $\omega \in \mathcal{C}([0,1]; \mathcal{M}_\beta^{\mathrm{S}}(\mathbb{R}))$ satisfying the following conditions:

(i) $\omega_0 = 0$;

(ii) the $\mathcal{D}^*$-valued map $t \mapsto \omega_t$ defined on [0,1] is absolutely continuous with respect to time and let $\dot{\omega}$ and $\Delta^*\omega$ be its generalized derivative and Laplacian, respectively;

(iii) for every $t \in [0,1]$, $\dot{\omega}_t - \Delta^*\omega_t/2 \in \mathcal{D}^*$ is absolutely continuous with respect to μ_t^0 with $\mathrm{d}(\dot{\omega}_t - \Delta^*\omega_t/2)/\mathrm{d}\mu_t^0$ being the (generalized) Radon–Nikodym derivative;

(iv) $\mathrm{d}(\dot{\omega}_t - \Delta^*\omega_t/2)/\mathrm{d}\mu_t^0$ is in $L^2([0,1] \times \mathbb{R},\ \mathrm{d}s\omega_t(\mathrm{d}y))$.

https://doi.org/10.1017/jpr.2015.24 Published online by Cambridge University Press

Let $\tilde{H}$ be the space for which conditions for H hold with $\mathcal{M}_\beta^{\mathrm{S}}(\mathbb{R})$ replaced by the space of measures $\mathcal{P}_\beta^{\mathrm{S}}(\mathbb{R})$, and with the additional assumption

$$\left\langle \mu_t^0, \frac{\mathrm{d}(\dot{\omega}_t - \Delta^*\omega_t/2)}{\mathrm{d}\mu_t^0} \right\rangle = 0,$$

where $\mathcal{P}_\beta^{\mathrm{S}}(\mathbb{R})$ is the set of signed measures μ with $\mu_\pm \in \mathcal{P}_\beta(\mathbb{R})$. Denoting $\omega_t^\varepsilon(\mathrm{d}y) := (a(\varepsilon)/\sqrt{\varepsilon})(\mu_t^\varepsilon(\mathrm{d}y) - \mu_t^0(\mathrm{d}y))$, we have the following two theorems.

Theorem 2. *Suppose that $F \in \mathbb{B}_{\alpha,\beta_0}$ then SBM, $\{\mu_t^\varepsilon\}$, obeys the MDP in $\mathcal{C}([0,1];\mathcal{M}_\beta^{\mathrm{S}}(\mathbb{R}))$ with rate function*

$$I(\omega) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^1 \int_{\mathbb{R}} \left| \frac{\mathrm{d}(\dot{\omega}_t - \Delta^*\omega_t/2)}{\mathrm{d}\mu_t^0}(y) \right|^2 \mu_t^0(\mathrm{d}y)\,\mathrm{d}t & \text{if } \mu \in H, \\ \infty & \text{otherwise.} \end{cases} \tag{13}$$

Theorem 3. *Suppose that $F \in \mathbb{B}_{\alpha,\beta_0}$ then FV, $\{\mu^\varepsilon\}$, satisfies the MDP on $\mathcal{C}([0,1];\mathcal{P}_\beta^{\mathrm{S}}(\mathbb{R}))$ with rate function*

$$I(\omega) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^1 \int_{\mathbb{R}} \left| \frac{\mathrm{d}(\dot{\omega}_t - \Delta^*\omega_t/2)}{\mathrm{d}\mu_t^0(y)} \right|^2 \mu_t^0(\mathrm{d}y)\,\mathrm{d}t & \text{if } \mu \in \tilde{H}, \\ \infty & \text{otherwise.} \end{cases} \tag{14}$$

Proofs of Theorems 1–3 are given in Sections 3 and 4.

3. Moderate deviations for the general SPDE

Our goal in this section is to establish the MDP for (7), referred to as the general SPDE. Note that by our assumption $F \in \mathbb{B}_{\alpha,\beta_0}$, we have

$$|u_s^0(y)| \le \int_{\mathbb{R}} p_s(x-y)|F(x)|\,\mathrm{d}x \le K\mathrm{e}^{\beta_0|y|},$$

where $p_t(x) = (1/\sqrt{2\pi t})\exp(-x^2/2t)$ is the heat kernel. Therefore, G_s^ε satisfies the following conditions:

$$\begin{gathered} \int_U |G_s^\varepsilon(a,y,v_1) - G_s^\varepsilon(a,y,v_2)|^2\lambda(\mathrm{d}a) \le K|v_1 - v_2|, \\ \int_U |G_s^\varepsilon(a,y,v)|^2\lambda(\mathrm{d}a) \le K(1 + v^2 + \mathrm{e}^{2\beta_0|y|}) \end{gathered} \tag{15}$$

for $y \in \mathbb{R}$ and $v, v_1, v_2 \in \mathbb{R}$ given by (9).

Since the proof of the uniqueness of strong solutions to (7) established in [14] uses only condition (8) then the same argument can be applied to (10) to achieve the uniqueness of strong solutions. SPDE (10) can therefore be presented by its mild form:

$$v_t^\varepsilon(y) = a(\varepsilon)\int_{\mathbb{R}}\int_0^t\int_U G_s^\varepsilon(a,x,v_s^\varepsilon(x))p_{t-s}(y-x)W(\mathrm{d}s\,\mathrm{d}a)\,\mathrm{d}x.$$

We show that this mild solution takes values in $\mathcal{C}([0,1];\mathbb{B}_\beta)$. To accomplish this we need the subsequent lemma.

Lemma 1. *For $\beta_1 \in (\beta_0, \beta)$ and every $n \geq 2$,*

$$\tilde{M} := \sup_{0<\varepsilon<1} \mathbb{E} \sup_{0\leq s\leq 1} \left(\int_{\mathbb{R}} |v_s^\varepsilon(x)|^2 \mathrm{e}^{-2\beta_1|x|}\,\mathrm{d}x \right)^n < \infty.$$

Proof. We adapt the argument in the proof of [14, Lemma 2.3] to the present setup. By Mitoma [12], if

$$\rho(x) = \begin{cases} C\exp\left(\dfrac{-1}{1-|x|^2}\right), & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

where C is determined by $\int_{\mathbb{R}} \rho(x)\mathrm{d}x = 1$, then $g(x) = \int_{\mathbb{R}} \mathrm{e}^{-|y|}\rho(x-y)\,\mathrm{d}y$ satisfies

$$K_1 \mathrm{e}^{-|x|} \leq g^{(n)}(x) \leq K_2 \mathrm{e}^{-|x|}, \tag{16}$$

where $g^{(n)}(x)$ is the nth derivative of $g(x)$. Note that if we replace $\mathrm{e}^{-|y|}$ with $\mathrm{e}^{-2\beta_1|y|}$ in the definition of $g(x)$ then the same estimates used to obtain (16) yield $K_1\mathrm{e}^{-2\beta_1|x|} \leq g^{(n)}(x) \leq K_2\mathrm{e}^{-2\beta_1|x|}$; therefore, we can consider $J(x) := \int \mathrm{e}^{-2\beta_1|y|}\rho(x-y)\,\mathrm{d}y < \infty$ instead of $\mathrm{e}^{-\beta|x|}$ in the definition of $\mathcal{M}_\beta(\mathbb{R})$ given by (1), where $0 < \beta \in \mathbb{R}$.

We denote the Hilbert space $L^2(\mathbb{R}, J(x)\,\mathrm{d}x)$ by $\mathcal{X}_0$. Then for every $f \in \mathcal{C}_c^\infty(\mathbb{R}) \cap \mathcal{X}_0$, we have

$$\langle v_t^\varepsilon, f\rangle_{\mathcal{X}_0} = a(\varepsilon) \int_{\mathbb{R}} \int_0^t \int_U G_s^\varepsilon(a, y, v_s^\varepsilon(y)) f(y) J(y) W(\mathrm{d}s\,\mathrm{d}a)\,\mathrm{d}y + \int_0^t \left\langle \frac{1}{2}\Delta v_s^\varepsilon, f \right\rangle_{\mathcal{X}_0} \mathrm{d}s. \tag{17}$$

Applying Itô's formula again, this time to (17), we obtain

$$\begin{aligned}
\langle v_t^\varepsilon, f\rangle_{\mathcal{X}_0}^2 &= 2a(\varepsilon) \int_0^t \langle v_s^\varepsilon, f\rangle_{\mathcal{X}_0} \int_U \int_{\mathbb{R}} G_s^\varepsilon(a, y, v_s^\varepsilon(y)) f(y) J(y)\,\mathrm{d}y W(\mathrm{d}s\,\mathrm{d}a) \\
&\quad + \int_0^t \langle v_s^\varepsilon, f\rangle_{\mathcal{X}_0} \langle \Delta v_s^\varepsilon, f\rangle_{\mathcal{X}_0}\,\mathrm{d}s \\
&\quad + a(\varepsilon)^2 \int_0^t \int_U \left(\int_{\mathbb{R}} G_s^\varepsilon(a, y, v_s^\varepsilon(y)) f(y) J(y)\,\mathrm{d}y \right)^2 \lambda(\mathrm{d}a)\,\mathrm{d}s.
\end{aligned}$$

Now we sum over a complete orthonormal system (CONS) of $\mathcal{X}_0$, $\{f_j\}_j$ to obtain

$$\begin{aligned}
\|v_t^\varepsilon\|_{\mathcal{X}_0}^2 &= 2a(\varepsilon) \int_0^t \int_U \langle v_s^\varepsilon, G_s^\varepsilon(a, \cdot, v_s^\varepsilon(\cdot))\rangle_{\mathcal{X}_0} W(\mathrm{d}s\,\mathrm{d}a) + \int_0^t \langle v_s^\varepsilon, \Delta v_s^\varepsilon\rangle_{\mathcal{X}_0}\,\mathrm{d}s \\
&\quad + a(\varepsilon)^2 \int_0^t \int_U \int_{\mathbb{R}} G_s^\varepsilon(a, y, v_s^\varepsilon(y))^2 J(y)\,\mathrm{d}y \lambda(\mathrm{d}a)\,\mathrm{d}s.
\end{aligned}$$

By Itô's formula,

$$\begin{aligned}
\|v_t^\varepsilon\|_{\mathcal{X}_0}^{2p} &= 2a(\varepsilon) p \int_U \int_0^t \|v_s^\varepsilon\|_{\mathcal{X}_0}^{2(p-1)} \langle v_s^\varepsilon, G_s^\varepsilon(a, \cdot, v_s^\varepsilon)\rangle_{\mathcal{X}_0} W(\mathrm{d}s\,\mathrm{d}a) \\
&\quad + \int_0^t p\|v_s^\varepsilon\|_{\mathcal{X}_0}^{2(p-1)} \langle v_s^\varepsilon, \Delta v_s^\varepsilon\rangle_{\mathcal{X}_0}\,\mathrm{d}s \\
&\quad + a(\varepsilon)^2 p \int_0^t \|v_s^\varepsilon\|_{\mathcal{X}_0}^{2(p-1)} \int_U \int_{\mathbb{R}} G_s^\varepsilon(a, y, v_s^\varepsilon(y))^2 J(y)\,\mathrm{d}y \lambda(\mathrm{d}a)\,\mathrm{d}s \\
&\quad + a(\varepsilon) p(p-1) \int_0^t \int_U \|v_s^\varepsilon\|_{\mathcal{X}_0}^{2(p-2)} \langle v_s^\varepsilon, G_s^\varepsilon(a, \cdot, v_s^\varepsilon(\cdot))\rangle_{\mathcal{X}_0}^2 \lambda(\mathrm{d}a)\,\mathrm{d}s.
\end{aligned}$$

Similar to Kurtz and Xiong [11], we can prove that

$$-\int_{\mathbb{R}} v_s^\varepsilon(y)(v_s^\varepsilon)'(y)J'(y)\,\mathrm{d}y = \frac{1}{2}\int_{\mathbb{R}} v_s^\varepsilon(y)^2 J''(y)\,\mathrm{d}y \le K\|v_s^\varepsilon\|_{\mathcal{X}_0}^2$$

and

$$\langle v_s^\varepsilon, \Delta v_s^\varepsilon\rangle_{\mathcal{X}_0} \le -\int_{\mathbb{R}} (v_s^\varepsilon)'(y)v_s^\varepsilon(y)J'(y)\,\mathrm{d}y \le K\|v_s^\varepsilon\|_{\mathcal{X}_0}^2.$$

Hence, with the help of the Doob and Burkholder–Davis–Gundy inequalities, we have

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t}\|v_s^\varepsilon\|_{\mathcal{X}_0}^{2p} \le{}& Ka(\varepsilon)\mathbb{E}\bigg(\int_0^t\int_U \|v_s^\varepsilon\|_{\mathcal{X}_0}^{4(p-1)}\langle v_s^\varepsilon, G_s^\varepsilon(a,y,v_s^\varepsilon(y))\rangle_{\mathcal{X}_0}^2\,\mathrm{d}s\lambda(\mathrm{d}a)\bigg)^{1/2}\\ &+K\mathbb{E}\int_0^t \|v_s^\varepsilon\|_{\mathcal{X}_0}^{2p}\,\mathrm{d}s\\ &+Ka(\varepsilon)\mathbb{E}\int_0^t\int_U\int_{\mathbb{R}}\|v_s^\varepsilon\|_{\mathcal{X}_0}^{2(p-1)}G_s^\varepsilon(a,y,v_s^\varepsilon(y))^2J(y)\,\mathrm{d}y\,\mathrm{d}s\lambda(\mathrm{d}a)\\ &+Ka(\varepsilon)\mathbb{E}\int_0^t\int_U\|v_s^\varepsilon\|_{\mathcal{X}_0}^{2(p-2)}\langle v_s^\varepsilon, G_s^\varepsilon(a,y,v_s^\varepsilon(y))\rangle_{\mathcal{X}_0}^2\,\mathrm{d}s\lambda(\mathrm{d}a).\end{aligned}$$

Now we apply Hölder's inequality and (15) to arrive at

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t}\|v_s^\varepsilon\|_{\mathcal{X}_0}^{2p} &\le Ka(\varepsilon)\mathbb{E}\bigg(\int_0^t\|v_s^\varepsilon\|_{\mathcal{X}_0}^{4p}\,\mathrm{d}s\bigg)^{1/2}+K\mathbb{E}\int_0^t\|v_s^\varepsilon\|_{\mathcal{X}_0}^{2p}\,\mathrm{d}s\\ &\le Ka(\varepsilon)\mathbb{E}\sup_{0\le s\le t}\|v_s^\varepsilon\|_{\mathcal{X}_0}^{p}\bigg(\int_0^t\|v_s^\varepsilon\|_{\mathcal{X}_0}^{2p}\,\mathrm{d}s\bigg)^{1/2}+K\mathbb{E}\int_0^t\|v_s^\varepsilon\|_{\mathcal{X}_0}^{2p}\,\mathrm{d}s\\ &\le \frac{1}{2}\mathbb{E}\sup_{0\le s\le t}\|v_s^\varepsilon\|_{\mathcal{X}_0}^{2p}+K_1\mathbb{E}\int_0^t\|v_s^\varepsilon\|_{\mathcal{X}_0}^{2p}\,\mathrm{d}s.\end{aligned}$$

The conclusion then follows from Gronwall's inequality. □

For our results, we also apply the lemma given below, the proof of which we have provided in [7].

Lemma 2. *Let $\{X_t^\varepsilon(y)\}$ be a family of random fields and suppose that $\beta_1 \in (\beta_0, \beta)$. If there exist constants $n, q, K > 0$ such that*

$$\mathbb{E}|X_{t_1}^\varepsilon(y_1) - X_{t_2}^\varepsilon(y_2)|^n \le K\mathrm{e}^{n\beta_1(|y_1|\vee|y_2|)}(|y_1-y_2|+|t_1-t_2|)^{2+q}, \tag{18}$$

then there exists a constant $\alpha > 0$ such that

$$\sup_{\varepsilon>0}\mathbb{E}\bigg|\sup_m \sup_{t_i\in[0,1],|y_i|\le m, i=1,2}\frac{|X_{t_1}^\varepsilon(y_1)-X_{t_2}^\varepsilon(y_2)|}{(|y_1-y_2|+|t_1-t_2|)^\alpha}\mathrm{e}^{-\beta m}\bigg|^n < \infty.$$

As a consequence $X_\cdot^\varepsilon \in \mathcal{C}([0,1];\mathbb{B}_\beta)$ almost surely. Furthermore, if (18) holds and

$$\sup_{\varepsilon>0}\mathbb{E}|X_{t_0}^\varepsilon(y_0)|^n < \infty \quad \textit{for some } (t_0,y_0)\in[0,1]\times\mathbb{R},$$

then

$$\sup_{\varepsilon>0}\mathbb{E}\bigg|\sup_{(t,y)\in[0,1]\times\mathbb{R}}\mathrm{e}^{-\beta|y|}|X_t^\varepsilon(y)|\bigg|^n < \infty,$$

and the family $\{X_\cdot^\varepsilon\}$ is tight in $\mathcal{C}([0,1];\mathbb{B}_\beta)$.

Lemma 3. *The solution to (10) takes values in* $\mathcal{C}([0,1];\mathbb{B}_\beta)$.

Proof. First we need the following inequalities established in [7]:

$$\mathbb{P}_1 := p_{t-s}(y_1 - x) - p_{t-s}(y_2 - x), \qquad \mathbb{P}_2 := p_{t_1-s}(y - x) - p_{t_2-s}(y - x),$$

$$\int_{\mathbb{R}} |\mathbb{P}_1|^2 \mathrm{e}^{2\beta_1|x|}\,\mathrm{d}x \le K\mathrm{e}^{2\beta_1(|y_1|\vee|y_2|)}(t-s)^{-(1/2+\alpha)}|y_1 - y_2|^\alpha, \tag{19}$$

$$\int_0^{t_1}\int_{\mathbb{R}} |\mathbb{P}_2|^2 \mathrm{e}^{2\beta_1|x|}\,\mathrm{d}x\,\mathrm{d}s \le K\mathrm{e}^{2\beta_1|y|}|t_1 - t_2|^\alpha, \tag{20}$$

and

$$\int_{t_1}^{t_2}\int_{\mathbb{R}} p^2_{t_2-s}(y-x)\mathrm{e}^{2\beta_1|x|}\,\mathrm{d}x\,\mathrm{d}s \le K|t_1 - t_2|^{\alpha/2}\mathrm{e}^{2\beta_1|y|}. \tag{21}$$

We proceed by demonstrating two cases. In the first case, we fix $t \in 0, 1]$ and let $y_1, y_2 \in \mathbb{R}$ be arbitrary such that $|y_i| \le m$ for all $i = 1, 2$. Applying the Burkholder–Davis–Gundy and Hölder inequalities, for $n > 0$, we obtain

$$\begin{aligned}
&\mathbb{E}|v_t^\varepsilon(y_1) - v_t^\varepsilon(y_2)|^n \\
&\quad= \mathbb{E}\bigg|a(\varepsilon)\int_0^t\int_U\int_{\mathbb{R}} \mathbb{P}_1 G_s^\varepsilon(a, x, v_s^\varepsilon(x))W(\mathrm{d}s\,\mathrm{d}a)\,\mathrm{d}x\bigg|^n \\
&\quad\le K\mathbb{E}\bigg(a(\varepsilon)^2\int_0^t\int_U\bigg(\int_{\mathbb{R}} \mathbb{P}_1 G_s^\varepsilon(a, x, v_s^\varepsilon(x))\,\mathrm{d}x\bigg)^2\mathrm{d}s\,\mathrm{d}a\bigg)^{n/2} \\
&\quad\le K\mathbb{E}\bigg(a(\varepsilon)^2\int_0^t\int_U\int_{\mathbb{R}} \mathbb{P}_1^2\mathrm{e}^{2\beta_1|x|}\,\mathrm{d}x\int_{\mathbb{R}} G_s^\varepsilon(a, x, v_s^\varepsilon(x))^2\mathrm{e}^{-2\beta_1|x|}\,\mathrm{d}x\lambda(\mathrm{d}a)\,\mathrm{d}s\bigg)^{n/2} \\
&\quad\le K\mathbb{E}\bigg(a(\varepsilon)^2\int_0^t\int_{\mathbb{R}} \mathbb{P}_1^2\mathrm{e}^{2\beta_1|x|}\,\mathrm{d}x\int_{\mathbb{R}}(1 + v_s^\varepsilon(x)^2 + \mathrm{e}^{2\beta_0|x|})\mathrm{e}^{-2\beta_1|x|}\,\mathrm{d}x\,\mathrm{d}s\bigg)^{n/2}.
\end{aligned}$$

By (19), we have

$$\begin{aligned}
&\mathbb{E}|v_t^\varepsilon(y_1) - v_t^\varepsilon(y_2)|^n \\
&\quad\le K\mathbb{E}\bigg(\int_0^t \mathrm{e}^{2\beta_1(|y_1|\vee|y_2|)}(t-s)^{-(1/2+\alpha)}|y_1 - y_2|^\alpha\int_{\mathbb{R}} v_s^\varepsilon(x)^2\mathrm{e}^{-2\beta_1|x|}\,\mathrm{d}x\,\mathrm{d}s\bigg)^{n/2} \\
&\quad\le \bar{M}K\mathrm{e}^{n\beta_1(|y_1|\vee|y_2|)}|y_1 - y_2|^{n\alpha/2}.
\end{aligned}$$

For the second case, we consider $y \in \mathbb{R}$ to be fixed and assume $t_1, t_2 \in [0, 1]$ to be arbitrary, then by (20) and (21),

$$\begin{aligned}
&\mathbb{E}|v_{t_1}^\varepsilon(y) - v_{t_2}^\varepsilon(y)|^n \\
&\quad\le K\mathbb{E}\bigg|a(\varepsilon)\int_0^{t_1}\int_{\mathbb{R}}\int_U \mathbb{P}_2 G_s^\varepsilon(a, x, v_s(x))W(\mathrm{d}s\,\mathrm{d}a)\,\mathrm{d}x\bigg|^n \\
&\qquad+ K\mathbb{E}\bigg|a(\varepsilon)\int_{t_1}^{t_2}\int_{\mathbb{R}}\int_U p_{t_2-s}(y-x)G_s^\varepsilon(a, x, v_s(x))W(\mathrm{d}s\,\mathrm{d}a)\,\mathrm{d}x\bigg|^n
\end{aligned}$$

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$$\leq K\mathbb{E}\bigg|\int_0^{t_1}\int_{\mathbb{R}}\mathbb{P}_2^2 e^{2\beta_1|x|}\,dx\int_{\mathbb{R}}(K+v_s^\varepsilon(x)^2)e^{-2\beta_1|x|}\,dx\,ds\bigg|^{n/2}$$
$$+K\mathbb{E}\bigg|\int_{t_1}^{t_2}\int_{\mathbb{R}}p_{t_2-s}^2(y-x)e^{2\beta_1|x|}\,dx\int_{\mathbb{R}}(K+v_s^\varepsilon(x)^2)e^{-2\beta_1|x|}\,dx\,ds\bigg|^{n/2}$$
$$\leq \bar{M}K\bigg|\int_0^{t_1}\int_{\mathbb{R}}\mathbb{P}_2^2 e^{2\beta_1|x|}\,dx\bigg|^{n/2}+\bar{M}K\bigg|\int_{t_1}^{t_2}\int_{\mathbb{R}}p_{t_2-s}^2(y-x)e^{2\beta_1|x|}\,dx\,ds\bigg|^{n/2}$$
$$\leq Ke^{n\beta_1|y|}|t_1-t_2|^{\alpha n/2}+Ke^{n\beta_1|y|}|t_1-t_2|^{\alpha n/4}$$
$$\leq Ke^{n\beta_1|y|}|t_1-t_2|^{\alpha n/4},$$

where in the last step we have used the fact that $|t_1-t_2|<1$. □

We now prove Theorem 1 by applying a technique offered by Budhiraja *et al.* [1]. To match the authors' setup, we write (10) as an infinite sum of independent Brownian motions as follows. Suppose that $\{\phi_j\}_j$ is a CONS of $L^2(U,\mathcal{U},\lambda)$ then

$$B_t^j:=\int_0^t\int_U\phi_j(a)W(ds\,da),\qquad j=1,2,\dots$$

is a sequence of independent Brownian motions by Lévy's characterization of Brownian motions. We can then present (10) in the following form:

$$v_t^\varepsilon(y)=a(\varepsilon)\sum_j\int_0^t G_s^{\varepsilon,j}(y,v_s^\varepsilon(y))\,dB_s^j+\frac{1}{2}\int_0^t\Delta v_s^\varepsilon(y)\,ds,\tag{22}$$

where $G_s^{\varepsilon,j}(y,v):=\int_U G_s^\varepsilon(a,y,v)\phi_j(a)\lambda(da)$. Similarly, the controlled PDE (11) can be written as

$$v_t(y)=\sum_j\int_0^t\int_U G(a,y,u_s^0(y))k_s^j\phi_j(a)\lambda(da)\,ds+\frac{1}{2}\int_0^t\Delta v_s(y)\,ds,$$

where $k_s^j:=\int_U h_s(a)\phi_j(a)\lambda(da)$. By the same argument as in [14], (22) has a strong solution so there exists a map $g^\varepsilon:\mathbb{B}_{\alpha,\beta_0}\times S\to\mathcal{C}([0,1];\mathbb{B}_\beta)$ such that $v^\varepsilon=g^\varepsilon(a(\varepsilon)B)$, where $B=\{B_t^j\}$. We now define

$$\mathcal{S}^N(\ell_2):=\bigg\{k\in L^2([0,1],\ell_2)\colon\int_0^1\|k_s\|_{\ell_2}^2\,ds\leq N\bigg\}.\tag{23}$$

To verify the assumption imposed by [1, Theorem 6], let $\{k^\varepsilon\}$ be a family of random variables taking values in $\mathcal{S}^N(\ell_2)$ such that $k^\varepsilon\to k$ in distribution as $\varepsilon\to 0$ and consider the SPDE

$$v_t^{\theta,\varepsilon}(y)=\theta\sum_j\int_0^t\int_{\mathbb{R}}p_{t-s}(y-x)G_s^{\varepsilon,j}(x,v_s^{\theta,\varepsilon}(x))\,dB_s^j\,dx$$
$$+\sum_j\int_0^t\int_{\mathbb{R}}p_{t-s}(y-x)G_s^{\varepsilon,j}(x,v_s^{\theta,\varepsilon}(x))k_s^{\varepsilon,j}\,dx\,ds.\tag{24}$$

We establish the tightness of $\{v^{\theta,\varepsilon}\}$ as follows.

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Lemma 4. *It holds that $v_t^{\theta,\varepsilon}(y)$ is tight in $\mathcal{C}([0,1],\mathbb{B}_\beta)$.*

Proof. According to Lemma 2, to achieve the tightness for $\{v_t^{\theta,\varepsilon}\}$ it is sufficient to show (18) for $v_t^{\theta,\varepsilon}$ and verify that $\sup_{\varepsilon>0}\mathbb{E}|v_{t_0}^{\theta,\varepsilon}(y_0)|^n<\infty$ for some $(t_0,y_0)\in[0,1]\times\mathbb{R}$. Following the same steps as in the proof of Lemma 1, we have

$$\tilde{M}:=\sup_{0<\varepsilon<1}\mathbb{E}\sup_{0\le s\le 1}\bigg(\int_{\mathbb{R}}|v_s^{\theta,\varepsilon}(x)|^2\mathrm{e}^{-2\beta_1|x|}\,\mathrm{d}x\bigg)^n<\infty. \tag{25}$$

Note that (18) can be attained for the first term on the right-hand side of (24) by exactly the same calculations performed in Lemma 3 with the use of $\tilde{M}$ given in (25) instead of $\bar{M}$ of Lemma 1. Thus, we focus on finding (18) for

$$\tilde{v}_t^{\theta,\varepsilon}(y):=\sum_j\int_0^t\int_{\mathbb{R}}p_{t-s}(y-x)G_j^{\varepsilon,j}(x,v_s^{\theta,\varepsilon}(x))k_s^{\varepsilon,j}\,\mathrm{d}x\,\mathrm{d}s.$$

Using the same method used in the proof of Lemma 3, we begin by fixing $t\in[0,1]$ and assuming y_1,y_2 to be any real numbers such that $|y_i|\le m$ for $i=1,2$ and $m\in\mathbb{N}$. Recall that $\mathbb{P}_1:=p_{t-s}(y_1-x)-p_{t-s}(y_2-x)$. With the help of the Cauchy–Schwartz inequality, (15), and our result (19), we obtain the following estimates:

$$\begin{aligned}
&\mathbb{E}|\tilde{v}_t^{\theta,\varepsilon}(y_1)-\tilde{v}_t^{\theta,\varepsilon}(y_2)|^n\\
&\quad=\mathbb{E}\bigg|\int_0^t\int_{\mathbb{R}}\mathbb{P}_1\sum_j G_s^{\varepsilon,j}(x,v_s^{\theta,\varepsilon}(x))k_s^{\varepsilon,j}\,\mathrm{d}x\,\mathrm{d}s\bigg|^n\\
&\quad\le\mathbb{E}\bigg|\int_0^t\int_{\mathbb{R}}\mathbb{P}_1\bigg(\sum_j G_s^{\varepsilon,j}(x,v_s^{\theta,\varepsilon}(x))^2\bigg)^{1/2}\|k_s^{\varepsilon}\|_{\ell_2}\,\mathrm{d}x\,\mathrm{d}s\bigg|^n\\
&\quad\le\mathbb{E}\bigg|\bigg(\int_0^t\bigg(\int_{\mathbb{R}}\mathbb{P}_1\sqrt{K(1+v_s^{\theta,\varepsilon}(x)^2+\mathrm{e}^{2\beta_0|x|})}\,\mathrm{d}x\bigg)^2\mathrm{d}s\bigg)^{1/2}\bigg(\int_0^t\|k_s^{\varepsilon}\|_{\ell_2}^2\,\mathrm{d}s\bigg)^{1/2}\bigg|^n\\
&\quad\le\mathbb{E}\bigg|\int_0^t\bigg(\int_{\mathbb{R}}\mathbb{P}_1\sqrt{K(1+v_s^{\theta,\varepsilon}(x)^2+\mathrm{e}^{2\beta_0|x|})}\,\mathrm{d}x\bigg)^2\mathrm{d}x\bigg|^{n/2}N^{n/2}\\
&\quad\le K\mathrm{e}^{n\beta_1(|y_1|\vee|y_2|)}|y_1-y_2|^{\alpha n/2},
\end{aligned}$$

where $N>0$ is the constant given by (23). Furthermore, the case for $0\le t_1<t_2\le 1$ arbitrary and $y\in\mathbb{R}$ fixed can be expressed as

$$\begin{aligned}
\mathbb{E}|\tilde{v}_{t_1}^{\theta,\varepsilon}(y)-\tilde{v}_{t_2}^{\theta,\varepsilon}(y)|^n&\le K\mathbb{E}\bigg|\int_{t_1}^{t_2}\int_{\mathbb{R}}\mathbb{P}_2\sum_j G_s^{\varepsilon,j}(x,v_s(x))k_s^{\varepsilon,j}\,\mathrm{d}x\,\mathrm{d}s\bigg|^n\\
&\quad+K\mathbb{E}\bigg|\int_0^{t_1}\int_{\mathbb{R}}p_{t_2-s}^2(y-x)\sum_j G_s^{\varepsilon,j}(x,v_s(x))k_s^{\varepsilon,j}\,\mathrm{d}x\,\mathrm{d}s\bigg|^n\\
&\le K\mathrm{e}^{n\beta_1|y|}|t_1-t_2|^{n\alpha/4},
\end{aligned}$$

where $\mathbb{P}_2:=p_{t_1-s}(y-x)-p_{t_2-s}(y-x)$. □

Thus, $\{v_t^{\theta,\varepsilon}\}$ is tight and for the assumption of [1, Theorem 6] to be satisfied we let $\theta=0$ for its first part and $\theta=a(\varepsilon)$ for the second part and apply the Prohorov theorem and so, by [1, Theorem 6], our Theorem 1 can be deduced.

4. Moderate deviations for SBM and FVP

We devote this section to the proofs of Theorems 2 and 3. Recall that

$$\omega_t^{\varepsilon}(\mathrm{d}y) := \frac{a(\varepsilon)}{\sqrt{\varepsilon}}(\mu_t^{\varepsilon}(\mathrm{d}y) - \mu_t^0(\mathrm{d}y)),$$

where in the case of SBM, $u_t^{\varepsilon}(y) := \int_0^y \mu_t^{\varepsilon}(\mathrm{d}x)$. Then, based on (9), we can write $v_t^{\varepsilon}(y) := \int_0^y \omega_t^{\varepsilon}(\mathrm{d}x)$. Similarly, for the FVP we have $u_t^{\varepsilon}(y) := \int_{-\infty}^y \mu_t^{\varepsilon}(\mathrm{d}x)$ which leads to $v_t^{\varepsilon}(y) := \int_{-\infty}^y \omega_t^{\varepsilon}(\mathrm{d}x)$. Analogous to [7, Lemma 6], it follows that for the set of functions with finite variations, $\mathcal{A}$, the map $\xi \colon \mathbb{B}_\beta \cap \mathcal{A} \to \mathcal{M}_\beta^{\mathrm{S}}(\mathbb{R})$ given as $\xi(u)(B) = \int \mathbf{1}_B(y)\,\mathrm{d}u(y)$ for all $B \in \mathbb{B}(\mathbb{R})$ is continuous. Therefore, map $\eta \colon \mathcal{C}([0,1];\mathbb{B}_\beta) \to \mathcal{C}([0,1];\mathcal{M}_\beta^{\mathrm{S}}(\mathbb{R}))$ defined as $\eta(v)_t = \xi(v_t)$ is also continuous. Since SBM and FVP can be written as $\omega_t^{\varepsilon}(\mathrm{d}y) = \eta(v^{\varepsilon})_t([0,y])$ and $\omega_t^{\varepsilon}(\mathrm{d}y) = \eta(v^{\varepsilon})_t((-\infty,y])$, respectively, then in both cases $\omega_t^{\varepsilon}(y)$ is a continuous function of v_t^{ε}. Based on our LDP result for v_t^{ε} given in Theorem 1, we can conclude by the contraction principle that $\{\omega_t^{\varepsilon}\}$ also satisfies the LDP for both models.

Our remaining task is to identify an explicit representation of the models' MDP rate functions. According to the contraction principle, rate functions for SBM and FVP are given by $\inf\{I(u) \colon u \in \eta^{-1}(\omega)\}$. Since η is injective, we then aim to find the rate functions following the form given by (12).

As for SBM, (4) satisfies the general SPDE (7) with the following properties:

$$U = \mathbb{R}, \qquad \lambda(\mathrm{d}a) = \mathrm{d}a, \qquad G(a,y,u) = \mathbf{1}_{\{0 \le a \le u\}} + \mathbf{1}_{\{u \le a \le 0\}}.$$

Then using the controlled PDE (11), we have

$$\begin{aligned}
\langle \omega_t, f\rangle &= \langle \partial_x v_t, f\rangle \\
&= -\langle v_t, f'\rangle \\
&= -\int_0^t\int_0^{\infty}\int_0^{u_s^0(y)} h_s(a) f'(y)\,\mathrm{d}a\,\mathrm{d}y\,\mathrm{d}s - \int_0^t\int_{-\infty}^0\int_{u_s^0(y)}^0 h_s(a) f'(y)\,\mathrm{d}a\,\mathrm{d}y\,\mathrm{d}s \\
&\quad - \int_0^t \left\langle \frac{1}{2}\Delta v_s, f'\right\rangle \mathrm{d}s \\
&= \int_0^t\int_0^{\infty} h_s(a) f((u_s^0)^{-1}(a))\,\mathrm{d}a\,\mathrm{d}s - \int_0^t\int_{-\infty}^0 h_s(a) f((u_s^0)^{-1}(a))\,\mathrm{d}a\,\mathrm{d}s \\
&\quad + \int_0^t \left\langle \frac{1}{2}\omega_s, \Delta f\right\rangle \mathrm{d}s \\
&= \int_0^t\int_0^{\infty} h_s(u_s^0(y)) f(y)\,\mathrm{d}u_s^0(y)\,\mathrm{d}s - \int_0^t\int_{-\infty}^0 h_s(u_s^0(y)) f(y)\,\mathrm{d}u_s^0(y)\,\mathrm{d}s \\
&\quad + \int_0^t \left\langle \frac{1}{2}\Delta^*\omega_s, f\right\rangle \mathrm{d}s \\
&= \int_0^t \langle h_s(u_s^0)\,\mathrm{sgn}(\cdot)\mu_s^0, f\rangle\,\mathrm{d}s + \frac{1}{2}\int_0^t \langle \Delta^*\omega_s, f\rangle\,\mathrm{d}s.
\end{aligned}$$

Thus,

$$h_t(u_t^0(y))\,\mathrm{sgn}(y) = \frac{\mathrm{d}(\dot{\omega}_t - \Delta^*\omega_t/2)}{\mathrm{d}\mu_t^0}(y).$$

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Note that

$$\int_{\mathbb{R}} |h_t(a)|^2 \, \mathrm{d}a = \int_{\mathbb{R}} |h_t(u_t^0(y))|^2 \, \mathrm{d}u_t^0(y) = \int_{\mathbb{R}} |h_t(u_t^0(y))|^2 \, \mathrm{d}\mu_t^0(y).$$

Letting the right-hand side of (13) be denoted as $I_0(\mu)$, if $I(\mu) < \infty$ then $I(\mu)$ given in (12) with $U = \mathbb{R}$ is equal to $I_0(\mu)$. For the $I_0(\mu) < \infty$ case we can reverse the above calculations to obtain $I_0(\mu) = I(\mu)$.

Similarly for the FVP, since FVP satisfies the general SPDE (7) with

$$U = [0, 1], \qquad \lambda(\mathrm{d}a) = \mathrm{d}a, \qquad G(a, y, u) = \mathbf{1}_{\{a<u\}} - u,$$

then

$$\begin{aligned}
\langle \omega_t, f\rangle &= -\langle v_t, f'\rangle \\
&= -\int_0^t \int_{\mathbb{R}} \int_0^{u_s^0(y)} h_s(a) f'(y) \, \mathrm{d}a \, \mathrm{d}y \, \mathrm{d}s \\
&\quad + \int_0^t \int_{\mathbb{R}} \int_0^1 u_s^0(y) h_s(a) f'(y) \, \mathrm{d}a \, \mathrm{d}y \, \mathrm{d}s - \int_0^t \left\langle \frac{1}{2}\Delta v_s(y), f'\right\rangle \mathrm{d}s \\
&= \int_0^t \langle h_s(u_s^0)\mu_s^0, f\rangle \, \mathrm{d}s - \int_0^t \left\langle \int_0^1 h_s(a) \, \mathrm{d}a \mu_s^0, f\right\rangle \mathrm{d}s + \int_0^t \left\langle \frac{1}{2}\Delta^* \omega_s, f\right\rangle \mathrm{d}s.
\end{aligned}$$

Thus,

$$\dot{\omega}_t - \frac{1}{2}\Delta^*\omega_t = h_t(u_t^0(y))\mu_t^0 - \int_0^1 h_t(a) \, \mathrm{d}a \mu_t^0.$$

Our goal is to find the infimum of $\int_0^1 |h_s(a)|^2 \, \mathrm{d}a$ over $h_s(a)$ satisfying (11). We note that if h satisfies (11) then $g_s(a) := h_s(a) - \int_0^1 h_s(a) \, \mathrm{d}a$ also satisfies the same equation. It is well known that the second moment is minimized when it is centralized. Therefore, we replace $h_s(a)$ by $g_s(a)$ in the definition of the rate function and write it as

$$\int_0^1 |g_s(a)|^2 \, \mathrm{d}a = \int_0^1 \left| \frac{\mathrm{d}(\dot{\omega}_t - \Delta^*\omega_t/2)}{\mathrm{d}\mu_t^0}(y)\right|^2 \mathrm{d}\mu_t^0(y),$$

in (12) to arrive at (14) for the $I(v) < \infty$ case and based on a similar argument as in the case of SBM we obtain (14). Thus, MDP is proved for the two models.

Acknowledgements

The authors would like to thank the anonymous referee for his/her detailed comments, which improved the paper. Also gratitude goes to the University of Tennessee, Knoxville where this article was written. J. Xiong was partially supported by FDCT 076/2012/A3.

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