

A STRUCTURED INVERSE SPECTRUM PROBLEM FOR INFINITE GRAPHS AND UNBOUNDED OPERATORS

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Abstract

Let G be an infinite graph on countably many vertices and let Λ be a closed, infinite set of real numbers. We establish the existence of an unbounded self-adjoint operator whose graph is G and whose spectrum is Λ .

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1. Introduction

Our main theorem states that given an infinite graph G on countably many vertices and a closed, infinite set Λ of real numbers, there is a real symmetric matrix whose graph is G and whose spectrum is Λ . More precisely, we construct an unbounded self-adjoint operator T on ℓ^2 with spectrum Λ such that the matrix of T with respect to the standard basis of ℓ^2 has the desired zero–nonzero pattern given by the graph G . Here and throughout, ℓ^2 denotes the Hilbert space of square-summable real sequences.

This inverse spectrum problem was considered in [2] under the assumption that the given spectrum Λ is compact. Since the spectrum of any bounded operator is a compact subset of the complex plane, the main result of [2] is in this sense optimal for bounded operators. Additionally, because the spectrum of any unbounded operator is a closed subset of the complex plane (see, for instance, [5, Proposition 2.6]), it is natural to ask whether one can replace the compactness assumption of Λ by closedness. This is precisely what we accomplish in this paper to settle the question of the possibility of such constructions in the most general setting of closed spectra and unbounded operators.

Throughout, all vector spaces will be over the field of real numbers making inner products $\langle v, w \rangle$ linear in both v and w .

2. Preliminaries

In this section we recall some definitions and establish a few basic results that we shall use later.

DEFINITION 2.1. Let \mathcal{H} be a Hilbert space.

- (1) A linear map $T : \mathcal{H} \rightarrow \mathcal{H}$ is called *bounded* if there exists a constant $C \geq 0$ such that $\|Tv\| \leq C\|v\|$ for all $v \in \mathcal{H}$. The smallest such C is called the *operator norm* $\|T\|_{\text{op}}$ of T .
- (2) An *unbounded* operator T on \mathcal{H} is a linear map of some dense subspace $\text{Dom}(T) \subset \mathcal{H}$ into \mathcal{H} .

According to this definition, ‘unbounded’ means ‘not necessarily bounded’, in the sense that we allow $\text{Dom}(T) = \mathcal{H}$ if T is bounded.

DEFINITION 2.2. Suppose that T is an unbounded operator on \mathcal{H} . Let $\text{Dom}(T^*)$ be the space of all $v \in \mathcal{H}$ for which the linear functional

$$w \mapsto \langle v, Tw \rangle, \quad w \in \text{Dom}(T),$$

is bounded. For $v \in \text{Dom}(T^*)$, we define T^*v to be the unique vector such that $\langle T^*v, w \rangle = \langle v, Tw \rangle$ for all $w \in \text{Dom}(T)$.

DEFINITION 2.3. An unbounded operator T on \mathcal{H} is:

- (1) *symmetric* if $\langle v, Tw \rangle = \langle Tv, w \rangle$ for all $v, w \in \text{Dom}(T)$; and, in particular,
- (2) *self-adjoint* if $\text{Dom}(T) = \text{Dom}(T^*)$ and $T^*v = Tv$ for all v in $\text{Dom}(T)$.

It is easy to check that T is symmetric if and only if T^* is an *extension* of T , that is, $\text{Dom}(T) \subset \text{Dom}(T^*)$ and $T = T^*$ on $\text{Dom}(T)$.

The following proposition, involving a ‘discrete version’ of the potential energy operator in quantum mechanics, will play a key role in proving our main result. Indeed, the spectral theorem implies that this multiplication operator is the prototype of all self-adjoint operators. See, for instance, [1, Chs. 9 and 10].

PROPOSITION 2.4. Let $\{\lambda_n\}_{n=1}^\infty$ be any sequence of real numbers. Let T be the unbounded operator on ℓ^2 with domain

$$\text{Dom}(T) = \{ \{a_n\}_{n=1}^\infty \in \ell^2 \mid \{ \lambda_n a_n \}_{n=1}^\infty \in \ell^2 \}$$

such that T maps $\{a_n\}_{n=1}^\infty \in \text{Dom}(T)$ to $\{ \lambda_n a_n \}_{n=1}^\infty$. Then T is self-adjoint.

PROOF. First, observe that $\text{Dom}(T)$ contains all finite sequences and hence it is dense in ℓ^2 . Next, since $\{\lambda_n\}_{n=1}^\infty$ is a sequence of real numbers, T is clearly symmetric and thus T^* is an extension of T . It remains to show that $\text{Dom}(T^*) \subset \text{Dom}(T)$.

Suppose that $\mathbf{b} \in \text{Dom}(T^*)$, so that

$$\mathbf{a} \mapsto \langle \mathbf{b}, T\mathbf{a} \rangle = \sum_{n=1}^\infty b_n \lambda_n a_n, \quad \mathbf{a} \in \text{Dom}(T),$$

is a bounded functional. This functional has a unique bounded extension to ℓ^2 and, therefore, by the Riesz representation theorem, it can be represented by a unique $\mathbf{c} \in \ell^2$. That is,

$$\sum_{n=1}^{\infty} b_n \lambda_n a_n = \sum_{n=1}^{\infty} c_n a_n$$

or

$$\sum_{n=1}^{\infty} (b_n \lambda_n - c_n) a_n = 0$$

for all $\mathbf{a} \in \ell^2$. This immediately implies that $b_n \lambda_n = c_n$ for all n and hence $\mathbf{b} \in \text{Dom}(T)$. Thus, $\text{Dom}(T^*) \subset \text{Dom}(T)$. \square

Let us recall the definition of the spectrum of an unbounded operator.

DEFINITION 2.5. Let T be an unbounded operator on \mathcal{H} . A number $\lambda \in \mathbb{C}$ is in the *resolvent set* of T if there exists a bounded operator S with the following properties: Sv belongs to $\text{Dom}(T)$ and $(T - \lambda I)Sv = v$ for all $v \in \mathcal{H}$, and $S(T - \lambda I)w = w$ for all $w \in \text{Dom}(T)$.

The complement in \mathbb{C} of the resolvent set of T is called the *spectrum* of T and is denoted by $\sigma(T)$.

For instance, one can easily check that the spectrum of the multiplication operator T in Proposition 2.4 is the closure of $\{\lambda_n \mid n \in \mathbb{N}\}$ as a subset of the real line.

DEFINITION 2.6. A sequence $\{T_n\}_{n=1}^{\infty}$ of unbounded operators on a Hilbert space \mathcal{H} is said to be *convergent* to an unbounded operator T if for each sufficiently large n , $T - T_n$ is bounded on $\text{Dom}(T_n) \cap \text{Dom}(T)$ and moreover $\|T - T_n\|_{\text{op}} \rightarrow 0$ as $n \rightarrow \infty$.

LEMMA 2.7. Suppose that $\{T_n\}_{n=1}^{\infty}$ is a sequence of self-adjoint operators that is convergent to an unbounded operator T on a Hilbert space \mathcal{H} . Assume that $\text{Dom}(T_n) = \mathcal{D}$ for all n , where \mathcal{D} is some dense subspace of \mathcal{H} . Then T is self-adjoint on \mathcal{D} .

PROOF. Clearly T is symmetric, because each T_n is symmetric on \mathcal{D} , and hence, for all $v, w \in \mathcal{D}$,

$$\langle w, Tv \rangle = \lim_{n \rightarrow \infty} \langle w, T_n v \rangle = \lim_{n \rightarrow \infty} \langle T_n w, v \rangle = \langle Tw, v \rangle.$$

Thus, T^* is an extension of T and $\text{Dom}(T^*) \supset \text{Dom}(T) = \mathcal{D}$.

Now let $w \in \text{Dom}(T^*)$, so that $v \mapsto \langle w, Tv \rangle$ is bounded for $v \in \mathcal{D}$. We claim that $w \in \mathcal{D}$. This is clear if $v \mapsto \langle w, T_n v \rangle$ is bounded on \mathcal{D} for some n , because in that case $w \in \text{Dom}(T_n^*) = \text{Dom}(T_n) = \mathcal{D}$. So, we assume that there exists a sequence of unit vectors $\{v_n\}_{n=1}^{\infty}$ in \mathcal{D} such that $|\langle w, T_n v_n \rangle| > n$ for each n . But then

$$\begin{aligned} |\langle w, Tv_n \rangle - \langle w, T_n v_n \rangle| &\leq |\langle w, Tv_n \rangle - \langle w, T_n v_n \rangle| \\ &\leq \|w\| \|T - T_n\|_{\text{op}}, \end{aligned}$$

by the Cauchy–Schwarz inequality. The right-hand side of the second inequality above tends to 0 as n goes to ∞ , implying that $|\langle w, Tv_n \rangle| \rightarrow \infty$. This is absurd; hence, $w \in \mathcal{D}$. Therefore, $\text{Dom}(T^*) \subset \mathcal{D}$, which finishes the proof that T is self-adjoint on \mathcal{D} . \square

Finally, to finish this section, we record a lemma whose easy proof we omit.

LEMMA 2.8. *Let \mathcal{H} be a Hilbert space with an orthonormal basis \mathfrak{B} . Suppose that \mathfrak{B}_1 and \mathfrak{B}_2 are two subsets of \mathfrak{B} that partition \mathfrak{B} and denote the Hilbert spaces generated by them by \mathcal{H}_1 and \mathcal{H}_2 , respectively. If T_i is an unbounded self-adjoint operator on \mathcal{H}_i , for $i = 1, 2$, then the operator T defined by $T_1 \oplus T_2$ is an unbounded self-adjoint operator on \mathcal{H} with $\text{Dom}(T) = \text{Dom}(T_1) \oplus \text{Dom}(T_2)$.*

3. Main theorem

In preparation for our main result, we introduce the notion of the graph of a symmetric matrix (or a self-adjoint operator).

DEFINITION 3.1. Let G be a (finite or infinite) graph whose vertices are indexed by $1, 2, \dots$. We say that G is the graph of a real symmetric matrix $A = [a_{ij}]$ if for any $i \neq j$, we have $a_{ij} \neq 0$ precisely when the vertices i and j are adjacent in G .

We say that G is the graph of a self-adjoint operator T on ℓ^2 if G is the graph of the standard matrix of T .

We mention in passing that the spectrum of a (locally finite) countable graph was defined in the 1982 paper of Mohar [4] and some of its basic properties were established.

EXAMPLE 3.2. Consider the graph P_3 , namely a path on three vertices, and a set of distinct real numbers $\{\lambda_1, \lambda_2, \lambda_3\}$. We shall construct a symmetric matrix M with the zero–nonzero pattern of P_3 and the spectrum $\sigma(M) = \{\lambda_1, \lambda_2, \lambda_3\}$. Let

$$M(x_1, x_2, x_3, y) = \begin{bmatrix} x_1 & y & 0 \\ y & x_2 & y \\ 0 & y & x_3 \end{bmatrix}.$$

We show the existence of (infinitely many) real numbers x_i and $y \neq 0$ such that

$$\sigma(M(x_1, x_2, x_3, y)) = \{\lambda_1, \lambda_2, \lambda_3\}.$$

Identifying the set of matrices $M = M(x_1, x_2, x_3, y)$ with $\mathbb{R}^3 \times \mathbb{R}$, we define

$$f: \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3$$

$$f(x_1, x_2, x_3, y) = (\text{Tr } M, \text{Tr } M^2, \text{Tr } M^3).$$

Note that the equation

$$F(x_1, x_2, x_3, y) := f(x_1, x_2, x_3, y) - \left(\sum \lambda_i, \sum \lambda_i^2, \sum \lambda_i^3 \right) = 0$$

has the trivial solution $(\lambda_1, \lambda_2, \lambda_3, 0)$ given by a diagonal matrix. We use the implicit function theorem to show that the x_i can be expressed as functions of y in a neighbourhood of 0 such that $F(x_1(y), x_2(y), x_3(y), y) = 0$. A short calculation of the Jacobian of F with respect to the x_i shows that

$$J(f)|_{x_i=\lambda_i, y=0} = \begin{bmatrix} 1 & 1 & 1 \\ 2\lambda_1 & 2\lambda_2 & 2\lambda_3 \\ 3\lambda_1^2 & 3\lambda_2^2 & 3\lambda_3^2 \end{bmatrix},$$

which is invertible because the λ_i are distinct. Thus, the implicit function theorem can be applied to obtain the desired perturbation of the diagonal matrix with nonzero y . Finally, since $\{\sum \lambda_i, \sum \lambda_i^2, \sum \lambda_i^3\}$ uniquely determines $\{\lambda_1, \lambda_2, \lambda_3\}$ via Newton’s identities, we have proved the existence of matrices with the zero–nonzero pattern of P_3 and a given spectrum.

The solution of the example above illustrates the construction used in [2] to prove the following theorem.

THEOREM 3.3 [2, Theorem 3.2]. *Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of distinct real numbers and suppose that $\{G_n\}_{n=1}^\infty$ is a sequence such that G_n is a graph on n vertices and also a subgraph of G_{n+1} for each $n \in \mathbb{N}$. Then, for any sequence of positive numbers $\{\varepsilon_n\}_{n=1}^\infty$, we can find a sequence of symmetric matrices $\{A_n\}_{n=1}^\infty$ such that for any $n \in \mathbb{N}$:*

- (i) A_n has graph G_n and spectrum $\{\lambda_1, \dots, \lambda_n\}$;
- (ii) $\|A_n \oplus [\lambda_{n+1}] - A_{n+1}\|_{\text{op}} < \varepsilon_n$; and
- (iii) A_{n+1} is obtained by perturbing the diagonal and the last row and column of $A_n \oplus [\lambda_{n+1}]$.

Now we proceed to the discussion of our inverse spectrum problem in the context of infinite graphs and unbounded operators. The next theorem is used in connection with the spectrum of perturbations of self-adjoint operators. First, we need a definition for the distance between two subsets of a metric space.

DEFINITION 3.4. Let A and B be two nonempty subsets of a metric space (X, ρ) . The Hausdorff distance between A and B , denoted $d(A, B)$, is defined by

$$d(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \rho(a, b), \sup_{b \in B} \inf_{a \in A} \rho(a, b) \right\}.$$

THEOREM 3.5. *Let T and A denote a self-adjoint operator and a bounded symmetric operator on a Hilbert space \mathcal{H} , respectively. Then $S = T + A$ is self-adjoint and the Hausdorff distance between the spectra of S and T , namely $d(\sigma(S), \sigma(T))$, satisfies*

$$d(\sigma(S), \sigma(T)) \leq \|A\|_{\text{op}}.$$

This theorem, whose proof can be found in [3, Theorem 4.10], implies the following corollaries that we shall refer to later on.

COROLLARY 3.6. *Let $\{T_n\}_{n=1}^\infty$ be a sequence of unbounded self-adjoint operators on a Hilbert space \mathcal{H} . Assume that $\{T_n\}_{n=1}^\infty$ converges to a self-adjoint operator T and that $\text{Dom}(T) \cap \text{Dom}(T_n)$ is dense in \mathcal{H} for all n . Then, for any $\lambda \in \sigma(T)$ and any neighbourhood U of λ , there exists an $N \in \mathbb{N}$ such that U intersects $\sigma(T_n)$ nontrivially for all $n > N$.*

PROOF. Since $\text{Dom}(T - T_n) = \text{Dom}(T) \cap \text{Dom}(T_n)$, the density of the right-hand side in \mathcal{H} guarantees that the difference $T - T_n$ of self-adjoint operators is symmetric. Also, $\|T - T_n\|_{\text{op}} \rightarrow 0$ implies that, for sufficiently large n , $T - T_n$ is bounded on $\text{Dom}(T) \cap \text{Dom}(T_n)$ and hence it can be extended to a bounded symmetric operator on \mathcal{H} . By definition of the Hausdorff distance,

$$d(\sigma(T - T_n), \{\lambda\}) \leq d(\sigma(T - T_n), \sigma(T))$$

for $\lambda \in \sigma(T)$. Now the corollary follows from Theorem 3.5. □

If $\{T_n\}_{n=1}^\infty$ is a sequence of noninvertible bounded operators on a Hilbert space and $\{T_n\}_{n=1}^\infty$ converges to an operator T , then T is also noninvertible. This is a well-known consequence of the openness of the invertibility condition in unital Banach algebras. Instead of explicitly referring to noninvertibility of T_n , one can equivalently assume that 0 belongs to $\sigma(T_n)$. This formulation has the advantage of making sense in more general contexts such as the next corollary.

COROLLARY 3.7. *Suppose that $\{T_n\}_{n=1}^\infty$ is a sequence of self-adjoint operators on a Hilbert space \mathcal{H} with $\text{Dom}(T_n) = \mathcal{D}$, for $n = 1, 2, \dots$, where \mathcal{D} is a dense subspace of \mathcal{H} . If $\{T_n\}_{n=1}^\infty$ is convergent to an operator T and $\lambda \in \sigma(T_n)$ for all n , then $\lambda \in \sigma(T)$.*

PROOF. Observe that T is a self-adjoint operator on \mathcal{D} by Lemma 2.7. Thus, $T - T_n$ is bounded and symmetric on \mathcal{D} for each n . If $\lambda \notin \sigma(T)$, then, since $\sigma(T)$ is a closed subset of \mathbb{R} , there exists an open subset U of \mathbb{R} containing λ that is disjoint from $\sigma(T)$; hence, $0 < d(\{\lambda\}, \sigma(T))$. This, together with Theorem 3.5, implies that, for each n ,

$$0 < d(\sigma(T_n), \sigma(T)) \leq \|T - T_n\|_{\text{op}},$$

which is in contradiction with the assumption that $\{T_n\}_{n=1}^\infty$ is convergent to T . Therefore, λ must be in $\sigma(T)$. □

Consider the standard Hilbert basis $\mathfrak{B} = \{e_1, e_2, \dots\}$ of ℓ^2 , where e_k is the sequence whose k th term is 1 and whose other entries are all 0. The *standard matrix* of a linear operator T on ℓ^2 is the infinite matrix whose k th column is $[Te_k]_{\mathfrak{B}}$, consisting of the terms of the sequence Te_k . Now we are ready to state and prove our main theorem. This is done by taking the limit, in a suitable sense, of the matrices that are constructed in Theorem 3.3.

THEOREM 3.8. *Given an infinite graph G on countably many vertices and a closed, infinite set Λ of real numbers, there exists an unbounded self-adjoint operator T on the Hilbert space ℓ^2 such that:*

- (i) *the (approximate point) spectrum of T equals Λ ; and*
- (ii) *the (real symmetric) standard matrix of T has graph G .*

PROOF. Let $\{\lambda_1, \lambda_2, \dots\}$ denote a countable dense subset of Λ . Suppose that the vertices of G are labelled by the numbers in \mathbb{N} and, for each $n \in \mathbb{N}$, let G_n be the induced subgraph of G on the first n vertices. By Theorem 3.3, for any $\varepsilon > 0$, we can find matrices $\{A_n\}_{n=1}^\infty$ such that A_n has graph G_n and spectrum $\{\lambda_1, \dots, \lambda_n\}$ and, moreover,

$$\|A_n \oplus [\lambda_{n+1}] - A_{n+1}\|_{\text{op}} < \frac{\varepsilon}{2^n}. \tag{3.1}$$

For each n define the unbounded linear operator T_n on the Hilbert space of square-summable sequences ℓ^2 with domain

$$\mathcal{D} = \{\{a_n\}_{n=1}^\infty \in \ell^2 \mid \{\lambda_n a_n\}_{n=1}^\infty \in \ell^2\}$$

such that the matrix representation of T_n with respect to the standard Hilbert basis $\mathfrak{B} = \{e_1, e_2, \dots\}$ of ℓ^2 is

$$M_n = A_n \oplus \text{diag}(\lambda_{n+1}, \lambda_{n+2}, \dots).$$

(Note that the definition of $\text{Dom}(T_n)$ does not depend on the value of n .) Proposition 2.4 and Lemma 2.8 imply that T_n is self-adjoint. It follows from (3.1) that, for any i in \mathbb{N} ,

$$\|M_n e_i - M_{n+1} e_i\|_2 < \frac{\varepsilon}{2^n}.$$

Thus, the sequence of partial sums $\{\sum_{k=1}^{n-1} (M_{k+1} e_i - M_k e_i)\}_{n=2}^\infty$ is absolutely convergent and therefore the sequence $\{M_n e_i\}_{n=1}^\infty$ satisfying

$$M_n e_i = M_1 e_i + \sum_{k=1}^{n-1} (M_{k+1} e_i - M_k e_i)$$

is convergent in ℓ^2 . Let M denote the matrix whose columns are obtained by this limiting process, that is, M is the matrix such that $M e_i = \lim_{n \rightarrow \infty} M_n e_i$ for each $i \in \mathbb{N}$. Note that for each $n = 1, 2, \dots$, the graph of A_n is the induced subgraph of G on the first n vertices. Thus, by construction, G is the graph of M ; this uses the fact that nonzero entries of M_n remain nonzero in the limit thanks to point (iii) of Theorem 3.3.

Our next objective is showing that M is indeed the standard matrix of an unbounded linear operator T on ℓ^2 . Observe that $T_m - T_n$ is a bounded operator on \mathcal{D} , by construction of T_m and T_n . Therefore, $T_m - T_n$ has a unique bounded extension

to $\mathcal{B}(\ell^2)$. We shall denote this extension by $T_m - T_n$ as well. Then

$$\begin{aligned} \|T_n - T_{n+1}\|_{\text{op}} &= \sup_{\|v\|_2=1} \|T_n v - T_{n+1} v\|_2 \\ &= \sup_{\|v\|_2=1} \left\| \begin{bmatrix} \begin{bmatrix} A_n & \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\ \hline \lambda_{n+1} v_{n+1} \end{bmatrix} - A_{n+1} \begin{bmatrix} v_1 \\ \vdots \\ v_{n+1} \end{bmatrix} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\|_2 \\ &= \sup_{\|v\|_2=1} \left\| \left(\begin{bmatrix} A_n & | & \hline \hline & | & \lambda_{n+1} \end{bmatrix} - A_{n+1} \right) \begin{bmatrix} v_1 \\ \vdots \\ v_{n+1} \end{bmatrix} \right\|_2 \\ &< \frac{\varepsilon}{2^n}, \end{aligned}$$

where the inequality in the last line is due to the submultiplicative property of the operator norm together with (3.1). This inequality immediately implies that the sequence of partial sums $\{\sum_{k=1}^{n-1} (T_{k+1} - T_k)\}_{n=2}^\infty$ is absolutely convergent in the Banach space of bounded operators $\mathcal{B}(\ell^2)$ and hence the sequence $\{T_n\}_{n=1}^\infty$ satisfying

$$T_n = T_1 + \sum_{k=1}^{n-1} (T_{k+1} - T_k)$$

is convergent to an unbounded operator T . This means that we can define T by

$$T = T_1 + \lim_{n \rightarrow \infty} \sum_{k=1}^{n-1} (T_{k+1} - T_k),$$

which is the sum of the unbounded self-adjoint operator T_1 and a bounded self-adjoint operator on ℓ^2 . Therefore, T is self-adjoint with domain given by $\mathcal{D} = \text{Dom}(T_1)$. Since, for each $i \in \mathbb{N}$, we have $T e_i = \lim_{n \rightarrow \infty} T_n e_i$ and $T_n e_i = M_n e_i$, we conclude that $T e_i = M e_i$ and thus M is the standard matrix of T .

It remains to prove that $\sigma(T) = \Lambda$. First, we claim that each $\lambda_i \in \{\lambda_1, \lambda_2, \dots\} \subset \Lambda$ is in the spectrum of T . To see this, note that T_n was defined so that $\{\lambda_1, \lambda_2, \dots\} \subset \sigma(T_n)$ for each n . Hence, by Corollary 3.7, we have $\{\lambda_1, \lambda_2, \dots\} \subset \sigma(T)$, as claimed. By taking closures, this inclusion implies that $\Lambda \subset \sigma(T)$, because $\{\lambda_1, \lambda_2, \dots\}$ is dense in Λ and $\sigma(T)$ is closed in \mathbb{R} .

Next, since the sequence $\{T_n\}_{n=1}^\infty$ is convergent to T and $\sigma(T_n) = \Lambda$ for all n , by Corollary 3.6, we conclude that for any $\lambda \in \sigma(T)$, every neighbourhood of λ intersects the closed set Λ . Hence, the reverse inclusion $\sigma(T) \subset \Lambda$ is also established.

Finally, to complete the proof of point (i) in the statement of the theorem, note that the spectrum of any self-adjoint operator equals its approximate point spectrum and, as shown above, T is self-adjoint. □

REMARK 3.9. At the end of the proof of Theorem 3.8, it is possible to give a direct argument to show that $\sigma(T) = \Lambda$. Indeed, by construction, the operators T_m are isospectral with $\sigma(T_m) = \Lambda$ for all $m \in \mathbb{N}$,

$$T = T_m + \lim_{n \rightarrow \infty} \sum_{k=m}^{n-1} (T_{k+1} - T_k), \quad (3.2)$$

and

$$\left\| \lim_{n \rightarrow \infty} \sum_{k=m}^{n-1} (T_{k+1} - T_k) \right\|_{\text{op}} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Thus, Theorem 3.5 applied to (3.2) implies that $d(\sigma(T), \Lambda) = 0$ and hence $\sigma(T) = \Lambda$, since both sets are closed in \mathbb{R} .

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