Bull. Aust. Math. Soc. **102** (2020), 439–450
doi:10.1017/S0004972720000192

ON (β, G_Π)-UNFAVOURABLE SPACES

HANFENG WANG ✉, WEI HE and JING ZHANG

(Received 7 January 2020; accepted 1 February 2020; first published online 16 March 2020)

Abstract

To study when a paratopological group becomes a topological group, Arhangel'skii *et al.* ['Topological games and topologies on groups', *Math. Maced.* **8** (2010), 1–19] introduced the class of (β, G_Π)-unfavourable spaces. We show that every μ-complete (or normal) (β, G_Π)-unfavourable semitopological group is a topological group. We prove that the product of a (β, G_Π)-unfavourable space and a strongly Fréchet (α, G_Π)-favourable space is (β, G_Π)-unfavourable. We also show that continuous closed irreducible mappings preserve the (β, G_Π)-unfavourableness in both directions.

2010 *Mathematics subject classification*: primary 22A05; secondary 54D40, 54E35.

Keywords and phrases: G_Π game, (β, G_Π)-unfavourable, strategy, topological group.

1. Introduction

All spaces considered here are assumed to be nonempty regular topological spaces unless explicitly stated to the contrary. A paratopological (respectively, semitopological) group is a group endowed with a topology such that the multiplication is jointly (respectively, separately) continuous. A topological group is a paratopological group such that the inversion is continuous. We write $\overline{A}^X$ for the closure of a set A in a space X and, when no confusion arises, $\overline{A}^X$ can be denoted by $\overline{A}$. For other terminology and notation, refer to [8].

When does a semitopological group become a paratopological group? What additional conditions are needed to be sure that a paratopological group is actually a topological one? There are many classical results and here we just list a few of them. In 1957, Ellis [7] showed that every locally compact semitopological group is a topological group. In [6], Bouziad proved that every Čech-complete semitopological group is a topological group. Later Kenderov *et al.* [9] proved that a strongly Baire semitopological group is a topological group. To study when the inverse operation of a paratopological group is continuous, Arhangel'skii, Choban and Kenderov introduced the class of (β, G_Π)-unfavourable spaces (see [2, 3]) and proved the following statement.

Project supported by SDNSF (No. ZR2018MA013) and NSFC (Nos. 11571175, 11801254).

THEOREM 1.1 [2]. *If a paratopological group X is a (β, G_Π)-unfavourable topological space, then the inversion is quasicontinuous and, therefore, X is a topological group.*

The class of (β, G_Π)-unfavourable spaces is defined by a topological game, G_Π, and it includes many important classes of topological spaces. Locally compact spaces, Čech-complete spaces and pseudocompact spaces are all (β, G_Π)-unfavourable, as well as (α, G_Π)-favourable. Every (β, G_Π)-unfavourable space has the Baire property, so not every metrisable space is in this class.

We study the properties of (β, G_Π)-unfavourable spaces, and establish that:

- the product space of a (β, G_Π)-unfavourable space and a strongly Fréchet (α, G_Π)-favourable space is (β, G_Π)-unfavourable;
- a space with a dense (β, G_Π)-unfavourable subspace is (β, G_Π)-unfavourable;
- a dense G_δ-subspace of a (β, G_Π)-unfavourable space is (β, G_Π)-unfavourable;
- every open subspace of a (β, G_Π)-unfavourable space is (β, G_Π)-unfavourable;
- every locally (β, G_Π)-unfavourable space is (β, G_Π)-unfavourable;
- a space that is the union of a family of locally finite (β, G_Π)-unfavourable subspaces is (β, G_Π)-unfavourable;
- continuous closed irreducible mappings preserve the (β, G_Π)-unfavourableness in both directions;
- any image of a (β, G_Π)-unfavourable space under a continuous open mapping is (β, G_Π)-unfavourable;
- every μ-complete (or normal) (β, G_Π)-unfavourable semitopological group is a topological group.

2. Preliminaries

Let us recall the topological game G_Π [2]. Suppose that X is a topological space and $\{H_n : n \in \omega\}$ is a decreasing sequence of nonempty open subsets of X. We say that $\{H_n : n \in \omega\}$ has the property Π if, whenever $\{W_n : n \in \omega\}$ is a decreasing sequence of nonempty open subsets of X such that $W_n \subset H_n$ for each $n \in \omega$, then $\{W_n : n \in \omega\}$ has at least one cluster point in X.

The topological game G_Π is a game played by two players α and β by selecting alternatively nonempty open subsets of a space X. Player β begins the game by selecting a nonempty open subset U_0 of X. Whenever β chooses an open nonempty subset U_n, player α responds by selecting a nonempty open subset V_n such that $V_n \subset U_n$. In turn, player β selects a nonempty open subset $U_{n+1} \subset V_n$ and the game goes on. Continuing this procedure indefinitely, the two players generate a sequence $\{(U_n, V_n) : n \in \omega\}$ of pairs of open nonempty subsets such that $U_{n+1} \subset V_n \subset U_n$ for $n \in \omega$. Every such sequence will be called a play. We say that player α wins the play $\{(U_n, V_n) : n \in \omega\}$ in the G_Π-game if the sequence $\{V_n : n \in \omega\}$ has the property Π. Otherwise player β is declared to be the winner of this play.

By a strategy t for player β in the game G_Π we mean a 'rule' that specifies each move of player β in every possible situation. More precisely, a strategy t for player β is a sequence of mappings $\{t_n : n \in \omega\}$ such that t_n determines the nth move of player β. The domain $\operatorname{Dom} t_0$ of t_0 is $\{X\}$. The value $t_0(X)$ of t_0 is some open nonempty subset of X. The domain $\operatorname{Dom} t_1$ of t_1 consists of all pairs (X, V_0), where V_0 is an arbitrary nonempty open subset of $U_0 := t_0(X)$. The values of t_1 satisfy the condition $t_1(X, V_0) \subset V_0$ for every $(X, V_0) \in \operatorname{Dom} t_1$. In general, $\operatorname{Dom} t_{n+1}$ for $n \geq 1$ consists of all finite sequences $(X, V_0, V_1, \ldots, V_{n-1}, V_n)$, where $(X, V_0, V_1, \ldots, V_{n-1}) \in \operatorname{Dom} t_n$ and V_n is an open nonempty subset of $t_n(X, V_0, V_1, \ldots, V_{n-1})$. The values of t_n satisfy the condition $t_{n+1}(X, V_0, V_1, \ldots, V_{n-1}, V_n) \subset V_n$. The play $\{(U_n, V_n) : n \in \omega\}$ is said to be a t-play if the moves of player β are made according to strategy t. The strategy for player α in G_Π can be defined similarly.

A strategy t for player β (respectively, α) is called a winning strategy in the game G_Π, if player β (respectively α) wins each t-play according to the winning rule of G_Π. If such a strategy t exists, then the space X is called (β, G_Π)-favourable (respectively, (α, G_Π)-favourable). If there does not exist a winning strategy t for player β (respectively α) in G_Π, then X is called (β, G_Π)-unfavourable (respectively, (α, G_Π)-unfavourable), that is, for each strategy t for player β (respectively α), there exists a t-play which is won by α (respectively β). Clearly, every (α, G_Π)-favourable space is (β, G_Π)-unfavourable, but the converse is not true.

Let us recall another topological game named G_{S_Y}, where Y is a dense subspace of a regular space X. The playing rule of G_{S_Y} is the same as that of G_Π: player β begins the game by selecting a nonempty open subset U_0 of X. Whenever β chooses an open nonempty subset U_n of X, player α responds by selecting a nonempty open subset $V_n \subset U_n$. In turn, player β selects a nonempty open subset $U_{n+1} \subset V_n$. Continuing in this way, the two players generate a play $\{(U_n, V_n) : n \in \omega\}$ on X. The winning rule is as follows: if every sequence $\{x_n : n \in \omega\}$ with $x_n \in V_n \cap Y$ for each $n \in \omega$ has at least one cluster point in X, then α is said to be the winner of the play $\{(U_n, V_n) : n \in \omega\}$; otherwise β wins this play. In particular, when $Y = X$, this game is denoted by G_S.

A regular space X is strongly Baire if there exists a dense subspace Y such that player β has no winning strategy in the game G_{S_Y}. In particular, every (β, G_S)-unfavourable space is strongly Baire.

3. Main results

LEMMA 3.1. *If a space X contains a dense (β, G_Π)-unfavourable subspace, then X is (β, G_Π)-unfavourable.*

PROOF. Assume that $\mathcal{B}$ is a base of the space X and fix some well-ordering on $\mathcal{B}$. Let Y be a dense subspace of X such that Y is (β, G_Π)-unfavourable and let t be an arbitrary strategy for player β in the game G_Π played on X. It suffices to show that t is not a winning strategy. To prove this, we will use the G_Π game played by the same players α and β on the space Y, and construct a strategy t' for player β in this game.

Let U_0 be the first move of player β under t. Clearly, U_0 is an open nonempty subset of X. Let U_0'' be the first element of $\mathcal{B}$ such that $\overline{U_0''}^X \subset U_0$. Such an element U_0'' of $\mathcal{B}$ does exist because $\mathcal{B}$ is a base of the regular space X. Now let $U_0' := U_0'' \cap Y$ be the first move of player β under t'. With a response to the first move of β under t', player α takes an open nonempty subset V_0' of Y. Suppose that V_0 is the first element of $\mathcal{B}$ satisfying $\overline{V_0}^X \cap Y \subset V_0'$. Since Y is dense in X,

$$V_0 \subset \overline{V_0}^X \subset \overline{V_0'}^X \subset \overline{U_0'}^X \subset \overline{U_0''}^X \subset U_0.$$

Let V_0 be the response of α to the first move of β under t. In turn, player β chooses an open nonempty subset U_1 of X under strategy t. Proceeding indefinitely, we have a t-play $\{(U_n, V_n) : n \in \omega\}$ and a t'-play $\{(U_n', V_n') : n \in \omega\}$ accompanied by a sequence $\{U_0'' : n \in \omega\}$ of open subsets of X, which satisfy the following conditions for each $n \in \omega$:

(1) $U_{n+1} \subset V_n \subset U_n$;

(2) $U_{n+1}' \subset V_n' \subset U_n'$;

(3) $U_n' = U_n'' \cap Y$, where U_n'' is the first element of $\mathcal{B}$ such that $\overline{U_n''}^X \subset U_n$;

(4) V_n is the first element of $\mathcal{B}$ such that $\overline{V_n}^X \cap Y \subset V_n'$.

This completes the construction of t'. Obviously, strategy t' is accompanied by t.

By the assumption, Y is a (β, G_Π)-unfavourable space, so there exists a t'-play $\{(U_n', V_n') : n \in \omega\}$ such that player α wins this play, that is, $\{V_n' : n \in \omega\}$ has the property Π in Y. It suffices to show that the corresponding sequence $\{V_n : n \in \omega\}$ has the property Π in X. To this end we take any decreasing sequence $\{W_n : n \in \omega\}$ of nonempty open subsets of X such that $W_n \subset V_n$ for each $n \in \omega$. Put $W_n' := W_n \cap Y$ for each $n \in \omega$. Since $V_n \cap Y \subset V_n'$, it follows that $W_n' \subset V_n'$ for $n \in \omega$. Hence $\{W_n' : n \in \omega\}$ has at least one cluster point in Y. Fix such a cluster point y. Then $y \in \bigcap_{n\in\omega} \overline{W_n'}^Y \subset \bigcap_{n\in\omega} \overline{W_n}^X$, which implies that the sequence $\{V_n : n \in \omega\}$ has the property Π. Therefore, X is a (β, G_Π)-unfavourable space. □

The inverse of the above lemma is not valid, that is, not every dense subspace of a (β, G_Π)-unfavourable space is (β, G_Π)-unfavourable. For instance, the space of real numbers $\mathbb{R}$ is Čech-complete, but its subspace $\mathbb{Q}$ of rational numbers is not (β, G_Π)-unfavourable, since it is of the first category.

The following result shows that for a dense G_δ-subspace the outcome will be different. Recall that a subspace Y of a space X is said to be a G_δ-subspace if there exist countably many open subspaces of X whose intersection coincides with Y.

Theorem 3.2. *If X is a (β, G_Π)-unfavourable space and Y is a dense G_δ-subspace of X, then Y is a (β, G_Π)-unfavourable space.*

Proof. Let $\{O_n : n \in \omega\}$ be a sequence of open subspaces of X which witnesses Y being a G_δ-subspace of X, that is, $Y = \bigcap_{n\in\omega} O_n$. Clearly, each O_n is dense in X. Assume that $\mathcal{B}$ is a base of the space X and fix some well-ordering on $\mathcal{B}$. Let $t := \{t_n : n \in \omega\}$ be an arbitrary strategy for player β in the game G_Π played on Y. Then it suffices to show that t is not a winning strategy. To prove this, we construct inductively another strategy $t' := \{t_n' : n \in \omega\}$ for β in the game G_Π played on X, which is accompanied by t, as follows.

Player β makes the first move to fix an open nonempty subset U_0 of Y under strategy t. Put $U'_0 := (X \setminus \overline{Y \setminus U_0}^X) \cap O_0$. Clearly, U'_0 is an open nonempty subset of X and is determined by U_0. Let U'_0 be the first move of player β under strategy t'. As a response to the first move of player β under t', player α chooses an open nonempty subset V'_0 of X. Let V''_0 be the first element of $\mathcal{B}$ such that $\overline{V''_0}^X \subset V'_0$. Clearly, $V''_0 \cap Y \subset U'_0 \cap Y = U_0$. So let $V_0 := V''_0 \cap Y$ be the first move of player α following the first move of player β under t. In turn, player β fixes an open nonempty subset U_1 of Y in the second move under t, and so on indefinitely. Then we have a t-play $\{(U_n, V_n) : n \in \omega\}$ and a t'-play $\{(U'_n, V'_n) : n \in \omega\}$, which are accompanied by a sequence $\{V''_n : n \in \omega\}$ of nonempty open subsets of X. The following conditions are satisfied for each $n \in \omega$:

(a) $U_{n+1} \subset V_n \subset U_n$;
(b) $U'_{n+1} \subset V'_n \subset U'_n$;
(c) $U'_n = (X \setminus \overline{Y \setminus U_n}^X) \cap O_n$;
(d) V''_n is the first element of $\mathcal{B}$ such that $\overline{V''_n}^X \subset V'_n$;
(e) $V_n = V''_n \cap Y$.

Since X is a (β, G_Π)-unfavourable space, t' is not a winning strategy for player β. So there exists a t'-play $\{(U'_n, V'_n) : n \in \omega\}$ such that the sequence $\{V'_n : n \in \omega\}$ has the property Π in X. It remains to verify that the corresponding sequence $\{V_n : n \in \omega\}$ has the property Π in Y. Indeed, take any decreasing sequence $\{W_n : n \in \omega\}$ of nonempty open subsets of Y such that $W_n \subset V_n$ for each $n \in \omega$. Furthermore, for each $n \in \omega$ choose an open subset W'_n of X such that $W'_n \cap Y = W_n$ and $W'_{n+1} \subset W'_n$. Since Y is dense in X, for each $n \in \omega$,

$$W'_n \subset \overline{W'_n}^X \subset \overline{W_n}^X \subset \overline{V_n}^X \subset \overline{V''_n}^X \subset V'_n.$$

Hence the set of cluster points of the sequence $\{W'_n : n \in \omega\}$, denoted by F, is nonempty. Observe that $\overline{W'_n}^X \subset V'_n \subset O_n$ for each $n \in \omega$. Thus,

$$F = \bigcap_{n\in\omega} \overline{W'_n}^X \subset \bigcap_{n\in\omega} O_n = Y.$$

Clearly, the set of cluster points of $\{W_n : n \in \omega\}$ is F too. Therefore, Y is a (β, G_Π)-unfavourable space. □

Arhangel'skii and Reznichenko (see [5] or [4, Theorem 2.4.1]) established that every paratopological group that is a dense G_δ-subspace of some regular feebly compact space is a topological group. Recall that a regular space is feebly compact if every locally finite family of open sets in X is finite. Clearly, 'feebly compact' is equivalent to 'pseudocompact' for Tychonoff spaces and every feebly compact space is (α, G_Π)-favourable.

By Lemma 3.1 we can improve Theorem 3.2 as follows.

THEOREM 3.3. *If a regular paratopological group X is a dense G_δ-subspace of some (β, G_Π)-unfavourable space, then X is a topological group.*

https://doi.org/10.1017/S0004972720000192 Published online by Cambridge University Press

Theorem 3.4. *If X is a (β, G_Π)-unfavourable space, then every open subspace is a (β, G_Π)-unfavourable space.*

Proof. Let Y be any open subspace of X and $t := \{t_n : n \in \omega\}$ be an arbitrary strategy for player β in the game G_Π played on Y. We will show that t is not a winning strategy. Indeed, since Y is open in X, we can construct a strategy $t' := \{t'_n : n \in \omega\}$ for player β in the game G_Π played on X which is accompanied by strategy t. To see this, let $t'_0(X) = t_0(Y) := U_0$ and $t'_n(X, V_0, \ldots, V_{n-1}) = t_n(Y, V_0, \ldots, V_{n-1}) := U_n$ for each $n \geq 1$. Then $\mathrm{Dom}\, t'_0 = \{X\}$, $\mathrm{Dom}\, t_0 = \{Y\}$ and, for each $n \geq 1$, $V_n \subset t'_n(X, V_0, \ldots, V_{n-1}) = t_n(Y, V_0, \ldots, V_{n-1}) \subset V_{n-1}$. Since X is (β, G_Π)-unfavourable, there exists a t'-play $\{(U_n, V_n) : n \in \omega\}$ such that player α wins this play, that is, the sequence $\{V_n : n \in \omega\}$ has the property G_Π. Obviously, the sequence $\{(U_n, V_n) : n \in \omega\}$ is also a t-play. Therefore, Y is a (β, G_Π)-unfavourable space. □

Remark 3.5. The above statement shows that the (β, G_Π)-unfavourableness property is hereditary to open subspaces. However, this property is not closed hereditarily. To see this we recall that every Tychonoff space can be embedded into a pseudocompact space as a closed subspace [1, Theorem 2.18].

Theorem 3.6. *Let X be a space that is the union of a locally finite family ζ of (β, G_Π)-unfavourable subspaces. Then X is (β, G_Π)-unfavourable.*

Proof. Let t be an arbitrary strategy for player β in the game G_Π played on X. We show that t is not a winning strategy.

Let U_0 $(= t_0(X))$ be the first move of player β under t. Clearly, U_0 is an open nonempty subset of X. Since ζ is locally finite in X, there exist a finite family η of ζ and a nonempty open subset O of U_0 such that $\eta = \{M \in \zeta : M \cap O \neq \emptyset\}$. Since η is finite, we can fix an open nonempty subset V_0 of O and some $M \in \eta$ such that $M \cap V_0$ is dense in V_0. Since the space M is (β, G_Π)-unfavourable and $M \cap V_0$ is open in M, it follows from Theorem 3.3 that the space $M \cap V_0$ is (β, G_Π)-unfavourable. Then Lemma 3.1 implies that V_0 is (β, G_Π)-unfavourable. Let V_0 be the move of player α following the first move of player β under t. In turn, player β takes an open nonempty subset U_1 of V_0 in the second move under t. Proceeding indefinitely, we have t-plays $\{(U_n, V_n) : n \in \omega\}$. Observe that the t-plays constructed above have one thing in common: each of the second moves of β is the unique set $t_1(X, V_0)$, which is contained in V_0. Now it remains to find such a t-play which is won by α.

For this purpose, we construct by t a strategy $t' := \{t'_n : n \in \omega\}$ for player β in the game G_Π played on the space V_0 as follows:

$$t'_0(V_0) = t_1(X, V_0) \quad \text{and} \quad t'_n(V_0, V_1, \ldots, V_n) = t_{n+1}(X, V_0, V_1, \ldots, V_n), \quad \text{for } n \geq 1.$$

In these equations, for each $n \in \omega$, V_n is an open nonempty subset of V_0 chosen by player α in the nth move and the $(n+1)$th move of a play in the game played on V_0 and X, respectively. Since V_0 is (β, G_Π)-unfavourable, there exists a t'-play $\{(t'_n(V_0, V_1, \ldots, V_n), V_{n+1}) : n \in \omega\}$ which is won by player α. Thus the corresponding t-play $\{((t_0(X), V_0), (t_{n+1}(X, V_0, V_1, \ldots, V_n), V_{n+1})) : n \in \omega\}$ is also won by α. This proves that the space X is (β, G_Π)-unfavourable. □

Corollary 3.7. *Let X be a space that is the union of a finite family of (β, G_Π)-unfavourable subspaces. Then X is (β, G_Π)-unfavourable.*

Corollary 3.8. *If a paratopological group X is the union of a locally finite family of pseudocompact subspaces, then X is a topological group.*

The following result shows that (β, G_Π)-unfavourableness is a 'local implies global' property.

Theorem 3.9. *If X is a locally (β, G_Π)-unfavourable space, then X is (β, G_Π)-unfavourable.*

Proof. Let t be an arbitrary strategy for player β in the game G_Π on X and let U_0 be the first move of β under t. By the assumption, X is locally (β, G_Π)-unfavourable, and by Theorem 3.3, the (β, G_Π)-unfavourableness property is open hereditary. So we can take an open nonempty subspace V_0 of X such that $V_0 \subset U_0$ and V_0 is (β, G_Π)-unfavourable. Now let V_0 be the response of player α to the first move of β under t. The rest of the proof is similar to that of Theorem 3.4, that is, by a similar argument we can find a t-play which is won by player α. Therefore, X is (β, G_Π)-unfavourable. □

Corollary 3.10. *If a Baire paratopological group X is the union of a countable family ζ of (β, G_Π)-unfavourable (or pseudocompact) subspaces, then X is a topological group.*

Proof. Suppose that $\zeta = \{X_k : k \in \omega\}$. Since X has the Baire property, there exists X_k such that X_k is somewhere dense, that is, we can find a nonempty open subspace U of X such that $X_k \cap U$ is dense in U. By the assumption, X_k is (β, G_Π)-unfavourable, so it follows Theorem 3.3 that $X_k \cap U$ is a (β, G_Π)-unfavourable space. Then Lemma 3.1 implies that the space U is (β, G_Π)-unfavourable. Since X is homogeneous, it follows that X is locally (β, G_Π)-unfavourable. Hence, from Theorem 3.6, X is (β, G_Π)-unfavourable. Therefore, X is a topological group. □

The condition that X is Baire in the corollary above cannot be dropped: to see this one only needs to take a countable paratopological nontopological group.

Theorem 3.11. *Let X, Y be two topological spaces and X be (β, G_Π)-unfavourable. If there exists a continuous open mapping $f : X \to Y$ of X onto Y, then Y is (β, G_Π)-unfavourable.*

Proof. Let $t := \{t_n : n \in \omega\}$ be an arbitrary strategy for player β in the game G_Π played on Y. It suffices to show that t is not a winning strategy. For this purpose we will construct inductively a strategy $t' := \{t'_n : n \in \omega\}$ for player β in the game G_Π played on X as follows.

Let U_0 be the first move of player β under t. Then $U'_0 := f^{-1}(U_0)$ is a nonempty open subset of X. Let U'_0 be the first move of β under t'. As a response to the first move of β under t', player α takes a nonempty open subset V'_0 of X. Put $V_0 := f(V'_0)$. Then V_0 is a nonempty open subset of Y. Let V_0 be the response of α to the first move of β

under t. In turn, player β takes a nonempty open subset U_1 of V_0 in the second move under t. Let $U_1' := f^{-1}(U_1) \cap V_0'$ be the second move of β under t'. As a response to the second move of β under t', player α takes a nonempty open subset V_1' of X. Continuing indefinitely, we can obtain a t-play $\{(U_n, V_n) : n \in \omega\}$ and a t'-play $\{(U_n', V_n') : n \in \omega\}$, and the following conditions are satisfied for each $n \in \omega$:

(a) $U_{n+1} \subset V_n \subset U_n$;
(b) $U_{n+1}' \subset V_n' \subset U_n'$;
(c) $U_n' = f^{-1}(U_n) \cap V_{n-1}'$, where $V_{-1}' = X$;
(d) $V_n = f(V_n')$.

Since X is (β, G_Π)-unfavourable, there exists a t'-play $\{(U_n', V_n') : n \in \omega\}$ which is won by α. It remains to show that the corresponding t-play $\{(U_n, V_n) : n \in \omega\}$ is also won by α. Indeed, take any decreasing sequence $\{W_n : n \in \omega\}$ of nonempty open subsets of Y such that $W_n \subset V_n$ for each $n \in \omega$. For each $n \in \omega$, put $W_n' := f^{-1}(W_n) \cap V_n'$. Then $\{W_n' : n \in \omega\}$ is a decreasing sequence of open subsets of X such that $W_n = f(W_n')$, since $V_n = f(V_n')$ for each $n \in \omega$. Clearly, the sequence $\{W_n' : n \in \omega\}$ has at least one cluster point in X, since the sequence $\{V_n' : n \in \omega\}$ has the property Π. Then it follows from the continuity of f that $\{W_n : n \in \omega\}$ has at least one cluster point in Y. Therefore, Y is (β, G_Π)-unfavourable. □

Corollary 3.12. *If X is the product space of a family ζ of topological spaces such that X is (β, G_Π)-unfavourable, then every member of ζ is a (β, G_Π)-unfavourable space.*

Recall that a mapping $f : X \to Y$ of a space X onto a space Y is irreducible if $Y \setminus f(X \setminus U) \neq \emptyset$ for any nonempty open subset U of X.

Theorem 3.13. *Let X, Y be two topological spaces and $f : X \to Y$ be a continuous closed irreducible mapping of X onto Y. Then X is (β, G_Π)-unfavourable if and only if Y is (β, G_Π)-unfavourable.*

Proof. The proof of the 'only if' part is similar to the proof of Theorem 3.6. Since f is closed and irreducible, the open subsets V_n in Theorem 3.6 can be obtained by $V_n := Y \setminus f(X \setminus V_n')$. So we omit the proof.

We now prove the 'if' part. Let $t := \{t_n : n \in \omega\}$ be an arbitrary strategy for player β in the game G_Π played on X. To verify t is not a winning strategy we need to construct inductively a strategy $t' := \{t_n' : n \in \omega\}$ for β in the game G_Π played on Y. The construction is as follows.

Let U_0 be the first move of player β under t. Put $U_0' := Y \setminus f(X \setminus U_0)$. Since f is closed and irreducible, it follows that U_0' is a nonempty open subset of Y. Let U_0' be the first move of β under t'. Following the first move of β under t', player α takes a nonempty open subset V_0' of Y. Then $V_0 := f^{-1}(V_0')$ is a nonempty open subset of X. Clearly, V_0 is contained in U_0. Let V_0 be the first move of α under t. In turn, player β takes a nonempty open subset U_1 in the second move under t. Continuing indefinitely, we can obtain a t-play $\{(U_n, V_n) : n \in \omega\}$ and a t'-play $\{(U_n', V_n') : n \in \omega\}$ which satisfy the following conditions for each $n \in \omega$:

https://doi.org/10.1017/S0004972720000192 Published online by Cambridge University Press

(a) $U_{n+1} \subset V_n \subset U_n$;
(b) $U'_{n+1} \subset V'_n \subset U'_n$;
(c) $U'_n = Y \setminus f(X \setminus U_n)$;
(d) $V_n = f^{-1}(V'_n)$.

Since Y is (β, G_Π)-unfavourable, there exists a t'-play $\{(U'_n, V'_n) : n \in \omega\}$ which is won by α. It remains to verify that the corresponding t-play $\{(U_n, V_n) : n \in \omega\}$ is also won by α. Indeed, take any decreasing sequence $\{W_n : n \in \omega\}$ of nonempty open subsets of X such that $W_n \subset V_n$ for each $n \in \omega$. Obviously, each $W'_n := Y \setminus f(X \setminus W_n)$ is a nonempty open subset of Y contained in V'_n and the sequence $\{W'_n : n \in \omega\}$ is decreasing. Since $\{V'_n : n \in \omega\}$ has the property Π, it follows that $\{W'_n : n \in \omega\}$ has at least one cluster point of Y. Fix such a cluster point y. Since f is closed and each $f^{-1}(W'_n) \subset W_n$, it is easy to verify that the sequence $\{W_n : n \in \omega\}$ has at least one cluster point located in the set $f^{-1}(y)$. Therefore, X is (β, G_Π)-unfavourable. □

It is known that every pseudocompact space with countable pseudocharacter is first countable. For a (β, G_Π)-unfavourable space we have the following result.

Theorem 3.14. *If X is a (β, G_Π)-unfavourable space with countable pseudocharacter, then X has a dense first-countable subspace.*

Proof. Fix an arbitrary nonempty open subspace U of X. By Theorem 3.3, U is (β, G_Π)-unfavourable, so there exists a decreasing sequence $\{V_n : n \in \omega\}$ of nonempty open subsets of U such that $\{V_n : n \in \omega\}$ has the property G_Π. Now take another sequence $\{W_n : n \in \omega\}$ of nonempty open subsets of U satisfying $\overline{W_{n+1}} \subset W_n \subset V_n$ for each $n \in \omega$. The set of cluster points of $\{W_n : n \in \omega\}$ is not empty. Fix such a cluster point x. Choose a sequence $\{O_n : n \in \omega\}$ of open neighbourhoods of x such that $\bigcap\{O_n : n \in \omega\} = \{x\}$ and $\overline{O_{n+1}} \subset O_n \subset W_n$ for each $n \in \omega$. Then it is easy to verify that $\{O_n : n \in \omega\}$ is a base of open neighbourhoods of X at x. Clearly, $x \in U$ and this implies that the subspace consisting of points with countable character in X is dense in X. □

Corollary 3.15. *If X is a (β, G_Π)-unfavourable paratopological group with countable pseudocharacter, then X is a metrisable topological group.*

A space X is said to be strongly Fréchet if for each $x \in X$ and every sequence $\xi = \{A_n : n \in \omega\}$ of subsets of X such that $x \in \bigcap_{n\in\omega} \overline{A_n}$, there exists a sequence $\{x_n : n \in \omega\}$ in X converging to x and intersecting infinitely many members of ξ.

Theorem 3.16. *Let Z be the product space of two topological spaces X and Y such that X is (β, G_Π)-unfavourable and Y is strongly Fréchet and (α, G_Π)-favourable. Then Z is a (β, G_Π)-unfavourable space.*

Proof. Assume that $\mathcal{B}$ is a base of the space Z such that the members of $\mathcal{B}$ have the form $W \times O$, where W and O are open subsets of X and Y, respectively. In addition, fix some well-ordering on $\mathcal{B}$.

Let $t := \{t_n : n \in \omega\}$ be an arbitrary strategy for player β in the game G_Π played on Z, and let s be a fixed winning strategy for player α in the game G_Π played on Y. To prove

https://doi.org/10.1017/S0004972720000192 Published online by Cambridge University Press

that t is not a winning strategy we will construct a strategy $t' := \{t'_n : n \in \omega\}$ for β in the game G_Π on X as follows.

Player β starts the game by choosing an open nonempty subset U_0 of Z under strategy t. Let $U'_0 \times U''_0$ be the first element of $\mathcal{B}$ such that $\overline{U'_0}^X \times \overline{U''_0}^Y \subset U_0$. Now let U'_0 be the first move of β under t' and U''_0 be the first move of β in the game G_Π played on Y. Then player α takes two open nonempty subsets V'_0 and V''_0 of X and Y, respectively, such that V'_0 is the response of α to the first move of β under t' and V''_0 is the first move of α under s following U''_0. Let $V_0 := V'_0 \times V''_0$ be the response of α to the first move of β under t. Then player β takes an open nonempty subset U_1 of Z in the second move under t. Continuing this procedure indefinitely, we can obtain a t-play $\{(U_n, V_n) : n \in \omega\}$, a t'-play $\{(U'_n, V'_n) : n \in \omega\}$ and an s-play $\{(U''_n, V''_n) : n \in \omega\}$, and the following conditions are satisfied for each $n \in \omega$:

(a) $U_{n+1} \subset V_n \subset U_n$;
(b) $U'_{n+1} \subset V'_n \subset U'_n$;
(c) $U''_{n+1} \subset V''_n \subset U''_n$;
(d) $V_n = V'_n \times V''_n$;
(e) $U'_n \times U''_n$ is the first element of $\mathcal{B}$ such that $\overline{U'_n}^X \times \overline{U''_n}^Y \subset U_n$.

Observing that strategies t, s and the base $\mathcal{B}$ of Z are fixed, we can see from the above construction that for each $n \in \omega$ the values $t'_n(X, V'_0, V'_1, \ldots, V'_{n-1})$ of the nth move of β under t' depend only on finite sequences of nonempty open subsets of X, that is, $t' := \{t'_n : n \in \omega\}$ is really a strategy for β in the game G_Π played on X.

By the assumption, X is (β, G_Π)-unfavourable and so t' is not a winning strategy. So there exists a t'-play $\{(U'_n, V'_n) : n \in \omega\}$ which is won by α, that is, the sequence $\{V'_n : n \in \omega\}$ has the property Π. Since s is a winning strategy for α, the corresponding s-play $\{(U''_n, V''_n) : n \in \omega\}$ is also won by α.

Now it remains to verify that the corresponding sequence $\{V'_n \times V''_n : n \in \omega\}$ has the property Π. For this purpose, take any decreasing sequence $\{W_n : n \in \omega\}$ of nonempty open subsets of Z such that $W_n \subset V'_n \times V''_n$ for each $n \in \omega$, and for each W_n choose a nonempty open subset W'_n of X and a nonempty open subset W''_n of Y such that $W'_n \times W''_n \subset W_n$. Clearly, the sequence $\{\bigcup_{k\geq n} W''_k : n \in \omega\}$ of open subsets of Y is decreasing and $\bigcup_{k\geq n} W''_k \subset \bigcup_{k\geq n} p_Y(W_k) \subset \bigcup_{k\geq n}(V''_k) = V''_n$ for each $n \in \omega$, where p_Y is the projection of Z onto Y. Hence, $\{\bigcup_{k\geq n} W''_k : n \in \omega\}$ has at least one cluster point in Y. Fix such a cluster point y. Obviously $y \in \bigcap\{\overline{\bigcup_{k\geq n} W''_k}^Y : n \in \omega\}$. Since Y is a strongly Fréchet space, for each $n \in \omega$ there exists $y_n \in \bigcup_{k\geq n} W''_k$ such that the sequence $\{y_n : n \in \omega\}$ converges to y. For each $n \in \omega$ fix $k_n \in \omega$ such that $k_n < k_{n+1}$ and $y_n \in W''_{k_n}$. Clearly, $\{\bigcup_{m\geq n} W'_{k_m} : n \in \omega\}$ is decreasing and each $\bigcup_{m\geq n} W'_{k_m} \subset \bigcup_{m\geq n} W'_m \subset V'_n$. Hence $\{\bigcup_{m\geq n} W'_{k_m} : n \in \omega\}$ has at least one cluster point in X. Fix such a cluster point x. Then it is easy to see that for any neighbourhood O of (x, y) in Z, there exists $W'_{k_m} \times W''_{k_m}$ such that $O \cap W'_{k_m} \times W''_{k_m} \neq \emptyset$, which implies that the set of cluster points of the sequence $\{W'_n \times W''_n : n \in \omega\}$ is not empty. Since $W'_n \times W''_n \subset W_n \subset V'_n \times V''_n$, for $n \in \omega$, it follows that $\{V'_n \times V''_n : n \in \omega\}$ has the property Π. Therefore, Z is a (β, G_Π)-unfavourable space. $\square$

https://doi.org/10.1017/S0004972720000192 Published online by Cambridge University Press

PROBLEM 3.17. Can the condition 'Y is strongly Fréchet' in Theorem 3.9 be dropped?

PROBLEM 3.18. Is it true that the product space of any two (β, G_Π)-unfavourable spaces is also (β, G_Π)-unfavourable, or at least has the Baire property?

A subset L of a space X is called bounded if for every locally finite family ξ of open subsets in X the set $\{U \in \xi : U \cap L \neq \emptyset\}$ is finite. It is known that a subset L of a Tychonoff space X is bounded if and only if every continuous function on X is bounded on L.

THEOREM 3.19. *If X is a normal (β, G_Π)-unfavourable topological space, then X is strongly Baire. In particular, if X is a normal (β, G_Π)-unfavourable semitopological group, then X is a topological group.*

PROOF. Let $t := \{t_n : n \in \omega\}$ be a strategy for player β in the game G_S played on X. We will show that t is not a winning strategy. Clearly, t is also a strategy for β in the game G_Π, since the two games have the same playing rules. Since X is (β, G_Π)-unfavourable, there exists a t-play $\{(U_n, V_n) : n \in \omega\}$ in the game G_Π which is won by α. Hence, the set of cluster points of the sequence $\{V_n : n \in \omega\}$, denoted by K, is not empty. Clearly, $K = \bigcap_{n\in\omega} \overline{V_n}$. Since $\{V_n : n \in \omega\}$ has the property G_Π, every locally finite family η of nonempty open subsets of X such that each element of η has nonempty intersection with K is finite. Hence, K is a bounded subset of X. Since K is a closed subset of the normal space X, every continuous real-valued function on K can be extended onto the whole space X. Thus K is a pseudocompact space. Hence, K is countably compact, since X is normal. For each $n \in \omega$ take a point $x_n \in V_n$.

Claim. The sequence $\{x_n : n \in \omega\}$ has at least one cluster point. If $K \cap \{x_n : n \in \omega\}$ is infinite, then the conclusion comes immediately from the countable compactness of K. So it remains to consider the case with $K \cap \{x_n : n \in \omega\} = \emptyset$. Suppose that $K \cap \overline{\{x_n : n \in \omega\}} = \emptyset$. Since X is normal, one can take an open subset O of X such that $\overline{\{x_n : n \in \omega\}} \subset O \subset \overline{O} \subset X \setminus K$. Now fix a decreasing sequence $\{O_n : n \in \omega\}$ of open subsets of X such that $\{x_k : k \geq n\} \subset O_n \subset V_n \cap O$, $n \in \omega$. Since the sequence $\{V_n : n \in \omega\}$ has the property Π, it follows that the set of cluster points of $\{O_n : n \in \omega\}$ is not empty. Fix such a cluster point x. Then x must belong to K, which contradicts the choice of the sets O_n. Hence, $K \cap \overline{\{x_n : n \in \omega\}} \neq \emptyset$, which implies that $\{x_n : n \in \omega\}$ has at least one cluster point.

The above argument shows that the sequence $\{V_n : n \in \omega\}$ has the property S. Thus the t-play $\{(U_n, V_n) : n \in \omega\}$ is also won by α in the game G_S. Hence, X is a (β, G_S)-unfavourable space.

Obviously, every (β, G_S)-unfavourable space is strongly Baire. As mentioned earlier every strongly Baire semitopological group is in fact a topological group [9]. Therefore, X is a topological group. □

A space X is μ-complete if the closure of any bounded subset of it is compact. All Dieudonné complete spaces are μ-complete, as well as all paracompact spaces and all submetrisable spaces.

With a proof similar to that of Theorem 3.19 we have the following result.

https://doi.org/10.1017/S0004972720000192 Published online by Cambridge University Press

THEOREM 3.20. *If X is a μ-complete (β, G_{Π})-unfavourable topological space, then X is strongly Baire. In particular, if X is a μ-complete (β, G_{Π})-unfavourable semitopological group, then X is a topological group.*

PROOF. We use the sets and the symbols from Theorem 3.19. Since X is μ-complete, the bounded closed subset K is compact. Then it follows that the open subsets O and O_n satisfying the conditions in Theorem 3.19 can also be constructed, although X is not normal. Hence, we can find a t-play $\{(U_n, V_n) : n \in \omega\}$ which is won by player α in the game G_S. Therefore, X is a topological group. □

Acknowledgement

The authors would like to thank the referee for helpful and valuable comments.

References

[1] A. V. Arhangelskii, 'Two types of remainders of topological groups', *Comment. Math. Univ. Carolin.* **49**(4) (2008), 119–126.
[2] A. V. Arhangel'skii, M. M. Choban and P. S. Kenderov, 'Topological games and topologies on groups', *Math. Maced.* **8** (2010), 1–19.
[3] A. V. Arhangel'skii, M. M. Choban and P. S. Kenderov, 'Topological games and continuity of group operations', *Topology Appl.* **157** (2010), 2542–2552.
[4] A. V. Arhangel'skii and E. A. Reznichenko, 'Paratopological and semitopological groups versus topological groups', *Topology Appl.* **151** (2005), 107–119.
[5] A. V. Arhangelskii and M. G. Tkachenko, *Topological Groups and Related Structures* (Atlantis Press, Amsterdam, 2008).
[6] A. Bouziad, 'Every Čech-analytic Baire semitopological group is a topological group', *Proc. Amer. Math. Soc.* **124**(3) (1996), 953–959.
[7] R. Ellis, 'A note on the continuity of the inverse', *Proc. Amer. Math. Soc.* **8** (1957), 119–125.
[8] R. Engelking, *General Topology*, revised and completed edition (Heldermann, Berlin, 1989).
[9] P. Kenderov, I. S. Kortezov and W. B. Moors, 'Topological games and topological groups', *Topology Appl.* **109** (2001), 157–165.

HANFENG WANG, Department of Mathematics,
Shandong Agricultural University, Taian 271018, China
e-mail: whfeng@sdau.edu.cn

WEI HE, Institute of Mathematics, Nanjing Normal University,
Nanjing 210046, China
e-mail: weihe@njnu.edu.cn

JING ZHANG, School of Mathematics and Statistics,
Minnan Normal University, Zhangzhou 363000, China
e-mail: zhangjing86@126.com