

ON SOME CONGRUENCES INVOLVING CENTRAL BINOMIAL COEFFICIENTS

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Abstract

We prove the following conjecture of Z.-W. Sun [‘On congruences related to central binomial coefficients’, *J. Number Theory* **13**(11) (2011), 2219–2238]. Let p be an odd prime. Then

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k2^k} \equiv -\frac{1}{2}H_{(p-1)/2} + \frac{7}{16}p^2B_{p-3} \pmod{p^3},$$

where H_n is the n th harmonic number and B_n is the n th Bernoulli number. In addition, we evaluate $\sum_{k=0}^{p-1} (ak+b)\binom{2k}{k}/2^k$ modulo p^3 for any p -adic integers a, b .

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1. Introduction

In 2006, Adamchuk [1] proposed the following congruence involving central binomial coefficients: for any prime $p \equiv 1 \pmod{3}$,

$$\sum_{k=1}^{2(p-1)/3} \binom{2k}{k} \equiv 0 \pmod{p^2}.$$

This conjecture was confirmed by the author [6]. Many researchers studied congruences for sums of binomial coefficients (see, for instance, [7, 10, 14, 17]). Pan and Sun [10] used a combinatorial identity to deduce that if p is a prime, then

$$\sum_{k=0}^{p-1} \binom{2k}{k+d} \equiv \left(\frac{p-d}{3}\right) \pmod{p} \quad \text{for } d = 0, 1, \dots, p,$$

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where (\cdot) is the Jacobi symbol. They also showed that for any odd prime p ,

$$\sum_{k=0}^{p-1} (3k+1) \binom{2k}{k} \equiv -\left(\frac{p}{3}\right) \pmod{p}.$$

In 2018, Apagodu [2] conjectured that for any odd prime p ,

$$\sum_{k=0}^{p-1} (5k+1) \binom{4k}{2k} \equiv -\left(\frac{p}{3}\right) \pmod{p}.$$

Mao and Cao [7] confirmed this conjecture and also showed that for any odd prime p ,

$$\sum_{k=0}^{p-1} (15k+5) \binom{4k}{2k} \equiv -\left(\frac{p}{5}\right) \pmod{p}.$$

The Bernoulli numbers $\{B_n\}$ and the Bernoulli polynomials $\{B_n(x)\}$ are given by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (0 < |x| < 2\pi), \quad B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k} \quad (n \in \mathbb{N}).$$

Mattarei and Tauraso [8] deduced that for any prime $p > 3$,

$$\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left(\frac{p}{3}\right) - \frac{1}{3} p^2 B_{p-2} \left(\frac{1}{3}\right) \pmod{p^3}.$$

Sun [13] obtained many congruences involving central binomial coefficients and proposed many conjectures. Our first goal is to prove a conjecture of Sun ([13, Conjecture 5.2] or [16, Conjecture 6]).

THEOREM 1.1. *Let p be an odd prime. Then*

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k2^k} \equiv -\frac{1}{2} H_{(p-1)/2} + \frac{7}{16} p^2 B_{p-3} \pmod{p^3}.$$

Sun [14] proved that for any odd prime p ,

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2} - p^2 E_{p-3} \pmod{p^3}, \quad (1.1)$$

where the Euler numbers $\{E_n\}$ are given by

$$E_0 = 1 \quad \text{and} \quad E_{2n} = - \sum_{k=1}^n \binom{2n}{2k} E_{2n-2k} \quad (n \geq 1).$$

Our second goal is to generalise (1.1) as follows.

THEOREM 1.2. For any odd prime p and p -adic integers a, b ,

$$\sum_{k=0}^{p-1} (ak + b) \frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2} (b - a) + ap(2 - 2^{p-1}) - (b - a)p^2 E_{p-3} \pmod{p^3}.$$

REMARK 1.3. If we set $a = 0, b = 1$, we obtain (1.1), and with $a = b = 1$, we obtain

$$\sum_{k=0}^{p-1} (k + 1) \frac{\binom{2k}{k}}{2^k} \equiv p(2 - 2^{p-1}) \pmod{p^3}. \tag{1.2}$$

Finally, we prove the following result.

THEOREM 1.4. Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} (k + 1)^2 \frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2} + p(2^{p-1} - 2) + p^2 - p^2 E_{p-3} \pmod{p^3}.$$

REMARK 1.5. Combining (1.1), (1.2) and Theorem 1.4,

$$\sum_{k=0}^{p-1} (k^2 + 3k + 1) \frac{\binom{2k}{k}}{2^k} \equiv p^2 \pmod{p^3}.$$

Combining Theorem 1.2 with $a = 2, b = 1$ and Theorem 1.4,

$$\sum_{k=0}^{p-1} k^2 \frac{\binom{2k}{k}}{2^k} \equiv 2(-1)^{(p-1)/2} + 3p(2^{p-1} - 2) + p^2 - 2p^2 E_{p-3} \pmod{p^3}.$$

Consequently, we can evaluate $\sum_{k=0}^{p-1} (ak^2 + bk + c) \binom{2k}{k} / 2^k$ modulo p^3 .

We prove Theorem 1.1 in Section 2. Sections 3 and 4 are devoted to proving Theorems 1.2 and 1.4. Our proofs make use of some congruences involving harmonic numbers and combinatorial identities which can be found and proved by the package Sigma [11] via the software Mathematica and some known congruences.

2. Proof of Theorem 1.1

For $n, m \in \{1, 2, 3, \dots\}$, define the harmonic numbers of order m by

$$H_n^{(m)} := \sum_{1 \leq k \leq n} \frac{1}{k^m}, \quad H_0^{(m)} := 0.$$

When $m = 1$, these numbers are the classical harmonic numbers.

To prove Theorem 1.1, we need some lemmas and identities. For each positive integer n ,

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2k}{k} \binom{n}{2k} 2^{n-2k} = \binom{2n}{n}, \tag{2.1}$$

$$\sum_{k=1}^n \frac{(-1)^k}{k} \binom{n}{k} (H_k^2 - H_k^{(2)}) = 2 \sum_{k=1}^n \frac{H_{k-1}}{k^2}, \tag{2.2}$$

$$\sum_{k=1}^n \frac{(-1)^k}{k} \binom{n}{k} = -H_n, \tag{2.3}$$

$$\sum_{k=1}^n \frac{(-1)^k}{k} \binom{n}{k} H_k = -H_n^{(2)}. \tag{2.4}$$

REMARK 2.1. Equation (2.1) follows from [3, (3.99)]; (2.2) can be proved by induction on n ; (2.3) and (2.4) follow from [3, (1.45)] and an identity of Hernández [4] (or [15, (3.4)]), respectively.

LEMMA 2.2 [12, Theorems 5.1 and 5.2]. *Let $p > 3$ be a prime. Then*

$$H_{p-1} \equiv -\frac{1}{3}p^2 B_{p-3} \pmod{p^3}, \quad H_{(p-1)/2}^{(3)} \equiv -2B_{p-3} \pmod{p},$$

$$H_{(p-1)/2} \equiv -2q_p(2) + pq_p(2)^2 \pmod{p^2}, \quad H_{(p-1)/2}^{(2)} \equiv \frac{7}{3}pB_{p-3} \pmod{p^2}.$$

LEMMA 2.3 [15, Lemma 4.2]. *Let $p = 2n + 1$ be an odd prime and $k \in \{0, \dots, n\}$. Then*

$$\frac{\binom{n}{k}(-4)^k}{\binom{2k}{k}} \equiv 1 - p \sum_{j=1}^k \frac{1}{2j-1} + \frac{p^2}{2} \left(\sum_{j=1}^k \frac{1}{2j-1} \right)^2 - \frac{p^2}{2} \sum_{j=1}^k \frac{1}{(2j-1)^2} \pmod{p^3}.$$

LEMMA 2.4. *Let $p = 2n + 1$ be an odd prime and $k \in \{0, \dots, n\}$. Then*

$$\frac{\binom{2k}{k} \binom{p-1}{2k}}{4^k} \equiv \binom{n}{k} (-1)^k \left(1 - \frac{p}{2} H_k + \frac{p^2}{8} (H_k^2 - H_k^{(2)}) \right) \pmod{p^3}.$$

PROOF. It is easy to check that

$$\binom{p-1}{2k} = \prod_{j=1}^{2k} \binom{p-j}{j} = \prod_{j=1}^{2k} \left(1 - \frac{p}{j} \right)$$

$$\equiv 1 - pH_{2k} + \frac{p^2}{2} (H_{2k}^2 - H_{2k}^{(2)}) \pmod{p^3}.$$

By Lemma 2.3, modulo p^3 ,

$$\frac{\binom{2k}{k}}{4^k} \equiv \frac{\binom{n}{k}(-1)^k}{1 - p(H_{2k} - \frac{1}{2}H_k) + \frac{1}{2}p^2(H_{2k} - \frac{1}{2}H_k)^2 - \frac{1}{2}p^2(H_{2k}^{(2)} - \frac{1}{4}H_k^{(2)})}$$

$$\equiv \binom{n}{k}(-1)^k \left(1 + p \left(H_{2k} - \frac{1}{2}H_k \right) + \frac{p^2}{2} \left(H_{2k} - \frac{1}{2}H_k \right)^2 + \frac{p^2}{2} \left(H_{2k}^{(2)} - \frac{1}{4}H_k^{(2)} \right) \right).$$

From this, we immediately obtain the desired result. □

PROOF OF THEOREM 1.1. The cases $p = 3, 5$ can be checked directly. We will assume $p > 5$ from now on. By (2.1),

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k2^k} = \sum_{k=1}^{p-1} \frac{1}{k2^k} \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{2j}{j} \binom{k}{2j} 2^{k-2j} = H_{p-1} + \sum_{j=1}^{(p-1)/2} \frac{\binom{2j}{j}}{4^j} \sum_{k=2j}^{p-1} \frac{\binom{k}{2j}}{k}.$$

By Sigma [11], we find the following identity which can be proved by induction on n :

$$\sum_{k=2j}^{n-1} \frac{1}{k} \binom{k}{2j} = \frac{1}{2j} \binom{n-1}{2j}.$$

This, with Lemma 2.4, yields

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k2^k} - H_{p-1} &= \frac{1}{2} \sum_{j=1}^{(p-1)/2} \frac{\binom{2j}{j} \binom{p-1}{2j}}{j4^j} \\ &\equiv \frac{1}{2} \sum_{j=1}^{(p-1)/2} \frac{(-1)^j}{j} \binom{\frac{1}{2}(p-1)}{j} \left(1 - \frac{p}{2} H_j + \frac{p^2}{8} (H_j^2 - H_j^{(2)}) \right) \pmod{p^3}. \end{aligned}$$

Substituting $n = (p - 1)/2$ into (2.2)–(2.4),

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k2^k} - H_{p-1} \equiv \frac{1}{2} \left(-H_{(p-1)/2} + \frac{p}{2} H_{(p-1)/2}^{(2)} + \frac{p^2}{4} \sum_{k=1}^{(p-1)/2} \frac{H_{k-1}}{k^2} \right) \pmod{p^3}.$$

In view of [5, Lemma 3.2] and Lemma 2.2,

$$\sum_{k=1}^{(p-1)/2} \frac{H_{k-1}}{k^2} = \sum_{k=1}^{(p-1)/2} \frac{H_k}{k^2} - H_{(p-1)/2}^{(3)} \equiv \frac{3}{2} B_{p-3} \pmod{p}.$$

This, with Lemma 2.2, yields the desired result. □

3. Proof of Theorem 1.2

LEMMA 3.1. For any prime $p > 3$,

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{(p-1)/2}{k} (-1)^k}{(2k+1)(2k+2)} \equiv p-1 + (-1)^{(p-1)/2} (1 - q_p(2) - p + pq_p(2)^2) \pmod{p^2},$$

where $q_p(2) = (2^{p-1} - 1)/p$ stands for the Fermat quotient.

PROOF. By Sigma, we find the following identity which can be proved by induction on n :

$$\sum_{k=0}^{n-1} \frac{\binom{n}{k}(-1)^k}{(2k+1)(2k+2)} = -\frac{1}{2n+2} - \frac{(-1)^n}{(2n+1)(2n+2)} + \frac{4^n}{(2n+1)\binom{2n}{n}}.$$

Setting $n = (p - 1)/2$,

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{(p-1)/2}{k}(-1)^k}{(2k+1)(2k+2)} = -\frac{1}{p+1} - \frac{(-1)^{(p-1)/2}}{p(p+1)} + \frac{2^{p-1}}{p\binom{p-1}{(p-1)/2}}.$$

The well-known Morley’s congruence [9] gives

$$\binom{p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2} 4^{p-1} \pmod{p^3} \quad \text{for } p > 3. \tag{3.1}$$

This, with $2^{p-1} = 1 + pq_p(2)$, yields

$$\begin{aligned} \sum_{k=0}^{(p-3)/2} \frac{\binom{(p-1)/2}{k}(-1)^k}{(2k+1)(2k+2)} &\equiv -\frac{1}{p+1} + \frac{(-1)^{(p-1)/2}}{p} \left(\frac{1}{2^{p-1}} - \frac{1}{p+1} \right) \\ &\equiv p-1 + (-1)^{(p-1)/2} \frac{1 - q_p(2)}{2^{p-1}(p+1)} \\ &\equiv p-1 + (-1)^{(p-1)/2} (1 - q_p(2) - p + pq_p^2(2)) \pmod{p^2}. \end{aligned}$$

This proves Lemma 3.1. □

LEMMA 3.2. *Let $p > 3$ be a prime. Then*

$$\begin{aligned} \sum_{k=0}^{(p-3)/2} \frac{\binom{(p-1)/2}{k} H_k (-1)^k}{(2k+1)(2k+2)} &\equiv -2q_p(2) + 2E_{p-3} + 2(-1)^{(p-1)/2} (q_p(2)^2 - q_p(2)) \pmod{p}, \\ \sum_{k=0}^{(p-3)/2} \frac{\binom{(p-1)/2}{k} k H_k (-1)^k}{(2k+1)(2k+2)} &\equiv 2q_p(2) - E_{p-3} + (-1)^{(p-1)/2} (2q_p(2) - q_p(2)^2) \pmod{p}. \end{aligned}$$

PROOF. By Sigma, we find the following identity which can be proved by induction on n ,

$$\sum_{k=0}^{n-1} \frac{\binom{n}{k} H_k (-1)^k}{(2k+1)(2k+2)} = \frac{H_n}{2n+2} - \frac{(-1)^n H_n}{(2n+1)(2n+2)} - \frac{4^n}{(2n+1)\binom{2n}{n}} \sum_{k=1}^n \frac{\binom{2k}{k}}{k4^k}.$$

Substituting $n = (p - 1)/2$ into the above identity and by (3.1),

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{(p-1)/2}{k} H_k (-1)^k}{(2k+1)(2k+2)} \equiv \frac{H_{(p-1)/2}}{p+1} - \frac{(-1)^{(p-1)/2} H_{(p-1)/2}}{p(p+1)} - \frac{(-1)^{(p-1)/2}}{p2^{p-1}} \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k4^k} \pmod{p}.$$

Tauraso [18] and Sun [14, (1.5)] respectively proved

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k4^k} \equiv -H_{(p-1)/2} \pmod{p^3}, \tag{3.2}$$

$$\sum_{k=\frac{p+1}{2}}^{p-1} \frac{\binom{2k}{k}}{k4^k} \equiv (-1)^{(p-1)/2} 2pE_{p-3} \pmod{p^2}. \tag{3.3}$$

These, with Lemma 2.2, yield

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{(p-1)/2}{k} H_k (-1)^k}{(2k+1)(2k+2)} \equiv -2q_p(2) + 2E_{p-3} + 2(-1)^{(p-1)/2} (q_p(2)^2 - q_p(2)) \pmod{p}.$$

Similarly, by Sigma, we find the following identity which can be proved by induction on n :

$$\sum_{k=0}^{n-1} \frac{\binom{n}{k} k H_k (-1)^k}{(2k+1)(2k+2)} = -\frac{H_n}{2n+2} - \frac{n(-1)^n H_n}{(2n+1)(2n+2)} + \frac{4^n}{2(2n+1)\binom{2n}{n}} \sum_{k=1}^n \frac{\binom{2k}{k}}{k4^k}.$$

Setting $n = (p - 1)/2$, and invoking (3.1), (3.2), (3.3) and Lemma 2.2,

$$\begin{aligned} \sum_{k=0}^{(p-3)/2} \frac{\binom{(p-1)/2}{k} k H_k (-1)^k}{(2k+1)(2k+2)} &\equiv -\frac{H_{(p-1)/2}}{p+1} - \frac{(p-1)(-1)^{(p-1)/2} H_{(p-1)/2}}{2p(p+1)} \\ &\quad + \frac{(-1)^{(p-1)/2}}{2p2^{p-1}} \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k4^k} \\ &\equiv 2q_p(2) - E_{p-3} + (-1)^{\frac{p-1}{2}} (2q_p(2) - q_p(2)^2) \pmod{p}. \end{aligned}$$

This completes the proof of Lemma 3.2. □

PROOF OF THEOREM 1.2. We can check the case $p = 3$ directly. From now on, we assume that $p > 3$. By Sigma, we find the following identity which can be proved by induction on n :

$$\sum_{k=2j}^{n-1} (ak + b) \binom{k}{2j} = \frac{an(2j+1) + 2bj + 2b - a}{2j+2} \binom{n}{2j+1}.$$

Substituting $n = p$ into this identity and using (2.1),

$$\begin{aligned} \sum_{k=0}^{p-1} (ak + b) \frac{\binom{2k}{k}}{2^k} &= \sum_{j=0}^{(p-1)/2} \frac{\binom{2j}{j} \binom{p}{2j+1} ap(2j+1) + 2bj + 2b - a}{4^j (2j+2)} \\ &= p \sum_{j=0}^{(p-1)/2} \frac{\binom{2j}{j} \binom{p-1}{2j} ap(2j+1) + 2bj + 2b - a}{4^j (2j+1)(2j+2)} \\ &= \frac{\binom{p-1}{(p-1)/2}}{2^{p-1}} (ap + b - a) + p \sum_{j=0}^{(p-3)/2} \frac{\binom{2j}{j} \binom{p-1}{2j} (ap + b)(2j+1) + b - a}{4^j (2j+1)(2j+2)}. \end{aligned}$$

This, with Lemma 2.4, yields, modulo p^3 ,

$$\begin{aligned} \sum_{k=0}^{p-1} (ak + b) \frac{\binom{2k}{k}}{2^k} - \frac{\binom{p-1}{(p-1)/2}}{2^{p-1}} (ap + b - a) \\ \equiv p \sum_{j=0}^{(p-3)/2} \binom{(p-1)/2}{j} (-1)^j \frac{(1 - \frac{1}{2}pH_j)((ap + b)(2j+1) + b - a)}{(2j+1)(2j+2)} \\ \equiv p \sum_{j=0}^{(p-3)/2} \binom{(p-1)/2}{j} (-1)^j \frac{(ap + b)(2j+1) + b - a - \frac{1}{2}p(2b - a)H_j - bpjH_j}{(2j+1)(2j+2)}. \end{aligned}$$

It is easy to check that

$$\sum_{k=0}^{n-1} \frac{\binom{n}{k} (-1)^k}{2k+2} = \frac{1}{2n+2} \sum_{k=0}^{n-1} \binom{n+1}{k+1} (-1)^k = -\frac{1}{2n+2} \sum_{k=1}^n \binom{n+1}{k} (-1)^k = \frac{1 - (-1)^n}{2n+2}.$$

Setting $n = (p-1)/2$,

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{(p-1)/2}{k} (-1)^k}{2k+2} = \frac{1 - (-1)^{(p-1)/2}}{p+1} \equiv (1 - (-1)^{(p-1)/2})(1-p) \pmod{p^2}.$$

This, with Lemmas 3.1 and 3.2, yields, modulo p^3 ,

$$\begin{aligned} \sum_{k=0}^{p-1} (ak + b) \frac{\binom{2k}{k}}{2^k} - \frac{\binom{p-1}{(p-1)/2}}{2^{p-1}} (ap + b - a) \\ \equiv pa(1 - (-1)^{(p-1)/2}) - p(b-a)(-1)^{(p-1)/2} q_p(2) \\ - ap^2 q_p(2)(1 + (-1)^{(p-1)/2}) - p^2(b-a)E_{p-3}. \end{aligned}$$

Simplifying this congruence using (3.1) and $2^{p-1} = 1 + pq_p(2)$ gives Theorem 1.2. \square

4. Proof of Theorem 1.4

LEMMA 4.1. *Let $p > 3$ be a prime. Then*

$$p(p + 1) \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} \binom{p-1}{2k}}{(k + 1)4^k} \equiv 2p - 2p^2 q_p(2) \pmod{p^3}.$$

PROOF. By Sigma, we can find and prove the identity

$$\sum_{k=0}^n \frac{\binom{n}{k} (-1)^k H_k}{k + 1} = -\frac{H_n}{n + 1},$$

and it is easy to see that

$$\sum_{k=0}^n \frac{\binom{n}{k} (-1)^k}{k + 1} = \frac{1}{n + 1} \sum_{k=0}^n \binom{n + 1}{k + 1} (-1)^k = -\frac{1}{n + 1} \sum_{k=1}^{n+1} \binom{n + 1}{k} (-1)^k = \frac{1}{n + 1}.$$

Substituting $n = (p - 1)/2$ into these identities and using Lemmas 2.4 and 2.2,

$$\begin{aligned} p(p + 1) \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} \binom{p-1}{2k}}{(k + 1)4^k} &\equiv p(p + 1) \sum_{k=0}^{(p-1)/2} \frac{\binom{(p-1)/2}{k} (-1)^k (1 - \frac{1}{2} p H_k)}{k + 1} \\ &= p(p + 1) \frac{2}{p + 1} + \frac{p^2(p + 1)}{2} \frac{2H_{(p-1)/2}}{p + 1} \equiv 2p + p^2 H_{(p-1)/2} \pmod{p^3}. \end{aligned}$$

This proves Lemma 4.1. □

LEMMA 4.2. *For any prime $p > 3$,*

$$\sum_{k=0}^{(p-1)/2} \frac{p \binom{2k}{k} \binom{p-1}{2k}}{(2k + 3)4^k} \equiv (-1)^{(p-1)/2} \left(\frac{1}{2} - \frac{3p}{4} + \frac{7p^2}{8} \right) + \frac{p^2}{2} - \frac{p^2}{2} E_{p-3} \pmod{p^3}.$$

PROOF. By Sigma, we can find and prove the following identity:

$$\sum_{k=0}^n \frac{\binom{n}{k} (-1)^k}{2k + 3} = \frac{4^n}{(2n + 1)(2n + 3) \binom{2n}{n}}.$$

Substituting $n = (p - 1)/2$ into this identity and using Lemma 2.4,

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \frac{p \binom{2k}{k} \binom{p-1}{2k}}{(2k + 3)4^k} &\equiv \frac{\binom{p-3}{(p-3)/2} \binom{p-1}{p-3}}{2^{p-3}} + p \sum_{k=0}^{(p-1)/2} \frac{\binom{(p-1)/2}{k} (-1)^k (1 - \frac{1}{2} p H_k)}{2k + 3} \\ &\quad - \left(\frac{(p - 1)/2}{(p - 3)/2} \right) (-1)^{(p-3)/2} \left(1 - \frac{p}{2} H_{(p-3)/2} \right) = S_1 - \frac{p}{2} S_2 \pmod{p^3}, \end{aligned}$$

where

$$S_1 = \frac{\binom{p-3}{(p-3)/2} \binom{p-1}{p-3}}{2^{p-3}} + \frac{1}{p+2} \frac{2^{p-1}}{\binom{p-1}{(p-1)/2}} - \binom{(p-1)/2}{(p-3)/2} (-1)^{(p-3)/2} \left(1 - \frac{p}{2} H_{(p-3)/2}\right),$$

$$S_2 = p \sum_{k=0}^{(p-1)/2} \frac{\binom{(p-1)/2}{k} (-1)^k H_k}{2k+3}.$$

In view of (3.1) and Lemma 2.2,

$$\begin{aligned} S_1 &= \frac{1}{2} (p-1)^2 \frac{\binom{p-1}{(p-1)/2}}{2^{p-1}} + \frac{(-1)^{(p-1)/2}}{(p+2)2^{p-1}} + \frac{1}{2} (p-1) (-1)^{(p-1)/2} \left(1 - \frac{p}{2} H_{(p-1)/2} + \frac{p}{p-1}\right) \\ &\equiv \frac{1-2p+p^2}{2} (1 + pq_p(2)) (-1)^{(p-1)/2} + \frac{(-1)^{(p-1)/2}}{2} \left(1 - \frac{p}{2} + \frac{p^2}{4}\right) \\ &\quad \cdot (1 - pq_p(2) + p^2 q_p(2)^2) - \frac{1}{2} (-1)^{(p-1)/2} \left(1 - 2p + pq_p(2) - p^2 q_p(2) - \frac{p^2}{2} q_p(2)^2\right) \\ &\equiv \frac{(-1)^{(p-1)/2}}{2} \left(1 - \frac{p}{2} + \frac{5p^2}{4} - pq_p(2) - \frac{p^2}{2} q_p(2) + \frac{3p^2}{2} q_p(2)^2\right) \pmod{p^3}. \end{aligned}$$

By Sigma, we can find and prove the identity

$$\sum_{k=0}^n \frac{\binom{n}{k} (-1)^k H_k}{2k+3} = -\frac{2}{2n+3} + \frac{4^n}{(2n+1)(2n+3) \binom{2n}{n}} \left(2 - \sum_{k=1}^n \frac{\binom{2k}{k}}{k4^k}\right).$$

Setting $n = (p-1)/2$ in this identity and using (3.1), (3.2), (3.3) and Lemma 2.2,

$$\begin{aligned} S_2 &= -\frac{2p}{p+2} + \frac{1}{p+2} \frac{2^{p-1}}{\binom{p-1}{(p-1)/2}} \left(2 - \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k4^k}\right) \\ &\equiv -p + \frac{1}{2} (-1)^{(p-1)/2} \left(1 - \frac{p}{2} - pq_p(2)\right) (2 - 2q_p(2) + pq_p(2)^2) + (-1)^{(p-1)/2} 2pE_{p-3} \\ &\equiv (-1)^{(p-1)/2} \left(1 - q_p(2) - \frac{p}{2} - \frac{p}{2} q_p(2) + \frac{3p}{2} q_p(2)^2\right) - p + pE_{p-3} \pmod{p^2}. \end{aligned}$$

Hence,

$$\sum_{k=0}^{(p-1)/2} \frac{p \binom{2k}{k} \binom{p-1}{2k}}{(2k+3)4^k} \equiv (-1)^{(p-1)/2} \left(\frac{1}{2} - \frac{3p}{4} + \frac{7p^2}{8}\right) + \frac{p^2}{2} - \frac{p^2}{2} E_{p-3} \pmod{p^3}.$$

This completes the proof of Lemma 4.2. □

PROOF OF THEOREM 1.4. It is easy to check by (2.1) that

$$\sum_{k=0}^{p-1} \frac{(k+1)^2 \binom{2k}{k}}{2^k} = \sum_{k=0}^{p-1} (k+1)^2 \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j} \frac{\binom{2j}{j}}{4^j} = \sum_{j=0}^{(p-1)/2} \frac{\binom{2j}{j}}{4^j} \sum_{k=2j}^{p-1} (k+1)^2 \binom{k}{2j}.$$

By Sigma, we find the following identity which can be proved by induction on n :

$$\sum_{k=2j}^{n-1} (k+1)^2 \binom{k}{2j} = \frac{n(n+1)(2nj+2n+2j+1)}{2(2j+1)(2j+3)} \binom{n-1}{2j}.$$

Substituting $n = p$ into this identity and using Lemmas 4.1 and 4.2,

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{(k+1)^2 \binom{2k}{k}}{2^k} &= \sum_{j=0}^{(p-1)/2} \frac{\binom{2j}{j}}{4^j} \frac{p(p+1)(2pj+2p+2j+1)}{2(2j+1)(2j+3)} \binom{p-1}{2j} \\ &= \sum_{j=0}^{(p-1)/2} \frac{\binom{2j}{j} \binom{p-1}{2j}}{4^j} \frac{p^2(p+1)}{2j+3} + \sum_{j=0}^{(p-1)/2} \frac{\binom{2j}{j} \binom{p-1}{2j}}{4^j} \left(\frac{2p(p+1)}{2j+3} - \frac{p(p+1)}{2(j+1)} \right) \\ &\equiv p^2 \binom{(p-1)/2}{(p-3)/2} (-1)^{(p-3)/2} + 3p^2 \sum_{j=0}^{(p-1)/2} \frac{\binom{2j}{j} \binom{p-1}{2j}}{(2j+3)4^j} \\ &\quad + 2p \sum_{j=0}^{(p-1)/2} \frac{\binom{2j}{j} \binom{p-1}{2j}}{(2j+3)4^j} - \frac{p(p+1)}{2} \sum_{j=0}^{(p-1)/2} \frac{\binom{2j}{j} \binom{p-1}{2j}}{(j+1)4^j} \\ &\equiv (-1)^{(p-1)/2} - p + p^2 - p^2 E_{p-3} + p^2 q_p(2) \pmod{p^3}, \end{aligned}$$

which gives the result in Theorem 1.4. □

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