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## ABSTRACT

Fix  $d \geq 2$  and a field  $k$  such that  $\text{char } k \nmid d$ . Assume that  $k$  contains the  $d$ th roots of 1. Then the irreducible components of the curves over  $k$  parameterizing preperiodic points of polynomials of the form  $z^d + c$  are geometrically irreducible and have gonality tending to  $\infty$ . This implies the function field analogue of the strong uniform boundedness conjecture for preperiodic points of  $z^d + c$ . It also has consequences over number fields: it implies strong uniform boundedness for preperiodic points of bounded eventual period, which in turn reduces the full conjecture for preperiodic points to the conjecture for periodic points. Our proofs involve a novel argument specific to finite fields, in addition to more standard tools such as the Castelnuovo–Severi inequality.

## 1. Introduction

### 1.1 Dynatomic curves

Fix an integer  $d \geq 2$ . Let  $k$  be a field such that  $\text{char } k \nmid d$ . View  $f = f_c := z^d + c$  as a polynomial in  $z$  with coefficients in  $k[c]$ . Let  $f^n(z)$  be the  $n$ th iterate of  $f$ ; in particular,  $f^0(z) := z$ . If  $n$  and  $m$  are nonnegative integers with  $n > m$ , then any irreducible factor of  $f^n(z) - f^m(z) \in k[z, c]$  defines an affine curve over  $k$ . By a dynatomic curve over  $k$ , we mean any such curve, or its smooth projective model. Any  $k$ -point on such a curve yields  $c_0 \in k$  equipped with a preperiodic point in  $k$ , that is, an element  $z_0 \in k$  that under iteration  $z^d + c_0$  eventually enters a cycle; the length of the cycle is called the eventual period. We consider two dynatomic curves to be different if the corresponding closed subschemes of  $\mathbb{A}_k^2$  are distinct. Section 2 describes all dynatomic curves in characteristic 0 explicitly.

### 1.2 Gonality

Let  $\bar{k}$  be an algebraic closure of  $k$ . Let  $\mu_d = \{x \in \bar{k} : x^d = 1\}$ .

For a curve  $X$  over  $k$ , let  $X_{\bar{k}} = X \times_k \bar{k}$ . If  $X$  is irreducible, define the gonality  $\gamma(X)$  of  $X$  as the least possible degree of a dominant rational map  $X \dashrightarrow \mathbb{P}_k^1$ . If  $X$  is geometrically irreducible, define its  $\bar{k}$ -gonality as  $\gamma(X_{\bar{k}})$ .

The following theorem, and its consequence, Theorem 1.7, are our main results.

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**THEOREM 1.1.** Fix  $d \geq 2$  and  $k$  such that  $\text{char } k \nmid d$ . Suppose that  $\mu_d \subset k$ .

- (a) Every dynatomic curve over  $k$  is geometrically irreducible.
- (b) If the dynatomic curves over  $k$  are listed in any order, their gonalitys tend to  $\infty$ .

*Remark 1.2.* Part (a) of Theorem 1.1 can fail if  $\mu_d \not\subset k$ . See Remark 2.2.

*Remark 1.3.* In proving Theorem 1.1 in positive characteristic, we face the challenge that we do not know explicitly what the dynatomic curves are, since we are not sure whether the known factors of the polynomials  $f^n(z) - f^m(z)$  are irreducible. Some results regarding the irreducibility of dynatomic curves in positive characteristic may be found in [DKO<sup>+</sup>19], but they are not sufficient for proving the full result. Instead, to overcome the difficulties, we use a novel argument that is specific to finite fields to prove that the degrees of dynatomic curves over the  $c$ -line must tend to infinity even though we do not know precisely what these curves are; see § 4.

Let us now introduce notation for our next result. Let  $\mu(n)$  denote the Möbius  $\mu$ -function. Then  $f^n(z) - z = \prod_{e|n} \Phi_e(z, c)$ , where

$$\Phi_n(z, c) := \prod_{e|n} (f^e(z) - z)^{\mu(n/e)} \in k[z, c]. \tag{1}$$

Let  $Y_1^{\text{dyn}}(n)$  be the curve defined by  $\Phi_n(z, c) = 0$  in  $\mathbb{A}_k^2$ , and let  $X_1^{\text{dyn}}(n)$  be the normalization of its projective closure. To simplify notation, we omit the superscript *dyn* from now on. General points of  $X_1(n)$  parameterize polynomials of the form  $z^d + c$  equipped with a point of exact order  $n$ .

The morphism  $(z, c) \mapsto (f(z), c)$  restricts to an order- $n$  automorphism of  $Y_1(n)$ , so it induces an order- $n$  automorphism  $\sigma$  of  $X_1(n)$ . The quotient of  $X_1(n)$  by the cyclic group generated by  $\sigma$  is called  $X_0(n)$ . If  $\text{char } k = 0$ , it is known that  $X_1(n)$  is geometrically irreducible (see [Bou92, LS94, Mor96]), so  $X_0(n)$  is too.

**THEOREM 1.4.** Fix  $d \geq 2$  and a field  $k$  of characteristic 0. Then

$$\gamma(X_0(n)) > \left( \frac{1}{2} - \frac{1}{2d} - o(1) \right) n$$

as  $n \rightarrow \infty$ . In particular,  $\gamma(X_0(n)) \rightarrow \infty$ .

*Remark 1.5.* Our definition of dynatomic curve does not include quotient curves such as  $X_0(n)$ , so the conclusion  $\gamma(X_0(n)) \rightarrow \infty$  of Theorem 1.4 does not follow from Theorem 1.1(b). In fact, our logic runs in the opposite direction: we use Theorem 1.4 in the proof of the characteristic-0 case of Theorem 1.1(b).

*Remark 1.6.* Although the lower bound in Theorem 1.4 is linear in  $n$ , the best upper bound we know,  $\gamma(X_0(n)) \leq (1 + o(1))d^n/n$  (see Proposition 3.1(b)), is exponential in  $n$ .

To prove Theorem 1.4, we use that  $X_0(n)$  already has a morphism to  $\mathbb{P}^1$  of degree lower than expected for its genus, namely  $X_0(n) \xrightarrow{c} \mathbb{P}^1$ . If it also had a morphism to  $\mathbb{P}^1$  of bounded degree, then the Castelnuovo–Severi inequality would make the genus of  $X_0(n)$  smaller than it actually is, a contradiction. See § 3 for details.

To prove Theorem 1.1(b), we use different arguments in characteristic 0 and characteristic  $> 0$ .

In characteristic 0, we use that each dynatomic curve dominates  $X_1(n)$  and hence also  $X_0(n)$  for some  $n$ , so by Theorem 1.4 its gonality is large when  $n$  is large; this lets us reduce to proving a gonality lower bound for the dynatomic curves above  $X_1(n)$  for each fixed  $n$ . The latter curves for fixed  $n$  come in towers, and we use the Castelnuovo–Severi inequality to work our way up each tower. See §3.

In characteristic  $p$ , we prove that the irreducible components of  $f^n(z) - f^m(z)$  have large degree over the  $c$ -line, and we use that to prove that over the finite field  $\mathbb{F}_q := \mathbb{F}_p(\mu_d)$  their smooth projective models have so many  $\mathbb{F}_q$ -points over  $c = \infty$  that their  $\mathbb{F}_q$ -gonalities must be large. Finally, we use a result controlling how gonality of a curve changes when the base field is enlarged. See §4.

### 1.3 Uniform boundedness of preperiodic points

The growth of gonality of classical modular curves implies the strong uniform boundedness theorem for torsion points on elliptic curves over function fields (the function field analogue of Merel’s theorem [Mer96]); see [NS96, Theorem 0.3]. Similarly, from Theorem 1.1 we will deduce the following function field analogue of a case of the Morton–Silverman conjecture [MS94, p. 100].

**THEOREM 1.7** (Strong uniform boundedness theorem for preperiodic points over function fields). *Fix  $d \geq 2$  and a field  $k$  such that  $\text{char } k \nmid d$ . Let  $K$  be the function field of an integral curve over  $k$ . Fix a positive integer  $D$ . Then there exists  $B = B(d, K, D) > 0$  such that, for every field extension  $L \supseteq K$  of degree  $\leq D$  and every  $c \in L$  not algebraic over  $k$ , the number of preperiodic points of  $z^d + c$  in  $L$  is at most  $B$ . If  $k$  is finite, the same holds with the words ‘not algebraic over  $k$ ’ deleted.*

*Remark 1.8.* As far as we know, Theorem 1.7 is the first theorem proving strong uniform boundedness of preperiodic points for all members of a nontrivial algebraic family of maps over a global field. See [DKO<sup>+</sup>19] for some related results and arguments; in particular, that article also explains the connection between uniform boundedness and gonality and geometric irreducibility, and proves geometric irreducibility of curves such as  $\Phi_n(z, c) = 0$  modulo  $p$  for many primes  $p$ . Our innovation, as mentioned above, is to show how to use finite fields to get the gonality and uniform boundedness results even without knowing that  $\Phi_n(z, c) = 0$  modulo  $p$  is geometrically irreducible.

*Remark 1.9.* Theorem 1.7 is best possible in the sense that the words ‘not algebraic over  $k$ ’ cannot be removed in general. For example, if  $k$  is algebraically closed and  $c \in k$ , then all preperiodic points of  $z^d + c$  lie in  $k$  and hence also in  $K$  and  $L$ ; the number of them is generally not even finite, let alone uniformly bounded!

Another application of Theorem 1.1 is the following result, which over *number fields* provides a uniform bound on preperiodic points having a bounded eventual period.

**THEOREM 1.10.** *Fix integers  $d \geq 2$ ,  $D \geq 1$ , and  $N \geq 1$ . Then there exists  $B = B(d, D, N) > 0$  such that, for every number field  $K$  satisfying  $[K : \mathbb{Q}] \leq D$  and every  $c \in K$ , the number of preperiodic points of  $z^d + c$  in  $K$  with eventual period at most  $N$  is at most  $B$ .*

Theorems 1.7 and 1.10 are proved in §5. Theorem 1.10 implies that the strong uniform boundedness conjecture for *periodic* points over number fields implies the strong uniform boundedness conjecture for *preperiodic* points over number fields, as we now explain.

**COROLLARY 1.11.** Fix integers  $d \geq 2$  and  $D \geq 1$ . Suppose that there exists a bound  $N = N(d, D)$  such that, for every number field  $K$  satisfying  $[K : \mathbb{Q}] \leq D$  and every  $c \in K$ , every periodic point of  $z^d + c$  in  $K$  has period at most  $N$ . Then there exists a bound  $B' = B'(d, D)$  such that, for every number field  $K$  satisfying  $[K : \mathbb{Q}] \leq D$  and every  $c \in K$ , the number of preperiodic points of  $z^d + c$  in  $K$  is at most  $B'$ .

*Proof.* By assumption, if  $[K : \mathbb{Q}] \leq D$  and  $c \in K$ , then the preperiodic points of  $z^d + c$  in  $K$  having eventual period at most  $N$  are all the preperiodic points in  $K$ . Therefore the bound  $B(d, D, N)$  of Theorem 1.10 is actually a bound on the total number of preperiodic points in  $K$ . Take  $B' = B(d, D, N) = B(d, D, N(d, D))$ .  $\square$

*Remark 1.12.* Eliminating the hypothesis that  $N(d, D)$  exists in Corollary 1.11 seems extremely difficult. Proving that  $N(2, 1)$  exists is a problem that has been studied for over two decades, and even this smallest case seems far out of reach of existing methods. It amounts to ruling out all sufficiently large periods  $n$  for quadratic polynomials over  $\mathbb{Q}$ , and entire articles have been devoted to ruling out just periods 4 and 5 [Mor98b, FPS97]. Going further has so far relied on major conjectures: the Birch and Swinnerton-Dyer conjecture rules out period 6 [Sto08], and the  $abc$  conjecture and its generalizations contained in one of Vojta’s conjectures imply uniform boundedness for preperiodic points of  $z^d + c$  over any fixed number field [Loo19].

### 2. Classification of dynatomic curves

For  $m, n \geq 1$ , let  $Y_1(m, n)$  be the curve over  $k$  whose general points parameterize polynomials  $z^d + c$  equipped with a preperiodic point that after exactly  $m$  steps enters an  $n$ -cycle. This curve is the zero locus in  $\mathbb{A}_k^2$  of the polynomial

$$\Phi_{m,n}(z, c) := \frac{\Phi_n(f^m(z), c)}{\Phi_n(f^{m-1}(z), c)}.$$

For a general point  $(z, c) \in Y_1(1, n)$ , the elements  $z$  and  $f^n(z)$  are distinct preimages of  $f(z)$ , so  $z = \zeta f^n(z)$  for some  $\zeta \in \mu_d - \{1\}$ . Suppose that  $\mu_d \subseteq k$ . For each  $\zeta \in \mu_d - \{1\}$ , let  $Y_1(1, n)^\zeta$  be the subscheme of  $Y_1(1, n)$  defined by the condition  $z = \zeta f^n(z)$ , so

$$Y_1(1, n) = \bigcup_{\zeta \in \mu_d - \{1\}} Y_1(1, n)^\zeta.$$

Both  $(z, c) \mapsto (f(z), c)$  and  $(z, c) \mapsto (\zeta^{-1}z, c)$  define isomorphisms  $Y_1(1, n)^\zeta \rightarrow Y_1(1, n)$ . In particular,  $Y_1(1, n)^\zeta$  equals the curve  $\Phi_n(\zeta^{-1}z, c) = 0$  in  $\mathbb{A}_k^2$ . For  $m \geq 2$ , let  $Y_1(m, n)^\zeta$  be the inverse image of  $Y_1(1, n)^\zeta$  under

$$\begin{aligned} Y_1(m, n) &\longrightarrow Y_1(1, n) \\ (z, c) &\longmapsto (f^{m-1}(z), c). \end{aligned}$$

Then, for any  $m, n \geq 1$ ,

$$Y_1(m, n) = \bigcup_{\zeta \in \mu_d - \{1\}} Y_1(m, n)^\zeta, \tag{2}$$

and  $Y_1(m, n)^\zeta$  equals the curve  $\Phi_n(\zeta^{-1}f^{m-1}(z), c) = 0$  in  $\mathbb{A}_k^2$ . The decomposition (2) corresponds to a factorization

$$\Phi_{m,n}(z, c) = \prod_{\zeta \in \mu_d - \{1\}} \Phi_n(\zeta^{-1}f^{m-1}(z), c). \tag{3}$$

The following theorem is a collection of results from [Bou92, LS94, Mor96, Gao16].

**THEOREM 2.1.** *Let  $k$  be a field of characteristic 0 such that  $\mu_d \subset k$ . Then the curves  $Y_1(n)$  for  $n \geq 1$  and the curves  $Y_1(m, n)^\zeta$  for  $m, n \geq 1$  and  $\zeta \in \mu_d - \{1\}$  are irreducible, so they are all the dynatomic curves over  $k$ .*

Computer experiments (at least for  $d = 2$ ) suggest that Theorem 2.1 holds for any field  $k$  such that  $\text{char } k \nmid d$  and  $\mu_d \subset k$ , but in positive characteristic this remains unproved. See [DKO<sup>+</sup>19], especially Theorems B and D, for progress in this direction.

*Remark 2.2.* If in Theorem 2.1 we drop the hypothesis that  $\mu_d \subset k$ , then the irreducible components of  $Y_1(m, n)$  over  $k$  are in bijection with the  $\text{Gal}(k(\mu_d)/k)$ -orbits in  $\mu_d - \{1\}$ . An irreducible component corresponding to an orbit of size greater than 1 is not geometrically irreducible.

### 3. Gonality in characteristic 0

Given a geometrically irreducible curve  $X$  over  $k$ , let  $g(X)$  denote the genus of its smooth projective model. Let  $D_0(n)$  be the degree of the morphism  $X_0(n) \xrightarrow{c} \mathbb{P}^1$ . Similarly, let  $D_1(n) = \text{deg}(X_1(n) \xrightarrow{c} \mathbb{P}^1)$ .

**PROPOSITION 3.1.** *Fix  $d \geq 2$ , and fix a field  $k$  of characteristic 0.*

- (a) *We have  $D_1(n) = (1 + o(1))d^n$  as  $n \rightarrow \infty$ .*
- (b) *We have  $D_0(n) = (1 + o(1))d^n/n$  as  $n \rightarrow \infty$ .*
- (c) *We have  $g(X_0(n)) > (1/2 - 1/2d - o(1))d^n$  as  $n \rightarrow \infty$ .*
- (d) *The Galois group of (the Galois closure of) the covering  $X_0(n) \xrightarrow{c} \mathbb{P}^1$  is the full symmetric group  $S_{D_0(n)}$ .*

*Proof.* We may assume  $k = \mathbb{C}$ .

(a) See the proof of [Mor96, Theorem 13(d)]. The estimate is deduced from  $D_1(n) = \text{deg}_z \Phi_n = \sum_{e|n} \mu(n/e)d^e$ , which follows from (1).

(b) The first morphism in the tower  $X_1(n) \rightarrow X_0(n) \xrightarrow{c} \mathbb{P}^1$  has degree  $n$ , so  $D_0(n) = D_1(n)/n$ . Substitute (a) into this.

(c) More precisely,  $g(X_0(n)) > (1/2 - 1/2d - 1/n)d^n + O(nd^{n/2})$  as  $n \rightarrow \infty$ , by [Mor96, Theorem 13(d)].

(d) This is a consequence of work by Bousch [Bou92] for  $d = 2$ , and Lau and Schleicher [LS94] for  $d > 2$ . See also [Mor98a, Theorem B] and [Sch17]. □

A well-known strategy for obtaining lower bounds on gonality (cf. [NS96] and [Poo07, §2]) involves the following result, which will tell us roughly that if a high-genus curve has a relatively low-degree map to a low-genus curve, it cannot have a second such map that is independent of the first.

**PROPOSITION 3.2** (Castelnuovo–Severi inequality). *Let  $F, F_1, F_2$  be function fields of curves over  $k$ , of genera  $g, g_1, g_2$ , respectively. Suppose that  $F_i \subseteq F$  for  $i = 1, 2$  and the compositum of  $F_1$  and  $F_2$  in  $F$  equals  $F$ . Let  $d_i = [F : F_i]$  for  $i = 1, 2$ . Then*

$$g \leq d_1g_1 + d_2g_2 + (d_1 - 1)(d_2 - 1).$$

*Proof.* See [Sti09, Theorem 3.11.3]. □

*Proof of Theorem 1.4.* Let  $X_0(n) \xrightarrow{h} \mathbb{P}^1$  be a dominant rational map of minimal degree.

*Case I:*  $h$  factors through  $X_0(n) \xrightarrow{c} \mathbb{P}^1$ . Then

$$\deg h \geq D_0(n) = (1 + o(1)) \frac{d^n}{n}$$

by Proposition 3.1(b), so  $\deg h$  is much larger than  $n$  when  $n$  is large.

*Case II:*  $h$  does not factor through  $X_0(n) \xrightarrow{c} \mathbb{P}^1$ . Then the compositum of  $k(c)$  and  $k(h)$  in the function field  $k(X_0(n))$  is strictly larger than  $k(c)$ . Because of the Galois group (Proposition 3.1(d)), the only nontrivial extension of  $k(c)$  in  $k(X_0(n))$  is the whole field  $k(X_0(n))$ . Thus  $k(c)$  and  $k(h)$  generate  $k(X_0(n))$ . By Proposition 3.2,

$$g(X_0(n)) \leq (D_0(n) - 1)(\deg h - 1).$$

Thus

$$\deg h \geq 1 + \frac{g(X_0(n))}{D_0(n) - 1} = \left( \frac{1}{2} - \frac{1}{2d} - o(1) \right) n$$

as  $n \rightarrow \infty$ , by Proposition 3.1(b),(c). □

The following lemma, which says in particular that  $Y_1(m, n)^\zeta \xrightarrow{z} \mathbb{A}^1$  is étale above 0, will yield genus inequalities to combine with the Castelnuovo–Severi inequality in the proof of Theorem 1.1.

**LEMMA 3.3.** *Let  $k$  be a field of characteristic 0 such that  $\mu_d \subseteq k$ . Let  $m$  and  $n$  be positive integers, and let  $\zeta \in \mu_d - \{1\}$ . Then the polynomial  $\Phi_n(\zeta^{-1}f^{m-1}(0), c) \in k[c]$  has only simple roots, and their number is  $d^{m-2}D_1(n)$  if  $m \geq 2$ .*

*Proof.* First suppose that  $n$  does not divide  $m - 1$ . In this case, the roots of  $\Phi_{m,n}(0, c)$  are distinct by [HT15, Theorem 1.1], and therefore the roots of  $\Phi_n(\zeta^{-1}f^{m-1}(0), c)$  are distinct by (3).

Now suppose that  $n$  divides  $m - 1$ . By [Eps12, Proposition A.1], the roots of  $\Phi_n(0, c)$  are simple. If  $c$  is such a root, then the polynomial  $f = f_c$  satisfies  $f^n(0) = 0$ , so  $f^{m-1}(0) = 0$  and  $\Phi_n(\zeta^{-1}f^{m-1}(0), c) = 0$ . Thus  $\Phi_n(0, c)$  divides  $\Phi_n(\zeta^{-1}f^{m-1}(0), c)$ . The factorization (3) yields

$$\frac{\Phi_{m,n}(0, c)}{\Phi_n(0, c)^{d-1}} = \prod_{\zeta \in \mu_d - \{1\}} \frac{\Phi_n(\zeta^{-1}f^{m-1}(0), c)}{\Phi_n(0, c)}. \tag{4}$$

By [HT15, Theorem 1.1],  $\Phi_{m,n}(0, c)/\Phi_n(0, c)^{d-1}$  has only simple roots, none of which are also roots of  $\Phi_n(0, c)$ . Combining this with (4) shows that  $\Phi_n(\zeta^{-1}f^{m-1}(0), c)$  has only simple roots.

It remains to prove  $\deg \Phi_n(\zeta^{-1}f^{m-1}(0), c) = d^{m-2}D_1(n)$ . In fact, this is [Gao16, Lemma 4.8]. In our notation, the argument is as follows. By induction on  $m$ , the degree of the polynomial  $f^{m-1}(0) \in k[c]$  is  $d^{m-2}$  if  $m \geq 2$ . Hence, by induction on  $e$ , we have  $\deg f^e(\zeta^{-1}f^{m-1}(0)) = d^e d^{m-2}$  for each  $e \geq 0$ . Thus the  $c$ -degree of  $f^e(\zeta^{-1}f^{m-1}(0)) - \zeta^{-1}f^{m-1}(0)$  is  $d^{m-2}$  times the  $z$ -degree of  $f^e(z) - z$  for each  $e \geq 1$ . Substituting  $\zeta^{-1}f^{m-1}(0)$  for  $z$  in (1) shows that

$$\deg_c \Phi_n(\zeta^{-1}f^{m-1}(0), c) = d^{m-2} \deg_z \Phi_n(z, c) = d^{m-2}D_1(n). \tag{5} \quad \square$$

*Proof of Theorem 1.1 in characteristic 0.*

(a) By Theorem 2.1, the dynatomic curves over  $k$  are the curves  $Y_1(n)$  and  $Y_1(m, n)^\zeta$ , and they are geometrically irreducible.

(b) The curves  $Y_1(n)$  and  $Y_1(m, n)^\zeta$  dominate  $X_0(n)$ , so their gonality is at least the gonality of  $X_0(n)$ , by [Poo07, Proposition A.1(vii)]. In light of Theorem 1.4, it remains to prove that in each tower

$$\dots \longrightarrow Y_1(m, n)^\zeta \longrightarrow Y_1(m - 1, n)^\zeta \longrightarrow \dots \longrightarrow Y_1(1, n)^\zeta$$

for fixed  $n$  and  $\zeta$ , the gonality tends to  $\infty$  as  $m \rightarrow \infty$ .

Let  $g_m = g(Y_1(m, n)^\zeta)$  and  $\gamma_m = \gamma(Y_1(m, n)^\zeta)$ . Let  $Y_1(m, n)^\zeta \xrightarrow{h} \mathbb{P}^1$  be a dominant rational map of the minimal degree  $\gamma_m$ .

*Case I:  $h$  factors through a curve  $Z$  with  $k(Y_1(m - 1, n)^\zeta) \subseteq k(Z) \subsetneq k(Y_1(m, n)^\zeta)$ .* These inclusions imply  $\gamma_{m-1} \leq \gamma(Z)$  (by [Poo07, Proposition A.1(vii)]) and  $2\gamma(Z) \leq \deg h = \gamma_m$ , respectively. Combining these yields  $\gamma_m \geq 2\gamma_{m-1}$ . Thus  $\gamma_m$  grows, by induction on  $m$ .

*Case II:  $h$  does not factor through any such curve  $Z$ .* Denote by  $\pi_m$  the degree- $d$  map

$$\begin{aligned} \pi_m : Y_1(m, n)^\zeta &\longrightarrow Y_1(m - 1, n)^\zeta \\ (z, c) &\longmapsto (f(z), c), \end{aligned}$$

and let  $\tilde{\pi}_m : X_1(m, n)^\zeta \rightarrow X_1(m - 1, n)^\zeta$  be its extension to the smooth projective models. Let  $R_m$  be the ramification divisor of  $\tilde{\pi}_m$ .

Applying Proposition 3.2 to  $\pi_m$  and  $h$  yields

$$g_m \leq dg_{m-1} + (d - 1)(\gamma_m - 1),$$

so it suffices to show that  $g_m - dg_{m-1} \rightarrow \infty$  as  $m \rightarrow \infty$ . By Riemann–Hurwitz, this is equivalent to showing that  $\deg R_m \rightarrow \infty$  as  $m \rightarrow \infty$ . In the fiber product diagram

$$\begin{array}{ccc} Y_1(m, n)^\zeta & \xrightarrow{\pi_m} & Y_1(m - 1, n)^\zeta \\ z \downarrow & & \downarrow z \\ \mathbb{A}^1 & \xrightarrow{f} & \mathbb{A}^1 \end{array}$$

both vertical morphisms are étale above 0 by Lemma 3.3, while  $f$  has ramification index  $d$  at 0, so  $\tilde{\pi}_m$  has ramification index  $d$  at each point of  $Y_1(m, n)^\zeta$  where  $z = 0$ . For  $m \geq 2$ , the number of such points is  $d^{m-2}D_1(n)$  by Lemma 3.3. Thus  $\deg R_m \geq (d - 1)d^{m-2}D_1(n)$ , which tends to  $\infty$  as  $m \rightarrow \infty$ . □

### 4. Gonality in characteristic $p$

#### 4.1 Reduction to the case of a finite field

Let  $\mathbb{F}_q = \mathbb{F}_p(\mu_d)$ .

LEMMA 4.1. *Theorem 1.1 for  $\mathbb{F}_q$  implies Theorem 1.1 for any characteristic  $p$  field  $k$  containing  $\mu_d$ .*

*Proof.* Theorem 1.1(a) for  $\mathbb{F}_q$  implies that the dynatomic curves over  $k$  are just the base extensions of the dynatomic curves  $X$  over  $\mathbb{F}_q$ , and that they are geometrically irreducible too. We will prove in § 4.4 that the smooth projective model of each  $X$  has an  $\mathbb{F}_q$ -point. Then [Poo07, Theorem 2.5(iii) and Proposition A.1(ii)] imply  $\gamma(X_k) \geq \sqrt{\gamma(X)}$ , which implies Theorem 1.1(b) for  $k$ . □



**4.2 Symbolic dynamics**

In order to prove the finiteness of the set of dynatomic curves of bounded degree over the  $c$ -line, and to prove that the dynatomic curves of higher degree have many  $\mathbb{F}_q$ -points above  $c = \infty$ , we need to analyze the splitting above  $c = \infty$  in dynatomic curves. This splitting is given in Lemma 4.2 below. The symbolic dynamics approach yielding Lemma 4.2 is standard (cf. [Mor96, Lemma 1]), but the ways that Lemma 4.2 will be applied in subsequent sections are not standard.

View  $f(z)$  as a polynomial in  $z$  over the local field  $\mathbb{F}_q((c^{-1}))$ . Normalize the valuation  $v$  on  $\mathbb{F}_q((c^{-1}))$  so that  $v(c^{-1}) = 1$ , and extend  $v$  to an algebraic closure. Let  $t = c^{-1/d}$ , so  $\mathbb{F}_q((t))$  is a totally tamely ramified extension of  $\mathbb{F}_q((c^{-1}))$  of degree  $d$ .

LEMMA 4.2.

- (a) For any nonnegative integers  $n > m$ , the polynomial  $f^n(z) - f^m(z)$  over  $\mathbb{F}_q(c)$  is separable and splits completely over  $\mathbb{F}_q((t))$ .
- (b) Each zero of  $f^n(z) - f^m(z)$  has valuation  $-1/d$  and generates  $\mathbb{F}_q((t))$  over  $\mathbb{F}_q((c^{-1}))$ .

*Proof.* The ideas in the following argument are well known; cf. [Mor96, Lemma 1].

(a) For each  $d$ th root of  $-c$ , interpreting  $(-c + z)^{1/d}$  as  $(-c)^{1/d}(1 - c^{-1}z)^{1/d}$  and expanding  $(1 - c^{-1}z)^{1/d}$  in a binomial series defines a branch of the inverse of  $z^d + c$  on the open disk  $D := \{z \in \mathbb{F}_q((t)) : v(z) > v(c)\}$ . Taking the derivative of  $(-c + z)^{1/d}$  shows that each branch is a contracting map  $D \rightarrow D$ . These branches have disjoint images, each a smaller open disk around a different  $d$ th root of  $-c$ . Let  $S$  be the set of these  $d$  functions. Each finite sequence of elements of  $S$  defines a composition of functions, and for each  $m$ , the images of the different  $m$ -fold compositions are disjoint open disks. For each infinite sequence  $s_1, s_2, \dots$  of elements of  $S$ , the images of  $s_1 \cdots s_m$  for  $m \geq 1$  are nested open disks whose radii tend to 0, so they have a unique point in their intersection; denote it  $[s_1 s_2 \cdots]$ . Any two distinct infinite sequences yield two points in disjoint disks, so these points are distinct. Since  $f \circ s_1$  is the identity,  $f$  maps  $[s_1 s_2 \cdots]$  to  $[s_2 s_3 \cdots]$ . For fixed nonnegative integers  $n > m$ , any  $n$ -long sequence  $s_1, \dots, s_n$  in  $S$  extends uniquely to an infinite sequence  $(s_i)$  satisfying  $s_{i+n} = s_{i+m}$  for all  $i \geq 1$ , and then  $[s_1 s_2 \cdots]$  is a zero of  $f^n(z) - f^m(z)$ . There are  $d^n$  of these, so they are all the zeros. In particular, these zeros are distinct elements of  $\mathbb{F}_q((t))$ . This implies that  $f^n(z) - f^m(z)$  is separable.

(b) The image of each  $s \in S$  consists of elements of valuation exactly  $-1/d$ . Thus each element  $[s_1 s_2 \cdots]$  has valuation  $-1/d$ . In particular, each zero of  $f^n(z) - f^m(z)$  generates an extension field of  $\mathbb{F}_q((c^{-1}))$  of ramification index divisible by  $d$ ; this extension field can only be the whole field  $\mathbb{F}_q((t))$ . □

**4.3 Dynatomic curves of low degree**

It is here that we use the trick requiring the ground field to be finite.

LEMMA 4.3. For each  $e \geq 1$ , the set of dynatomic curves  $X$  over  $\mathbb{F}_q$  such that  $\deg(X \xrightarrow{c} \mathbb{P}^1) = e$  is finite.

*Proof.* Suppose that  $Q(z) = \sum_{r=0}^e q_r z^{e-r}$  is a monic degree- $e$  factor of  $f^n(z) - f^m(z)$  over  $\mathbb{F}_q(c)$  for some  $n$  and  $m$ . For each  $r$ , the coefficient  $q_r$  is the  $r$ th elementary symmetric polynomial evaluated at the negatives of the zeros of  $Q$ ; those zeros have valuation  $-1/d$  by Lemma 4.2(b), so  $v(q_r) \geq -r/d$ . On the other hand, by Gauss’s lemma,  $q_r \in \mathbb{F}_q[c]$ , so  $\deg q_r \leq r/d$ . Thus there are only finitely many possibilities for each  $q_r$ , and hence finitely many possibilities for  $Q$ , each of which yields one dynatomic curve. □

**4.4 Gonality of dynatonic curves**

*Proof of Theorem 1.1.* By Lemma 4.1, we may assume that  $k = \mathbb{F}_q$ .

(a) Let  $X$  be the smooth projective model of a dynatonic curve, corresponding to a factor of  $f^n(z) - f^m(z)$  for some  $n$  and  $m$ . By Lemma 4.2(b),  $f^n(z) - f^m(z)$  splits completely over  $\mathbb{F}_q((t))$ , and the preimage of  $\infty$  under  $X \xrightarrow{c} \mathbb{P}^1$  consists of  $\mathbb{F}_q$ -points, each of ramification index  $d$ . Every irreducible component  $Z$  of  $X_{\overline{\mathbb{F}}_q}$  dominates  $\mathbb{P}^1$  via  $c$  and hence must contain one of those  $\mathbb{F}_q$ -points, say  $x$ . Then each  $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ -conjugate of  $Z$  contains  $x$ . On the other hand, since  $X_{\overline{\mathbb{F}}_q}$  is smooth, its irreducible components are disjoint. Thus  $Z$  equals each of its conjugates, so  $Z$  descends to an irreducible component of  $X$ , which must be  $X$  itself. (This proves also that  $X$  has an  $\mathbb{F}_q$ -point, as promised in the proof of Lemma 4.1.)

(b) To bound gonality from below, we use Ogg’s method of counting points over a finite field; cf. [Ogg74], [Poo07, §3], and [DKO<sup>+</sup>19, §4]. Let  $X$  be the smooth projective model of a dynatonic curve. Let  $e = \deg(X \xrightarrow{c} \mathbb{P}^1)$ . Each preimage of  $\infty$  in  $X$  is an  $\mathbb{F}_q$ -point of ramification index  $d$ , so there are  $e/d$  such points. On the other hand,  $\mathbb{P}^1$  has only  $q + 1$  points over  $\mathbb{F}_q$ , so any nonconstant morphism  $X \rightarrow \mathbb{P}^1$  has degree at least  $e/(d(q + 1))$ . As  $X$  varies,  $e \rightarrow \infty$  by Lemma 4.3. □

**5. Strong uniform boundedness of preperiodic points**

*Proof of Theorem 1.7.* Without loss of generality,  $K = k(u)$  for some indeterminate  $u$ . Given  $L$ , let  $Y$  be the smooth projective integral curve over  $k$  with function field  $L$ . The condition  $[L : K] \leq D$  implies that  $Y$  has gonality at most  $D$ , so each irreducible component of  $Y_{\overline{k}}$  has gonality at most  $D$ . For  $c \in L$  not algebraic over  $k$ , if  $z \in L$  and  $n > m$  satisfy  $f^n(z) - f^m(z) = 0$ , then  $(z, c)$  is a nonconstant and hence smooth  $L$ -point of the curve  $f^n(z) - f^m(z) = 0$  in  $\mathbb{A}^2$ , so it yields an  $L$ -point on a dynatonic curve  $X$  corresponding to a factor of  $f^n(z) - f^m(z)$ . This  $L$ -point defines a nonconstant  $k$ -morphism  $Y \rightarrow X$ , so  $\gamma(X) \leq \gamma(Y) \leq D$ . By Theorem 1.1(b), this places a uniform bound on  $n$ . For each  $n$ , the number of preperiodic points of  $z^d + c$  corresponding to that value of  $n$  is uniformly bounded by  $\sum_{m=0}^{n-1} \deg(f^n(z) - f^m(z)) = nd^n$ , so bounding  $n$  bounds the number of preperiodic points too.

If  $k$  is finite and  $c$  lies in the maximal algebraic extension  $\ell$  of  $k$  in  $L$ , then all the preperiodic points of  $z^d + c$  in  $L$  are in  $\ell$ , but  $[\ell : k] \leq D$ , so the number of preperiodic points is uniformly bounded by  $(\#k)^D$ . □

To prove Theorem 1.10, we need the following result of Frey [Fre94, Proposition 2], which in turn relies on Faltings’s theorems on rational points on subvarieties of abelian varieties.

LEMMA 5.1. *Let  $C$  be a curve defined over a number field  $K$ . Let  $D \geq 1$ . If there are infinitely many points  $P \in C(\overline{K})$  of degree  $\leq D$  over  $K$ , then  $\gamma(C) \leq 2D$ .*

*Proof of Theorem 1.10.* For each  $(m, n)$ , let  $S_{m,n}$  be the set of  $(z_0, c) \in \overline{\mathbb{Q}} \times \overline{\mathbb{Q}}$  such that  $z_0$  and  $c$  belong to some number field  $K$  of degree  $\leq D$ , and iteration of  $z^d + c$  maps  $z_0$  into a cycle of length exactly  $n$  after exactly  $m$  steps; thus  $S_{m,n} \subseteq Y_1(m, n)(\overline{\mathbb{Q}})$ . Suppose that the conclusion fails; then, for some  $n \leq N$ , there exist infinitely many  $m \geq 1$  for which  $S_{m,n}$  is nonempty. Fix such an  $n$ .

Let  $m_0 \geq 1$ . By choice of  $n$ , the disjoint union  $\coprod_{m \geq m_0} S_{m,n}$  is infinite. For each  $m \geq m_0$ , the  $c$ -coordinate map  $Y_1(m, n) \rightarrow \mathbb{A}^1$  factors through  $Y_1(m_0, n)$ , so we obtain maps of sets

$$\coprod_{m \geq m_0} S_{m,n} \longrightarrow S_{m_0,n} \longrightarrow \overline{\mathbb{Q}}. \tag{5}$$

On the other hand, by Northcott's theorem [Nor50, Theorem 3], for any given  $c \in \overline{\mathbb{Q}}$ , the set of preperiodic points of  $z^d + c$  of degree  $\leq D$  over  $\mathbb{Q}$  is finite; thus each  $c \in \overline{\mathbb{Q}}$  has finite preimage under the composition (5). Hence the image of (5) is infinite, so  $S_{m_0, n}$  is infinite. Thus some  $\mathbb{Q}(\mu_d)$ -irreducible component  $Y_1(m_0, n)^\zeta$  of  $Y_1(m_0, n)$  contains infinitely many points of degree  $\leq D$  over  $\mathbb{Q}(\mu_d)$ . By Lemma 5.1,  $\gamma(Y_1(m_0, n)^\zeta) \leq 2D$ .

The previous paragraph applies for every integer  $m_0 \geq 1$ , contradicting Theorem 1.1(b).  $\square$

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