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ON *f*-PREFRATTINI SUBGROUPS

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1. **On a theorem of Gaschutz.** The Prefrattini subgroups of a finite soluble group were introduced by Gaschutz [3]. These are a conjugacy class of subgroups which avoid complemented chief factors and cover Frattini chief factors. Gaschutz [3, Satz 7.1] showed that if G has p-length 1 for each prime p, and if $U \leq G$ avoids all complemented chief factors and covers all Frattini factors, then U is a Prefrattini subgroup of G. We begin by proving the analogous result for the f-Prefrattini subgroups introduced by Hawkes [5]. If f is a saturated formation, then the f-Prefrattini subgroups of G are a conjugacy class of subgroups which avoid f-eccentric complemented chief factors of G and cover all other chief factors of G. We wish to prove

THEOREM 1. *If G has p-length 1 for each prime p, and if $U \leq G$ avoids all f-eccentric complemented chief factors of G and covers all other chief factors of G, then U is an f-Prefrattini subgroup of G.*

In proving this theorem we wish to make use of the notion of a p-normally embedded subgroup as introduced by Hartley [4] and as studied in [2]. We say $V \leq G$ is p-normally embedded in G if a Sylow p-subgroup, V_p, of V is also Sylow in some normal subgroup of G. We write V-*pne*-G. If V-*pne*-G for each prime p, then we will say V is strongly pronormal in G. All groups considered here are finite solvable groups. The theorems from [2] which we will need are

THEOREM 2.3. *Suppose $V \leq G$ and that V_p is Sylow in V. Suppose further that for each prime p, V_p is pronormal in G. Then V is pronormal in G. In particular if V is strongly pronormal in G, then V is pronormal in G.*

THEOREM 2.6. *Suppose V is strongly pronormal in G. If $U \leq G$ covers each chief factor of G that V covers and avoids each chief factor of G that V avoids, then U is conjugate to V in G.*

We now prove

THEOREM 2. *If G has p-length 1, if f is a saturated formation and if W^f is an f-Prefrattini subgroup of G, then W^f-pne-G.*

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Proof. If $K=O_{p'}(G)\neq 1$, then since W^fK/K is an f-Prefrattini subgroup of G/K, induction implies W^fK/K-*pne*-G/K. Since K is p', this implies W^f-*pne*-G. Thus we may assume $0_{p'}(G)=1$ so that G has a normal Sylow p-subgroup P. If $\Phi(G)\neq 1$, then $W^f/\Phi(G)$-*pne*-$G/\Phi(G)$ by induction and hence W^f-*pne*-G. Thus we may assume $\Phi(G)=1$. But then $P=Fit(G)$ is a direct product of minimal normal subgroups, say $P=N_1\times N_2\times\cdots\times N_k$. Each N_i is complemented and W^f avoids the f-eccentric ones and covers the f-central ones. Suppose W^f covers $N_1, N_2, \ldots, N_i$ and avoids $N_{i+1}, \ldots, N_k$. Then $N_1\times N_2\times\cdots\times N_i$ is Sylow in W^f and W^f-*pne*-G. Q.E.D.

If G has p-length 1 for each prime p, then Theorem 2 implies that the f-Prefrattini subgroups of G are strongly pronormal in G so that Theorem 1 follows from Theorem 2.6. Before proceeding we note that Theorem 1 will also be obtained as a special case of Theorem 7.

2. **A construction of Makan.** A. Makan has recently constructed interesting subgroups which are related to the Prefrattini subgroups. Makan's theorem states

THEOREM [6]. *Suppose that V is strongly pronormal in G, that S is a Sylow system of G which reduces into V, and that $W=W(S)$ is the Prefrattini subgroup corresponding to S. Then VW is a subgroup of G which avoids all partially V-complemented chief factors of G and which covers all other chief factors of G.* (A partially V-complemented chief factor of G is a complemented chief factor at least one of whose complements contains V.)

We note that in particular this theorem says that if V is strongly pronormal in G then there is a Prefrattini subgroup of G such that $VW=WV$.

Makan actually states his theorem only for the case in which V is an F-injector for some Fischer class F and in this case he refers to the subgroups he constructs as F_Φ-subgroups. However, Makan's construction of these subgroups depends only on the fact that V is strongly pronormal and not on the fact that V is an F-injector. The other properties of V which his proof requires, namely that V covers or avoids each chief factor of G, that Hartley's Lemma 4 [4] applies and that V is pronormal all hold if V is strongly pronormal. Thus Makan's theorem can be stated as above.

We also remark that if F is an arbitrary Fitting class, if G has p-length 1 for each prime p, and if V is an F-injector of G, then V is strongly pronormal in G [2]. Thus if G has p-length 1 for each prime p, Makan's theorem does construct F_Φ-subgroups even if F is not a Fischer class.

The obvious way to extend Makan's result as we have stated it would be to prove that if V is strongly pronormal then V permutes with some f-Prefrattini subgroup. To do this we first prove

THEOREM 3. *Suppose that V is strongly pronormal in G, that S is a Sylow system of G which reduces into V, that f is a saturated formation and that $D=D(S)$ is the*

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f-normalizer corresponding to S. Then VD is a subgroup of G which avoids all f-eccentric V-avoided chief factors of G and which covers all other chief factors of G.

Proof. We suppose that f is locally defined by formations $f(p)\leq f$ and that K_p is the $f(p)$ residual of G. If S^p is the Sylow p-complement of S we let $T^p=K_p \cap S^p$. Then Theorem 3.3 of [1] states that $N_G(T^p)$ covers all f-central p-chief factors of G and avoids the f-eccentric p-chief factors of G.

Let P be a Sylow p-subgroup of V and let P^G be its normal closure in G. V-*pne*-G means that P is Sylow in P^G. Now let $B_p=N_G(T^p)P^G$. Note that $D=\bigcap_q N_G(T^q)\leq B_p$. Also since S^p reduces into V, $V^p=S^p \cap V\leq S^p\leq N_G(T^p)$ so that $V=V^pP\leq N_G(T^p)P^G=B_p$. Thus $\langle D, V\rangle\leq B_p$. Suppose that H/K is a p-chief factor of G. If H/K is either f-central or V-covered then certainly B_p covers H/K. Suppose then that H/K is both f-eccentric and V-avoided. Then HP^G/KP^G is also an f-eccentric p-chief factor of G and so $N_G(T^p)$ avoids HP^G/KP^G. But then $B_p=N_G(T^p)P^G$ also avoids HP^G/KP^G so that $B_p \cap HP^G=B_p \cap KP^G$. That is, $P^G(B_p \cap H)=P^G(B_p \cap K)$. Since $B_p \cap H\geq B_p \cap K$, $B_p \cap H=(B_p \cap H) \cap P^G(B_p \cap K)=(B_p \cap K)$. $(B_p \cap H \cap P^G)$. But $H \cap P^G\leq K$ since P^G avoids H/K and so $B_p \cap H=(B_p \cap K)(B_p \cap H \cap P^G)=B_p \cap K$. Thus B_p avoids those p-chief factors which are both f-eccentric and V-avoided and covers the rest.

Let $Z=\bigcap_p B_p$. Then Z avoids all chief factors of G which are both f-eccentric and V-avoided. Also $\langle D, V\rangle\leq Z$ implies that Z covers all other chief factors of G. Finally $|VD|=|V|\,|D|/|V \cap D|\geq|Z|$ so that $VD=Z$.

THEOREM 4. *Suppose that V is strongly pronormal in G, that S is a Sylow system of G which reduces into V, that f is a saturated formation and that $W^f=W^f(S)$ is the f-Prefrattini subgroup corresponding to S. Then VW^f is a subgroup of G which avoids all partially V-complemented f-eccentric chief factors of G and covers all other chief factors of G.*

Proof. By Hawkes Theorem 4.1 [5] $W^f(S)=DW$ where $D=D(S)$ is the f-normalizer corresponding to S and $W=W(S)$ is the Prefrattini subgroup corresponding to S. By Makan's Theorem $VW=WV$ and by Theorem 3 $VD=DV$. Thus $VW^f=VDW=DVW=DWV=W^fV$ so that W^fV is a subgroup of G. Clearly W^fV covers each chief factor that is covered either by D or by VW. Thus the only chief factors which W^fV could possibly avoid are the partially V-complemented f-eccentric chief factors of G. Let H/K be such a chief factor. We wish to find a complement M/K of H/K such that both $V\leq M$ and S reduces into M. Since H/K is partially V-complemented there does exist a complement M/K such that $V\leq M$. Choose $g\in G$ such that S reduces into M^g. Since $V^g\leq M^g$ and S reduces into M^g there exists $h\in M^g$ such that S reduces into $V^{gh}\leq M^g$. By assumption S also reduces into V. Either by arguing directly from the fact that V is strongly pronormal or by using the fact that V is pronormal and invoking the corollary to the theorem

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in [7] we conclude that $V=V^{gh}$. Thus $V\leq M^g$ and M^g is the complement we want. M^g complements the f-eccentric chief factor H/K and so M^g is f-abnormal. Since S reduces into M^g, Corollary 3.4 [5] implies that $W^f=W^f(S)\leq M^g$. But then $VW^f\leq M^g$ and VW^f avoids H/K as required. Q.E.D.

THEOREM 5. *Let V and $W^f=W^f(S)$ be as in the statement of Theorem* 4, *but assume that in addition that G has p-length* 1 *for some prime p. Then VW^f-pne-G. In particular, if G has p-length* 1 *for each prime p, then VW^f is itself strongly pronormal in G.*

Proof. V-*pne*-G by assumption and W^f-*pne*-G by Theorem 2 and so Theorem 5 will follow from

LEMMA 6. *If A-pne-G, B-pne-G and $AB=BA$, then AB-pne-G.*

Proof. Let A_p be Sylow p in A and B_p Sylow p in B. If $A_p=1$, then B_p is Sylow in AB so that AB-*pne*-G. If $A_p\neq 1$, then A_p is Sylow in $R\trianglelefteq G$ for some $R\neq 1$. Consider AR/R-*pne*-G/R, BR/R-*pne*-G/R. $(AR/R)(BR/R)=(BR/R)(AR/R)$ and so $(AB)R/R$-*pne*-G/R by induction. Let $|(AB)R|_p$ and $|AB|_p$ denote the orders of the Sylow p-subgroups of $(AB)R$ and AB respectively. Then $|(AB)R|_p=|AB|_p|R|_p/|AB\cap R|_p$. Since A_p is Sylow in R, $|A|_p=|R|_p=|AB\cap R|_p$ and $|(AB)R|_p=|AB|_p$. Thus a Sylow p-subgroup P of AB is also Sylow in $(AB)R$. $(AB)R/R$-*pne*-G/R implies PR/R is Sylow in $L/R\trianglelefteq G/R$ for some $L\trianglelefteq G$. But then $[L:P]=[L:PR][PR:P]$ is p' and P is Sylow in $L\trianglelefteq G$ so that AB-*pne*-G.

Q.E.D.

Our final theorem contains Theorem 1 as the special case $V=1$.

THEOREM 7. *Again let V and $W^f=W^f(S)$ be as above. If G has p-length* 1 *for each prime p, and if $U\leq G$ avoids each partially V-complemented f-eccentric chief factor of G and covers all other chief factors of G then U is conjugate VW^f in G.*

Proof. Theorem 7 follows from Theorem 5 and Theorem 2.6 [2] quoted above. Q.E.D.

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