

Analytical solution of the Grad Shafranov equation in an elliptical prolate geometry

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The analytical solution in toroidal coordinates of the Grad Shafranov equation has been at the origin of the tokamak breakthrough in the fusion development. Unfortunately, the standard toroidal coordinates have a circular poloidal section, which does not fit the elongated cross-section of the present tokamak experiments. In axisymmetry, the vacuum Grad Shafranov equation coincides with the Laplace equation for the toroidal component of the vector potential. In the present paper the solutions for the Laplace equation and that for the vacuum Grad Shafranov equation are tackled in the elliptical prolate toroidal cap-cyclide coordinates framework. The following report of the geometrical properties and of the metric of these coordinates allows us to work out the analytical solution of both equations in terms of the Wangerin functions.

Key words: fusion plasma, plasma confinement

1. Introduction

In plasma physics the magnetic hydro dynamic (MHD) equilibrium equation $\nabla P = J \times B$ is one of the most assessed equations. The analytical solution to the toroidal axisymmetric Grad Shafranov equation (Shafranov 1960; Mukhovatov & Shafranov 1971) was at the basis of the breakthrough realized by the tokamak configuration along the path of fusion energy achievements. In axisymmetry the Grad Shafranov equation

$$\Delta^* \psi = \frac{\partial^2 \psi}{\partial R^2} - \frac{1}{R} \frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial z^2} = \mu_0 R J_\varphi \quad (1.1)$$

where J_φ is the toroidal plasma current, ψ is the flux function and (R, z, φ) are the standard cylindrical coordinates; in the vacuum ($J_\varphi = 0$) coincides with the Laplace equation for the toroidal component of the vector potential (Bateman 1978). The analytical solution to the related Laplace equation in several different geometries was deeply studied during the second half of the 19th century (Neumann 1864; Böcher 1894; Wangerin 1909), pointing to the possibility of finding an analytical solution also for oblate and prolate toroidal geometries. A good summary of all these works can be found in Moon and Spencer's book (1971). During the first half of the 20th century the problem of finding an analytical solution to the inductance of a torus

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was first tackled (Fock 1932), and a first analytical solution to the Laplace equation, in elliptical oblate coordinates, was found by Lebedev (1937). More recently Chance (1997) has solved the Laplace equation for a general azimuthal geometry by using a Green function technique, even in presence of internal currents. Eventually, a team of Japanese scientists found the analytical solution to the vacuum Grad Shafranov equation for an oblate toroidal system (Aikawa & Takahara 1975; Honma *et al.* 1979). In the meanwhile, a large amount of scientific papers have been written to solve the tokamak equilibrium problem, adopting several different approaches. Analytical solutions to the nonlinear Grad Shafranov equation have been studied and found for simple cases of the source term (Soloviev 1968; Greene 1988; Zheng, Wootton & Solano 1996; Cerfon & Freidberg 2010). The employment of a system of toroidal coordinates, which has the flux surfaces as the radial coordinate, has allowed us to develop robust codes with the mapping solution expanded on the momentum (Boozer 1980; Lao, Hirschmann & Wieland 1981; Brambilla 2003). Predictive equilibrium codes have been developed, through the numerical resolution (Feneberg & Lackner 1973; Cenacchi, Galvao & Taroni 1976; Blum, Le Foll & Thooris 1981; Albanese, Blum & De Barbieri 1987; Albanese, Ambrosino & Mattei 2015) and/or the adoption of a semi-analytical approach to deal with (Alladio & Crisanti 1986; Alladio *et al.* 1991) the Grad Shafranov equation, and have been then employed to design the planned plasma configuration of practically all ongoing tokamak experiments. The linearization of the problem (Albanese & Villone 1998) has allowed us to easily optimize a given equilibrium configuration in terms of different aspects (currents in the poloidal coils, forces on the coils, interaction with the first wall...). Moreover, the use of the linearized solver has allowed us to optimize the real time control of ongoing experiments (Sartori *et al.* 2005, 2008; Albanese *et al.* 2011) and to design an optimal control system for future devices (Albanese *et al.* 2009; Bachmann *et al.* 2015). Although it is an ill-posed problem (different internal current distributions can produce the same magnetic signal), the magnetic measurements have been used to constrain the equilibrium codes, in order to reconstruct the experimental plasma boundary and to work out, as much as possible, the most important internal features of the confined plasma; i.e. the internal magnetic and kinetic inductances β_p and l_i (Shafranov 1971), various plasma moments (Van Milligen 1990; Van Milligen & Lopez Fraguas 1994) (for instance the plasma column centroid and its elongation), the internal mapping of the magnetic surfaces and the identification of the plasma current density. As for the predictive equilibrium codes, both numerical and semi-analytical approaches have been used (Lao *et al.* 1985; Alladio & Crisanti 1986; Alladio *et al.* 1991). Eventually, the fact that the plasma equilibrium reconstruction is a mathematically ill-posed problem and that in some used codes it depends on the knowledge of external currents, and the fact that experimental noise must be accounted for, has always led to a poor reconstruction of the internal mapping of the magnetic topology. In several codes internal independent measurements have been used to constrain the solution. In particular, within the most used code, EFIT, the polarimetry (Li *et al.* 2011) and/or the motional Stark effect (Lao *et al.* 2005) have both been used to further constrain the problem and to improve the reconstructed poloidal flux mapping. This code has also been employed for real time reconstruction, which is to be used on the plasma shape control system (Ferron *et al.* 1998). This latter aspect has been strongly improved thanks to the use of powerful modern computer parallelization (Yue *et al.* 2013). Of course, the use of all these techniques and of modern computers has always led to result improvements. However, the details of the internal parameters, like the current density profile, are still

far from what would be necessary for a deeper understanding of the physics and/or for an accurate real time control of the internal profiles. This represents a serious concern when considering that in a DEMO class machine the available diagnostics will be strongly reduced, with the consequent necessity of having very robust and flexible plasma models. Reconstructive equilibrium codes (Alladio & Crisanti 1986; Alladio *et al.* 1991), based on a semi-analytical approach, are in principle more flexible and robust than the completely numerical ones, as they use the multipolar analytical solution of the electromagnetic problem on the most suitable function basis. In particular this approach gives the possibility to fully separate the external world (passive structures, coils, ...) from the internal plasma features. The most important advantage given by this approach is that any 'unphysical' solution, which could be worked out by a pure numerical code in the presence of measurement errors, is not intrinsically possible. This aspect was widely discussed in relation to the FTU experiment's circular toroidal geometry in Alladio & Crisanti (1986). In that paper the robustness of the approach was also verified on the old JET experiment, without the presence of internal divertor coils. The used reconstructive equilibrium code was the one that showed the possibility in JET of producing X point configurations and the consequent possibility of achieving an improved energy confinement regime (Alladio *et al.* 1984). Unfortunately, so far, the exact analytical solution of the Grad Shafranov equation for the vacuum region has been found only in relation to the standard circular shaped and for the elliptical oblate toroidal geometries (Aikawa & Takahara 1975; Honma *et al.* 1979). Both these geometries are not suitable to the present tokamak experiments, all of which are based on an elliptical prolate geometry. An interesting approach to analytically solving the Grad Shafranov equation through the separable variable technique and using not constant source functions has been used in Atanasiu *et al.* (2004) and Guazzotto & Freidberg (2007). In both papers the solution has been worked out by using an (x, y) coordinate system. Unluckily this 'not natural' system of coordinates results in a solution with a large expansion that does not always converge. Guazzotto & Freidberg partially overcome the problem by linking some geometrical property of the plasma boundary to the solution expansion. Nonetheless, the non-robustness of the solution remains. In the present paper the mathematical problem is summarized and the analytical solution of the vacuum Grad Shafranov equation is worked out for the cap-cyclide coordinate system, which is the natural one for the modern elongated and D-shaped tokamak. Moreover, the paper addresses how the solution could be used to realize a robust reconstructive equilibrium code, capable of better fitting the needs of a future reactor relevant tokamak, like DEMO.

2. Cap-cyclide geometry

In cylindrical coordinates (R, Z) the vacuum Grad Shafranov equation (Shafranov 1960) is

$$\Delta^* \psi = \frac{\partial^2 \psi}{\partial R^2} - \frac{1}{R} \frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial z^2} = 0. \quad (2.1)$$

This former equation is formally similar to the Laplace equation, but for a sign (minus instead of plus) in the first derivative part.

$$\Delta \psi = \frac{\partial^2 \psi}{\partial R^2} + \frac{1}{R} \frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial z^2} = 0. \quad (2.2)$$

Consequently, it is obvious that any analytical solution found for the Laplace equation will be correlated to the analytical solution of the Grad Shafranov equation. Since the

end of the 19th century (Böcher 1894; Wangerin 1909) analytical solutions to the Laplace equation have been worked out using a large set of system of coordinates, that allows writing of the solution in a ‘separable’ way, i.e.

$$\psi(\alpha, \beta, \gamma) = A(\alpha)B(\beta)G(\gamma), \tag{2.3}$$

where (α, β, γ) are independent orthogonal variables. For a subset of the system of coordinates a so-called ‘R-separable’ solution can be worked out. In particular this is the solution for the coordinate systems with a toroidal symmetry, such as the proper ones for the tokamak experiments

$$\psi(\alpha, \beta, \gamma) = A(\alpha)B(\beta)G(\gamma)/\mathcal{R}(\alpha, \beta, \gamma) \tag{2.4}$$

$\mathcal{R}(\alpha, \beta, \lambda)$ being a function determined by the coordinate system.

For the present toroidal elongated tokamak, the most appropriate coordinate system is the cap-cyclide one (Moon & Spencer 1971), (figure 1) with

$$\left. \begin{aligned} x &= \frac{\Lambda}{a_s \Gamma} \operatorname{dn}(v, k_1) \operatorname{sn}(\mu, k) \cos \varphi \\ y &= \frac{\Lambda}{a_s \Gamma} \operatorname{dn}(v, k_1) \operatorname{sn}(\mu, k) \sin \varphi \\ z &= \frac{k^{0.5} \Pi}{2a_s \Gamma}, \end{aligned} \right\} \tag{2.5}$$

where (x, y, z) are the standard Cartesian coordinates, $\operatorname{dn}, \operatorname{cn}, \operatorname{sn}$ are the Jacobi elliptical functions, a_s is a dimensional scale parameter and

$$\left. \begin{aligned} \Lambda &= 1 - \operatorname{dn}^2(\mu) \operatorname{sn}^2(v) \\ \Gamma &= \operatorname{sn}^2(\mu) \operatorname{dn}^2(v) + [(\Lambda/k^{0.5}) + \operatorname{cn}(\mu) \operatorname{dn}(\mu) \operatorname{sn}(v) \operatorname{cn}(v)]^2 \\ \Pi &= (\Lambda^2/k) - [\operatorname{sn}^2(\mu) \operatorname{dn}^2(v) + \operatorname{cn}^2(\mu) \operatorname{dn}^2(\mu) \operatorname{sn}^2(v) \operatorname{cn}^2(v)]. \end{aligned} \right\} \tag{2.6}$$

The new variables (μ, v, Π) are defined between

$$0 \leq \mu \leq K; \quad 0 \leq v \leq K'; \quad 0 \leq \varphi \leq 2\pi. \tag{2.7a-c}$$

Here K and iK' are respectively the real and the imaginary complete elliptical integrals

$$K(k) = \int_0^{\pi/2} \frac{d\vartheta}{(1 - k^2 \sin^2 \vartheta)^{1/2}}; \quad iK'(k_1) = i \int_0^{\pi/2} \frac{d\vartheta}{(1 - k_1^2 \sin^2 \vartheta)^{1/2}}; \quad k^2 + k_1^2 = 1 \tag{2.8a-c}$$

and k and k_1 are respectively the parameter and the complementary parameter of the elliptical integral.

The coordinate transformation of (2.5) can be also written in the complex plane as

$$(x^2 + y^2)^{1/2} + iz = \frac{1}{a_s \operatorname{sn}(\mu + iv) + ia_s k^{-0.5}} + \frac{1}{2a_s k^{-0.5}}. \tag{2.9}$$

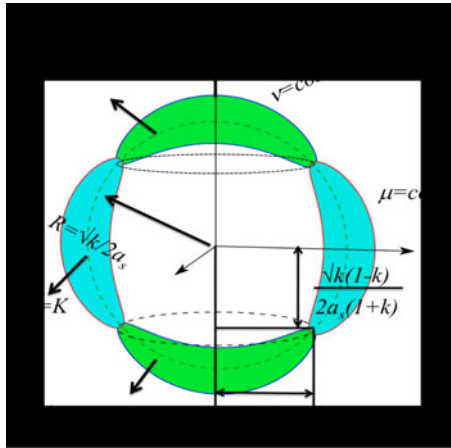


FIGURE 1. Cap-cyclide coordinate system. Surfaces at constant μ and v .

Writing from here on out $dn(\mu)$, $cn(\mu)$, $sn(\mu) = dn\mu$, $cn\mu$, $sn\mu$ and $dn(v)$, $cn(v)$, $sn(v) = dnv$, cnv , snv , for $\mu = \text{const.}$ the coordinate surfaces are ring-cyclides (see figure 1) with equation

$$(x^2 + y^2 + z^2)^2 + A_\mu(x^2 + y^2) + B_\mu z^2 + C_\mu = 0, \tag{2.10}$$

with

$$A_\mu = -(R_\mu^1 + R_\mu^2); \quad C_\mu = R_\mu^1 R_\mu^2 \tag{2.11a,b}$$

and

$$B_\mu = \frac{4a_s^2(sn^2\mu + 1/k)^2}{k(1/k - sn^2\mu)^2} \left[-R_\mu^1 R_\mu^2 + (R_\mu^1 + R_\mu^2) \frac{sn^2\mu}{a_s^2(sn^2\mu + 1/k)^2} - \frac{k^2}{16a_s^4} \right], \tag{2.12}$$

where (note the definitions reported in Moon & Spencer (1971) present a small mistake)

$$\left. \begin{aligned} R_1(\mu) &= \frac{ksn^2\mu \left(1 - \frac{dn^2\mu}{1+k}\right)^2}{a_s^2 \left\{ ksn^2\mu + \left[\left(1 - \frac{dn^2\mu}{1+k}\right) \frac{1}{k^{0.5}} + \frac{cn\mu \, dn\mu}{1+k} k^{0.5} \right]^2 \right\}^2} \\ R_2(\mu) &= \frac{ksn^2\mu \left(1 - \frac{dn^2\mu}{1+k}\right)^2}{a_s^2 \left\{ ksn^2\mu + \left[\left(1 - \frac{dn^2\mu}{1+k}\right) \frac{1}{k^{0.5}} - \frac{cn\mu \, dn\mu}{1+k} k^{0.5} \right]^2 \right\}^2} \end{aligned} \right\} \tag{2.13}$$

For $v = \text{const.}$ the coordinate surfaces are cap-cyclides (see figure 1) with equation

$$(x^2 + y^2 + z^2)^2 + A_v(x^2 + y^2) + B_v z^2 + C_v = 0, \tag{2.14}$$

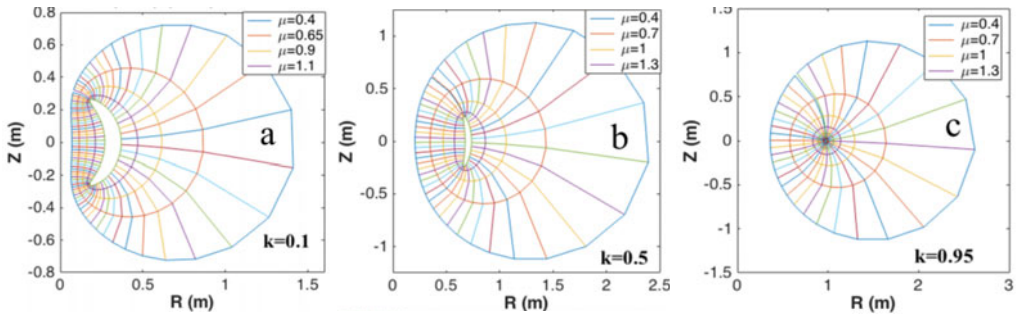


FIGURE 2. (a) For $k \rightarrow 0$ the constant μ surfaces assume a bean shape; (b) for intermediate k the surfaces can be either D or purely elliptical prolate; (c) for $k \rightarrow 1$ the surfaces tend to the standard toroidal ones.

where (note the definitions reported in Moon & Spencer (1971) present a small mistake)

$$\left. \begin{aligned} A_v &= \frac{1}{8a_s^2} \frac{(k + dn^2v)^2}{dn^2v} \left[\frac{(cn^4v + 6ksn^2v cn^2v + k^2sn^4v)(dn^2v - k)^2}{(cn^2v - ksn^2v)^2(k + dn^2v)^2} - 1 \right] \\ B_v &= \frac{k}{2a_s^2} \frac{cn^4v + 6ksn^2v cn^2v + k^2sn^4v}{(cn^2v - ksn^2v)^2} \\ C_v &= \frac{k^2}{16a_s^4}. \end{aligned} \right\} \quad (2.15)$$

By varying the parameter k , the coordinates transformation of equation (2.5) describes a large set of quite different geometries (figure 2). For $\mu \rightarrow 0$, independently of k , the geometry resembles the standard toroidal geometry; for larger values of μ the shape of the constant μ surfaces depend on the value of the k parameter. For $k \rightarrow 0$ (figure 2a) the surfaces tends to assume a bean shape, like the old PBX-M (England *et al.* 1997) experiment. For intermediate values of k (figure 2b) the surfaces can be either D or purely elliptical prolate shaped, like the largest part of the present ongoing tokamak experiments. For $k \rightarrow 1$ (figure 2c) all the surfaces are similar to the standard toroidal ones, independently of the μ value.

For $\mu \rightarrow K$ the surface at constant μ degenerates in a circumference arc of radius

$$\left. \begin{aligned} R_0 &= \sqrt{X_0^2(K, v) + Z_0^2(K, v)} = \frac{\sqrt{k}}{2a_s}, \quad \text{where} \\ X_0(K, v) &= \frac{k \, dnv}{a_s(k + dn^2v)}; \quad Z_0(K, v) = \frac{\sqrt{k}(dn^2v - k)}{2a_s(dn^2v + k)}. \end{aligned} \right\} \quad (2.16)$$

For $k \rightarrow 1$, $R_0 = 1/2a_s$, as is clear from the previous discussion, is the centre of the standard toroidal coordinates. For the constant μ contours it is possible to define the local major radius, the local minus radius and the local aspect ratio.

$$\left. \begin{aligned} R(\mu) &= \frac{\sqrt{k}(1 + ksn^2\mu)}{2a_s(1 + k)sn\mu}; \quad a(\mu) = \frac{\sqrt{k}cn\mu \, dn\mu}{2a_s(1 + k)sn\mu} \\ A(\mu) &= \frac{R(\mu)}{a(\mu)} = \frac{(1 + ksn^2\mu)}{cn\mu sn\mu}; \quad R_0 = \sqrt{R^2(\mu) + a^2(\mu)}. \end{aligned} \right\} \quad (2.17)$$

By using the expansion of the Jacobi elliptical functions in terms of the hyperbolic coordinates, when $k \rightarrow 1$, in the limit of large aspect ratio ($\mu \rightarrow 0$) we have that $A(\mu) \sim \cosh(\mu)$ as in the standard toroidal geometry. We can define the local elongation as $b(\mu)/a(\mu)$, where $b(\mu)$ is the local maximum height of any contour; after some heavy algebra we have

$$b(\mu) = \frac{\sqrt{k}(1 - ksn^2\mu)}{2a_s(1 + k)sn\mu}; \quad \frac{b(\mu)}{a(\mu)} = \frac{(1 - ksn^2\mu)}{cn\mu sn\mu}. \tag{2.18a,b}$$

For $\mu \rightarrow 0$, $b(\mu)/a(\mu) \rightarrow 1$ independently of the parameter k , i.e. the contours are circular as for the standard toroidal case. For $\mu \rightarrow K$, $b(\mu)/a(\mu) \rightarrow \infty$ independently of the parameter k (but $k = 1$, where $b(\mu)/a(\mu) \rightarrow 1$), this is because, in the centre of the ‘geometry’, the constant surface converges to a circumference arc and not to a point.

3. Geometry metric

For the Laplace equation (2.2) to admit a quasi-separable solution of type (2.4), in a rotational geometry like the cap-cyclides, it is necessary and sufficient that the three following conditions are satisfied (Moon & Spencer 1971)

$$\frac{1}{f_\mu} \frac{\partial}{\partial \mu} \left(f_\mu \frac{\partial \mathcal{R}}{\partial \mu} \right) + \frac{1}{f_v} \frac{\partial}{\partial v} \left(f_v \frac{\partial \mathcal{R}}{\partial v} \right) + \alpha_1 (\Phi_{11} + \Phi_{21}) \mathcal{R} = 0 \tag{3.1}$$

$$\frac{g_\mu}{g_v} = -(\Phi_{13} + \Phi_{23}); \quad \frac{\sqrt{g}}{g_i} = \mathcal{R}^2 f_i(u_i) F(u_1, u_2, u_3). \tag{3.2a,b}$$

Here \mathcal{R} is the function defined in (2.4) and g_i are the coordinate transformation metric factors

$$g_\mu = g_v = \frac{1}{a_s^2 \Gamma^2} \Lambda^2 \Omega^2; \quad g_\varphi = \frac{1}{a_s^2 \Gamma^2} \Lambda^2 sn^2 \mu \, dn^2 \mu; \quad g^{1/2} = \left(\frac{\Lambda}{a_s \Gamma} \right)^3 \Omega^2 sn \mu \, dn \mu, \tag{3.3a-c}$$

where Λ and Γ are given in (2.6) and $\Omega^2 = (1 - sn^2 \mu \, dn^2 v)(dn^2 v - k^2 sn^2 \mu)$ and $\Phi_{\alpha\beta}$ are elements of the Stäckel matrix (Morse & Feshbach 1953)

$$[S] = \begin{bmatrix} -k^2 sn^2 \mu & -1 & -(k^2 sn^2 \mu + sn^{-2} \mu) \\ dn^2 v & 1 & (dn^2 v + k^2 dn^{-2} v) \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.4}$$

Consequently, from the third condition of (3.2a,b)

$$\left. \begin{aligned} f_\mu = sn\mu; \quad f_v = dnv; \quad f_\varphi = 1; \quad \mathcal{R} = \left(\frac{\Lambda}{a_s \Gamma} \right)^{1/2} \\ F_\mu = dnv; \quad F_v = sn\mu; \quad F_\varphi = (\Omega^2)/(sn\mu \, dnv). \end{aligned} \right\} \tag{3.5}$$

Eventually, by solving the equation (3.1) we find $\alpha_1 = 1$.

4. Laplace equation

As a consequence of what was discussed in the previous paragraph, the Laplace equation (2.2) in the cap-cyclide coordinates admits a quasi-separable solution of the type of (2.4).

$$f(\mu, \nu, \varphi) = \left(\frac{\Gamma}{\Lambda}\right)^{0.5} M(\mu)N(\nu)\Phi(\varphi), \tag{4.1}$$

where the equations for the radial $M(\mu)$ and the angular part $N(\nu)$ can be written as

$$\left. \begin{aligned} \frac{d^2M}{d\mu^2} + \frac{cn\mu}{sn\mu} \frac{dn\mu}{d\mu} \frac{dM}{d\mu} + \left[k^2sn^2\mu - \alpha_2 - \alpha_3 \left(k^2sn^2\mu + \frac{1}{sn^2\mu} \right) \right] M &= 0 \\ \frac{d^2N}{d\nu^2} - \frac{k^2sn\nu cn\nu}{dn\nu} \frac{dN}{d\nu} + \left[-dn^2\nu + \alpha_2 + \alpha_3 \left(dn^2\nu + \frac{k^2}{dn^2\nu} \right) \right] N &= 0 \\ \frac{d^2\Phi}{d\varphi^2} + \alpha_3\Phi &= 0. \end{aligned} \right\} \tag{4.2}$$

By substituting $z_1 = sn^2\mu$ and $z_2 = dn^2\nu$ the two equations for $M(\mu)$ and $N(\nu)$ can be written as

$$\frac{d^2Z}{dz^2} + \frac{1}{2} \left[\frac{1}{z-a_1} + \frac{1}{z-a_2} + \frac{2}{z-a_3} \right] \frac{dZ}{dz} + \frac{1}{4} \left[\frac{A_0 + A_1z + A_2z^2}{(z-a_1)(z-a_2)(z-a_3)^2} \right] Z = 0. \tag{4.3}$$

Here, the variable z represents both the variables z_1 and z_2 and, consequently, the equations for the functions $M(\mu)$ and $N(\nu)$ are formally similar. In this equation $a_1 = (1; 1)$, $a_2 = (1/k^2; k^2)$, $a_3 = (0; 0)$, $A_0 = (-q^2/k^2; -q^2k^2)$, $A_1 = (-p^2; p^2)$, $A_2 = (1 - q^2; 1 - q^2)$, where the first term in the brackets refers to the equation for $M(\mu)$, the second to the equation for $N(\nu)$, p and q are integer numbers and $p' = p/k$. Equation (4.3) is a particular case of the general Böcher (1894), Moon & Spencer (1971) equation

$$\left. \begin{aligned} \frac{d^2\tilde{Z}}{dz^2} + P(z) \frac{d\tilde{Z}}{dz} + Q(z)\tilde{Z} &= 0 \quad \text{with} \\ P(z) &= \frac{1}{2} \left[\frac{m_1}{z-a_1} + \frac{m_2}{z-a_2} + \dots + \frac{m_{n-1}}{z-a_{n-1}} \right]; \\ Q(z) &= \frac{1}{4} \left[\frac{A_0 + A_1z + \dots + A_lz^l}{(z-a_1)^{m_1}(z-a_2)^{m_2} \dots (z-a_{n-1})^{m_{n-1}}} \right]. \end{aligned} \right\} \tag{4.4}$$

In the particular case of (4.3) the general Böcher equation is named a Wangerin equation. This equation has two singularities of the first order for $z = 1, a_2$ and two singularities of the second order for $z = 0$ and $z \rightarrow \infty$. Consequently the Wangerin equation is characterized by poles $\{1, 1, 2, 2\}$ and the Laplace equation has the solution (Moon & Spencer 1971).

$$\left. \begin{aligned} M(sn^2\mu) &= AD_p^q(k, sn^2\mu) + BF_p^q(k, sn^2\mu) \\ N(dn^2\nu) &= AD_{p'}^q(k, dn^2\nu) + BF_{p'}^q(k, dn^2\nu) \\ \Phi(\varphi) &= A \sin(q\varphi) + B \cos(q\varphi). \end{aligned} \right\} \tag{4.5}$$

5. Grad Shafranov equation

The Grad Shafranov operator of equation (2.1) in axisymmetry ($\partial/\partial\varphi = 0$) is similar to the toroidal part of the Laplace operator applied to the potential vector

$$(\Delta A)_\varphi = -\frac{1}{R} \Delta^* \psi = 0. \tag{5.1}$$

This expression for an orthogonal system of coordinates (u_1, u_2, u_3) assumes the explicit form

$$\begin{aligned} & \frac{\partial^2 A_\varphi}{\partial u_1^2} + \frac{\partial^2 A_\varphi}{\partial u_2^2} + \frac{1}{2g_3} \left(\frac{\partial g_3}{\partial u_1} \frac{\partial A_\varphi}{\partial u_1} + \frac{\partial g_3}{\partial u_2} \frac{\partial A_\varphi}{\partial u_2} \right) \\ & + \frac{A_\varphi}{2g_3} \left\{ \left(\frac{\partial^2 g_3}{\partial u_1^2} + \frac{\partial^2 g_3}{\partial u_2^2} \right) - \frac{1}{g_3} \left[\left(\frac{\partial g_3}{\partial u_1} \right)^2 + \left(\frac{\partial g_3}{\partial u_2} \right)^2 \right] \right\} = 0. \end{aligned} \tag{5.2}$$

That in our case of the cap-cyclide coordinates becomes

$$\begin{aligned} (\Delta A)_\varphi = & \frac{\partial}{\partial \mu} \left[\frac{\partial A_\phi}{\partial \mu} + \left(\frac{1}{f_\mu} \frac{\partial f_\mu}{\partial \mu} - \mathcal{R}^2 \frac{\partial \mathcal{R}^{-2}}{\partial \mu} \right) A_\phi \right] \\ & + \frac{\partial}{\partial \nu} \left[\frac{\partial A_\phi}{\partial \nu} + \left(\frac{1}{f_\nu} \frac{\partial f_\nu}{\partial \nu} - \mathcal{R}^2 \frac{\partial \mathcal{R}^{-2}}{\partial \nu} \right) A_\phi \right] = 0. \end{aligned} \tag{5.3}$$

where here \mathcal{R} is the separation function defined in (3.5). By remembering that in our coordinates the Laplace equation in axisymmetry can be written as

$$\Delta f = \frac{1}{\mathcal{R}^2 \Omega^2 f_\mu f_\nu} \left[f_\nu \frac{\partial}{\partial \mu} \left(\mathcal{R}^2 f_\mu \frac{\partial f}{\partial \mu} \right) + f_\mu \frac{\partial}{\partial \nu} \left(\mathcal{R}^2 f_\nu \frac{\partial f}{\partial \nu} \right) \right] = 0. \tag{5.4}$$

Equation (5.3) is identical to equation (5.4), plus a non-differential term in A_ϕ

$$\left[\frac{\partial}{\partial \mu} \left(\frac{1}{f_\mu} \frac{\partial f_\mu}{\partial \mu} - \mathcal{R}^2 \frac{\partial \mathcal{R}^{-2}}{\partial \mu} \right) + \frac{\partial}{\partial \nu} \left(\frac{1}{f_\nu} \frac{\partial f_\nu}{\partial \nu} - \mathcal{R}^2 \frac{\partial \mathcal{R}^{-2}}{\partial \nu} \right) \right] = \Phi_{13} + \Phi_{23}. \tag{5.5}$$

Where $\Phi_{\alpha\beta}$ are elements of the Stäckel matrix. Consequently the set of separable solutions of (5.4) for the toroidal component of the potential vector is the same as the one of set (4.2), when assuming that the axisymmetry is ($\partial/\partial\varphi = 0 \rightarrow q^2 = \alpha_3 = 0$), plus the non-differential term of (5.5).

$$\left. \begin{aligned} & \frac{d^2 M}{d\mu^2} + \frac{cn\mu \, dn\mu}{sn\mu} \frac{dM}{d\mu} + \left(\frac{1}{sn^2\mu} - \alpha_2 \right) M = 0 \\ & \frac{d^2 N}{d\nu^2} - \frac{k^2 sn\nu \, cn\nu}{dn\nu} \frac{dN}{d\nu} + \left(\alpha_2 + \frac{k^2}{dn^2\nu} \right) N = 0. \end{aligned} \right\}. \tag{5.6}$$

But this set of equations, when assuming $q^2 = \alpha_3 = 1$, is identical to the one of (4.2); consequently the two-dimensional (2-D) solution of the toroidal component of the vectorial Laplace equation for the potential vector is the same as the 3-D scalar Laplace equation when assuming $q^2 = \alpha_3 = 1$. If now we remember that

$$\psi = \oint A_\varphi \, dl = 2\pi A_\varphi \frac{\Lambda}{\alpha_s \Gamma} sn\mu \, dn\mu. \tag{5.7}$$

We get that the solution for the Grad Shafranov equation (2.1) can be expressed in terms of the Wangerin functions with $q^2 = 1$

$$\psi = [AD_p^1(k, sn^2\mu) + BF_p^1(k, sn^2\mu)][AD_p^1(k, dn^2\nu) + BF_p^1(k, dn^2\nu)]. \tag{5.8}$$

It is important to note that in (5.8) all terms, but $A_{p,p'}$ and $B_{p,p'}$, are geometrical terms. The physics information, regarding the topological structure of the magnetic field, depends only on the unknown moments $A_{p,p'}$ and $B_{p,p'}$, constant in the vacuum region. The moments $A_{p,p'}$ describe the field produced by all the currents flowing internally to the vacuum region, whilst the moments $B_{p,p'}$ describe the field produced by all the currents flowing externally to the vacuum region. Consequently, the contribution of all the currents flowing externally from the plasma region is intrinsically separated by the internal properties of the confined plasma, which are fully described by the moments $A_{p,p'}$. In a region with a current distribution, the solution of the Grad Shafranov equation remains formally the same as (2.1), but now the moments $A_{p,p'}$ and $B_{p,p'}$ are not constant anymore, and can be worked out by using a Green function technique, once the current distribution is known.

$$\left. \begin{aligned} A_p(r) &= \int_0^V G(r')(\mu_0 r' J_\varphi) d^3 r' \\ B_p(r) &= \int_V^\infty G(r')(\mu_0 r' J_\varphi) d^3 r'. \end{aligned} \right\} \tag{5.9}$$

On a given tokamak experiment we have a set of magnetic probes, measuring the local value of the magnetic field B and of the flux function ψ

$$\left. \begin{aligned} \psi_l &= \sum_{p,p'} [A_{p,p'} D_p^1(k, sn^2\mu_l) + B_{p,p'} F_p^1(k, sn^2\mu_l)] \\ &\times [A_{p,p'} D_{p'}^1(k, dn^2\nu_l) + B_{p,p'} F_{p'}^1(k, dn^2\nu_l)] \\ B_l &= \frac{1}{2\pi r_l} \left(\frac{\partial \psi}{\partial r} \right)_l \end{aligned} \right\} \tag{5.10}$$

here the subscript l indicates the generic position of a magnetic probe. Clearly this set of magnetic probes allows us to exactly determine the values of the moments $A_{p,p'}$ and $B_{p,p'}$ and, consequently, to have the exact analytical expression of the magnetic topology in the region between the plasma column and the closer poloidal coil. Moreover, this methodology allows us to discriminate between the contribution provided by the external equilibrium currents (given by the moments $B_{p,p'}$) and the one provided by the internal plasma currents distribution (given by the moments $A_{p,p'}$). As described in Alladio & Crisanti (1986) this approach, alongside the Green technique of (5.9) and the Grad Shafranov equation, allows us to build up a robust semi-analytical reconstructive equilibrium code. The robustness of using this approach to deal with the mathematically ill-posed problem of reconstructing the internal plasma equilibrium only relying on external magnetic measurements has been verified both in the context of the FTU circular tokamak and in the ‘old’ JET, where the divertor coils were missing. The robustness of the solution obtained, against the magnetic measurements error, was widely verified at JET and it allowed us to discover the possibility for JET to realize an X point configuration (Alladio *et al.* 1984; Alladio & Crisanti 1986). The drawback of the equilibrium code described in Alladio & Crisanti (1986) was that it was worked out by using the pure toroidal geometry,

with a circular poloidal section. When applied to the modern elongated tokamak, this reconstructive equilibrium code was not able to exactly discriminate between the contribution provided by the external coils and the one provided by the plasma. The solution written down in (5.10) will allow us to solve this problem and to build up a very robust semi-analytical equilibrium code, even capable of dealing with the next generation of tokamaks, where the measurements will be far from the plasma border and the signal produced by high internal magnetic moments will be from the very low up to becoming comparable with the intrinsic system noise.

6. Conclusions

Equilibrium reconstructive codes (Lao *et al.* 1985; Alladio & Crisanti 1986; Alladio *et al.* 1991) are probably the most important diagnostic tool in fusion-based experiments, and their importance will be even stronger in future DEMO class experiments, where the scarce availability of diagnostics will require very robust codes, based on fundamental physics principles. The equilibrium codes based on an analytical solution to the Grad Shafranov equation are very robust and, probably, the most suitable tools to accomplish the necessary quality in terms of robustness and reliability. Unfortunately, so far, an analytical solution to the Grad Shafranov equation was available only for circular and elliptical oblate shaped toroidal geometries, not adaptable to the present elongated tokamak. The analytical solution to the vacuum Grad Shafranov, in the cap-cyclide coordinates, as worked out in this paper, opens the possibility of realizing an equilibrium reconstructive code based on an elongated geometry. The Wangerin functions have been evaluated for the first time in Aikawai & Takahara (1975). However, in order to use them for an equilibrium code, an independent evaluation and cross-checking will be necessary. Moreover, it will be necessary to evaluate their derivatives, to be able to get the poloidal magnetic field and to open the possibility of finding the Green function for this geometry. Eventually, this will allow us to write a robust reconstructive equilibrium code, to be used on the present and future elongated tokamak experiments.

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