Alternative Proof of a Theorem in Change of Axes.

By Professor JACK.

If by any change of axes $ax^2 + 2hxy + by^2$ be changed into $a'x'^2 + 2h'x'y' + b'y'^2$, then will

$$\frac{a + b - 2h\cos\omega}{\sin^2\omega} = \frac{a' + b' - 2h'\cos\omega'}{\sin^2\omega'},$$

$$\text{and} \quad \frac{ab - h^2}{\sin^2\omega} = \frac{a'b' - h'^2}{\sin^2\omega'}.$$

FIGURE 3.

Let OX, OY be first axes ; OX′, OY′ second axes ;

$\angle XOY = \omega$, $\qquad \angle X'OY' = \omega'$.

Take OA a fixed line in the plane.

Let $\angle AOP = \theta$.

Let $\angle AOX = \alpha$, $\qquad \angle AOX' = \alpha'$.

$\therefore$ $\angle MOP = \theta - \alpha$, $\qquad \angle M'OP = \theta - \alpha'$;

and $\angle OPM = \omega - (\theta - \alpha) = \omega - \theta + \alpha$, $\qquad \angle OPM' = \omega' - \theta + \alpha'$.

Let $OP = r$ (same in both cases).

$$\text{Now} \quad \frac{OP}{\sin\omega} = \frac{OM}{\sin(\omega - \theta + \alpha)} = \frac{MP}{\sin(\theta - \alpha)};$$

$$\therefore \quad x = r\frac{\sin(\omega - \theta + \alpha)}{\sin\omega} \quad \text{and} \quad y = r\frac{\sin(\theta - \alpha)}{\sin\omega}.$$

$$\text{Similarly} \quad x' = r\frac{\sin(\omega' - \theta + \alpha')}{\sin\omega'} \quad \text{and} \quad y' = r\frac{\sin(\theta - \alpha')}{\sin\omega'}.$$

Substitute these values of xy, $x'y'$ in $ax^2 + 2hxy + by^2$ and $a'x'^2 + 2h'x'y' + b'y'^2$ respectively ; then

$$\frac{r^2}{\sin^2\omega}\left\{a\sin^2\overline{\omega - \theta + \alpha} + 2h\sin\overline{\omega - \theta + \alpha}\sin\overline{\theta - \alpha} + b\sin^2\overline{\theta - \alpha}\right\}$$

$$= \frac{r^2}{\sin^2\omega'}\left\{a'\sin^2\overline{\omega' - \theta + \alpha'} + 2h'\sin\overline{\omega' - \theta + \alpha'}\sin\overline{\theta - \alpha'} + b'\sin^2\overline{\theta - \alpha'}\right\}$$

Multiply by $\frac{2}{r^2}$ and simplify (by trigonometry),

$$\therefore \frac{1}{\sin^2\omega}\left\{a(1 - \cos\overline{2\omega - 2\theta + 2\alpha}) + 2h(\cos\overline{\omega - 2\theta + 2\alpha} - \cos\omega) + b(1 - \cos\overline{2\theta - 2\alpha})\right\}$$

$$= \frac{1}{\sin^2\omega'}\left\{a'(1 - \cos\overline{2\omega' - 2\theta + 2\alpha'}) + 2h'(\cos\overline{\omega' - 2\theta + 2\alpha'} - \cos\omega') + b'(1 - \cos\overline{2\theta - 2\alpha'})\right\}.$$

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Expand and arrange, grouping the coefficients of $\cos 2\theta$, $\sin 2\theta$;

$$\therefore \quad \frac{1}{\sin^2\omega}\left\{\begin{array}{l} a - 2h\cos\omega + b \\ -\cos 2\theta(a\cos\overline{2\omega+2\alpha} - 2h\cos\overline{\omega+2\alpha} + b\cos 2\alpha) \\ -\sin 2\theta(a\sin\overline{2\omega+2\alpha} - 2h\sin\overline{\omega+2\alpha} + b\sin 2\alpha) \end{array}\right\}$$

$$= \frac{1}{\sin^2\omega'}\left\{\begin{array}{l} a' - 2h'\cos\omega' + b' \\ -\cos 2\theta(a'\cos\overline{2\omega'+2\alpha'} - 2h'\cos\overline{\omega'+2\alpha'} + b'\cos 2\alpha') \\ -\sin 2\theta(a'\sin\overline{2\omega'+2\alpha'} - 2h'\sin\overline{\omega'+2\alpha'} + b'\sin 2\alpha') \end{array}\right\}.$$

Now if

$$l + m\cos\phi + n\sin\phi = l' + m'\cos\phi + n'\sin\phi$$

holds for all values of ϕ then must

$$l = l',$$

$$m = m',$$

$$n = n' \quad \text{and} \quad m^2 + n^2 = m'^2 + n'^2.$$

Hence from above

$$\frac{a - 2h\cos\omega + b}{\sin^2\omega} = \frac{a' - 2h'\cos\omega' + b'}{\sin^2\omega'}; \qquad (1)$$

and

$$\frac{1}{\sin^4\omega}\left\{\begin{array}{l} (a\cos\overline{2\omega+2\alpha} - 2h\cos\overline{\omega+2\alpha} + b\cos 2\alpha)^2 \\ +(a\sin\overline{2\omega+2\alpha} - 2h\sin\overline{\omega+2\alpha} + b\sin 2\alpha)^2 \end{array}\right\} = \frac{1}{\sin^4\omega'}\left\{\begin{array}{l} \text{a similar} \\ \text{expression} \end{array}\right\};$$

$$\therefore \quad \frac{1}{\sin^4\omega}\left\{\begin{array}{l} a^2 + 4h^2 + b^2 \\ -4h\cos\omega(a+b) + 2ab\cos 2\omega \end{array}\right\} = \frac{1}{\sin^4\omega'}\left\{\begin{array}{l} \text{a similar} \\ \text{expression} \end{array}\right\};$$

$$\therefore \quad \frac{1}{\sin^4\omega}\left\{\begin{array}{l} (a - 2h\cos\omega + b)^2 \\ +4(h^2 - ab)\sin^2\omega \end{array}\right\} = \frac{1}{\sin^4\omega'}\left\{\begin{array}{l} (a' - 2h'\cos\omega' + b')^2 \\ +4(h'^2 - a'b')\sin^2\omega' \end{array}\right\}.$$

Take from both sides the squares of the equals (1)

$$i.e., \quad \left(\frac{a - 2h\cos\omega + b}{\sin^2\omega}\right)^2 = \left(\frac{a' - 2h'\cos\omega' + b'}{\sin^2\omega'}\right)^2, \qquad (1)$$

$$\therefore \quad \frac{4(h^2 - ab)\sin^2\omega}{\sin^4\omega} = \frac{4(h'^2 - a'b')\sin^2\omega'}{\sin^4\omega'};$$

$$\therefore \quad \frac{h^2 - ab}{\sin^2\omega} = \frac{h'^2 - a'b'}{\sin^2\omega'}. \qquad (2)$$

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