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Upper bounds on measure-theoretic tail entropy for dominated splittings

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Abstract. For differentiable dynamical systems with dominated splittings, we give upper estimates on the measure-theoretic tail entropy in terms of Lyapunov exponents. As our primary application, we verify the upper semi-continuity of metric entropy in various settings with domination.

Key words: measure-theoretic tail entropy, dominated splitting, upper semi-continuity
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1. *Introduction*

Let f be a homeomorphism on a compact metric space M. For $K \subset M$, $n \in \mathbb{N}$ and any observable scale $\varepsilon > 0$, a subset $K_1 \subset K$ is called (n, ε)-spanning for K if for any $x \in K$ there exists $y \in K_1$ such that $d(f^i(x), f^i(y)) \leq \varepsilon$ for all $i \in [0, n)$. Let $r_n(f, K, \varepsilon)$ denote the smallest cardinality of any (n, ε)-spanning set of K. The ε-topological entropy of K is defined by

$$h(f, K, \varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log r_n(f, K, \varepsilon).$$

The topological entropy of f on K is defined by

$$h(f, K) = \lim_{\varepsilon\to 0} h(f, K, \varepsilon).$$

For $x \in M$, $n \in \mathbb{N}$ and $\varepsilon > 0$, let

$$B_n(f, x, \varepsilon) = \{y \in M : d(f^i(x), f^i(y)) < \varepsilon, |i| < n\},$$
$$B_\infty(f, x, \varepsilon) = \bigcap_{n\in\mathbb{N}} B_n(f, x, \varepsilon);$$

then the ε-tail entropy at x is defined by

$$h^*(f, x, \varepsilon) = h(f, B_\infty(f, x, \varepsilon)).$$

Tail entropy has been studied broadly since the pioneering works of Bowen [**5**] and Misiurewicz [**16**] in the 1970s, because of its fundamental role in the estimates of entropy in both the topological and measure-theoretic sense.

Given a compact f-invariant set $\Lambda \subset M$ and $\varepsilon > 0$, denote

$$h^*(f, \Lambda, \varepsilon) = \sup_{x\in\Lambda} h^*(f, x, \varepsilon).$$

We say that f on Λ is entropy expansive [**5**] if there exists $\delta > 0$ such that $h^*(f, \Lambda, \delta) = 0$ and asymptotically entropy expansive [**16**] if $\lim_{\delta\to 0} h^*(f, \Lambda, \delta) = 0$. Both of these properties imply the upper semi-continuity of metric entropy.

Denote by $\mathcal{M}_{\mathrm{inv}}(f, \Lambda)$ and $\mathcal{M}_{\mathrm{erg}}(f, \Lambda)$ the sets of f-invariant and ergodic f-invariant Borel probability measures on a compact f-invariant set $\Lambda \subset M$, respectively. Consider $\mu \in \mathcal{M}_{\mathrm{erg}}(f, \Lambda)$; then $h^*(f, x, \varepsilon)$ is a constant for μ-almost every (a.e.) x [**9**, Proposition 2.8], which we denote by $h^*(f, \mu, \varepsilon)$. In general, when $\mu \in \mathcal{M}_{\mathrm{inv}}(f, \Lambda)$, denoting its ergodic decomposition by $\mu = \int_{\mathcal{M}_{\mathrm{erg}}(f,\Lambda)} d\tau(m)$, we define the measure-theoretic tail entropy of μ as

$$h^*(f, \mu, \varepsilon) = \int_{\mathcal{M}_{\mathrm{erg}}(f,\Lambda)} h^*(f, m, \varepsilon)\, d\tau(m).$$

By the tail variational principle [**6**, **10**], one has

$$\lim_{\varepsilon\to 0} \sup_{\mu\in\mathcal{M}_{\mathrm{inv}}(f,\Lambda)} h^*(f, \mu, \varepsilon) = \lim_{\varepsilon\to 0} h^*(f, \Lambda, \varepsilon).$$

However, it is unknown if $\sup_{\mu\in\mathcal{M}_{\mathrm{inv}}(f,\Lambda)} h^*(f, \mu, \varepsilon) = h^*(f, \Lambda, \varepsilon)$ for any $\varepsilon > 0$.

Tail entropy measures the local dynamical complexity in the process of observation with respect to the evolutions of dynamical systems. It is known that uniformly hyperbolic systems are entropy expansive and so are all diffeomorphisms away from tangencies [**13**]. As a more general concept, dominated splitting exhibiting uniformly hyperbolic behavior on projective bundles is admitted by plenty of systems beyond uniform hyperbolic systems [**2–4**, **14**, **20**]. In the present paper, we attempt to study tail entropy in the setting of dominated splitting.

Let Diff(M) be the space of C^1 diffeomorphisms on a compact boundaryless Riemannian manifold M. For $f \in \mathrm{Diff}(M)$, a splitting $T_\Lambda M = E_1 \oplus_< \cdots \oplus_< E_\ell$ over a compact f-invariant set $\Lambda \subset M$ is said to be dominated if there exists $L \in \mathbb{N}$ such that for any $x \in \Lambda$, $v \in E_i(x)$, $w \in E_j(x)$ with $\|v\| = \|w\| = 1$ and $1 \le i < j \le \ell$,

$$\|D_x f^L v\| \le \tfrac{1}{2}\|D_x f^L w\|.$$

Taking an adapted metric [**17**], we may assume that $L = 1$ in the following discussion for dominated splittings.

Recall that the geometric divergent rate of any $x \in M$ relative to a direction $v \in T_x M$ is given by the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n v\|,$$

which exists and is called the Lyapunov exponent along v for almost every point x of every f-invariant measure by Oseledec's theorem [**18**]. For a dominated splitting $T_\Lambda M = E_1 \oplus_< \cdots \oplus_< E_\ell$ over Λ, for the purpose of studying the approximation process of Lyapunov exponents with respect to the evolution time N, we define, for any $1 \le i \le \ell$,

$$\Delta_f^\pm(x, E_i; N) = \lim_{n\to\pm\infty} \frac{1}{|nN|} \sum_{k=0}^{n-1} \log^+ \|(D_{f^{kN}(x)} f^{\pm N}|_{E_i})^\wedge\|,$$

$$\Delta_f(x, E_i; N) = \min\{\Delta_f^+(x, E_i, N), \Delta_f^-(x, E_i, N)\},$$

$$\Delta_f(x; N) = \min\{\Delta_f(x, E_i, N) : 1 \le i \le \ell\},$$

where $\log^+ t = \max\{0, \log t\}$ and, for a linear transformation $T : X_1 \to X_2$ between two finite-dimensional linear spaces X_1 and X_2, $T^\wedge$ denotes the map on the exterior algebra of X_1 (in this manner, $\|T^\wedge\|$ is the maximum of the absolute values of Jacobians of T on any linear subspace of X_1). Denote $\Delta_f^+(x, E_i)$ ($\Delta_f^-(x, E_i)$) as the sum of positive Lyapunov exponents on E_i (the sum of the absolute values of negative Lyapunov exponents on E_i); then by Oseledec's theorem [**18**] one could get that for μ-a.e. x of every $\mu \in \mathcal{M}_{\mathrm{inv}}(f, \Lambda)$,

$$\Delta_f(x, E_i; N) \to \Delta_f(x, E_i) := \min\{\Delta_f^+(x, E_i), \Delta_f^-(x, E_i)\} \quad \text{as } N \to +\infty,$$

$$\Delta_f(x; N) \to \Delta_f(x) := \min\{\Delta_f(x, E_i) : 1 \le i \le \ell\} \quad \text{as } N \to +\infty.$$

For $\mu \in \mathcal{M}_{\mathrm{inv}}(f, \Lambda)$, let

$$\Delta_f(\mu, E_i) = \int \Delta_f(x, E_i)\, d\mu(x),$$

$$\Delta_f(\mu; N) = \int \Delta_f(x; N)\, d\mu(x),$$

$$\Delta_f(\mu) = \int \Delta_f(x)\, d\mu(x).$$

By analyzing the approximation process of Lyapunov exponents, we can get the estimates concerning the relationship between the scale of measure-theoretic tail entropy and the evolution time.

THEOREM 1.1. *Let $f \in \mathrm{Diff}(M)$ and $T_\Lambda M = E_1 \oplus_< \cdots \oplus_< E_\ell$ be a dominated splitting over a compact f-invariant set Λ, then there exists a sequence $\{\varepsilon_N\}_{N\in\mathbb{N}}$ of positive numbers with $\lim_{N\to+\infty} \varepsilon_N = 0$ such that*

$$\lim_{N\to+\infty} \sup_{\mu\in\mathcal{M}_{\mathrm{inv}}(f,\Lambda)} (h^*(f, \mu, \varepsilon_N) - \Delta_f(\mu; N)) \le 0.$$

In particular, we have

$$\lim_{\varepsilon\to 0} h^*(f, \mu, \varepsilon) \le \Delta_f(\mu)$$

for any $\mu \in \mathcal{M}_{\mathrm{inv}}(f, \Lambda)$.

Remark. The tail entropy was studied with respect to a dominated splitting over the manifold M by Buzzi, Crovisier and Fisher [**8**, Theorem 7.6]. In Theorem 1.1, we focus on the uniform difference between the tail entropy of measures and the Lyapunov exponents of themselves relative to the evolution time.

In order to use the measure-theoretic tail entropy to estimate the difference between the full metric entropy $h_\mu(f)$ and the metric entropy $h_\mu(f, \mathcal{P})$ with respect to some partition $\mathcal{P}$, we further establish the following theorem, which is a strengthening version of [**12**, Proposition 2.1] for the use of infinite Bowen balls in the definition of tail entropy here.

THEOREM 1.2. *Let M be a compact metric space and $f : M \to M$ a homeomorphism with finite topological entropy. For any $\mu \in \mathcal{M}_{\mathrm{inv}}(f, M)$, we have*

$$h_\mu(f) - h_\mu(f, \mathcal{P}) \le h^*(f, \mu, \rho)$$

for any finite measurable partition $\mathcal{P}$ with $\mathrm{diam}(\mathcal{P}) \le \rho$.

In what follows, applying Theorems 1.1 and 1.2, we may deduce the upper semi-continuity property of metric entropy in case that $\Delta_f(\mu) = 0$.

COROLLARY 1.3. *Let $f \in \mathrm{Diff}(M)$ and $T_\Lambda M = E_1 \oplus_< \cdots \oplus_< E_\ell$ be a dominated splitting over a compact f-invariant set Λ. Then the metric entropy map in $\mathcal{M}_{\mathrm{inv}}(f, \Lambda)$ is upper semi-continuous at any μ with $\Delta_f(\mu) = 0$.*

In fact, Corollary 1.3 could give rise to the upper semi-continuity of metric entropy for plenty of systems with domination.

Combining with [**15**] and [**1**, Theorem 3.3], for a C^1 generic $f \in \mathrm{Diff}(M)$, a generic element μ in $\mathcal{M}_{\mathrm{erg}}(f, M)$ admits dominated Oseledec splittings, so the corresponding $\Delta_f(\mu) = 0$, which implies, by Corollary 1.3, the upper semi-continuity of metric entropy at μ in $\mathcal{M}_{\mathrm{inv}}(f, \mathrm{supp}(\mu))$, where $\mathrm{supp}(\mu)$ is the support of μ. Moreover, given a homoclinic class H, if we denote by $\mathcal{M}_{\mathrm{per}}(H)$ the closure of the convex hull of periodic measures supported on H, then, by [**1**, Theorem 3.1'], $\mathrm{supp}(\mu) = H$ and $h_\mu(f) = 0$ for generic $\mu \in \mathcal{M}_{\mathrm{per}}(H)$; thus, we can obtain the following result.

COROLLARY 1.4. *For a C^1 generic f in* $\mathrm{Diff}(M)$ *and any homoclinic class H of f, the set of continuity points of metric entropy in $\mathcal{M}_{\mathrm{inv}}(f, H)$ includes a residual subset of $\mathcal{M}_{\mathrm{per}}(H)$.*

In the setting of conservative systems, for a C^1 generic f in $\mathrm{Diff}_{\mathrm{vol}}(M)$, which denotes the space of C^1 diffeomorphisms on M preserving the volume measure vol, by [**2**, **3**], the Oseledec splitting of vol is dominated. Thus, we have the following result.

COROLLARY 1.5. *For a C^1 generic f in* $\mathrm{Diff}_{\mathrm{vol}}(M)$*, the volume measure* vol *is an upper semi-continuity point of metric entropy in $\mathcal{M}_{\mathrm{inv}}(f, M)$.*

Besides, by Corollary 1.3, we may also get an alternative criterion for the upper semi-continuity of metric entropy for dominated splittings consisting of bundles without mixed behavior or of one dimension in [**9**, **13**, **22**], since $\Delta_f(\mu) = 0$ is satisfied in those contexts.

2. *Dynamics of foliations*

Let $f \in \mathrm{Diff}(M)$, Λ be a compact f-invariant set and there exists a dominated splitting $T_\Lambda M = E_1 \oplus_< \cdots \oplus_< E_\ell$ over Λ. For $1 \le i \le j \le \ell$, denote

$$E_{i(i+1)\cdots j} = E_i \oplus \cdots \oplus E_j.$$

Let ξ_0 be a positive lower bound for the angles between any pair of bundles E_i and E_j, $1 \le i \ne j \le \ell$. By [**7**, **11**], with respect to the given domination structure, one may have a family of local invariant fake foliations. In the following content, given a foliation $\mathcal{F}$ and a point y in the domain, we denote by $\mathcal{F}(y)$ the leaf through y and by $\mathcal{F}(y, \rho)$ the neighborhood of radius ρ around y inside the leaf.

PROPOSITION 2.1. *For any $\xi \in (0, \xi_0/4)$, there exist $0 < \rho_2 < \rho_1$ such that the neighborhood $B(x, \rho_1)$ of every $x \in \Lambda$ admits foliations*

$$\{\mathcal{F}_x^* : x \in \Lambda\}, \quad * \in \{i(i+1)\cdots j : 1 \le i \le j \le \ell\}$$

such that for any $y \in B(x, \rho_1)$ and $* \in \{i(i+1)\cdots j : 1 \le i \le j \le \ell\}$:
(i) *almost tangency: $T_y\mathcal{F}_x^*(y)$ lies in a cone of width ξ of $E_*(x)$;*
(ii) *local invariance: $f^{\pm}\mathcal{F}_x^*(y, \rho_2) \subset \mathcal{F}^*_{f^{\pm}(x)}(f^{\pm}(y))$;*
(iii) *coherence: $\mathcal{F}_x^*$ is subfoliated by $\mathcal{F}_x^{\#}$ whenever # is a subsentence of $*$.*

Along the leaves of foliations $\mathcal{F}_x^*$, we could define the projections as follows: for $y \in B(x, \rho_1)$, $1 \le i \le \ell - 1$, let

$$[y]_x^{1\cdots i} = \mathcal{F}_x^{(i+1)\cdots\ell}(y) \cap \mathcal{F}_x^{1\cdots i}(x),$$
$$[y]_x^{(i+1)\cdots\ell} = \mathcal{F}_x^{1\cdots i}(y) \cap \mathcal{F}_x^{(i+1)\cdots\ell}(x),$$

wherever they are well defined. The almost tangency property (i) and the uniform positive lower bound among angles of different bundles E_* allow us to be able to choose a constant $C_1 > 0$ such that for any $\rho \in (0, \rho_1/C_1)$, $y \in B(x, \rho)$ and

$$* \in \{1\cdots i, (i+1)\cdots\ell : 1 \le i \le \ell - 1\},$$

one has

$$[y]_x^* \in \mathcal{F}_x^*(x, C_1\rho).$$

By taking some local trivialization of the tangent bundle, for any $N \in \mathbb{N}$ and $\rho \in (0, \rho_1/C_1)$, we define

$$\sigma(N, \rho) = \max\bigg\{\log\bigg(\frac{\|(D_{x_1}f^{\pm N})^{\wedge_k}\|}{\|(D_{x_2}f^{\pm N})^{\wedge_k}\|}\bigg) : x_j \in \mathcal{F}_x^*(x, C_1\rho),\ j = 1, 2,$$
$$1 \le k \le \dim E_*(x),\ * \in \{1\cdots i, (i+1)\cdots\ell : 1 \le i \le \ell-1\},\ x \in \Lambda\bigg\}.$$

Denote $e^P = \max\{\|D_x^{\pm}f\| : x \in M\}$. For any $N \in \mathbb{N}$, one may let ξ and ρ_1 be small such that $\rho(N) = \rho_1 e^{-NP}/C_1$ satisfying $\sigma(N, \rho(N)) < 1/N$.

LEMMA 2.2. (Pliss [**19**]) *Let $b_0 \le c_2 < c_1$ and $\theta = (c_1 - c_2)/(c_1 - b_0)$. Given real numbers $b_1, \ldots, b_T$ with $\sum_{i=1}^T b_i \le c_2 T$ and $b_i \ge b_0$ for every i, there exist $\tau \ge T\theta$ and $1 \le k_1 < k_2 < \cdots < k_\tau \le T$ such that*

$$\sum_{i=k+1}^{k_j} b_i \le c_1(k_j - k), \quad 0 \le k < k_j,\ 1 \le j \le \tau.$$

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LEMMA 2.3. *There exists $N_0 > 0$ such that for any $N \geq N_0$ and $\mu \in \mathcal{M}_{\mathrm{inv}}(f, \Lambda)$, for μ-a.e. x, $B_\infty(f, x, \rho(N)) = \{x\}$ or $B_\infty(f, x, \rho(N)) \subset \mathcal{F}^i_x(x, C_1\rho(N))$ for some $i \in \{1, \ldots, \ell\}$.*

Proof. Let $N_0 = [16/(\log 2)] + 1$. So, $\sigma(N, \rho(N)) < (\log 2)/16$ for any $N \geq N_0$. For $\mu \in \mathcal{M}_{\mathrm{inv}}(f, \Lambda)$ and $1 \leq i \leq \ell$, we denote for μ-a.e. x,

$$a^+_{E_i}(x) = \lim_{n\to\infty} \frac{1}{|nN|} \sum_{k=0}^{n-1} \log m(D_{f^{kN}(x)} f^N |_{E_i}),$$

$$a^-_{E_i}(x) = \lim_{n\to\infty} \frac{1}{|nN|} \sum_{k=0}^{n-1} \log \|D_{f^{-kN}(x)} f^{-N} |_{E_i}\|.$$

Let

$$i_0(x) = \min\left\{1 \leq i \leq \ell : a^+_{E_i}(x) > \frac{\log 2}{2}\right\};$$

then by the domination $T_\Lambda M = E_1 \oplus_< \cdots \oplus_< E_{i_0(x)-1} \oplus_< E_{i_0} \oplus_< \cdots \oplus_< E_\ell$, one has

$$\begin{cases} a^-_{E_{i_0(x)-1}}(x) \leq \dfrac{\log 2}{2}, & \\ a^-_{E_j}(x) \leq -\dfrac{\log 2}{2} & \text{for all } j \in [1, i_0(x) - 2], \\ a^+_{E_j}(x) \geq \dfrac{\log 2}{2} & \text{for all } j \in [i_0(x), \ell]. \end{cases}$$

Hence, there exist $1 < n_1 < n_2 < \cdots < n_t < \cdots$ such that for any $t \in \mathbb{N}$,

$$\frac{1}{n_t} \sum_{k=0}^{n_t-1} \log m(D_{f^{kN}(x)} f^N |_{E_{i_0(x)} \oplus\cdots\oplus E_\ell}) > \frac{\log 2}{4},$$

i.e.,

$$\frac{1}{n_t} \sum_{k=1}^{n_t} \log \|D_{f^{kN}(x)} f^{-N} |_{E_{i_0(x)} \oplus\cdots\oplus E_\ell}\| < -\frac{\log 2}{4}.$$

Let $b_0 = -NP$, $c_1 = -(\log 2)/4$, $c_2 = -(\log 2)/8$ and $\theta = (c_1 - c_2)/(c_1 - b_0)$. Applying Lemma 2.2, for each n_t, we can find $\tilde{n}_t \in [\theta n_t, n_t]$ such that

$$\sum_{k=j+1}^{\tilde{n}_t} \log \|D_{f^{kN}(x)} f^{-N} |_{E_{i_0(x)} \oplus\cdots\oplus E_\ell}\| \leq -\frac{\log 2}{8}(\tilde{n}_t - j), \quad 0 \leq j < \tilde{n}_t.$$

By the choice of $N \geq N_0$, we have

$$\|Df_z^{-N} |_{T_z \mathcal{F}_y^{i_0(x)(i_0(x)+1)\cdots\ell}(z)}\| \leq 2^{1/16} \|Df_y^{-N} |_{E_{i_0} \oplus\cdots\oplus E_\ell}\|$$
$$\text{for all } z \in \mathcal{F}_y^{i_0(i_0+1)\cdots\ell}(y, C_1\rho(N)) \text{ for all } y \in \Lambda.$$

Therefore, for $1 \leq j \leq \tilde{n}_t$,

$$f^{-jN_0}(\mathcal{F}^{i_0(i_0+1)\cdots\ell}_{f^{\tilde{n}_t}(x)}(f^{\tilde{n}_t}(x), C_1\rho(N))) \subset \mathcal{F}^{i_0(x)(i_0(x)+1)\cdots\ell}_{f^{\tilde{n}_t-j}(x)}(f^{\tilde{n}_t-j}(x), 2^{-(\tilde{n}_t-j)/16} C_1\rho(N)).$$

For any $y \in B_\infty(f, x, \rho(N))$,

$$[f^n(y)]^{i_0(x)(i_0(x)+1)\cdots\ell} \in \mathcal{F}^{i_0(x)(i_0(x)+1)\cdots\ell}_{f^n(x)}(f^n(x), C_1\rho(N)).$$

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By the local invariance of fake foliations,

$$[y]^{i_0(x)(i_0(x)+1)\cdots\ell} = f^{-n}([f^n(y)]^{i_0(i_0+1)\cdots\ell}) \quad \text{for all } n \in \mathbb{N}.$$

Especially,

$$[y]^{i_0(x)(i_0(x)+1)\cdots\ell} = f^{-\tilde{n}_t}([f^{\tilde{n}_t}(y)]^{i_0(x)(i_0(x)+1)\cdots\ell}) \in \mathcal{F}_x^{i_0(x)(i_0(x)+1)\cdots\ell}(x, 2^{-\tilde{n}_t/16}C_1\rho(N)).$$

Letting $t \to +\infty$, we get that

$$[y]^{i_0(x)(i_0(x)+1)\cdots\ell} = \{x\}.$$

Similarly, one can deduce that $[y]^{1\cdots(i_0(x)-2)} = \{x\}$, since $a^-_{E_j}(x) \le -(\log 2)/2$ for any $j \in [1, i_0(x)-2]$. Then

$$\begin{aligned} B_\infty(f, x, \rho(N)) &\subset \mathcal{F}_x^{1\cdots(i_0(x)-1)}(x, C_1\rho(N)) \cap \mathcal{F}_x^{(i_0(x)-1)\cdots\ell}(x, C_1\rho(N)) \\ &\subset \mathcal{F}_x^{i_0(x)-1}(x, C_1\rho(N)). \end{aligned}$$

Furthermore, if $a^-_{E_{i_0(x)-1}}(x) \le -(\log 2)/2$, then $[y]^{1\cdots(i_0(x)-1)} = \{x\}$; thus,

$$B_\infty(f, x, C_1\rho(N)) = \{x\}. \qquad \square$$

3. *Tail entropy along leaves*

By Lemma 2.3, given $N \ge N_0$, $\mu \in \mathcal{M}_{\mathrm{inv}}(f, \Lambda)$, without loss of generality, for μ-a.e. x, we may assume that $B_\infty(f, x, C_1\rho(N)) \subset \mathcal{F}^i_x(x, C_1\rho(N))$ for some i. Therefore, in what follows, we only need analyze the dynamics on leaves $\mathcal{F}^*_y(y)$, $* \in \{1, \ldots, \ell\}$, $y \in \Lambda$. For the simplicity of symbols, we write $V^*_y = \mathcal{F}^*_y(y)$. Moreover, we denote by $B_{V^*_y}(z, \rho)$ the ball in V^*_y centered at z with radius ρ and define Bowen balls along leaves as follows:

$$B_{V^*_y,n}(z, \rho) = \{p \in V^*_y : d_{V^*_{f^j(y)}}(f^j(p), f^j(z)) < \rho, |j| < n\},$$

where d_V denotes the distance in a submanifold $V \subset M$. For the convenience of computations, we intend to approximate the local complexity of dynamical systems by that of their linearity. Taking local trivializations, we may assume that $V^*_y \subset \mathbb{R}^{\dim E_*}$. Note that there exists a constant $C_2 > 0$ depending only on $\dim M$ such that for any $1 \le j \le \dim M$ and any linear map $X : \mathbb{R}^j \to \mathbb{R}^j$, one has

$$\Gamma(X(B_{\mathbb{R}^j}(0, 1)), \mathbb{R}^j, 1/2) \le C_2 \|X^\wedge\|^+,$$

where $\Gamma(U, V, \rho)$ denotes the minimal cardinality of covers for U whose elements are balls with radius ρ in a manifold V, and $\|X^\wedge\|^+ = e^{\log^+\|X^\wedge\|}$.

LEMMA 3.1. *There exists $\eta_1 > 0$ such that for any $y \in \Lambda$, $* \in \{1, \ldots, \ell\}$, $z \in B_{V^*_y}(y, C_1\rho(N))$ and $\eta \in (0, \eta_1)$,*

$$\Gamma(f^{\pm N}(B_{V^*_y}(z, \eta)), V^*_{f^{\pm N}(y)}, \eta/2)) \le C_2 e^{2/N} \|(D_y f^{\pm N}|_{E_i})^\wedge\|^+.$$

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Proof. From the definition of $\rho(N)$, for $* \in \{1, \ldots, \ell\}$, $z \in B_{V_y^*}(y, C_1\rho(N))$,

$$\|(D_z f^{\pm N}|_{T_z V_y^*})^\wedge\|^+ \le e^{1/N}\|(D_y f^{\pm N}|_{E_i})^\wedge\|^+.$$

For $\eta > 0$, define $g_{\eta,z}(p) = \eta p + z$, $z \in V_y^*$. Let $F_{\pm N,\eta,z}(p) = g^{-1}_{\eta, f^{\pm N}(z)} \circ f^{\pm N} \circ g_{\eta,z}(p)$. Then

$$\|F_{\pm N,\eta,z}(p) - D_z f^{\pm N}|_{T_z V_y^*}(p)\| \text{ converges to } 0 \quad \text{as } \eta \to 0$$

uniformly for $p \in B_{\mathbb{R}^{\dim E_*}}(0, 1)$, $z \in B_{V_y^*}(y, C_1\rho(N))$. Observe that

$$\Gamma(f^{\pm N}(B_{V_y^*}(z, \eta)), V^*_{f^{\pm N}(y)}, \eta/2) = \Gamma(F_{\pm N,\eta,y}(B_{\mathbb{R}^{\dim E_*}}(y, 1)), \mathbb{R}^{\dim E_*}, 1/2).$$

So, there exists $\eta_1 > 0$ uniformly such that for any $\eta \in (0, \eta_1)$,

$$\begin{aligned}
&\Gamma(f^{\pm N}(B_{V_y^*}(z, \eta)), V^*_{f^{\pm N}(y)}, \eta/2)\\
&\quad \le e^{1/N}\Gamma((D_z f^{\pm N}|_{T_z V_y^*}(B_{\mathbb{R}^{\dim E_*}}(0, 1)), \mathbb{R}^{\dim E_*}, 1/2)\\
&\quad \le C_2 e^{1/N}\|(D_z f^{\pm N}|_{T_z V_y^*})^\wedge\|^+\\
&\quad \le C_2 e^{2/N}\|(D_y f^{\pm N}|_{E_*})^\wedge\|^+.
\end{aligned}$$

□

Let $N \ge N_0$, $\mu \in \mathcal{M}_{\mathrm{inv}}(f, \Lambda)$; then, for μ-a.e. x, there exists i such that $B_\infty(f, x, C_1\rho(N)) \subset V_x^i(x, C_1\rho(N))$. For $\eta \in (0, \eta_1)$, let $\{y_1, \ldots, y_{k(0)}\}$ be a finite η-net of $B_{V_x^i,n}(f^N, x, C_1\rho(N))$. Let $R_{j_0} = B_{V_x^i}(y_{j_0}, \eta) \cap B_{V_x^i,n}(f^N, x, C_1\rho(N))$, $1 \le j_0 \le k(0)$. By induction, for $0 \le s \le n-2$, suppose that

$$y_{j_0,\ldots,y_{j_s}}, \quad R_{j_0,\ldots,j_s}: \quad 1 \le j_0 \le k(0),\ 1 \le j_t \le k(0, j_0, \ldots, j_{t-1}),\ 1 \le t \le s$$

have been defined. Given $y_{j_0,\ldots,y_{j_s}}$, using Lemma 3.1, one may take a set D which is an $\eta/2$-net of $f^N(B_{V^i_{f^{sN}(x)}}(y_{j_0,\ldots,y_{j_s}}, \eta))$ and has cardinality not more than $C_2 e^{2/N}\|(D_{f^{sN}(x)} f^N|_{E_i})^\wedge\|^+$. Observe that from the $\eta/2$-net D, we can choose a set

$$\{y_{j_0,\ldots,j_s,j_{s+1}} : 1 \le j_{s+1} \le k(0, j_0, \ldots, j_s)\}$$

with $k(0, j_0, \ldots, j_s) \le \sharp D$, which forms an η-net of $f^N(B_{V^i_{f^{sN}(x)}}(y_{j_0,\ldots,y_{j_s}}, \eta)) \cap f^N(R_{j_0,\ldots,y_{j_s}})$. For $1 \le j_{s+1} \le k(0, j_0, \ldots, j_s)$, denote

$$R_{j_0,\ldots,j_s,j_{s+1}} = B_{V^i_{f^{(s+1)N}(x)}}(y_{j_0,\ldots,j_s,j_{s+1}}, \eta) \cap f^N(R_{j_0,\ldots,j_s}).$$

In this way we could define all situations for $0 \le s \le n-1$.

For $1 \le j_0 \le k(0)$, $1 \le j_t \le k(0, j_0, \ldots, j_{t-1})$, $1 \le t \le n-1$, define

$$U_{j_0,\ldots,j_{n-1}} = \{y \in B_{V_x^i,n}(f^N, x, C_1\rho(N)) : f^{tN}(y) \in R_{j_0,\ldots,j_t}, 0 \le t \le n-1\}.$$

Then

$$\bigcup_{j_0,\ldots,j_{n-1}} U_{j_0,\ldots,j_{n-1}} = B_{V_x^i,n}(f^N, x, C_1\rho(N)).$$

Note that for any $y, z \in U_{j_0,\ldots,j_{n-1}}$, $0 \le t \le n-1$,

$$d_{V^i_{f^{tN}(x)}}(f^{tN}(y), f^{tN}(z)) \le d_{V^i_{f^{tN}(x)}}(f^{tN}(y), y_{j_0,\ldots,j_t}) + d_{V^i_{f^{tN}(x)}}(f^{tN}(z), y_{j_0,\ldots,j_t}) \le 2\eta.$$

Therefore,

$$\begin{aligned}
&r_n(f^N, B_{V_x^i,n}(f^N, x, C_1\rho(N)), 2\eta)\\
&\quad \le \sum k(0, j_0, \ldots, j_{n-2}) \le k(0)\cdot \Pi_{t=0}^{n-2}(C_2 e^{2/N}\|(D_{f^{tN}(x)} f^N|_{E_i})^{\wedge}\|^{+}),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n}\log r_n(f, B_{V_x^i,n}(f, x, C_1\rho(N)), 2\eta)\\
&\quad \le \limsup_{n\to\infty} \frac{1}{nN}\log r_n(f^N, B_{V_x^i,n}(f^N, x, C_1\rho(N)), 2\eta)\\
&\quad \le \limsup_{n\to\infty} \frac{1}{nN}\log(\Pi_{t=0}^{n-2}(C_2 e^{2/N}\|(D_{f^{tN}(x)} f^N|_{E_i})^{\wedge}\|^{+}))\\
&\quad \le \frac{2+\log C_2}{N} + \Delta_f^{+}(x, E_i, N).
\end{aligned}$$

By the arbitrariness of η, we obtain

$$\begin{aligned}
h^*(f, x, \rho(N)) &\le \lim_{\eta\to 0}\limsup_{n\to\infty} \frac{1}{n}\log r_n(f, B_{V_x^i,n}(f, x, C_1\rho(N)), 2\eta)\\
&\le \frac{2+\log C_2}{N} + \Delta_f^{+}(x, E_i, N).
\end{aligned}$$

Similarly, considering the inverse f^{-1}, we get

$$\begin{aligned}
h^*(f^{-1}, x, \rho(N)) &\le \lim_{\eta\to 0}\limsup_{n\to\infty} \frac{1}{n}\log r_n(f^{-1}, B_{V_x^i,n}(f^{-1}, x, C_1\rho(N)), 2\eta)\\
&\le \frac{2+\log C_2}{N} + \Delta_f^{-}(x, E_i, N).
\end{aligned}$$

4. *Measure-theoretic tail entropy and upper semi-continuity*

In this section, we first analyze the relationship between the scale of measure-theoretic tail entropy and the evolution time and hence give the proof of Theorem 1.1.

Proof of Theorem 1.1. For $N \ge N_0$, let $\varepsilon_N = \rho(N)$. If $\mu \in \mathcal{M}_{\text{erg}}(f, \Lambda)$, then $h^*(f^{\pm}, x, \varepsilon_N)$ are constants for μ-a.e. x, which we denote by $h^*(f^{\pm 1}, \mu, \varepsilon_N)$. By [**9**, Proposition 2.7], one further obtains

$$h^*(f, \mu, \varepsilon_N) = h^*(f^{-1}, \mu, \varepsilon_N).$$

Hence,

$$\begin{aligned}
h^*(f, \mu, \varepsilon_N) &\le \frac{2+\log C_2}{N} + \min\{\Delta_f^{\pm}(\mu, E_i; N) : 1 \le i \le \ell\}\\
&= \frac{2+\log C_2}{N} + \Delta_f(\mu, N).
\end{aligned}$$

When $\mu \in \mathcal{M}_{\text{inv}}(f, \Lambda)$, using the ergodic decomposition $\mu = \int_{\mathcal{M}_{\text{erg}}(f,\Lambda)} d\tau(m)$, we deduce that

$$\begin{aligned}
h^*(f, \mu, \varepsilon_N) &\le \frac{2+\log C_2}{N} + \int_{\mathcal{M}_{\text{erg}}(f,M)} \Delta_f(m, N)\, d\tau(m)\\
&= \frac{2+\log C_2}{N} + \Delta_f(\mu, N),
\end{aligned}$$

which gives rise to

$$\sup_{\mu\in\mathcal{M}_{\rm inv}(f,\Lambda)} (h^*(f,\mu,\varepsilon_N)-\Delta_f(\mu,N)) \le \frac{2+\log C_2}{N} \to 0 \quad \text{as } N\to+\infty.$$

In particular, since $\Delta_f(\mu,N)\to\Delta_f(\mu)$ as $N\to+\infty$, we have

$$\lim_{\varepsilon\to 0} h^*(f,\mu,\varepsilon) \le \Delta_f(\mu)$$

for any $\mu\in\mathcal{M}_{\rm inv}(f,\Lambda)$. □

Next we are going to prove Theorem 1.2.

Proof of Theorem 1.2. By Jacobs' theorem (see [**21**, Theorem 8.4]), it suffices to consider μ to be ergodic. Moreover, by [**12**, Proposition 2.1] (note that the finiteness of topological entropy is used in the proof there), it is in fact enough to prove that for μ-a.e. x,

$$\begin{aligned}\lim_{\delta\to 0}\limsup_{n\to 0}\frac{1}{n}\log r_n(f,B_n(f,x,\rho),\delta) &\le \lim_{\delta\to 0}\limsup_{n\to 0}\frac{1}{n}\log r_n(f,B_\infty(f,x,\rho),\delta)\\ &= h^*(f,\mu,\rho).\end{aligned}$$

Note that, given $\gamma>0$, for μ-a.e. x, there exist $L(x)\in\mathbb{N}$ and a finite subset $D_{L(x)}(x)\subset B_\infty(f,x,\rho)$ with $\bigcup_{y\in D_{L(x)}(x)} B_{L(x)}(f,y,\delta)\supset B_\infty(f,x,\rho)$ satisfying

$$\sharp D_{L(x)}(x) = r_{L(x)}(f,B_\infty(f,x,\rho),\delta) \le e^{L(x)(h^*(f,\mu,\rho)+\gamma)}.$$

Furthermore, one may choose $T(x)\in\mathbb{N}$ such that

$$\bigcup_{y\in D_{L(x)}(x)} B_{L(x)}(f,y,\delta)\supset B_{T(x)}(f,x,\rho),$$

which implies that

$$r_{L(x)}(f,B_{T(x)}(f,x,\rho),\delta)\le \sharp D_{L(x)}(x)\le e^{L(x)(h^*(f,\mu,\rho)+\gamma)}.$$

For any $j\in\mathbb{N}$, denote $Y_j=\{x : L(x)\le j,\ T(x)\le j\}$; then $\mu(Y_j)\to 1$ as $j\to+\infty$. For μ-a.e. x, by the ergodicity of μ, for large n, one has

$$\frac{\sharp\{0\le k<n : f^k(x)\notin Y_j\}}{n}\le 1-\mu(Y_j)+\frac{1}{j}.$$

We define a sequence $0=n_0<n_1<\cdots<n_{k-1}<n_k=n$ of integers by induction. Suppose that n_s is defined; then

$$\begin{cases} n_{s+1}=n_s+L(f^{n_s}(x)) & \text{if } f^{n_s}(x)\in Y_j \text{ and } n_s+j\le n,\\ n_{s+1}=\min\{t>n_s : f^t(x)\in Y_j\} & \text{if } f^{n_s}(x)\notin R_j \text{ and } \min\{t>n_s : f^t(x)\in Y_j\}\le n,\\ n_{s+1}=n & \text{otherwise.}\end{cases}$$

Since the elements of $\{x, f(x),\ldots,f^{n-1}(x)\}$ outside Y_j do not exceed $n(1-\mu(Y_j)+1/j)$, by [**5**, Lemma 2.1],

$$r_n(B_n(f,x,\rho),2\delta)\le e^{n(h^*(f,\mu,\rho)+\gamma)}\cdot r_1(f,M,\delta)^{n(1-\mu(Y_j)+1/j)+j},$$

which implies that

$$\begin{aligned}
&\limsup_{n\to+\infty} \frac{1}{n} \log r_n(f, B_n(f, x, \rho), 2\delta) \\
&\quad \le h^*(f, \mu, \rho) + \gamma + \left(1 - \mu(Y_j) + \frac{1}{j}\right) \log r_1(f, M, \delta).
\end{aligned}$$

Since j and γ are arbitrary, it follows that

$$\limsup_{n\to+\infty} \frac{1}{n} \log r_n(f, B_n(f, x, \rho), 2\delta) \le h^*(f, \mu, \rho).$$

Letting $\delta \to 0$, we finish the proof of Theorem 1.2. □

Now, together with the uniform arguments in Theorems 1.1 and 1.2, we are in a position to prove Corollary 1.3.

Proof of Corollary 1.3. If $\Delta_f(\mu) = 0$, then, given $\delta > 0$, for large $N \in \mathbb{N}$ one has $\Delta_f(\mu, N) \le \delta$. Besides, by Theorem 1.1, taking N sufficiently large in advance, we have

$$h^*(f, \nu, \varepsilon_N) \le \Delta_f(\nu, N) + \delta$$

for any $\nu \in \mathcal{M}_{\mathrm{inv}}(f, \Lambda)$. Note that $\Delta_f(\nu, N)$ is continuous relative to $\nu \in \mathcal{M}_{\mathrm{inv}}(f, \Lambda)$, so, for ν close to μ, we have $\Delta_f(\nu, N) \le 2\delta$ and hence

$$h^*(f, \nu, \varepsilon_N) \le 3\delta.$$

Let $\mathcal{P}$ be a finite measurable partition with $\mu(\partial(\mathcal{P})) = 0$ and $\mathrm{diam}(\mathcal{P}) \le \varepsilon_N$. By Theorem 1.2,

$$h_\nu(f) - h_\nu(f, \mathcal{P}) \le h^*(f, \nu, \varepsilon_N) \le 3\delta.$$

Moreover, for the fixed $\mathcal{P}$, $h_\nu(f, \mathcal{P})$ is upper semi-continuous at μ, which implies that

$$h_\nu(f, \mathcal{P}) \le h_\mu(f, \mathcal{P}) + \delta,$$

when ν is close to μ. Therefore,

$$h_\nu(f) \le h_\nu(f, \mathcal{P}) + 3\delta \le h_\mu(f, \mathcal{P}) + 4\delta,$$

which, consequently, combining with the arbitrariness of δ, gives the upper semi-continuity of metric entropy at μ in $\mathcal{M}_{\mathrm{inv}}(f, \Lambda)$. The proof of Corollary 1.3 is completed. □

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