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Vortex structures for an $SO(5)$ model of high-T_C superconductivity and antiferromagnetism

S. Alama, **L. Bronsard** and **T. Giorgi**
McMaster University, Department of Mathematics and Statistics, Hamilton, Ontario, Canada L8S 4K1 (alama@mcmaster.ca; bronsard@math.mcmaster.ca; giorgi@math.mcmaster.ca)

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We study the structure of symmetric vortices in a Ginzburg–Landau model based on Zhang's $SO(5)$ theory of high-temperature superconductivity and antiferromagnetism. We consider both a full Ginzburg–Landau theory (with Ginzburg–Landau scaling parameter $\kappa < \infty$) and a $\kappa \to \infty$ limiting model. In all cases we find that the usual superconducting vortices (with normal phase in the central core region) become unstable (not energy minimizing) when the chemical potential crosses a threshold level, giving rise to a new vortex profile with antiferromagnetic ordering in the core region. We show that this phase transition in the cores is due to a bifurcation from a simple eigenvalue of the linearized equations. In the limiting large-κ model, we prove that the antiferromagnetic core solutions are always non-degenerate local energy minimizers and prove an exact multiplicity result for physically relevant solutions.

1. Introduction

In 1986, Bednorz and Müller announced their discovery of high-critical-temperature (T_C) superconductors and promptly received the 1987 Nobel Prize for their efforts. This discovery has led to a new flowering of superconductivity (abbreviated SC) theory, since the high-temperature phenomenon cannot be explained by the accepted models for conventional superconductors. In particular, many physicists have come to the conclusion that the microscopic BCS theory (see [18]) does not correctly describe the interactions which produce SC at high temperatures. At the present time, there are several competing theories which attempt to explain these interactions. One theory is based on the observation that high-T_C compounds also exhibit an ordered phase called *antiferromagnetism* when physical parameters (such as temperature, chemical potential or 'doping', and magnetic field) are varied. Antiferromagnetism (abbreviated AF) is an insulating phase of matter in which electron spins orient themselves in the direction opposite to their nearest neighbours. The coexistence of these two phases (AF and SC) in the phase diagram of the high-T_C compounds has led to the speculation that high-temperature SC and AF could be explained by the same type of interaction.

Following in this direction, Zhang [19] proposed a quantum statistical mechanics model which incorporates AF and high-temperature SC. The model is based on a broken $SO(5)$ symmetry tying the complex-order parameter of SC to the Néel vector which describes AF. The interactions between the SC and AF order

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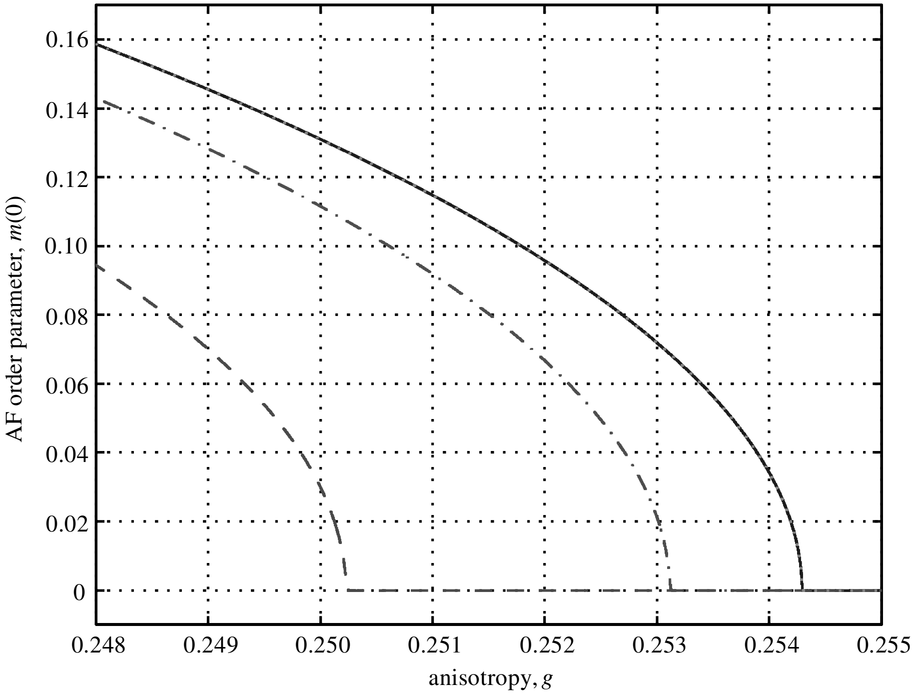

Figure 1. A numerical bifurcation curve, $m(0)$ versus g, for values $\kappa = 20$ (dotted), 40 (dot–dash), 120 (solid) and $d = 1$, indicates a second-order transition to AF cores in model ($\mathrm{GL}_{\kappa,g}$). For the high-κ model ($\mathrm{GL}_{\infty,g}$), we prove that the above image correctly depicts the solution set (see theorem 4.5). Numerical simulations indicate that the bifurcation occurs at $g^*_\infty \simeq 0.2545$ (see [2]).

parameters in this model should have some effect on the familiar constructions from conventional SC theory. In a recent paper, Arovas *et al*. [4] introduced a phenomenological Ginzburg–Landau model based on the $SO(5)$ theory and studied isolated vortex solutions in the plane. Recall that in a conventional superconductor the magnetic field is expelled from the superconducting bulk, and only penetrates in thin tubes (the vortices) where SC is suppressed. Hence, in the conventional theory, the magnetic field is constrained to a small *core* of normal (non-SC) phase. Using a simplified model, Arovas *et al*. predicted a new kind of vortex structure in the $SO(5)$ model: vortices with antiferromagnetic cores, which should be observed for small values of the chemical potential. They also predicted that (as the chemical potential is gradually decreased) the transition from normal core to AF core vortices occurs in a discontinuous fashion. In other words, AF cores should be produced via a *first-order* phase transition.

In this paper we rigorously analyse vortex cores in the full $SO(5)$ Ginzburg–Landau model and in an 'extreme type II' limiting model (also called 'high-κ model') to understand the nature of the transition between normal core and AF core solutions. For both models we show that the vortex solutions with normal cores become unstable (within the class of radial functions—see (1.1) below) and vortices with AF cores are produced by bifurcation from the normal core solutions. In the extreme

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type II model we prove that the transition is continuous (i.e. *second order*), contrary to the prediction of [4] (see figure 1). Furthermore, we show that for each value of the chemical potential there exists a unique stable vortex profile (see theorem 4.5).

The full $SO(5)$ Ginzburg–Landau free energy is written in terms of the SC order parameter $\psi \in \mathbb{C}$ and the AF order parameter (Néel vector) $\boldsymbol{m} = (m_1, m_2, m_3)$. In non-dimensional form, the free energy is

$$\mathcal{F} = \frac{1}{2}\int_{\Omega}\biggl\{\tfrac{1}{2}\kappa^2(1-|\psi|^2-|\boldsymbol{m}|^2)^2+g\kappa^2|\boldsymbol{m}|^2+\left|\left(\frac{1}{\mathrm{i}}\nabla-\boldsymbol{A}\right)\psi\right|^2+|\nabla\boldsymbol{m}|^2+|\nabla\times\boldsymbol{A}|^2\biggr\}\,\mathrm{d}x.$$

(We refer to the paper by Alama *et al.* [2], where the free energy is written in dimensional form.) In these variables, the penetration depth $\lambda = 1$ and the Ginzburg–Landau parameter κ is the reciprocal of the correlation length ξ. The parameter g measures the strength of doping (chemical potential) of the material. It is this term which breaks the $SO(5)$ symmetry of the potential term. We take $g > 0$; with this assumption, SC is preferred in the bulk of the sample.

To study isolated vortex solutions in the plane $\Omega = \mathbb{R}^2$ we seek critical points of $\mathcal{F}$ of the form

$$\psi = f(r)\mathrm{e}^{\mathrm{i}d\theta}, \qquad \boldsymbol{A} = S(r)\left(-\frac{y}{r^2}, \frac{x}{r^2}\right), \qquad \boldsymbol{m} = m(r)\boldsymbol{m}_0, \tag{1.1}$$

where $\boldsymbol{m}_0$ a fixed unit vector and $d \in \mathbb{Z}\setminus\{0\}$ represents the degree of the vortex. As for conventional SC vortices, we expect that only the solutions with $d = \pm 1$ will be energy minimizers (see [11,13]). Critical points of $\mathcal{F}$ with this ansatz solve the system of equations

$$\left.\begin{aligned} -f'' - \frac{1}{r}f' + \frac{(d-S)^2}{r^2}f &= \kappa^2(1-f^2-m^2)f,\\ -S'' + \frac{1}{r}S' &= (d-S)f^2,\\ -m'' - \frac{1}{r}m' + \kappa^2 gm &= \kappa^2(1-f^2-m^2)m, \end{aligned}\right\} \tag{$\mathrm{GL}_{\kappa,g}$}$$

with $f(r) \geqslant 0$, $f(r), S(r) \to 0$ as $r \to 0$ and $f(r) \to 1$, $S(r) \to d$ as $r \to \infty$, and $m'(0) = 0$, $m(r) \to 0$ as $r \to \infty$.

In addition, we study the following 'extreme type II' model:

$$\left.\begin{aligned} -f'' - \frac{1}{r}f' + \frac{d^2}{r^2}f &= (1-f^2-m^2)f,\\ -m'' - \frac{1}{r}m' + gm &= (1-f^2-m^2)m. \end{aligned}\right\} \tag{$\mathrm{GL}_{\infty,g}$}$$

The system $(\mathrm{GL}_{\infty,g})$ is obtained in the limit $\kappa \to \infty$ after rescaling solutions to $(\mathrm{GL}_{\kappa,g})$ by the correlation length $\xi = 1/\kappa$. For high-T_{C} superconductors, κ is very large and hence the vortex cores are very narrow compared to the penetration depth, which measures the length scale for magnetic fields. By rescaling, we capture the structure of the vortex cores and decouple the magnetic field, which lives on a much larger length scale. Indeed, the calculations which led Arovas *et al.* [4] to predict AF vortex cores are mostly based on $(\mathrm{GL}_{\infty,g})$ and its associated free energy functional.

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We observe that when the AF order parameter $m = 0$, the two systems $(\mathrm{GL}_{\kappa,g})$ and $(\mathrm{GL}_{\infty,g})$ reduce to the familiar Ginzburg–Landau vortex equations, well studied in the mathematical literature (see, for example, [5–7, 13, 14]). We call these the *normal core* solutions. In a previous paper [1], we have proven that when $\kappa^2 \geqslant 2d^2$, there is a unique normal core solution, which is a non-degenerate minimizer of the appropriate free energy functional. This characterization will be essential for our analysis of the normal-to-AF core transition.

We now discuss our results. We define a reduced energy functional defined for functions satisfying the symmetric vortex ansatz (1.1), as well as appropriate function spaces in which that functional is smooth. We find that, for every κ (including the extreme type II model), there exists $g^*_\kappa > 0$ such that the conventional normal core vortex solutions of $(\mathrm{GL}_{\kappa,g})$ (and $(\mathrm{GL}_{\infty,g})$) are strict local minimizers of the reduced energy for $g > g^*_\kappa$, but are not local minimizers when $0 < g < g^*_\kappa$. In particular, energy minimizers must have AF order in the vortex core for $0 < g < g^*_\kappa$. When $\kappa^2 \geqslant 2d^2$, we show that the AF core solutions bifurcate from the normal core solution at a simple eigenvalue of the linearized system $(\mathrm{GL}_{\kappa,g})$ (or $(\mathrm{GL}_{\infty,g})$). The bifurcating solutions remain bounded for $g > 0$ and lose compactness as $g \to 0+$ with $f \to 0$ and $m \to 1$.

For the limiting problem $(\mathrm{GL}_{\infty,g})$, we obtain a complete picture of the phase transition to AF cores. This is because all AF core vortex solutions are non-degenerate minima of the reduced energy (see theorem 3.1). Stable (locally minimizing) solutions with $m(r) > 0$ bifurcate from $m = 0$ at $g = g^*_\infty$ to values $g < g^*_\infty$. Moreover, for each $g < g^*_\infty$, there exists exactly one solution with $m(r) > 0$.

In the language of physics, our results indicate a *second-order* (or continuous) phase transition between normal and AF vortex cores in $(\mathrm{GL}_{\infty,g})$. This information concerning the nature of the transition was not derived in the paper by Arovas *et al.* [4], and hence the result is new to the physics literature as well. For $(\mathrm{GL}_{\kappa,g})$, Alama *et al.* [2] present numerical simulations (based on gradient flow for a finite-elements approximation of the free energy) which suggest that the transition is also second order for $\kappa < \infty$ (see figure 1). However, we were not able to extend the arguments used in studying the bifurcation curves of $(\mathrm{GL}_{\infty,g})$ to the more complicated system $(\mathrm{GL}_{\kappa,g})$ (see remark 4.3 for further discussion).

Here is an outline of the content of the paper. In the second section we introduce the reduced energy and function spaces, we treat briefly the questions of existence, regularity and decay of solutions, and we present properties of physically relevant ('admissible') solutions. We also prove the monotonicity of the solution profiles (f, S, m) under the hypothesis that the solution is a local reduced energy minimizer. This result (theorem 2.9) is done in the spirit of the weak maximum principle (see theorem 8.1 of [10]).

Section 3 contains the proof that all solutions of $(\mathrm{GL}_{\infty,g})$ with $m > 0$ represent non-degenerate local minima of the reduced energy. This result is the key to understanding the bifurcation diagram for $(\mathrm{GL}_{\infty,g})$. The bifurcation analysis itself occupies § 4.

The last two sections contain the *a priori* estimates used in rigorously passing to the limit $\kappa \to \infty$ and in studying the global behaviour of bifurcating continua. In both cases, we require estimates on solutions which are energy independent. For the limit $\kappa \to \infty$, this is because the reduced energy of minimizers behaves like $\log\kappa$,

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and in studying global bifurcation we require estimates valid for any physically relevant solution (whether it is energy minimizing or not). The starting point for these estimates is a Pohozaev-type identity (see proposition 5.4). The proof of convergence to $(\mathrm{GL}_{\infty,g})$ as $\kappa \to \infty$ is presented in § 5; other *a priori* estimates are derived in § 6.

2. Solutions of the Ginzburg–Landau system

2.1. Preliminaries

Here and in the rest of the paper, we fix the value of $d \in \mathbb{Z} \setminus \{0\}$. In this section, $\kappa \in \mathbb{R}$ is fixed. Note that, without loss of generality, we may take $d > 0$, since the free energy and the corresponding Euler–Lagrange equations are invariant under the transformation $(\psi, \boldsymbol{A}, \boldsymbol{m}) \to (\bar{\psi}, -\boldsymbol{A}, \boldsymbol{m})$.

Following our previous work [1] on symmetric vortices, we define a function space for which the free energy will be a smooth functional. First we fix some notation. We denote by L^p_r, H the Lebesgue and Sobolev spaces (respectively) of radially symmetric functions in $\mathbb{R}^2$, that is,

$$L^p_r = \left\{ u(r) : \int_0^\infty |u(r)|^p r \,\mathrm{d}r < \infty \right\}, \quad p < \infty,$$
$$H := H^1_r = \left\{ u(r) : \int_0^\infty [(u'(r))^2 + (u(r))^2] r \,\mathrm{d}r < \infty \right\},$$

and analogously for L^∞_r. We also denote

$$\int u(r) \,\mathrm{d}r = \int_0^\infty u(r) r \,\mathrm{d}r.$$

Define the Hilbert space

$$X = \left\{ u \in H : \int \frac{u^2}{r^2} r \,\mathrm{d}r < \infty \right\},$$

with norm

$$\|u\|_X = \sqrt{\int \left[(u'(r))^2 + u^2 + \frac{u^2}{r^2} \right] r \,\mathrm{d}r}.$$

The following density and imbedding properties for the space X are proven in [1].

Lemma 2.1.

(i) *X is compactly embedded in L^p_r for each $p \in (2, \infty)$.*

(ii) *X is compactly embedded in $L^2_{r,\mathrm{loc}}$.*

(iii) *For every $u \in X$,*

$$\|u\|^2_\infty \leqslant \int \left[(u')^2 + \frac{u^2}{r^2} \right] r \,\mathrm{d}r.$$

In particular, X embeds continuously into L^∞_r.

(iv) *$C^\infty_0((0, \infty))$ is dense in X.*

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We note that the compactness of the embedding of H into $L^p_{r,\mathrm{loc}}$ $(1 \leqslant p < \infty)$ is just the classical Rellich–Kondrachov theorem, and the compact embedding of H into L^p_r for $2 < p < \infty$ is due to Strauss [17].

2.2. Energy

We now define our energy functionals, using the space X defined above. To keep the appropriate boundary condition at infinity, we fix any function $\eta \in C^\infty([0,\infty))$ with $\eta(r) = 0$ for $0 \leqslant r \leqslant 1$, $\eta(r) = 1$ for all $r \geqslant 2$ and $0 < \eta < 1$. Then set $f_0 = \eta$, $S_0 = d\eta$, and seek solutions (f, S, m) of $(\mathrm{GL}_{\kappa,g})$ with $f = f_0 + u$, $S = S_0 + rv$, $u, v \in X$, $m \in H$. (Later we will see that this choice poses no restriction on solutions which are physically relevant.) We denote by $Y_0 = X \times X \times H$ and by Y the affine space

$$Y = \{(f, S, m) : f = f_0 + u,\ S = S_0 + rv,\ u, v \in X,\ m \in H\} = Y_0 + (f_0, S_0, 0).$$

For $(f, S, m) \in Y$, we define

$$\mathcal{E}_{\kappa,g}(f, S, m) = \frac{1}{2} \int \left\{ (f')^2 + \left[\frac{S'}{r}\right]^2 + (m')^2 + \kappa^2 g m^2 + \frac{(d - S)^2 f^2}{r^2} + \tfrac{1}{2}\kappa^2 (1 - f^2 - m^2)^2 \right\} r\,\mathrm{d}r \tag{2.1}$$

and the functional $I_{k,g} : Y_0 \to \mathbb{R}$ by

$$I_{k,g}(u, v, m) = \mathcal{E}_{\kappa,g}(f_0 + u, S_0 + rv, m) - \mathcal{E}_{\kappa,g}(f_0, S_0, 0).$$

Throughout the paper we will take advantage of these two representations of our spaces and energies, and use the formulation which is more convenient at the given moment.

Defining an energy functional for the limiting problem $(\mathrm{GL}_{\infty,g})$ is trickier, since the naive choice for the energy (namely (2.1) with $S = 0$ and $\kappa = 1$) would be infinite for all f satisfying the desired boundary condition at $r = \infty$. Our solution is to subtract off the offending term from the energy density. Let $\tilde{f}_\infty$ be the (unique) positive solution to the high-κ vortex equation,

$$-\tilde{f}''_\infty - \frac{1}{r}\tilde{f}'_\infty + \frac{d^2}{r^2}\tilde{f}_\infty = (1 - \tilde{f}^2_\infty)\tilde{f}_\infty,$$

with $\tilde{f}_\infty(0) = 0$, $\tilde{f}_\infty(r) \to 1$ as $r \to \infty$. The uniqueness of $\tilde{f}_\infty$ was established by Chen *et al*. [7]. The estimates in [7] ensure that $\tilde{f}_\infty$ is smooth, $\tilde{f}_\infty(r) \sim r^d$ near $r = 0$ and $(1 - \tilde{f}_\infty) \in H$.

We define the appropriate spaces for the free energy $\mathcal{E}_{\infty,g}$ based on $\tilde{f}_\infty$. Let $Z_0 = X \times H$ and

$$Z = \{(f, m) : f = \tilde{f}_\infty + u,\ u \in X,\ m \in H\} = Z_0 + (\tilde{f}_\infty, 0).$$

Then the energy for the high-κ model is

$$\mathcal{E}_{\infty,g}(f, m) = \frac{1}{2} \int \left\{ (f')^2 + (m')^2 + g m^2 + \frac{d^2}{r^2}[f^2 - \tilde{f}^2_\infty] + \tfrac{1}{2}(1 - f^2 - m^2)^2 \right\} r\,\mathrm{d}r. \tag{2.2}$$

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If we write $f = \tilde{f}_\infty + u$, we reduce to the equivalent functional

$$\begin{aligned} I_{\infty,g}(u,m) &= \mathcal{E}_{\infty,g}(\tilde{f}_\infty + u, m) - \mathcal{E}_{\infty,g}(\tilde{f}_\infty, 0) \\ &= \frac{1}{2}\int\bigg\{(u')^2 + \frac{d^2}{r^2}u^2 + (m')^2 + gm^2 + \tfrac{1}{2}(1-(\tilde{f}_\infty+u)^2 - m^2)^2 \\ &\qquad - \tfrac{1}{2}(1-\tilde{f}_\infty^2)^2 + 2(1-\tilde{f}_\infty^2)\tilde{f}_\infty u\bigg\}\, r\,\mathrm{d}r. \end{aligned} \tag{2.3}$$

By a direct expansion of the energy in powers of u, v, m, we see that $I_{k,g} : Y_0 \to \mathbb{R}$ and $I_{\infty,g} : Z_0 \to \mathbb{R}$ are smooth (C^∞) functionals.

When $g > 0$ is fixed, we obtain solutions of $(\mathrm{GL}_{\kappa,g})$ and $(\mathrm{GL}_{\infty,g})$ as global minimizers for $\mathcal{E}_{\kappa,g}$ and $\mathcal{E}_{\infty,g}$ (in the appropriate spaces, Y and Z).

THEOREM 2.2. *For every fixed $g > 0$, $\kappa \in \mathbb{R}$ and $d \in \mathbb{Z} - 0$, the functional $I_{\kappa,g}$ admits a minimizer $(u, v, m) \in X \times X \times H$. Moreover, $(f, S, m) = (f_0 + u, S_0 + rv, m)$ is a smooth solution of the system $(GL_{\kappa,g})$.*

THEOREM 2.3. *For every fixed $g > 0$ and $d \in \mathbb{Z} - 0$, the functional $I_{\infty,g}$ admits a minimizer $(u, m) \in X \times H$. Moreover, $(f, m) = (\tilde{f}_\infty + u, m)$ is a smooth solution of the system $(GL_{\infty,g})$.*

The proofs of theorems 2.2 and 2.3 are straightforward but technical, and are deferred to § 6.

2.3. Admissible solutions

As in [1], we define a natural class of solutions to the system $(\mathrm{GL}_{\kappa,g})$.

DEFINITION 2.4. We call (f_*, S_*, m_*) an *admissible* solution to $(\mathrm{GL}_{\kappa,g})$ if the following hold.

(i) The system $(\mathrm{GL}_{\kappa,g})$ holds for all $r \in (0, \infty)$.

(ii) $\mathcal{E}_{\kappa,g}(f_*, S_*, m_*) < \infty$.

(iii) $f_*(r) \geqslant 0$ and $m_*(r) \geqslant 0$ for all $r \geqslant 0$.

(iv) $S_*(0) = 0$ and $m_*'(0) = 0$.

A solution (f_*, m_*) of $(\mathrm{GL}_{\infty,g})$ is called *admissible* if the above conditions hold, where we replace κ by ∞ and disregard S_*.

A solution to $(\mathrm{GL}_{\kappa,g})$ or $(\mathrm{GL}_{\infty,g})$ with $m_* \equiv 0$ is called a *normal core* solution.

The admissible solutions are those which are physically relevant in the context of the vortex core problem described in the introduction. We note that the normal core solutions are *unique* for $\kappa^2 \geqslant 2d^2$ (see [1] for the case $2d^2 \leqslant \kappa^2 < \infty$ and [7] for $\kappa = \infty$).

We now present some properties of admissible solutions. In the following, we will assume that $\kappa \in \mathbb{R} \cup \{\infty\}$, with the understanding that $S_* = 0$ when $\kappa = \infty$.

PROPOSITION 2.5. *Let (f_*, S_*, m_*) be any admissible solution of $(GL_{\kappa,g})$. Then we have the following.*

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(i) *For all* $r \in (0, \infty)$ *it holds* $0 < f_*(r) < 1$, $0 \leqslant m_*(r) < 1$, $f_*^2(r) + m_*^2(r) < 1$ *and, if* $\kappa \neq \infty$, $0 < S_*(r) < d$.

(ii) *Either* $m_*(r) > 0$ *for all* $r \in [0, \infty)$ *or* m_* *vanishes identically.*

(iii) $f_*(r) \to 1$, $m_*(r) \to 0$ *and, if* $\kappa \neq \infty$, $S_*(r) \to d$ *as* $r \to \infty$. *Moreover, there exist constants* $\sigma, C_0 > 0$ *such that, for* $\kappa \neq \infty$,

$$0 < 1 - f_*(r) \leqslant C_0 \mathrm{e}^{-\sigma r}, \quad 0 < d - S_*(r) \leqslant C_0 \mathrm{e}^{-\sigma r}, \quad 0 \leqslant m_*(r) \leqslant C_0 \mathrm{e}^{-\sigma r}$$

and, for $\kappa = \infty$,

$$0 < 1 - f_*(r) \leqslant \frac{d^2}{2r^2} + \frac{8d^2 + d^4}{8r^4} + \mathcal{O}(r^{-6}), \qquad 0 \leqslant m_*(r) \leqslant C_0 \mathrm{e}^{-\sigma r}$$

for all $r > 0$.

(iv) $f_*(r) \sim r^d$, $S_*(r) \sim r^2$ *for* $r \sim 0$.

(v) *If* $\kappa \neq \infty$, $S_*'(r) > 0$ *for all* $r > 0$.

Proof. The proof is very similar to that of proposition 2.3 of [1], so we provide only a sketch. From the finiteness of the free energy, we immediately conclude that $m_* \in H$ and hence $m_* \in L^p_r$ for any $p \in [2, \infty]$, and $m_*(r) \to 0$ as $r \to \infty$. Since $f_* \geqslant 0$, finiteness of energy again implies $1 - f_* \in L^2_r$ (see (6.10) for details) and therefore the bound $0 < f_*(r) < 1$ follows exactly as in proposition 2.3 of [1]. When $\kappa < \infty$, the bound $0 < S_*(r) < d$ and the proof that $S_*'(r) > 0$ are also unchanged from [1]. To show $z = f_*^2 + m_*^2 < 1$, we use the equation satisfied by z; this argument is already presented in [2]. Statement (ii) is a simple consequence of the strong maximum principle.

The exponential decay in (iii) for m_* is consequence of proposition 7.4 in [12], and so are the ones for f_* and S_* if $\kappa \neq \infty$. If $\kappa = \infty$, the polynomial decay of f_* can be proven as in lemma 3.3 in [7], since $m_*(r) \leqslant C(R)/r^6$ for any $r > R$, with $C(R)$ a big enough constant.

The behaviour at zero given in (iv) can be proven as in [14]. □

We now connect admissible solutions to our space X.

PROPOSITION 2.6. *Let* (f_0, S_0, m_0), (f_1, S_1, m_1) *be admissible solutions to* $(GL_{\kappa,g})$. *Then* $(f_1 - f_0) \in X$, $[(S_1 - S_0)/r] \in X$ *and* $m_1, m_0 \in H$.

Proof. As already remarked, condition (ii) of the definition of admissible solutions implies $m_1, m_0 \in H$ and $m_1, m_0 \in L^p_r$ for any $p \in [2, \infty)$. Then the rest of the proposition for $\kappa \neq \infty$ is proven as in proposition 2.4 of [1]. When $\kappa = \infty$, we note that $(1 - f_i) \in H$ for $i = 1, 2$ and that, by (iv) of proposition 2.5, we have $(f_1 - f_2)^2 \leqslant cr^{2d}$ for $r \sim 0$ and, again by finiteness of energy, we conclude our statement. □

REMARK 2.7. In light of proposition 2.6, we observe that the choice of f_0, S_0 in the definition of the space Y may be replaced by any fixed admissible solution of $(\mathrm{GL}_{\kappa,g})$. It will be convenient to choose instead the 'basepoint' $(\tilde{f}_\kappa, \tilde{S}_\kappa, 0)$ to be

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a 'normal core' solution to $(\mathrm{GL}_{\kappa,g})$. In other words, an equivalent definition of the space Y is

$$Y = \{(f, S, m) : f = \tilde{f}_\kappa + u,\ S = \tilde{S}_\kappa + rv,\ u, v \in X,\ m \in H\}. \tag{2.4}$$

We recall that the normal core solutions are uniquely determined for $\kappa^2 \geqslant 2d^2$. When $\kappa^2 < 2d^2$, we fix any one.

REMARK 2.8. Proposition 2.6 also implies that the admissible solutions are exactly those which arise from minimization problems for $\mathcal{E}_{\kappa,g}$ and $\mathcal{E}_{\infty,g}$ in the space Y. In particular, as an immediate corollary, we obtain the following statement:

> (f_*, S_*, m_*) is an admissible solution to $(\mathrm{GL}_{\kappa,g})$ if and only if $f_* \geqslant 0$, $m_* \geqslant 0$, $(f_*, S_*, m_*) \in Y$ and $\mathcal{E}'_{\kappa,g}(f_*, S_*, m_*)[u, v, w] = 0$ for all $u, v \in X$ and $w \in H$.

An analogous statement holds for the problem $(\mathrm{GL}_{\infty,g})$.

With this choice of representation for our spaces Y, Z, we now look at the second variation of energy with respect to the variables $(u, v, w) \in X \times X \times H$. We define

$$\begin{aligned}
\mathcal{E}''_{\kappa,g}&(f_*, S_*, m_*)[u, v, w] \\
&= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \mathcal{E}_{\kappa,g}(f_* + tu, S_* + trv, m_* + tw) \\
&= \int \bigg\{ (u')^2 + (w')^2 + \frac{(d - S_*)^2}{r^2} u^2 + \kappa^2 g w^2 \\
&\qquad + (v')^2 + \frac{v^2}{r^2} - 4\frac{(d - S_*)}{r} f_* uv + f_*^2 v^2 \\
&\qquad - \kappa^2(1 - f_*^2 - m_*^2)(u^2 + w^2) + 2\kappa^2 (f_* u + m_* w)^2 \bigg\} r\,\mathrm{d}r.
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
\mathcal{E}''_{\infty,g}&(f_*, m_*)[u, w] \\
&= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \mathcal{E}_{\infty,g}(f_* + tu, m_* + tw) \\
&= \int \bigg\{ (u')^2 + (w')^2 + \frac{d^2}{r^2} u^2 + g w^2 \\
&\qquad - (1 - f_*^2 - m_*^2)(u^2 + w^2) + 2(f_* u + m_* w)^2 \bigg\} r\,\mathrm{d}r.
\end{aligned} \tag{2.6}$$

Note that if we write $f_* = \tilde{f}_\kappa + u_*$, $S_* = \tilde{S}_\kappa + rv_*$, then

$$\mathcal{E}''_{\kappa,g}(f_*, S_*, m_*)[u, v, w] = \mathrm{D}^2\, I_{\kappa,g}(u_*, v_*, m_*)[u, v, w],$$

the usual second Fréchet derivative.

For admissible solutions which are stable, in the sense that the second variation of energy about the solution is a non-negative quadratic form, we have monotonicity of the profiles $f(r)$, $m(r)$.

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Theorem 2.9. *Suppose (f,S,m) is an admissible solution of $(GL_{\kappa,g})$ and that $\mathcal{E}''_{\kappa,g}(f,S,m)[u,v,w] \geqslant 0$ for all $(u,v,w) \in Y_0$. Then $f'(r) > 0$ and (if it is not identically zero) $m'(r) < 0$ for all $r > 0$.*

For the problem $(\mathrm{GL}_{\infty,g})$, the same theorem holds, with exactly the same proof. We will see later that all admissible solutions of $(\mathrm{GL}_{\infty,g})$ with $m(r) > 0$ are stable (in the above sense), and hence we will obtain the stronger result announced in corollary 3.2.

Proof. Let $\tilde{u}(r) = f'(r)$, $\tilde{w}(r) = m'(r)$. Then, differentiating the first and third equations of $(\mathrm{GL}_{\kappa,g})$, we get

$$-\tilde{u}'' - \frac{1}{r}\tilde{u}' + \frac{(d-S)^2}{r^2}\tilde{u} - \kappa^2(1-3f^2-m^2)\tilde{u} + 2\kappa^2 mf\tilde{w} = -\frac{1}{r^2}\tilde{u} + 2\frac{d-S}{r}f\left[\frac{S'}{r} + \frac{d-S}{r^2}\right]$$

and

$$-\tilde{w}'' - \frac{1}{r}\tilde{w}' + g\kappa^2\tilde{w} - \kappa^2(1-f^2-3m^2)\tilde{w} + 2\kappa^2 fm\tilde{u} = -\frac{1}{r^2}\tilde{w}.$$

Suppose there exist intervals (a,b), (c,d) such that

$$\tilde{u}(r) < 0, \quad r \in (a,b), \quad \tilde{u}(a) = 0 = \tilde{u}(b)$$

or

$$\tilde{w}(r) > 0 \quad r \in (c,d), \quad \tilde{w}(c) = 0 = \tilde{w}(d).$$

Note that, by properties (i), (iii) and (iv) of admissible solutions in proposition 2.5, $a \neq 0$ and $b, d < +\infty$. Let

$$u(r) = \begin{cases} \tilde{u}(r) & \text{if } r \in (a,b), \\ 0 & \text{otherwise,} \end{cases}$$

$$w(r) = \begin{cases} \tilde{w}(r) & \text{if } r \in (c,d), \\ 0 & \text{otherwise.} \end{cases}$$

Then $u \leqslant 0$, $w \geqslant 0$ and an integration by parts shows that

$$\int (u')^2 r\,\mathrm{d}r = -\int_a^b \tilde{u}\frac{1}{r}(r\tilde{u}')' r\,\mathrm{d}r,$$

and similarly for w. If we now use $(u,0,w)$ as a test function in the second variation of energy and recall from proposition 2.5 that $S(r) < d$, $S'(r) > 0$ for all $r > 0$, we obtain

$$\begin{aligned} 0 &\leqslant \mathcal{E}''_{\kappa,g}(f,S,m)[u,0,w] \\ &\leqslant \int\left[-\frac{1}{r^2}u^2 + 2\frac{d-S}{r}f\left[\frac{S'}{r} + \frac{d-S}{r^2}\right]u - \frac{1}{r^2}w^2\right] r\,\mathrm{d}r < 0, \end{aligned}$$

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unless $u, w \equiv 0$. Consequently, $\tilde{u} = f' \geqslant 0$ and $\tilde{w} = m' \leqslant 0$. Strict inequality follows from the strong maximum principle, since $\tilde{u}$, $\tilde{w}$ satisfy equations of the form

$$-\Delta_r \tilde{u} + c_1(r)\tilde{u} \geqslant -2\kappa^2 m f \tilde{w} \geqslant 0,$$
$$-\Delta_r \tilde{w} + c_2(r)\tilde{w} = -2\kappa^2 m f \tilde{u} \leqslant 0.$$

□

3. Non-degeneracy of solutions of $(\mathrm{GL}_{\infty,g})$

THEOREM 3.1. *For any admissible solution (f_*, m_*) of $(GL_{\infty,g})$ with $m_* > 0$, there exists a constant $\sigma_* > 0$ such that*

$$\mathcal{E}''_{\infty,g}(f_*, m_*)[u, w] \geqslant \sigma_*(\|u\|_X^2 + \|w\|_H^2)$$

for all $u \in X$, $w \in H$.

COROLLARY 3.2. *For any admissible solution (f_*, m_*) of $(GL_{\infty,g})$, $f_*'(r) > 0$ for all $r \geqslant 0$. If m_* is not identically zero, then $m_*'(r) < 0$ for all $r > 0$.*

The corollary follows from theorem 3.1 and the argument of theorem 2.9 when $m_* > 0$. Note that when $m_* \equiv 0$, the system $(\mathrm{GL}_{\infty,g})$ reduces to the single equation studied in [7] and the strict monotonicity of f_* is part of their result. Also, in the case that $m_* \equiv 0$, the theorem reduces to $\mathcal{E}''_{\infty,g}(f_*)[u] \geqslant \sigma_* \|u\|_X^2$.

The key step in proving theorem 3.1 is the following identity.

THEOREM 3.3. *For any admissible solution (f_*, m_*) of $(GL_{\infty,g})$ with $m_* > 0$ and any $u \in X$, $w \in H$,*

$$\mathcal{E}''_{\infty,g}(f_*, m_*)[u, w] = \int \left\{ f_*^2\left[\left(\frac{u}{f_*}\right)'\right]^2 + m_*^2\left[\left(\frac{w}{m_*}\right)'\right]^2 + 4(f_* u + m_* w)^2 \right\} r\,\mathrm{d}r. \tag{3.1}$$

Proof of theorem 3.3. First we prove the identity for $u \in C_0^\infty((0, \infty))$ and $w \in C_0^\infty([0, \infty))$. First note that, using $f_* > 0$ and $m_* > 0$, we have

$$f_*^2\left[\left(\frac{u}{f_*}\right)'\right]^2 = (u')^2 - 2\frac{uu'f_*'}{f_*} + u^2\frac{(f_*')^2}{f_*^2}, \tag{3.2}$$

with a similar identity holding for m_*, w. Hence

$$\begin{aligned} 0 &= \mathcal{E}'_{\infty,g}(f_*, m_*)\left[\frac{u^2}{f_*}, \frac{w^2}{m_*}\right] \\ &= \int \left\{ (u')^2 + (w')^2 + \frac{d^2}{r^2}u^2 + gw^2 \right. \\ &\qquad \left. - (1 - f_*^2 - m_*^2)(u^2 + w^2) - f_*^2\left[\left(\frac{u}{f_*}\right)'\right]^2 - m_*^2\left[\left(\frac{w}{m_*}\right)'\right]^2 \right\} r\,\mathrm{d}r. \end{aligned}$$

Substituting this in the formula for $\mathcal{E}''_{\infty,g}(f_*, m_*)[u, w]$, we obtain

$$\mathcal{E}''_{\infty,g}(f_*, m_*)[u, w] = \int \left\{ f_*^2\left[\left(\frac{u}{f_*}\right)'\right]^2 + m_*^2\left[\left(\frac{w}{m_*}\right)'\right]^2 + 2[f_* u + m_* w]^2 \right\} r\,\mathrm{d}r.$$

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To obtain the result for any $(u,w)\in X\times H$, let u_n be a sequence of $C_0^\infty((0,\infty))$ functions converging to u in X, and w_n a sequence in $C_0^\infty([0,\infty))$ converging to w in H. By continuity of $\mathcal{E}''_{\infty,g}(f_*,S_*)$, the limit passes in the second variation of $\mathcal{E}_{\infty,g}$. For the right-hand side we expand,

$$\int f_*^2\biggl(\biggl(\frac{u}{f_*}\biggr)'\biggr)^2 r\,\mathrm{d}r=\int\biggl\{(u')^2-2\frac{f_*'}{f_*}uu'+\biggl(\frac{f_*'}{f_*}\biggr)^2u^2\biggr\}r\,\mathrm{d}r \tag{3.3}$$

and note that

$$\biggl(\frac{f_*'}{f_*}\biggr)^2\leqslant c\biggl(1+\frac{1}{r^2}\biggr),$$

since $f_*\sim r^d$ for $r\sim 0$. Hence each term is controlled by the X-norm and can be passed to the limit. A similar argument may be applied for the second term in the right-hand side of (3.1). The quotient is expanded as in (3.3) above, with m_*, w replacing f_*, u. Then we claim that $m'(r)/m(r)$ is uniformly bounded for $r\in[0,\infty)$. Indeed, by the basic gradient bound for solutions of the Poisson equation (see § 3.4 of [10]), we have, for any $r_0>1$,

$$|m'(r_0)|\leqslant 2\sup_{|r-r_0|\leqslant 1}m(r)+\tfrac{1}{2}\sup_{|r-r_0|\leqslant 1}|\kappa^2(1-g-f^2-m^2)m|\leqslant C_1\sup_{|r-r_0|\leqslant 1}m(r).$$

Applying the Harnack inequality (corollary 9.25 of [10]), we then obtain

$$\biggl|\frac{m'(r_0)}{m(r_0)}\biggr|\leqslant C_1\frac{\sup_{|r-r_0|\leqslant 1}m(r)}{m(r_0)}\leqslant C_1\frac{\sup_{|r-r_0|\leqslant 1}m(r)}{\inf_{|r-r_0|\leqslant 1}m(r)}\leqslant C_1'$$

for all $r_0>1$. Therefore, m'/m is uniformly bounded and we may pass to the H^1_r limit in the second term in (3.1). The last term is clearly continuous in the L^2_r-norm in both u and w. In conclusion, we may pass to the limit $u_n\to u$, $w_n\to w$ and obtain (3.1) for $u\in X$, $w\in H$. □

Proof of theorem 3.1. Define

$$\sigma_*=\inf\{\mathcal{E}''_{\infty,g}(f_*,m_*)[u,w]:u\in X,\ w\in H,\ \|u\|_X^2+\|w\|_H^2=1\}.$$

We must show that $\sigma_*>0$.

By theorem 3.3, $\sigma_*\geqslant 0$. To obtain a contradiction, assume instead that $\sigma_*=0$. We claim that in this case the infimum is attained at a non-trivial (u_*,w_*), with $\mathcal{E}''_{\kappa,g}(f_*,m_*)[u_*,w_*]=\sigma_*=0$. But this contradicts theorem 3.3, and hence $\sigma_*>0$.

We now claim that the infimum $\sigma_*=0$ is attained in Z_0. Take any minimizing sequence, $(u_n,w_n)\in X\times H$ with $\|u_n\|_X^2+\|w_n\|_H^2=1$ and

$$\mathcal{E}''_{\infty,g}(f_*,m_*)[u_n,w_n]\to\sigma_*=0.$$

By Sobolev embedding, there exists a subsequence (still denoted by u_n, w_n) and $u_*\in X$, $w_*\in H$ so that $u_n\to u_*$, $w_n\to w_*$, weakly in X, H (respectively) and strongly in L^2_{loc}.

First we claim that $(u_*,w_*)\neq(0,0)$. Indeed, if both u_*, w_* vanish identically, then, by weak convergence, $(u_n,w_n)\rightharpoonup(u_*,w_*)=(0,0)$ and the compact embed-

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dings, we obtain

$$\int\left((u_n')^2 + \frac{d^2}{r^2}u_n^2 + 2\kappa^2 f_*^2 u_n^2 + (w_n')^2 + g\kappa^2 w_n^2\right) r\,\mathrm{d}r$$
$$= \mathcal{E}''_{\infty,g}(f_*, m_*)[u_n, w_n] + \int [\kappa^2(1 - f_*^2 - m_*^2)(u_n^2 + w_n^2) - 2\kappa^2 m_*^2 w_n^2 - 4\kappa^2 f_* m_* u_n w_n] r\,\mathrm{d}r \to 0.$$

In particular, $(u_n, w_n) \to (0,0)$ in the norm on $X \times H$, which contradicts the fact that $\|u_n\|_X^2 + \|w_n\|_H^2 = 1$. Thus the claim holds, and $(u_*, w_*) \neq (0,0)$.

Next we use lower semicontinuity in the norm and L^2_{loc} convergence to pass to the limit

$$\mathcal{E}''_{\infty,g}(f_*, m_*)[u_*, w_*] \leqslant \liminf_{n\to\infty} \mathcal{E}''_{\infty,g}(f_*, m_*)[u_n, w_n] = 0. \tag{3.4}$$

This contradicts theorem 3.3, since $\mathcal{E}''_{\infty,g}(f_*, m_*)[u_*, w_*] > 0$. (Note that u/f_* is non-constant since $u \in X$ but $f_* \notin X$.) We conclude that $\sigma_* > 0$, as desired. □

We note that the same result holds when $m_* \equiv 0$. Hence, following the method of [1], we obtain another proof of uniqueness for the solution to the high-κ equation for f_* studied in [7].

4. Bifurcation from the normal cores

In this section we show that (when $\kappa^2 \geqslant 2d^2$) AF core solutions are nucleated by means of a bifurcation from the normal core solution family at a simple eigenvalue of the linearized equations. We will also require *a priori* estimates (whose proof we will present in § 6) to obtain global information about the solutions set for all $\kappa^2 \geqslant 2d^2$, and the stronger result of theorem 3.1 to fully categorize solutions in the extreme type-II model ($\mathrm{GL}_{\infty,g}$). We present the detailed argument for the problem ($\mathrm{GL}_{\kappa,g}$). The functional analytic framework is entirely similar for the problem ($\mathrm{GL}_{\infty,g}$) and so we omit it and concentrate instead on the more precise global characterization of solutions which we prove for ($\mathrm{GL}_{\infty,g}$).

4.1. Local bifurcation at g_κ^*

We define a map $\mathcal{F} : Y \times \mathbb{R} \to Y_0^*$ by

$$\langle (u, v, w), \mathcal{F}(f_*, S_*, m_*, g)\rangle_{Y_0, Y_0^*} = \mathcal{E}'_{\kappa,g}(f_*, S_*, m_*)[u, v, w],$$

$(u, v, w) \in Y_0$, $(f_*, S_*, m_*) \in Y$. Its linearization is the operator $\mathcal{F}'(f_*, S_*, m_*, g) \in L(Y_0, Y_0^*)$, defined by

$$\langle (u, v, w), \mathcal{F}'(f_*, S_*, m_*, g)[\phi, \psi, \xi]\rangle_{Y_0, Y_0^*} = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathcal{E}'_{\kappa,g}(f_* + t\phi, S_* + rt\psi, m_* + t\xi)[u, v, w].$$

We remark that the explicit expansion of the energy $I_{\kappa,g}$ in terms of $u_* = f_* - \tilde{f}_\kappa$, $v_* = (S_* - \tilde{S}_\kappa)/r$, w_* ensures that $\mathcal{F}$ is a C^2 map in all arguments u_*, v_*, w_*, g.

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By the natural identification $Y_0 \simeq Y_0^*$ of a Hilbert Space with its dual, we may also represent $\mathcal{F}'$ by $\mathcal{L}_g \in L(Y_0, Y_0)$ as

$$((u,v,w), \mathcal{L}_g[\phi,\psi,\xi])_{Y_0} = \langle (u,v,w), \mathcal{F}'(f_*, S_*, m_*, g)[\phi,\psi,\xi]\rangle_{Y_0,Y_0^*}.$$

If $i : Z^* \to Z$ is the isomorphism, then $\mathcal{L}_g = i \circ \mathcal{F}'(f_*, S_*, m_*, g)$.

LEMMA 4.1. *For all $g > 0$, $\mathcal{L}_g$ is a Fredholm operator of index zero.*

Proof. Define an equivalent inner product on Y_0,

$$((u,v,w),(\phi,\psi,\xi))_{Y_0} = \int \biggl\{ u'\phi' + 2\kappa^2 u\phi + \frac{(d-S_*)^2}{r^2} u\phi + v'\psi' + v\psi + \frac{1}{r^2} v\psi + w'\xi' + g\kappa^2 w\xi \biggr\} r\,\mathrm{d}r.$$

Then we write

$$((u,v,w), \mathcal{L}_g[\phi,\psi,\xi])_{Y_0} = ((u,v,w),(\phi,\psi,\xi))_{Y_0} + ((u,v,w), K[\phi,\psi,\xi])_{Y_0},$$

where K is defined by

$$\begin{aligned}((u,v,w), K[\phi,\psi,\xi])_{Y_0} &= \int \biggl[2\kappa^2(f_*^2-1)u\phi + 2\kappa^2 f_* m_*(u\xi + w\phi) \\ &\quad + 2\kappa^2 m_*^2 w\xi - \kappa^2(1-f_*^2-m_*^2)(u\phi + w\xi) \\ &\quad + (f_*^2-1)v\psi - 2\frac{d-S_*}{r} f_*(u\psi + v\phi) \biggr] r\,\mathrm{d}r.\end{aligned}$$

Recalling the decay properties of f_*, S_*, m_* and the embedding properties of H, X, we observe that K is compact, and hence $\mathcal{L}_g = Id_{Y_0} + K$ is Fredholm with index zero. □

As a direct consequence of lemma 4.1,

$$\dim\ker(\mathcal{F}') = \dim\ker(\mathcal{L}_g) = \operatorname{codim}\operatorname{Ran}(\mathcal{L}_g) = \operatorname{codim}\operatorname{Ran}(\mathcal{F}').$$

Now we may apply the standard bifurcation theory of Crandall and Rabinowitz [8] at an eigenvalue g^* of $\mathcal{F}'(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g^*)$. Indeed, note that when $m_* = 0$, the linearization of $\mathcal{F}$ decouples into two components,

$$\begin{aligned}&\langle (u,v,w), \mathcal{F}'(f_*, S_*, 0, g)[\phi,\psi,\xi]\rangle_{Y_0,Y_0^*} \\ &\qquad = \langle (u,v), \mathcal{F}_{1,2}'(f_*, S_*)[\phi,\psi]\rangle_{X^2,(X^2)^*} + \langle w, \mathcal{F}_3'(f_*, g)\xi\rangle_{H,H^*},\end{aligned}$$

where

$$\begin{aligned}&\langle (u,v), \mathcal{F}_{1,2}'(f_*, S_*)[\phi,\psi]\rangle_{X^2,(X^2)^*} \\ &\quad = \langle (u,v,0), \mathcal{F}'(f_*, S_*, 0, g)[\phi,\psi,0]\rangle_{Y_0,Y_0^*} \\ &\quad = \int \biggl[u'\phi' + \frac{(d-S_*)^2}{r^2} u\phi + v'\psi' + \frac{v\psi}{r^2} \\ &\qquad + f_*^2 u\psi - 2\frac{d-S_*}{r} f_*(u\psi + v\phi) - \kappa^2(1-3f_*^2)u\phi \biggr] r\,\mathrm{d}r\end{aligned}$$

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and

$$\begin{aligned}\langle w, \mathcal{F}_3'(f_*, g)\xi\rangle_{H,H^*} &= \langle (0,0,w), \mathcal{F}'(f_*, S_*, 0, g)[0,0,\xi]\rangle_{Y_0,Y_0^*} \\ &= \int \{w'\xi' + g\kappa^2 w\xi - \kappa^2(1-f_*^2)w\xi\} r\,\mathrm{d}r.\end{aligned}$$

By theorem 3.1 of [1], when $\kappa^2 \geqslant 2d^2$, the operator $\mathcal{F}_{1,2}' \geqslant \sigma_* > 0$ is bounded away from zero (in quadratic form sense). Hence, if $(\phi, \psi, \xi) \in \ker(\mathcal{F}'(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g_\kappa^*))$, we take $(u, v, w) = (\phi, \psi, 0)$ and obtain

$$\begin{aligned}0 &= \langle (\phi, \psi, 0), \mathcal{F}'(f_*, S_*, 0, g_\kappa^*)[\phi, \psi, \xi]\rangle_{Y_0,Y_0^*} \\ &= \langle (\phi, \psi), \mathcal{F}_{1,2}'(f_*, S_*)[\phi, \psi]\rangle_{X^2,(X^2)^*} \\ &\geqslant \sigma_*(\|\phi\|_X^2 + \|\psi\|_X^2).\end{aligned}$$

In particular, $\phi, \psi = 0$.

The operator $\mathcal{F}_3'(f_*, g) = L + g\kappa^2$, where $L = -\Delta_r - V(r)$, is a Schrödinger operator with potential $V(r) = \kappa^2(1 - f_*^2(r)) \geqslant 0$ and $V(r) \to 0$ as $r \to \infty$. It is a well-known fact in mathematical physics that in dimension two, such operators have at least one negative eigenvalue.

Lemma 4.2. *Suppose $V : [0, \infty) \to \mathbb{R}$ is continuous, non-negative, $V(r) \to 0$ as $r \to \infty$ and V is not identically zero, and define $L = -\Delta - V(r)$ as a self-adjoint operator on the space $L^2(\mathbb{R}^2)$. Then the ground state energy,*

$$\lambda_0 = \inf\left\{ \int [(u')^2 - V(r)u^2] r\,\mathrm{d}r \Big/ \int u^2 r\,\mathrm{d}r : u \neq 0,\ u \in H \right\} < 0$$

and is attained at an eigenfunction $u_0 \in H$. Moreover, λ_0 is an isolated non-degenerate eigenvalue and $u_0 > 0$.

The proof follows as an application of the Birman–Schwinger principle in [16]. We provide an elementary variational proof for the reader's convenience.

Proof. Let

$$u_n(r) = \begin{cases} 1 & \text{if } r \leqslant n, \\ \ln(r/n^2)/\ln(1/n) & \text{if } n \leqslant r \leqslant n^2, \\ 0 & \text{if } r \geqslant n. \end{cases}$$

Then

$$\int (u_n')^2 r\,\mathrm{d}r = \frac{1}{\ln n} \to 0,$$

while

$$\int V(r)u_n^2 r\,\mathrm{d}r \geqslant \int_0^n V(r) r\,\mathrm{d}r \to \int_0^\infty V(r) r\,\mathrm{d}r > 0$$

(possibly infinite). Hence, for $n = N$ large but fixed, we have

$$\int [(u_N')^2 - V(r)u_N^2] r\,\mathrm{d}r < 0,$$

and hence $\lambda_0 < 0$. Since L is a relatively compact perturbation of $-\Delta$, λ_0 is a discrete eigenvalue with associated eigenfunction u_0 contained in the form domain

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of L, H. By standard arguments, $u_0 > 0$ and λ_0 is a simple (non-degenerate) eigenvalue. □

By lemma 4.2,

$$-g_\kappa^* = \inf_{w\in H-\{0\}} \left\{ \int \left[\frac{1}{\kappa^2}(w')^2 - (1-\tilde{f}_\kappa^2)w^2 \right] r\,\mathrm{d}r \bigg/ \int w^2 r\,\mathrm{d}r \right\} < 0$$

and we have that $\lambda_0 = -\kappa^2 g_\kappa^*$ is the ground state eigenvalue of the Schrödinger operator $-\Delta_r - \kappa^2(1-\tilde{f}_\kappa^2)$. Since λ_0 is a simple eigenvalue,

$$\dim\ker(\mathcal{F}_3'(\tilde{f}_\kappa, g_\kappa^*)) = \dim\ker(-\Delta_r - \kappa^2(1-\tilde{f}_\kappa^2) + \kappa^2 g_\kappa^*) = 1.$$

In conclusion, when $g = g_\kappa^*$, the operator $\mathcal{F}'(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g_\kappa^*)$ has a simple eigenvalue and the eigenvector is of the form $(0, 0, w_\kappa)$, with w_κ the (positive) eigenfunction of $\mathcal{F}_3'$.

Finally, we observe that the operator $(\partial/\partial g)\mathcal{F}'(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g) \in L(Y_0, Y_0^*)$,

$$\left\langle (u, v, w), \frac{\partial}{\partial g}\mathcal{F}'(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g)[\phi, \psi, \xi] \right\rangle_{Y_0, Y_0^*} = \int \kappa^2 w\xi r\,\mathrm{d}r.$$

At the eigenvalue $g = g_\kappa^*$, we have

$$\frac{\partial}{\partial g}\mathcal{F}'(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g_\kappa^*)[0, 0, w_\kappa] = \kappa^2 w_\kappa \notin \mathrm{Ran}(\mathcal{F}'(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g_\kappa)).$$

Therefore, theorem 1.7 of [8] applies and g_κ^* is a bifurcation point for $\mathcal{F}$ in $Y \times \mathbb{R}$; there exists a neighbourhood U of $(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g_\kappa^*)$ in $Y_0 \times \mathbb{R}$ such that the set of non-trivial solutions of $\mathcal{F}(f, S, m, g) = 0$ in U is a unique C^1 curve parametrized by $\ker(\mathcal{F}'(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g_\kappa^*))$.

Remark 4.3. Since $\mathcal{F}$ is a smooth (C^∞) map, we may calculate various derivatives of the bifurcation curve through the normal core solutions at g_κ^*. If we parametrize $g = \gamma(t)$, with $\gamma(0) = g_\kappa^*$, then we follow Crandall and Rabinowitz [8] or Ambrosetti and Prodi [3] (see remark 4.3) to calculate derivatives of $\gamma(t)$ and determine the direction of the bifurcation curve locally at $g = g_\kappa^*$. We obtain that $\gamma'(0) = 0$ and

$$\gamma''(0) = -2\left[\int [\tilde{f}_\kappa u_* w_\kappa^2 + w_\kappa^4] r\,\mathrm{d}r \bigg/ \int w_\kappa^2 r\,\mathrm{d}r \right],$$

where u_κ is obtained from the (unique) solution to the linear system

$$\mathcal{F}'(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g_\kappa^*)[u_*, v_*, w_*] = -(2\kappa^2 \tilde{f}_\kappa w_\kappa^2, 0, 0),$$

with $(u_*, v_*, w_*) \perp \ker \mathcal{F}'(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g_\kappa^*)$. Taking the scalar product of the above system with $(u_*, v_*, 0)$ (and recalling that $\mathcal{F}'(\tilde{f}_\kappa, \tilde{S}_\kappa, 0, g_\kappa^*)$ is positive definite in the complement of its kernel), we obtain

$$\int \tilde{f}_\kappa u_* w_\kappa^2 r\,\mathrm{d}r < 0,$$

and hence the expression for $\gamma''(0)$ is indefinite in sign. In a joint paper with Berlinsky [2], we present computational evidence that solutions bifurcate to the left to

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smaller values $g < g_\kappa^*$. By standard bifurcation theory (see, for example, [9]), the direction of bifurcation indicates the stability of the solutions, and indeed we observe numerically that the AF core solutions which bifurcate at g_κ^* are stable (local energy minimizers).

4.2. Global bifurcation for $(\mathrm{GL}_{\infty,g})$

We obtain the same abstract bifurcation result for the extreme type-II model $(\mathrm{GL}_{\infty,g})$. Namely, the value

$$g_\infty^* = -\inf_{w\in H-\{0\}}\left\{\int[(w')^2-(1-\tilde{f}_\infty^2)w^2]r\,\mathrm{d}r\bigg/\int w^2 r\,\mathrm{d}r\right\} > 0$$

is a bifurcation point for non-trivial $(m>0)$ solutions from the (trivial) curve of normal core solutions $(\tilde{f}_\infty, 0, g)$. But in this case we can make a much more precise statement.

Proposition 4.4. *Let*

$$\Sigma = \{(f,m,g) : (f,m) \text{ is an admissible solution to } (GL_{\infty,g}) \text{ with } m>0\}.$$

Then $\mathcal{C} = \Sigma \cup \{(\tilde{f}_\infty, 0, g_\infty^*)\}$ *is a connected* C^1 *curve, parametrized by* g. *Moreover, for any* $g_0 > 0$, $\mathcal{C}\cap\{g \geqslant g_0\}$ *is compact.*

As a consequence, we have the following exact solvability theorem for $(\mathrm{GL}_{\infty,g})$.

Theorem 4.5. *For* $g \geqslant g_\infty^*$, *the normal core solutions* $(\tilde{f}_\infty, 0)$ *are the only admissible solutions of* $(GL_{\infty,g})$.

For $0 < g < g_\infty^*$, *there is a unique solution with* $m > 0$. *This solution is the global minimizer of* $\mathcal{E}_{\infty,g}$.

The proofs of these two results hinge on the powerful theorem 3.1 and the following compactness theorem, which will be proven in § 6.

Theorem 4.6. *Let* $0 < a < b$. *Then the set of all admissible solutions of* $(GL_{\infty,g})$ *with* $g \in [a,b]$ *is compact in* Z.

Proof of theorem 4.4. Let $\mathcal{C}'$ be a maximally connected component of $\mathcal{C}$ and suppose that $(f_0, m_0, g_0) \in \mathcal{C}'$ but $(f_0, m_0, g_0) \neq (\tilde{f}_\infty, 0, g_\infty^*)$. Since $m=0$ only when $g = g_\infty^*$, we must have $m_0 > 0$. By theorem 3.1, (f_0, m_0, g_0) is a non-degenerate zero of $\mathcal{F}$ in $Z\times\mathbb{R}$, so by the implicit function theorem, there exists a neighbourhood U of (f_0, m_0, g_0) in $Z \times \mathbb{R}$, an interval $J = (g_0 - \delta, g_0 + \delta)$ and a C^1 function $\Phi : J \to Z$ so that all solutions of $\mathcal{F} = 0$ in U are of the form $(\Phi(g), g)$ with $g \in J$.

Let

$$\hat{g} = \sup\{g : \text{there exists a solution } (f,m,g)\in\mathcal{C}'\} > g_0.$$

Note first that any solution must satisfy

$$0 \leqslant \int (m')^2 r\,\mathrm{d}r \leqslant \int (1-g)m^2 r\,\mathrm{d}r,$$

and hence $g < 1$ for any solution with $m \not\equiv 0$. Since, by proposition 4.6,

$$\mathcal{C}'\cap\{g\geqslant g_0\} = \mathcal{C}'\cap\{g_0 \leqslant g \leqslant 1\}$$

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is compact, there exists a solution at $g = \hat{g}$, $(\hat{f}, \hat{m}, \hat{g}) \in \mathcal{C}'$. First we claim that $\hat{m} = 0$. If not, then by proposition 2.5, $\hat{m}(r) > 0$ for all $r > 0$, so by theorem 3.1, $(\hat{f}, \hat{m})$ is a non-degenerate minimum of $(\mathrm{GL}_{\infty,\hat{g}})$. By the implicit function theorem argument above, there exists a C^1 curve of non-trivial solutions through $(\hat{f}, \hat{m}, \hat{g})$, parametrized by g. In particular, we contradict the definition of $\hat{g}$ is the supremum of all g for solutions in the connected component $\mathcal{C}'$. Hence $\hat{m} = 0$, as desired.

Now we show that $\hat{g} = g^*_\infty$. Take a sequence $(f_n, m_n, g_n) \in \mathcal{C}'$ with $g_n \to \hat{g}$, so the above arguments imply that $f_n - \tilde{f}_\infty \to 0$ in X and $m_n \to 0$ in H. Let

$$t_n = \int (1 - f_n^2) m_n^2 r \, \mathrm{d}r \to 0.$$

Then $w_n = m_n / t_n$ solves

$$-w_n'' - \frac{1}{r} w_n' + g_n w_n = (1 - f_n^2 - m_n^2) w_n. \tag{4.1}$$

Since

$$\int ((w_n')^2 + g w_n^2) r \, \mathrm{d}r = \int (1 - f_n^2 - m_n^2) w_n^2 r \, \mathrm{d}r \leqslant \int (1 - f_n^2) w_n^2 r \, \mathrm{d}r = 1 \tag{4.2}$$

(by the choice of t_n), we have $\|w_n\|_H \leqslant 1/g$ and we may extract a subsequence (which we continue to call w_n) which converges $w_n \rightharpoonup w_\infty$ weakly in H and strongly in L^2_{loc}. By the strong convergence of $f_n \to \tilde{f}_\infty$, we have

$$\int (1 - \tilde{f}_\infty^2) w_\infty^2 r \, \mathrm{d}r = 1,$$

so $w_\infty \not\equiv 0$ and $w_\infty \geqslant 0$. Passing to the limit in (4.1), we have

$$\int \Big(w_\infty' \phi' + \hat{g} w_\infty \phi - (1 - \tilde{f}_\infty^2) w_\infty \phi \Big) r \, \mathrm{d}r = 0, \tag{4.3}$$

for all $\phi \in H$. This can only occur when $\hat{g} = g^*_\infty$, the ground state eigenvalue of the above Schrödinger operator.

We have just shown that the point $(\tilde{f}_\infty, 0, g^*_\infty)$ belongs to every connected component of $\mathcal{C}$, and hence $\mathcal{C}$ is connected. The solution set $\mathcal{C}$ is everywhere a C^1 curve; for $g > g^*_\infty$, this results from the implicit function theorem argument in the first paragraph, and at g^*_∞ it is a consequence of bifurcation from a simple eigenvalue [8]. We now claim that there exists exactly one solution in $\mathcal{C}$ for every $g \leqslant g^*_\infty$. Suppose not, and consider

$$D = \{ g \in (0, g^*_\infty) : \text{there exist two distinct solutions } (f_{g,1}, m_{g,1}),\ (f_{g,2}, m_{g,2}) \text{ in } \mathcal{C} \text{ at } g \}$$

and

$$g_0 = \sup D.$$

First we note that $g_0 < g^*_\infty$. To see this, we note that the only solution in $\mathcal{C}$ with $g = g^*_\infty$ is the normal core solution, and the bifurcation theorem ensures that the solution set in a neighbourhood of the bifurcation point $(\tilde{f}_\infty, 0, g^*_\infty)$ is a single smooth curve.

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Next we claim that $g_0 \notin D$. Indeed, if $g_0 \in D$, there exist two distinct solutions $(f_{g_0,1}, m_{g_0,1})$ and $(f_{g_0,2}, m_{g_0,2})$ for $g = g_0$. By the implicit function theorem argument of the first paragraph, there exist neighbourhoods U_1 (of $(f_{g_0,1}, m_{g_0,1}, g_0)$) and U_2 (of $(f_{g_0,2}, m_{g_0,2}, g_0)$) in $Z \times \mathbb{R}$ such that all solutions of $\mathcal{F} = 0$ in U_1, U_2 are given by smooth curves parametrized by g. In particular, $\mathcal{C}$ contains two distinct solutions for g in an interval to the right of g_0, contradicting the definition of g_0 as the supremum.

Hence $g^*_\infty > g_0 \notin D$ and there exists a sequence $g_k \to g_0$ for which $\mathcal{C}$ contains two distinct solutions, $(f_{g_k,1}, m_{g_k,1})$, $(f_{g_k,2}, m_{g_k,2})$. By theorem 4.6, along some subsequence these solutions converge and since $g_0 \notin D$, they both converge to a single solution, (f_{g_0}, m_{g_0}). But this contradicts the implicit function theorem argument, which implies that the solution set near (f_{g_0}, m_{g_0}, g_0) is a single curve parametrized by g. We conclude that the AF core solutions are *unique* for each $g \in (0, g^*_\infty)$. □

4.3. Behaviour for $g \to 0$, $\kappa < \infty$

For the problem $(\mathrm{GL}_{\kappa,g})$, we do not have the strong information provided by theorem 3.1 which determines the global structure of the solution set, and hence we cannot make the same elegant conclusion about the uniqueness of AF core solutions. However, we may still say something about the global structure of the continuum bifurcating from the normal cores at $g = g^*_\kappa$. When $\kappa^2 \geqslant 2d^2$, we may apply the global bifurcation theorem of Rabinowitz [15] to conclude that the continuum Σ_κ of zeros of $\mathcal{F}(f, S, m, g) = 0$ with $m > 0$ is unbounded in the space $Y \times \mathbb{R}$. (Note that Σ_κ cannot contain any other eigenvalues of the linearization about the normal core solutions, as is easily seen from the calculations (4.1)–(4.3) above.) In the next section we will prove the following *a priori* estimate, which has as a direct consequence the fact that Σ_κ can only become unbounded as $g \to 0^+$.

THEOREM 4.7. *Let d, κ be fixed. For any compact interval $J \in (0, \infty)$, there exists $C_0 = C_0(\kappa, d, J) > 0$ such that every admissible solution (f, S, m) of $(GL_{\kappa,g})$ with $g \in J$ satisfies $\|(f, S, m)\|_Y \leqslant C$.*

Let us now concentrate on this loss of compactness in the continuum Σ_κ as $g \to 0^+$. We prove the following.

THEOREM 4.8. *For any sequence of (absolute) minimizers $(f_g, S_g, m_g) \in Y$ with $g \to 0^+$, we have $f_g \to 0$ in X_{loc}, $S_g \to 0$ locally uniformly and $m_g \to 1$ in H_{loc}.*

Fix $\kappa \in \mathbb{R}$ and for any $g > 0$ consider a minimizer $(f_g, S_g, m_g) \in Y$ of $\mathcal{E}_{\kappa,g}$.

LEMMA 4.9. *$\mathcal{E}_{\kappa,g}(f_g, S_g, m_g) \to 0$ as $g \to 0$.*

Proof. We will show that for any $\epsilon > 0$, there exist $g_\epsilon > 0$ and H radial functions $(f_\epsilon, S_\epsilon, m_\epsilon) \in Y$ such that $0 < \mathcal{E}_{\kappa,g}(f_\epsilon, S_\epsilon, m_\epsilon) < \epsilon$ for any $g < g_\epsilon$.

For a fixed $\rho > 0$, we define

$$u_\rho(r) = \begin{cases} 1 & \text{if } r \leqslant \rho, \\ \ln(r/\rho^2)/\ln(1/\rho) & \text{if } \rho \leqslant r \leqslant \rho^2, \\ 0 & \text{if } r \geqslant \rho^2 \end{cases}$$

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and consider

$$f_\rho(r) = \cos(u_\rho(r)\pi/2), \qquad m_\rho(r) = \sin(u_\rho(r)\pi/2)$$

and

$$S_\rho(r) = \begin{cases} 0 & \text{if } r \in (0, \rho/2), \\ d & \text{if } r \in (\rho, \infty). \end{cases}$$

A direct computation shows that

$$\mathcal{E}_{\kappa,g}(f_\rho, S_\rho, m_\rho) \leqslant \frac{C}{\rho^2} + \frac{\pi^2}{4\ln\rho} + \tfrac{1}{2}\kappa^2 g\rho^4$$

for any $g > 0$.

For a given $\epsilon > 0$, we choose a ρ_ϵ such that

$$\frac{C}{\rho_\epsilon^2} + \frac{\pi^2}{4\ln\rho_\epsilon} < \tfrac{1}{2}\epsilon$$

and a $g_\epsilon = g_\epsilon(\rho_\epsilon)$ for which $\frac{1}{2}\kappa^2 g_\epsilon \rho_\epsilon^4 < \frac{1}{2}\epsilon$, i.e.

$$g_\epsilon < \tfrac{1}{2}\epsilon\frac{2}{\kappa^2\rho_\epsilon^4}.$$

Then $\mathcal{E}_{\kappa,g}(f_g, S_g, m_g) \leqslant \mathcal{E}_{\kappa,g}(f_{\rho_\epsilon}, S_{\rho_\epsilon}, m_{\rho_\epsilon}) < \epsilon$ for any $g < g_\epsilon$. □

Proof of theorem 4.8. By lemma 4.9, each term in the energy tends to zero as $g \to 0$. First note that

$$\int (S_g'/r)^2 r\,\mathrm{d}r \to 0,$$

combined with (1.4) in [5], implies that

$$S_g(r)/r \to 0 \quad \text{uniformly.} \tag{4.4}$$

For any $R_0 > 0$, we then have

$$\begin{aligned} o(1) &= \int \frac{(d - S_g)^2}{r^2} f_g^2 r\,\mathrm{d}r \\ &\geqslant \int_0^{R_0} \frac{(d - S_g)^2}{r^2} f_g^2 r\,\mathrm{d}r \\ &= \int_0^{R_0} \frac{d^2}{r^2} f_g^2 r\,\mathrm{d}r + o(1). \end{aligned}$$

In particular, $f_g \to 0$ in L^2_{loc}, X_{loc}. Finally, by the reverse triangle inequality,

$$\begin{aligned} o(1) &= \sqrt{\tfrac{1}{2}\kappa^2 \int_0^{R_0} (1 - f_g^2 - m_g^2)^2 r\,\mathrm{d}r} \\ &= \frac{\kappa}{\sqrt{2}}\|1 - f_g^2 - m_g^2\|_{L^2([0,R_0])} \\ &\geqslant \frac{\kappa}{\sqrt{2}}[\|1 - m_g^2\|_{L^2([0,R_0])} - \|f_g^2\|_{L^2([0,R_0])}] \\ &\geqslant \frac{\kappa}{\sqrt{2}}\|1 - m_g\|_{L^2([0,R_0])} + o(1), \end{aligned}$$

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where we have also used $0 \leqslant f_g < 1$, $0 < m_g < 1$ and $f_g \to 0$ in L^2_{loc}. In conclusion, $m_g \to 1$ in L^2_{loc} and, in fact, in H^1_{loc}, since

$$\int (m_g')^2 r \,\mathrm{d}r \to 0$$

by the energy estimate. □

5. The limit $\kappa \to \infty$

In this section we show that the problem $(\mathrm{GL}_{\infty,g})$ arises as a limiting case of $(\mathrm{GL}_{\kappa,g})$ as $\kappa \to \infty$. For any solution $(f_\kappa, S_\kappa, m_\kappa)$ of $(\mathrm{GL}_{\kappa,g})$, define

$$\hat{f}_\kappa(r) = f_\kappa\left(\frac{r}{\kappa}\right), \qquad \hat{S}_\kappa(r) = S_\kappa\left(\frac{r}{\kappa}\right), \qquad \hat{m}_\kappa(r) = m_\kappa\left(\frac{r}{\kappa}\right). \tag{5.1}$$

We prove the following result.

THEOREM 5.1. *Let $(f_\kappa, S_\kappa, m_\kappa)$ be any family of solutions of $(GL_{\kappa,g})$ for $\kappa > 0$ and $(\hat{f}_\kappa, \hat{S}_\kappa, \hat{m}_\kappa)$ defined as in (5.1). For any sequence $\kappa_n \to \infty$, there exists a subsequence and a solution (f_∞, m_∞) of $(GL_{\infty,g})$ so that (as $\kappa_{n_k} \to \infty$) $\hat{f}_{\kappa_n} - f_\infty \to 0$ in X, $\hat{m}_{\kappa_n} - m_\infty \to 0$ in H and $\hat{S}_{\kappa_n} \to 0$ locally uniformly. Moreover we have the following.*

(i) *If $g \geqslant g^*_\infty$, then $m_\kappa \to 0$.*

(ii) *If $m_\kappa \not\equiv 0$ for all large κ and $g \neq g^*_\infty$, then $\lim_{\kappa\to\infty} \hat{m}_\kappa = m_\infty > 0$.*

As a simple consequence of the uniform convergence of $\hat{f}_\kappa \to \tilde{f}_\infty$, we have the following.

COROLLARY 5.2. $g^*_\infty = \lim_{\kappa\to\infty} g^*_\kappa$.

REMARK 5.3. This implies that the bifurcation diagram for $(\mathrm{GL}_{\kappa,g})$ with κ very large should strongly resemble the very precise image given for $(\mathrm{GL}_{\infty,g})$ by theorem 4.5. In particular, for any fixed $g > g^*_\infty$, the system $(\mathrm{GL}_{\kappa,g})$ cannot have solutions $(f_{\kappa,g}, S_{\kappa,g}, m_{\kappa,g})$ with $m_{\kappa,g} > 0$ for κ large.

Simple calculations using the energy $\mathcal{E}_{\kappa,g}$ show that $\inf_Y \mathcal{E}_{\kappa,g} \sim \ln\kappa$, and hence we require require energy-independent estimates for our solutions $(\hat{f}_\kappa, \hat{S}_\kappa, \hat{m}_\kappa)$. To obtain these estimates, we begin with a simple version of the celebrated Pohozaev identity. This identity will also be essential for proving the *a priori* estimates used in the bifurcation analysis in the previous section.

PROPOSITION 5.4. *For any finite energy solution (f, S, m) of $(GL_{\kappa,g})$, we have*

$$g\kappa^2 \int m^2 r\,\mathrm{d}r + \tfrac{1}{2}\kappa^2 \int (1 - f^2 - m^2)^2 r\,\mathrm{d}r = \int \left[\frac{S'}{r}\right]^2 r\,\mathrm{d}r.$$

For any finite energy solution (f, m) of $(GL_{\infty,g})$, we have

$$g \int m^2 r\,\mathrm{d}r + \frac{1}{2} \int (1 - f^2 - m^2)^2 r\,\mathrm{d}r = \tfrac{1}{2} d^2.$$

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Proof. We multiply the first equation in $(\mathrm{GL}_{\kappa,g})$ by $f'(r)r$ and integrate $r\,\mathrm{d}r$ to obtain

$$\begin{aligned}\tfrac{1}{2}\kappa^2\int(1-f^2-m^2)(f^2)'r^2\,\mathrm{d}r &= \int(d-S)^2(\tfrac{1}{2}f^2)'\,\mathrm{d}r\\ &= \int(d-S)S'f^2\,\mathrm{d}r = -\int\left(\frac{S'}{r}\right)'S'r\,\mathrm{d}r\\ &= \int\frac{S'}{r}(S'r)'\,\mathrm{d}r = \int\left(\frac{S'}{r}\right)^2 r\,\mathrm{d}r,\end{aligned}$$

using the equation for $S(r)$ and integrating by parts whenever necessary. We also multiply the third equation in $(\mathrm{GL}_{\kappa,g})$ by $m'(r)r$ and integrate $r\,\mathrm{d}r$ to obtain

$$\begin{aligned}\tfrac{1}{2}\kappa^2\int(1-f^2-m^2)(m^2)'r^2\,\mathrm{d}r &= \tfrac{1}{2}g\kappa^2\int(m^2)'r^2\,\mathrm{d}r\\ &= -g\kappa^2\int m^2 r\,\mathrm{d}r.\end{aligned}$$

Together,

$$\begin{aligned}\int\left(\frac{S'}{r}\right)^2 r\,\mathrm{d}r &= g\kappa^2\int m^2 r\,\mathrm{d}r + \tfrac{1}{2}\kappa^2\int(1-f^2-m^2)(m^2+f^2)'r^2\,\mathrm{d}r\\ &= g\kappa^2\int m^2 r\,\mathrm{d}r + \tfrac{1}{2}\kappa^2\int(1-f^2-m^2)^2 r\,\mathrm{d}r.\end{aligned}$$

For the case $\kappa=\infty$, we proceed in the same way, except the equation for f yields

$$\int(1-f^2-m^2)(\tfrac{1}{2}f^2)'r^2\,\mathrm{d}r = \tfrac{1}{2}d^2.$$

The calculation then continues as above. □

Proof of theorem 5.1.

STEP 1 (bounding the sequence). From the Pohozaev identity (proposition 5.4) and lemma 4.2 of [5] after rescaling, we have

$$\begin{aligned}d^2 &\geqslant \int\left(\frac{S'_\kappa}{r}\right)^2 r\,\mathrm{d}r\\ &= \kappa^2\int\left(\frac{\hat{S}'_\kappa}{r}\right)^2 r\,\mathrm{d}r \qquad (5.2)\\ &= g\int\hat{m}_\kappa^2 r\,\mathrm{d}r + \frac{1}{2}\int(1-\hat{f}_\kappa^2-\hat{m}_\kappa^2)^2 r\,\mathrm{d}r. \qquad (5.3)\end{aligned}$$

Using (5.2) and lemma 1.2 (ii) in [5], we have

$$\sup_{r\in[0,\infty)}\left|\frac{\hat{S}_\kappa(r)}{r}\right| \to 0, \qquad (5.4)$$

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and hence $\hat{S}_\kappa \to 0$ locally uniformly. From (5.3) we obtain the uniform bound $\|\hat{m}_\kappa\|_2 \leqslant C$ (depending on g, which we assume is fixed). From the equation for m_κ, after a change of scale, we obtain

$$\begin{aligned}\int [(\hat{m}_\kappa')^2 + g\hat{m}_\kappa^2] r\,\mathrm{d}r &= \int (1 - \hat{f}_\kappa^2 - \hat{m}_\kappa^2) r\,\mathrm{d}r \\ &\leqslant \int \hat{m}_\kappa^2 r\,\mathrm{d}r \\ &\leqslant C\end{aligned}$$

and therefore $\|\hat{m}_\kappa\|_H \leqslant C$ uniformly in κ.

Recalling proposition 2.5, any solution satisfies $0 < \hat{m}_\kappa(r) < 1$ and we may conclude that $\|\hat{m}_\kappa\|_q \leqslant \|\hat{m}_\kappa\|_2 \leqslant C$ for all $p \in [2, \infty]$. By the triangle inequality,

$$\begin{aligned}\|1 - \hat{f}_\kappa^2\|_2 &\leqslant \|1 - \hat{f}_\kappa^2 - \hat{m}_\kappa^2\|_2 + \|\hat{m}_\kappa^2\|_2 \\ &\leqslant |d| + C,\end{aligned}$$

and hence we obtain

$$\begin{aligned}\int (1 - \hat{f}_\kappa)^2 r\,\mathrm{d}r &\leqslant \int (1 - \hat{f}_\kappa^2)^2 r\,\mathrm{d}r \\ &\leqslant C\end{aligned}$$

(since $\hat{f}_\kappa \geqslant 0$).

Choose a function $\eta \in C^\infty(\mathbb{R})$ with

$$\eta(r) = \begin{cases} 1 & \text{if } r \leqslant 2, \\ 0 & \text{if } r \geqslant 3 \end{cases}$$

and $0 \leqslant \eta(r) \leqslant 1$ for all r. Using $\eta^2 \hat{f}_\kappa$ as a test function in the weak form of the rescaled equation for $\hat{f}_\kappa$,

$$\begin{aligned}\int \eta^2\biggl[(\hat{f}_\kappa')^2 + \frac{d^2}{r^2}\hat{f}_\kappa^2\biggr] r\,\mathrm{d}r &= \int [(1 - \hat{f}_\kappa^2 - \hat{m}_\kappa^2)\hat{f}_\kappa^2\eta^2 - \eta\eta'\hat{f}_\kappa\hat{f}_\kappa'] r\,\mathrm{d}r \\ &\leqslant \int [\tfrac{1}{2}(1 - \hat{f}_\kappa^2)_2 + \tfrac{1}{2}\eta^4 + \tfrac{1}{2}\eta^2(\hat{f}_\kappa')^2 + 2\hat{f}_\kappa^2(\eta')^2] r\,\mathrm{d}r \\ &\leqslant C + \frac{1}{2}\int \eta^2(\hat{f}_\kappa')^2 r\,\mathrm{d}r.\end{aligned}$$

Absorbing the last term back to the left-hand side,

$$\int \eta^2\biggl[(\hat{f}_\kappa')^2 + \frac{d^2}{r^2}\hat{f}_\kappa^2\biggr] r\,\mathrm{d}r \leqslant C. \tag{5.5}$$

Now choose another smooth function f_0 with

$$f_0(r) = \begin{cases} 0 & \text{if } r \geqslant 1, \\ 1 & \text{if } r \geqslant 2 \end{cases}$$

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and $0 \leqslant f_0(r) \leqslant 1$. Note that, with this choice, $f_0^2 + \eta^2 \geqslant 1$. We use $(\hat{f}_\kappa - 1)f_0^2$ as a test function in the equation for $\hat{f}_\kappa$ to obtain

$$\begin{aligned}
\int \biggl[(\hat{f}_\kappa')^2 + \frac{d^2}{r^2}(\hat{f}_\kappa^2 - 1)^2\biggr] r\,\mathrm{d}r
&= \int [(1 - \hat{f}_\kappa^2 - \hat{m}_\kappa^2)\hat{f}_\kappa(\hat{f}_\kappa - 1)f_0^2 - (\hat{f}_\kappa - 1)\hat{f}_\kappa' f_0' f_0] r\,\mathrm{d}r \\
&\quad + \int \biggl[-\frac{d^2}{r^2}(\hat{f}_\kappa - 1)f_0^2 + \frac{\hat{S}_\kappa(2d - \hat{S}_\kappa)}{r^2}(\hat{f}_\kappa - 1)^2 f_0^2\biggr] r\,\mathrm{d}r \\
&\leqslant C\int_1^2 (\hat{f}_\kappa - 1)|\hat{f}_\kappa'| f_0 r\,\mathrm{d}r + \int_1^\infty \biggl[\frac{d^2}{r^2}(1 - \hat{f}_\kappa) + \frac{4d}{r^2}(\hat{f}_\kappa - 1)^2\biggr] r\,\mathrm{d}r \\
&\leqslant \frac{1}{2}\int (\hat{f}_\kappa')^2 f_0^2 r\,\mathrm{d}r + C\|\tilde{f}_\kappa - 1\|_2^2 + \int \biggl[\frac{C}{r^4} + (\hat{f}_\kappa^2 - 1)^4\biggr] r\,\mathrm{d}r \\
&\leqslant C + \frac{1}{2}\int (\hat{f}_\kappa')^2 f_0^2 r\,\mathrm{d}r.
\end{aligned}$$

(Note that in the first line, the first integrand is non-positive.) In conclusion,

$$\int \biggl[(\hat{f}_\kappa')^2 + \frac{d^2}{r^2}(\hat{f}_\kappa^2 - 1)^2\biggr] r\,\mathrm{d}r \leqslant C. \tag{5.6}$$

Now define $u_\kappa = \hat{f}_\kappa - f_0 \in X$. Then, from (5.5) and (5.6), we obtain

$$\begin{aligned}
\int \biggl[(u_\kappa')^2 + u_\kappa^2 + \frac{d^2}{r^2}u_\kappa^2\biggr] r\,\mathrm{d}r
&\leqslant 2\int [(\hat{f}_\kappa'^2) + (f_0)^2] r\,\mathrm{d}r + \int_0^2 \frac{2d^2}{r^2}\hat{f}_\kappa^2 r\,\mathrm{d}r + \int_2^\infty (d^2 + 1)(\hat{f}_\kappa - 1)^2 \\
&\leqslant 2\int \biggl[(\eta^2 + f_0^2)(\hat{f}_\kappa')^2 + \frac{d^2}{r^2}\hat{f}_\kappa^2\eta^2\biggr] r\,\mathrm{d}r + C \\
&\leqslant C.
\end{aligned}$$

In other words, u_κ is uniformly bounded in X and we may extract weakly convergent subsequences $u_n = u_{\kappa_n} \rightharpoonup u_*$ (in X), $m_n = m_{\kappa_n} \to m_*$ (in H).

STEP 2 (strong convergence). We next show that the sequences u_n, m_n converge in norm. Let $f_n = f_0 + u_n$ and $S_n = \hat{S}_{\kappa_n}$. First note that

$$(1 - f_n^2 - m_n^2)m_n - (1 - f_p^2 - m_p^2)m_p = (1 - f_n^2)w - (m_n^2 + m_n m_p + m_p^2)w + (f_n^2 - f_p^2)m_p.$$

Hence, using compact embeddings of X, H into L^q for $2 < q < \infty$,

$$\begin{aligned}
\int [((m_n - m_p)')^2 + g(m_n - m_p)^2] r\,\mathrm{d}r
&= \int [(1 - f_n^2) - (m_n^2 + m_n m_p + m_p^2)(m_n - m_p)](m_n - m_p)^2 r\,\mathrm{d}r \\
&\quad + \int (f_n + f_p)(u_n - u_p)m_p(m_n - m_p) r\,\mathrm{d}r \\
&= o(1).
\end{aligned}$$

Therefore, $m_n \to m_*$ in norm.

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We proceed in the same way with u_n,

$$\int\left\{(u_n'-u_p')^2+\left[\frac{(d-S_n)^2}{r^2}f_n-\frac{(d-S_p)^2}{r^2}f_p\right](u_n-u_p)\right\}r\,\mathrm{d}r$$
$$=\int\{(1-f_n^2-m_n^2)f_n-(1-f_p^2-m_p^2)f_p\}(u_n-u_p)r\,\mathrm{d}r. \quad (5.7)$$

Now we expand,

$$\frac{(d-S_n)^2}{r^2}f_n-\frac{(d-S_p)^2}{r^2}f_p$$
$$=\left[\frac{d^2}{r^2}-\frac{S_n}{r^2}(2d-S_n)\right](u_n-u_p)+f_p\left[\frac{S_p}{r^2}(2d-S_p)-\frac{S_n}{r^2}(2d-S_n)\right].$$

Now we take each term separately,

$$\int_0^1\frac{S_n}{r^2}(2d-S_n)(u_n-u_p)^2r\,\mathrm{d}r\leqslant\sup_{r\in[0,1]}|S_n|\int_0^1 2d\frac{(u_n-u_p)^2}{r^2}r\,\mathrm{d}r\to 0,$$

since $S_n\to 0$ locally uniformly and u_n are uniformly bounded. By (5.4),

$$\int_1^\infty\frac{S_n}{r^2}(2d-S_n)(u_n-u_p)^2r\,\mathrm{d}r\leqslant\sup\left|\frac{S_n}{r}\right|\int_1^\infty 2d(u_n-u_p)^2r\,\mathrm{d}r\to 0.$$

Choose $r_0>0$ so that $\int_{r_0}^\infty r^{-2}\,\mathrm{d}r<\frac{1}{4}\varepsilon^2$ and κ sufficiently large so that

$$dr_0^2\sup\left|\frac{S_p}{r}\right|\|u_n-u_p\|_X<\tfrac{1}{2}\varepsilon.$$

Then

$$\int_0^{r_0}f_p\frac{S_p}{r^2}(2d-S_p)(u_n-u_p)r\,\mathrm{d}r\leqslant dr_0^2\sup_{0\leqslant r\leqslant r_0}\left|\frac{S_p}{r}\right|\sqrt{\int_0^{r_0}\frac{(u_n-u_p)^2}{r^2}r\,\mathrm{d}r}\leqslant\tfrac{1}{2}\varepsilon$$

and

$$\int_{r_0}^\infty f_p\frac{S_p}{r^2}(2d-S_p)(u_n-u_p)r\,\mathrm{d}r\leqslant 2d^2\left[\int_{r_0}^\infty\frac{dr}{r^3}\right]^{1/2}\left[\int_0^\infty(u_n-u_p)^2r\,\mathrm{d}r\right]^{1/2}<\tfrac{1}{2}\varepsilon.$$

We return to (5.7) and substitute the above estimates,

$$\int\left\{(u_n'-u_p')^2+\frac{d^2}{r^2}(u_n-u_p)^2\right\}r\,\mathrm{d}r+o(1)$$
$$=\int\{(1-f_n^2-m_n^2)f_n-(1-f_p^2-m_p^2)f_p\}(u_n-u_p)r\,\mathrm{d}r$$
$$=\int\{(1-3f_0^2)(u_n-u_p)^2-3f_0(u_n^2-u_p^2)(u_n-u_p)$$
$$-(u_n^3-u_p^3)(u_n-u_p)-f_0(m_n^2-m_p^2)(u_n-u_p)$$
$$-m_n^2(u_n-u_p)^2-u_p(m_n^2-m_p^2)(u_n-u_p)\}r\,\mathrm{d}r$$
$$=-2\int(u_n-u_p)^2r\,\mathrm{d}r+o(1),$$

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where we use the facts that $m_n \to m_*$ strongly in H, u_n is bounded in X and $u_n \to u_*$ in L^2_{loc}. In conclusion, the subsequence $u_n \to u_*$ strongly in X.

STEP 3 (determining when $m_\infty = 0$). Since all solutions of $(\mathrm{GL}_{\infty,g})$ with $g \geqslant g^*_\infty$ have $m_\infty = 0$, we have $\hat{m}_\kappa \to 0$ when $g \geqslant g^*_\infty$. On the other hand, suppose $\hat{m}_\kappa > 0$ for all sufficiently large κ, but $\hat{m}_\kappa \to 0$. By uniqueness of the normal core solution, $\hat{f}_\kappa \to \tilde{f}_\infty$, the unique solution of

$$-\Delta_r \tilde{f}_\infty + \frac{d^2}{r^2}\tilde{f}_\infty = (1 - \tilde{f}^2_\infty)\tilde{f}_\infty.$$

Let

$$t_\kappa = \int (1 - \hat{f}^2_\kappa)\hat{m}^2_\kappa r \,\mathrm{d}r \to 0$$

and set $w_\kappa = \hat{m}_\kappa / t_\kappa$. Then

$$-\Delta_r w_\kappa + g w_\kappa = (1 - \hat{f}^2_\kappa - \hat{m}^2_\kappa) w_\kappa.$$

Since

$$\int [(w'_\kappa)^2 + g w^2_\kappa] r \,\mathrm{d}r = \int (1 - \hat{f}^2_\kappa - \hat{m}^2_\kappa) w^2_\kappa r \,\mathrm{d}r \leqslant 1$$

(by the choice of t_κ), the bound $\|w_\kappa\|_H \leqslant 1/g$ results. We extract a subsequence (which we still denote by w_κ) with $w_\kappa \rightharpoonup w_\infty$ weakly in H. Note that $w_\infty \geqslant 0$. By the choice of t_κ, the uniform convergence $\hat{f}_\kappa \to \tilde{f}_\infty$ and the L^2_{loc} convergence of $w_\kappa \to w_\infty$, we have

$$\begin{aligned}\int (1 - \tilde{f}^2_\infty) w^2_\infty r \,\mathrm{d}r &= \int [(1 - \tilde{f}^2_\infty)(w^2_\infty - w^2_\kappa) + (\hat{f}^2_\kappa - \tilde{f}^2_\infty) w^2_\kappa + (1 - \hat{f}^2_\kappa) w^2_\kappa] r \,\mathrm{d}r \\ &= 1 + o(1).\end{aligned}$$

In particular, $w_\infty \not\equiv 0$. By weak convergence, we may pass to the limit in the equation for w_κ, and hence w_∞ is a non-trivial non-negative solution of

$$-\Delta_r w_\infty + g w_\infty = (1 - \tilde{f}^2_\infty) w_\infty.$$

This can only occur when $g = g^*_\infty$.

This completes the proof of theorem 5.1. □

6. Estimates and existence

In this section we derive the technical estimates which were needed in our analysis of the bifurcation problem in § 4. We also provide the details of the proof of existence of minimizers of the energies $\mathcal{E}_{\kappa,g}$ and $\mathcal{E}_{\infty,g}$.

6.1. *A priori* estimates

We may now prove *a priori* estimates for the solutions of our system $(\mathrm{GL}_{\kappa,g})$, theorem 4.7, as well as the compactness result for solutions of $(\mathrm{GL}_{\infty,g})$ (both theorems as stated in the previous section). Note that both theorems are stated for all solutions, not only energy minimizers, and hence we will use our Pohozaev identity

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(proposition 5.4) to obtain energy-independent estimates. As before, we denote by $\tilde{f}_\kappa$, $\tilde{S}_\kappa$ a normal core solution at κ, and $u = f - \tilde{f}_\kappa$, $v = (S - \tilde{S}_\kappa)/r$.

By the Pohozaev identity and lemma 4.2 of [5], we have

$$\kappa^2 \int [gm^2 + \tfrac{1}{2}(1 - f^2 - m^2)^2] r \,\mathrm{d}r = \int \left(\frac{S'}{r}\right)^2 r \,\mathrm{d}r \leqslant \tfrac{1}{2} d^2. \tag{6.1}$$

In particular, we obtain

$$\int (1-f)^2 r \,\mathrm{d}r \leqslant C + \frac{C}{g}, \qquad \int m^2 r \,\mathrm{d}r \leqslant \frac{C}{g},$$

with constant C depending on κ, d. From the first estimate, we obtain

$$\|u\|_2 \leqslant \|\tilde{f}_\kappa + u - 1\|_2 + \|\tilde{f}_\kappa - 1\|_2 \leqslant C + C/g.$$

The equation for m, together with the second estimate, gives

$$\int [(m')^2 + \kappa^2 g m^2] r \,\mathrm{d}r = \kappa^2 \int (1 - f^2 - m^2) m^2 r \,\mathrm{d}r \leqslant \kappa^2 \int m^2 r \,\mathrm{d}r \leqslant \frac{C}{g}.$$

In particular, $\|m\|_H \leqslant C$, $\|u\|_2 \leqslant C$ and the constant depending on κ, d may be chosen uniformly for $g \in J$.

Using the right half of (6.1), we have

$$\tfrac{1}{2} d^2 \geqslant \int \left(\frac{S'}{r}\right)^2 r \,\mathrm{d}r = \int \left[\left(\frac{\tilde{S}'_\kappa}{r}\right)^2 + 2 \frac{\tilde{S}'_\kappa}{r} \frac{(rv)'}{r} + \left(\frac{(rv)'}{r}\right)^2\right] r \,\mathrm{d}r. \tag{6.2}$$

Since

$$\left| 2 \int \frac{\tilde{S}'_\kappa}{r} \frac{(rv)'}{r} r \,\mathrm{d}r \right| \leqslant 2 \int \left(\frac{\tilde{S}'_\kappa}{r}\right)^2 + \frac{1}{2} \int \left(\frac{(rv)'}{r}\right)^2 r \,\mathrm{d}r$$

and

$$\int \left(\frac{(rv)'}{r}\right)^2 r \,\mathrm{d}r = \int \left[(v')^2 + \frac{v^2}{r^2}\right] r \,\mathrm{d}r,$$

we may conclude from (6.2) that

$$\int \left[(v')^2 + \frac{v^2}{r^2}\right] r \,\mathrm{d}r \leqslant C, \tag{6.3}$$

with constant depending only on d. From the embedding properties of X, lemma 2.1, we conclude that $\|v\|_\infty \leqslant C$.

We now use v as a test function in the weak form of the equation for S to obtain an estimate,

$$\begin{aligned} \left| \int \frac{d - S}{r} f^2 v r \,\mathrm{d}r \right| &= \left| \int \frac{S'}{r^2} (rv)' r \,\mathrm{d}r \right| \\ &\leqslant \frac{1}{2} \int \left(\frac{S'}{r^2}\right)^2 r \,\mathrm{d}r + \frac{1}{2} \int \left[(v')^2 + \frac{v^2}{r^2}\right] r \,\mathrm{d}r \\ &\leqslant C. \end{aligned} \tag{6.4}$$

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On the other hand, expanding the left-hand side of (6.4),

$$\int \left(\frac{d-S}{r}\right) f^2 vr\,\mathrm{d}r = \int \left[\frac{d-\tilde{S}_\kappa}{r} - v\right](\tilde{f}_\kappa + u)^2 vr\,\mathrm{d}r$$
$$= \int \left(\frac{d-\tilde{S}_\kappa}{r}\right)[\tilde{f}_\kappa^2 + 2\tilde{f}_\kappa u + u^2]vr\,\mathrm{d}r$$
$$- \int 2\tilde{f}_\kappa uv^2 r\,\mathrm{d}r - \int v^2\tilde{f}_\kappa^2 r\,\mathrm{d}r - \int v^2u^2r\,\mathrm{d}r. \tag{6.5}$$

To bound the term

$$\int v^2\tilde{f}_\kappa^2 r\,\mathrm{d}r,$$

we need to evaluate the other terms,

$$\left|\int \frac{d-\tilde{S}_\kappa}{r}\tilde{f}_\kappa^2 vr\,\mathrm{d}r\right| \leqslant 2\int \left[\frac{d-\tilde{S}_\kappa}{r}\right]^2 \tilde{f}_\kappa^2 r\,\mathrm{d}r + \frac{1}{8}\int \tilde{f}_\kappa^2 v^2 r\,\mathrm{d}r$$
$$\leqslant C + \frac{1}{8}\int \tilde{f}_\kappa^2 v^2 r\,\mathrm{d}r,$$
$$2\left|\int \frac{d-\tilde{S}_\kappa}{r}\tilde{f}_\kappa uvr\,\mathrm{d}r\right| \leqslant \int \left[\frac{d-\tilde{S}_\kappa}{r}\right]^2 \tilde{f}_\kappa^2 r\,\mathrm{d}r + \|v\|_\infty^2\|u\|_2^2$$
$$\leqslant C,$$
$$\left|\int \frac{d-\tilde{S}_\kappa}{r}u^2vr\,\mathrm{d}r\right| \leqslant \tfrac{1}{2}\|u\|_2^2\|v\|_\infty^2 + \frac{1}{2}\int \left[\frac{d-\tilde{S}_\kappa}{r}\right]^2 u^2r\,\mathrm{d}r,$$
$$2\left|\int \tilde{f}_\kappa uv^2r\,\mathrm{d}r\right| \leqslant 8\|u\|_2^2\|v\|_\infty^2 + \frac{1}{8}\int \tilde{f}_\kappa^2 v^2 r\,\mathrm{d}r$$
$$\leqslant C + \frac{1}{8}\int \tilde{f}_\kappa^2 v^2 r\,\mathrm{d}r,$$
$$\int v^2u^2r\,\mathrm{d}r \leqslant \|v\|_\infty^2\|u\|_2^2$$
$$\leqslant C.$$

Hence

$$\frac{3}{4}\int v^2\tilde{f}_\kappa^2 r\,\mathrm{d}r \leqslant C + \frac{1}{2}\int \left[\frac{d-\tilde{S}_\kappa}{r}\right]^2 u^2r\,\mathrm{d}r. \tag{6.6}$$

Finally, we use u as a test function in the weak form of the equation for f. Recalling the definition of $\tilde{f}_\kappa$ as a normal core solution, we expand and cancel terms to arrive at

$$\int \left[(u')^2 + \left(\frac{d-\tilde{S}_\kappa}{r}\right)^2\right]r\,\mathrm{d}r = \int \left[2\left(\frac{d-\tilde{S}_\kappa}{r}\right)(\tilde{f}_\kappa+u)uv - (\tilde{f}_\kappa+u)uv^2\right]r\,\mathrm{d}r$$
$$+ \kappa^2\int \left[(1-3\tilde{f}_\kappa^2)u^2 - 3\tilde{f}_\kappa u^3 - u^4 - m^2fu\right]r\,\mathrm{d}r. \tag{6.7}$$

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Each term on the right-hand side may be controlled as follows,

$$\left|\int m^2 f u r\,\mathrm{d}r\right| \leqslant \frac{1}{2}\int m^4 + \frac{1}{2}\int u^2 r\,\mathrm{d}r$$
$$\leqslant C,$$
$$\left|\int (1-3\tilde{f}_\kappa^2)u^2 r\,\mathrm{d}r\right| \leqslant 3\|u\|_2^2$$
$$\leqslant C,$$
$$2\left|\int \left(\frac{d-\tilde{S}_\kappa}{r}\right)\tilde{f}_\kappa u v r\,\mathrm{d}r\right| \leqslant \|u\|_2^2\|v\|_\infty^2 + \int \left(\frac{d-\tilde{S}_\kappa}{r}\right)^2 \tilde{f}_\kappa^2 r\,\mathrm{d}r$$
$$\leqslant C,$$
$$\left|\int \tilde{f}_\kappa u^3 r\,\mathrm{d}r\right| \leqslant \frac{3}{2}\int \tilde{f}_\kappa^2 u^2 r\,\mathrm{d}r + \frac{1}{2}\int u^4 r\,\mathrm{d}r,$$
$$2\left|\int \left(\frac{d-\tilde{S}_\kappa}{r}\right)u^2 v r\,\mathrm{d}r\right| \leqslant 6\|u\|_2^2\|v\|_\infty^2 + \frac{1}{6}\int \left(\frac{d-\tilde{S}_\kappa}{r}\right)^2 u^2 r\,\mathrm{d}r$$
$$\leqslant C + \frac{1}{6}\int \left(\frac{d-\tilde{S}_\kappa}{r}\right)^2 u^2 r\,\mathrm{d}r,$$
$$\left|\int \tilde{f}_\kappa u v^2 r\,\mathrm{d}r\right| \leqslant \tfrac{1}{2}\|u\|_2^2\|v\|_\infty^2 + \frac{1}{2}\int \tilde{f}_\kappa^2 v^2 r\,\mathrm{d}r$$
$$\leqslant C + \frac{1}{3}\int \left(\frac{d-\tilde{S}_\kappa}{r}\right)^2 u^2 r\,\mathrm{d}r,$$

where, in the last estimate, we apply (6.6). Using (6.7), we have

$$\int \left[(u')^2 + \frac{1}{2}\left(\frac{d-\tilde{S}_\kappa}{r}\right)^2 u^2\right] r\,\mathrm{d}r \leqslant C.$$

Consequently, $\|u\|_X \leqslant C$. Returning to (6.6), it follows that

$$\int \tilde{f}_\kappa^2 v^2 r\,\mathrm{d}r \leqslant C + \frac{2}{3}\int \left(\frac{d-\tilde{S}_\kappa}{r}\right)^2 u^2 r\,\mathrm{d}r \leqslant C,$$

and hence (6.3) yields $\|v\|_X \leqslant C$. This concludes the proof of theorem 4.7.

An analogous result may be proven for solutions of $(\mathrm{GL}_{\infty,g})$.

THEOREM 6.1. *Let d be fixed. For any compact interval $J \in (0,\infty)$, there exists $C_0 = C_0(d,J) > 0$ such that every admissible solution (f,m) of $(GL_{\infty,g})$ with $g \in J$ satisfies $\|(f,m)\|_{Z_0} \leqslant C$.*

The proof of theorem 6.1 is similar to (and simpler than) the previous one and is left to the reader.

6.2. Compactness

Here we prove theorem 4.6, which asserts that the family of solutions to $(\mathrm{GL}_{\infty,g})$ with g bounded away from zero is a compact set. The same result holds for $(\mathrm{GL}_{\kappa,g})$, although the proof is more complicated due to the additional terms involving $S(r)$.

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Proof of theorem 4.6. Suppose $f_n = \tilde{f}_\infty + u_n$, m_n, g_n are a sequence of solutions of $(\mathrm{GL}_{\infty,g_n})$ with $g_n \in [a,b]$. By the theorem, we have $\|u_n\|_X, \|m_n\|_H \leqslant C$, and hence we may extract a subsequence with $u_n \rightharpoonup \tilde{u}$, $m_n \rightharpoonup \tilde{m}$ and $g_n \to \tilde{g} \in [a,b]$. Then we have

$$\int \biggl[(u_n' - u_k')^2 + \frac{d^2}{r^2}(u_n - u_k)^2\biggr] r\,\mathrm{d}r \\ = \int [(1 - f_n^2 - m_n^2)f_n - (1 - f_k^2 - m_k^2)f_k](u_n - u_k) r\,\mathrm{d}r \tag{6.8}$$

and

$$\int [(m_n' - m_k')^2 + \tilde{g}(m_n - m_k)^2] r\,\mathrm{d}r \\ = \int [(1 - f_n^2 - m_n^2)m_n - (1 - f_k^2 - m_k^2)m_k](m_n - m_k) r\,\mathrm{d}r + o(1). \tag{6.9}$$

We now expand the two right-hand side terms. First we use the embedding properties of X, H and the fact that $0 \leqslant f_n < 1$ for any solution to show

$$\begin{aligned} &[(1 - f_n^2 - m_n^2)f_n - (1 - f_k^2 - m_k^2)f_k](u_n - u_k) \\ &\quad = -\int 2\tilde{f}_\infty^2 (u_n - u_k)^2 r\,\mathrm{d}r + \int (1 - \tilde{f}_\infty^2)(u_n - u_k)^2 r\,\mathrm{d}r \\ &\qquad - 2\int [\tilde{f}_\infty (u_n + u_k)(u_n - u_k)^2 - f_n(u_n + u_k)(u_n - u_k)^2] r\,\mathrm{d}r \\ &\qquad - \int [f_n(m_n + m_k)(m_n - m_k)(u_n - u_k) + (u_k^2 + m_k^2)(u_n - u_k)^2] r\,\mathrm{d}r \\ &\quad = -\int 2\tilde{f}_\infty^2 (u_n - u_k)^2 r\,\mathrm{d}r + o(1). \end{aligned}$$

Applying the above estimate to (6.8), we have

$$\int \biggl[(u_n' - u_k')^2 + \biggl(\frac{d^2}{r^2} + 2\tilde{f}_\infty^2\biggr)(u_n - u_k)^2\biggr] r\,\mathrm{d}r \to 0$$

as $n, k \to \infty$, so $u_n \to \tilde{u}$ in norm on the space X.

Similarly, we estimate

$$\begin{aligned} &\int [(1 - f_n^2 - m_n^2)m_n - (1 - f_k^2 - m_k^2)m_k](m_n - m_k) r\,\mathrm{d}r \\ &\quad = \int (1 - \tilde{f}_\infty^2)(m_n - m_k)^2 r\,\mathrm{d}r \\ &\qquad - \int [2\tilde{f}_\infty u_n (m_n - m_k)^2 + 2\tilde{f}_\infty m_k (m_{-} m_k)(u_n - u_k) + u_n^2 (m_n - m_k)^2] r\,\mathrm{d}r \\ &\qquad - \int [m_k(u_n^2 - u_k^2)(m_n - m_k) + (m_n^3 - m_k^3)(m_n - m_k)] r\,\mathrm{d}r \\ &\quad = o(1). \end{aligned}$$

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Therefore, equation (6.9) implies that $m_n \to \tilde{m}$ in H. By passing to the limit in the weak formulation of $(\mathrm{GL}_{\infty,g_n})$, we easily obtain that $(\tilde{f}, \tilde{m})$ solve $(\mathrm{GL}_{\infty,\tilde{g}})$, and hence the specified solution set is compact. □

6.3. Existence

Let (u_n, v_n, m_n) be a minimizing sequence for $I_{\kappa,g}$, so

$$(f_n, S_n, m_n) = (f_0 + u_n, S_0 + rv_n, m_n)$$

is a minimizing sequence for $\mathcal{E}_{\kappa,g}$. To prove theorem 2.2, we first observe that the energy $\mathcal{E}_{\kappa,g}$ is a sum of positive terms, and hence each is individually bounded. In particular, m_n is uniformly bounded in H.

Now we must estimate u_n. First note that $\mathcal{E}_{\kappa,g}(|f_n|, S_n, m_n) = \mathcal{E}_{\kappa,g}(f_n, S_n m_n)$, and so we may assume that our minimizing sequence satisfies $f_n(r) \geqslant 0$ for all r. Next we observe

$$\|1 - f_n^2\|_2 \leqslant \|1 - f_n^2 - m_n^2\|_2 + \|m_n^2\|_2 \leqslant \|1 - f_n^2 - m_n^2\|_2 + C. \tag{6.10}$$

Hence we conclude that

$$\begin{aligned} C &\geqslant \mathcal{E}_{\kappa,g}(f_n, S_n, m_n) \\ &\geqslant \int \left\{ (f_n')^2 + \left[\frac{S_n'}{r}\right]^2 + \frac{(d - S_n)^2 f_n^2}{r^2} + \tfrac{1}{2}\kappa^2(1 - f_n^2)^2 \right\} r\,\mathrm{d}r. \end{aligned}$$

The right-hand side of the above inequality is the free energy of conventional Ginzburg–Landau vortices studied in [1]. The boundedness of $\|u_n\|_X$, $\|v_n\|_X$ then follows from the argument of proposition 4.2 of [1]. We may then pass to the limit in $\mathcal{E}_{\kappa,g}$ via lower semicontinuity of the norms and Fatou's lemma. □

To prove theorem 2.3, let (u_n, m_n) be a minimizing sequence for I_∞ in $X \times H$, so $(f_n, m_n) = (\tilde{f}_\infty + u_n, m_n)$ is a minimizing sequence for $\mathcal{E}_{\infty,g}$. Choose $r_g \geqslant 1$, so that $d^2/r_g^2 \leqslant g/2$. Then

$$\begin{aligned} &\mathcal{E}_{\infty,g}(f_n, m_n) \\ &\quad\geqslant \int_0^{r_g} \left[(m_n')^2 + g m_n^2 - \frac{d^2}{r^2}\tilde{f}_\infty^2\right] r\,\mathrm{d}r \\ &\qquad + \int_{r_g}^\infty \left[(m_n')^2 + \left(g - \frac{d^2}{r^2}\right) m_n^2 + \frac{d^2}{r^2}(f_n^2 + m_n^2 - 1) \right. \\ &\qquad\qquad \left. + \tfrac{1}{2}(f_n^2 + m_n^2 - 1)^2 + \frac{d^2}{r^2}(1 - \tilde{f}_\infty^2)\right] r\,\mathrm{d}r \\ &\quad\geqslant \int_0^\infty [(m_n')^2 + \tfrac{1}{2} g m_n^2] r\,\mathrm{d}r - \int_0^{r_g} \frac{d^2}{r^2}\tilde{f}_\infty^2 r\,\mathrm{d}r + \int_{r_g}^\infty \left[\frac{d^2}{r^2}(1 - \tilde{f}_\infty^2) - \frac{d^4}{2r^4}\right] r\,\mathrm{d}r, \end{aligned}$$

where we have used the elementary bound $ax + x^2/2 \geqslant -a^2/2$. In particular, $\mathcal{E}_{\infty,g}$ is bounded below and the minimizing sequence has $\|m_n\|_H \leqslant C$ uniformly in n. By Sobolev embedding, we also conclude that $\|m_n\|_p \leqslant C_p$ for all $p \in [2, \infty)$.

Now we must estimate u_n. As above, we note that $\mathcal{E}_{\infty,g}(|f_n|, m_n) = \mathcal{E}_{\infty,g}(f_n, m_n)$, and so we may assume that our minimizing sequence satisfies $f_n(r) \geqslant 0$ for all r,

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and the bound (6.10) holds. Note that we also have

$$\|u_n\|_2 \leqslant \|\tilde{f}_\infty - 1\|_2 + \|1 - f_n\|_2 \leqslant C + \|1 - f_n\|_2. \tag{6.11}$$

By the estimate on m_n, (6.10) and (6.11), we now have

$$\begin{aligned}
C &\geqslant \int\biggl[(f_n')^2 + \frac{d^2}{r^2}(f_n^2 - \tilde{f}_\infty^2) + \tfrac{1}{2}(1 - f_n^2)^2\biggr] r\,\mathrm{d}r \\
&= \int\biggl[(u_n')^2 + 2\tilde{f}_\infty' u_n' + (\tilde{f}_\infty')^2 + \frac{d^2}{r^2}(2\tilde{f}_\infty u_n + u_n^2) + \tfrac{1}{2}(1 - f_n^2)^2\biggr] r\,\mathrm{d}r \\
&= \int\biggl[(u_n')^2 + (\tilde{f}_\infty')^2 + \frac{d^2}{r^2}u_n^2 + \tfrac{1}{2}(1 - f_n^2)^2 + 2(1 - \tilde{f}_\infty^2)\tilde{f}_\infty u_n\biggr] r\,\mathrm{d}r \\
&\geqslant \int\biggl[(u_n')^2 + \frac{d^2}{r^2}u_n^2 + \tfrac{1}{4}u_n^2 - 8(1 - \tilde{f}_\infty^2)^2 - \tfrac{1}{8}\tilde{f}_\infty^2 u_n^2\biggr] r\,\mathrm{d}r - C \\
&\geqslant \int\biggl[(u_n')^2 + \frac{d^2}{r^2}u_n^2 + \tfrac{1}{8}u_n^2\biggr] r\,\mathrm{d}r - C.
\end{aligned}$$

In conclusion, $\|u_n\|_X \leqslant C$. We extract a subsequence for which both $u_n \rightharpoonup u_0$ and $m_n \rightharpoonup m_0$ weakly in X, H, respectively, and pointwise almost everywhere.

By semicontinuity of the norm, Fatou's lemma (for the positive terms) and the $L^2_{r,\mathrm{loc}}$ convergence of $u_n \to u_0$, we can pass to the limit in (2.3),

$$I_{\infty,g}(u_0, m_0) \leqslant \liminf_{n\to\infty} I_\infty(u_n, m_n) = \inf_{X\times H} I_\infty.$$

So the infimum of I_∞ is attained. □

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