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Generalized Casimir operators for loop Lie superalgebras

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Abstract. Let $\mathfrak{g}$ be the queer superalgebra $\mathfrak{q}(n)$ over the field of complex numbers $\mathbb{C}$. For any associative, commutative, and finitely generated $\mathbb{C}$-algebra A with unity, we consider the loop Lie superalgebra $\mathfrak{g} \otimes A$. We define a class of central operators for $\mathfrak{g} \otimes A$, which generalizes the classical Gelfand invariants. We show that they generate the algebra $U(\mathfrak{g} \otimes A)^{\mathfrak{g}}$. We also show that there are no non-trivial $\mathfrak{g}$-invariants of $U(\mathfrak{g} \otimes A)$ where $\mathfrak{g} = \mathfrak{p}(n)$, the periplectic Lie superalgebra.

1 Introduction

The theory of Lie superalgebras primarily arose from an attempt to understand the mathematical foundation of supersymmetry in theoretical physics. The literature has been developed substantially since then. Representation theory of Lie superalgebras plays a crucial role in quantum optics and many other areas of theoretical physics, notably in string theory. A comprehensive description of the mathematical theory of Lie superalgebras is given in [8], containing the complete classification of all finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero. For generalities of the theory of Lie superalgebras and supergeometry we refer to [11] and [5], respectively.

Loop (super)algebras are of great importance in the literature. Affine Kac Moody Lie (super)algebras can be realized by means of an affinization technique on Loop superalgebras. In superstring theory a particular type of Loop superalgebras, namely the superconformal algebras are invaluable tool. By definition, they are tensor product of finite dimensional simple Lie superalgebras with the algebra of Laurent polynomials. For a survey on finite dimensional representation theory of loop algebras we refer to [17].

In this article we deal with Loop superalgebras associated with the superalgebras belong to the strange series in Kac's classification, namely the queer and periplectic Lie superalgebras. By definition, a loop superalgebra is of the form $\mathfrak{g} \otimes A$, where $\mathfrak{g}$ is a Lie superalgebra and A is an associative commutative finitely generated $\mathbb{C}$-algebra with identity. When $\mathfrak{g}$ is a basic classical superalgebra and $A = \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$, the Laurent polynomial algebra in d commuting variables, a complete classification of irreducible finite dimensional modules for $\mathfrak{g} \otimes A$ is given in [17] and [12]. They turn out to be evaluation modules at finitely many points except in types $A(m, n)$ and $C(n)$. In [16],

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Savage classified irreducible finite-dimensional representations of equivariant map superalgebras in the case where $\mathfrak{g}$ is a basic classical Lie superalgebra. As a special case, his results give the classification of the irreducible finite-dimensional modules for the twisted loop superalgebras, and in particular generalize the classification results obtained in [15] and [12]. In [2], the authors classified finite-dimensional irreducible representations of equivariant map queer Lie superalgebras. While in the basic classical setting those irreducible modules were isomorphic to tensor products of generalized evaluation modules, in the queer case they are irreducible products of evaluation modules. In [1], the authors completed the classification of finite-dimensional irreducible modules of equivariant map superalgebras by describing these modules for the periplectic superalgebra.

In [14], Rao defined a set of operators for the Loop superalgebra $\mathfrak{g} \otimes A$, where $\mathfrak{g}$ is a contragredient Lie superalgebra, and called them the generalized Casimir operators, generalizing his work in [13] for Lie algebras. In particular, specializing to the case where $\mathfrak{g} = \mathfrak{gl}(m|n)$, he defined a set of central operators $T_k(a_1, \ldots, a_k)$ for $a_i \in A$, in the universal enveloping algebra $U(\mathfrak{g} \otimes A)$ of $\mathfrak{g} \otimes A$, using the construction of the Gelfand invariants in the center of $U(\mathfrak{g})$ and conjectured that these central operators generate the algebra of $\mathfrak{g}$-invariants of $U(\mathfrak{g} \otimes A)$. In [10], the authors gave a proof of this conjecture and gave a spanning set for the space of invariants $U(\mathfrak{osp}(m|2n) \otimes A)^{\mathrm{OSp}(m|2n)}$ where $\mathfrak{osp}(m|2n)$ is the orthosymplectic superalgebra and $\mathrm{OSp}(m|2n)$ is the corresponding supergroup.

In the current article, we study $U(\mathfrak{g} \otimes A)^{\mathfrak{g}}$ corresponding to the strange series of superalgebras, i.e., $\mathfrak{g}$ is either $\mathfrak{q}(n)$ and $\mathfrak{p}(n)$. Using the Schur–Weyl–Sergeev duality, we first derive the tensor version of the first fundamental theorem (FFT) of invariant theory for $\mathfrak{q}(n)$ which gives a spanning set for $(V^{\otimes k} \otimes (V^*)^{\otimes k})^{\mathfrak{q}(n)}$, $k \geq 1$, where $V = \mathbb{C}^n \oplus \mathbb{C}^n$ is the defining representation of $\mathfrak{q}(n)$. Using this version of FFT, we give a spanning set for $U(\mathfrak{q}(n) \otimes A)^{\mathfrak{q}(n)}$. We define a set of central operator for $U(\mathfrak{q}(n) \otimes A)$ which are similar to the central operators defined by Rao in [14] in the case of $\mathfrak{gl}(m|n)$. We show that these central operators generate $U(\mathfrak{q}(n) \otimes A)^{\mathfrak{q}(n)}$ as an algebra. Specializing to $A = \mathbb{C}$, these central operators give an algebra generating set for the center of the universal enveloping algebra of $\mathfrak{q}(n)$. We mention that a set of algebra generators of the center of $\mathfrak{q}(n)$ is given in [18] without a proof. These operators are helpful to understand the tensor product decompositions of $\mathfrak{g}$-modules. In fact these central operators when applied to a certain tensor product of $\mathfrak{q}(n)$-modules move one highest weight vector to another highest weight vector.

The periplectic Lie superalgebra $\mathfrak{p}(n)$ is a superanalog of the orthogonal or symplectic Lie algebra preserving an odd non-degenerate symmetric or skew-symmetric bilinear form. In [9], Moon proved a Schur–Weyl duality statement for $\mathfrak{p}(n)$ by introducing an algebra called the periplectic Brauer algebra which plays the role of Brauer algebra as in the case of orthogonal and symplectic Lie algebra or in the case of the encompassing orthosymplectic Lie superalgebra. In [4], Deligne et al., proved the tensor version of the FFT for the periplectic Lie supergroup, hence thereby giving a spanning set for $(V^{\otimes 2k})^{Pe(V)}$, where $V = \mathbb{C}^n \oplus \mathbb{C}^n$ and $Pe(V)$ is the periplectic supergroup which is by definition the subgroup of $GL(V)$ preserving a non-degenerate odd symmetric form on V. Using these results, for an associative, commutative and finitely generated $\mathbb{C}$-algebra A, we show that the only

$\mathfrak{p}(n)$-invariants of $U(\mathfrak{p}(n) \otimes A)$ are the elements of $\mathbb{C}$. We note that from the work of M. Gorelik it can also be derived that the centre of $\mathfrak{p}(n)$ is trivial (see [7]).

We briefly describe the contents of each sections. In Section 2, we give preliminaries of queer Lie superalgebra and recall the Schur–Weyl–Sergeev duality for $\mathfrak{q}(n)$. In Section 3, we define loop superalgebras and describe a spanning (resp. algebra generating) set for $U(\mathfrak{g} \otimes A)^{\mathfrak{g}}$ starting with a homogeneous basis (resp. a set of homogeneous algebra generators) of $[T(\mathfrak{g})]^{\mathfrak{g}}$. In Section 4, we prove the tensor FFT for the the queer superalgebra. In Section 5, we give a spanning set for $U(\mathfrak{q}(n) \otimes A)^{\mathfrak{q}(n)}$, define a set of central operators in $U(\mathfrak{q}(n) \otimes A)$ and show that these central operators generate the algebra $U(\mathfrak{q}(n) \otimes A)^{\mathfrak{q}(n)}$. In Section 6, after recalling the definition of the periplectic Lie superalgebra $\mathfrak{p}(n)$, we show that the only $\mathfrak{p}(n)$-invariants of $U(\mathfrak{p}(n) \otimes A)$ are the constants.

Notation: Throughout this article, we work over the field of complex numbers $\mathbb{C}$. All modules and algebras are defined over $\mathbb{C}$ and in addition all the modules are of finite dimension. We write $\mathbb{Z}_2 = \{0, 1\}$ and use its standard field structure. We put $(-1)^0 = 1$ and $(-1)^1 = -1$.

2 Preliminaries

2.1 Queer superalgebra

For a positive integer n, we set $I_{n|n} := \{-1, \ldots, -n, 1, 2, \ldots, n\}$, on which we define the parity of $i \in I_{n|n}$ to be $|i| = 0$ if $i > 0$ and $|i| = 1$ if $i < 0$.

Let $V := \mathbb{C}^n \oplus \mathbb{C}^n$ be the superspace with standard basis v_i of parity i for $i \in I_{n|n}$. Its endomorphism ring $\mathrm{End}_{\mathbb{C}}(V)$ is an associative superalgebra with standard basis E_{ij} of parity $|i| + |j|$ for $i, j \in I_{n|n}$. It is also a Lie superalgebra under the standard supercommutator

$$[x, y] = xy - (-1)^{|x||y|} yx,$$

for all homogeneous $x, y \in \mathrm{End}_{\mathbb{C}}(V)$. It is denoted by $\mathfrak{gl}(n|n)$.

The queer superalgebra $\mathfrak{q}(n)$ can be defined in two ways. The map $E_{ij} \mapsto E_{-i,-j}$ is an involutive automorphism of $\mathfrak{gl}(n|n)$ and we define $\mathfrak{q}(n)$ to be the fixed point sub-superalgebra of this automorphism.

Alternatively, we define $P : V \to V$ by $Pv_i = (-1)^{|v_i|} v_{-i}$ and $\mathfrak{q}(n) := \{f \in \mathrm{End}_{\mathbb{C}}(V) : [f, P] = 0\}$. It is closed under the superbracket and we call it the queer superalgebra.

With respect to the ordered basis $B := B_1 \cup B_0 = \{e_{-1}, \ldots e_{-n}, e_1, \ldots, e_n\}$, the operator P can be written in matrix form as

$$P = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and $\mathfrak{q}(n)$ can be expressed in the matrix form as

$$\mathfrak{q}(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} : A, B \text{ are arbitrary } n \times n \text{ matrices} \right\}.$$

In terms of basis elements of $\mathfrak{gl}(n|n)$ we have $P = E_{-n,n} + E_{-(n-1),n-1} + \cdots + E_{-1,1} - E_{1,-1} - \cdots - E_{n,-n}$. Note that $\mathfrak{q}(n)$ as the Lie sub-superalgebra of $\mathfrak{gl}(n|n)$ is spanned by $F_{ij} := E_{ij} + E_{-i,-j}$ for $i > 0$. Note that we have $F_{ij} = F_{-i,-j} = F_{-j,-i}$ for all i, j.

The superalgebra $\mathfrak{gl}(n|n)$ and hence $\mathfrak{q}(n)$ acts on V by matrix multiplication and on the k-fold tensor product $V^{\otimes k}$ by

(2.1)

$$x \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \sum_{j=1}^{k} (-1)^{(|v_1|+\cdots+|v_{j-1}|)|v|} v_1 \otimes v_2 \otimes \cdots \otimes x v_j \otimes v_{j+1} \otimes \cdots \otimes v_k,$$

where the elements $x \in \mathfrak{q}(n)$ and $v_i \in V$ are all homogeneous. We then extend to all of $\mathfrak{q}(n)$ acting on all of $V^{\otimes k}$ by linearity.

2.2 The Sergeev superalgebras

Let S_k be the symmetric group on k letters. It is generated by the transpositions $s_1, \ldots, s_{k-1}$, where $s_i = (i, i+1)$ for all i.

The Sergeev superalgebra Ser_k is the associative superalgebra generated by $s_1, \ldots, s_{k-1}$ and $c_1, \ldots, c_{k-1}, c_k$ with the following defining relations:

- $s_i^2 = 1,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ s_i s_j = s_j s_i \quad (|i-j| > 1)$,
- $c_i^2 = -1,\ c_i c_j = -c_j c_i, \quad (i \neq j)$
- $s_i c_i s_i = c_{i+1},\ s_i c_j = c_j s_i, \quad j \neq i, i+1$.

The generators $s_1, \ldots, s_{k-1}$ are regarded as even and the subalgebra generated by them is isomorphic to the group algebra $\mathbb{C}[S_k]$ of S_k; the generators $c_1, \ldots, c_{k-1}, c_k$ are called odd and the $\mathbb{C}$-subalgebra generated by them is isomorphic to the Clifford superalgebra $\mathbf{C}l_k$.

The Sergeev superalgebra Ser_k is isomorphic to the superalgebra $\mathbb{C}[S_k] \ltimes \mathbf{C}l_k$, which is $\mathbb{C}[S_k] \otimes \mathbf{C}l_k$ as a superspace with the multiplication given by

$$(\sigma \otimes c_{i_1} \ldots c_{i_l})(\tau \otimes c_{j_1} \ldots c_{j_m}) = \sigma\tau \otimes c_{\tau^{-1}(i_1)} \ldots c_{\tau^{-1}(i_l)} c_{j_1} \ldots c_{j_m},$$

where $1 \leq i_s, j_t \leq k$.

The superalgebra Ser_k has as basis the elements which can be written in the form $\sigma \otimes c_1^{\varepsilon_1} \ldots c_k^{\varepsilon_k}$ where $\sigma \in S_k$ and $\varepsilon_i \in \mathbb{Z}_2$ for all i.

2.3 Action of the Sergeev algebra and Schur–Weyl–Sergeev duality

The symmetric group S_k acts on the k-fold tensor product $V^{\otimes k}$ by

(2.2)

$$s_i \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_k) = (-1)^{|v_i||v_{i+1}|} v_1 \otimes v_2 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_k,$$

where v_i's are $\mathbb{Z}_2$-homogeneous. We then extend the action to $V^{\otimes k}$ by linearity.

More generally the action of S_k on $V^{\otimes k}$ is defined as follows: for $\sigma \in S_k$ and homogeneous $\boldsymbol{v} = v_1 \otimes v_2 \otimes \cdots \otimes v_k$, the action of σ on $\boldsymbol{v}$ is given by:

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_k) = (-1)^{\gamma(\boldsymbol{v},\sigma^{-1})} (v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}),$$

where $\gamma(\boldsymbol{v},\sigma^{-1}) = \prod_{(i,j)\in \mathrm{Inv}(\sigma)} |v_i||v_j|$, with $\mathrm{Inv}(\sigma) = \{(i,j): i < j \text{ and } \sigma(i) > \sigma(j)\}$. It can be easily verified that $\gamma(\boldsymbol{v},\sigma\tau) = \gamma(\sigma^{-1}\boldsymbol{v},\tau) + \gamma(\boldsymbol{v},\sigma)$ for two permutations σ and τ.

The generators c_j acts on $V^{\otimes k}$ by

$$c_j \cdot (v_1 \otimes \cdots \otimes v_k) = (-1)^{|v_1|+\cdots+|v_{j-1}|} v_1 \otimes \cdots \otimes v_{j-1} \otimes P(v_j) \otimes v_{j+1} \otimes v_{j+2} \otimes \cdots \otimes v_k,$$

where $v_1,\ldots,v_k \in V$ are homogeneous elements and we then extend the action to $V^{\otimes k}$ by linearity.

More generally, for $\varepsilon_i \in \mathbb{Z}_2$, $i = 1,\ldots,k$, we have

$$c_1^{\varepsilon_1}\ldots c_k^{\varepsilon_k} \cdot (v_1 \otimes \cdots \otimes v_k) = (-1)^{\sum_{i>j}\varepsilon_i |v_j|} P^{\varepsilon_1} v_1 \otimes \cdots \otimes P^{\varepsilon_k} v_k.$$

The actions of S_k and $\mathbf{C}l_k$ give rise to a left action of the Sergeev superalgebra Ser_k on $V^{\otimes k}$. So we get an algebra homomorphism

$$\Psi_k : \mathrm{Ser}_k \to \mathrm{End}_{\mathbb{C}}(V^{\otimes k}).$$

We record that for a basis element $\sigma^{-1} \otimes c_1^{\varepsilon_1}\ldots c_k^{\varepsilon_k} \in Ser_k$, we have

$$(\sigma^{-1} \otimes c_1^{\varepsilon_1}\ldots c_k^{\varepsilon_k}) \cdot (v_1 \otimes \cdots \otimes v_k) = (-1)^{\gamma(\boldsymbol{v},\sigma)+\sum_{i>j}\varepsilon_i|v_j|} P^{\varepsilon_{\sigma(1)}} v_{\sigma(1)} \otimes \cdots \otimes P^{\varepsilon_{\sigma(k)}} v_{\sigma(k)}.$$

Let $\mathrm{End}_{\mathfrak{q}(n)}(V^{\otimes k})$ be the centralizer algebra of the $\mathfrak{q}(n)$-action on $V^{\otimes k}$. That is

$$\mathrm{End}_{\mathfrak{q}(n)}(V^{\otimes k}) := \{f \in \mathrm{End}_{\mathbb{C}}(V^{\otimes k}) : xf = fx, \quad \text{for all } x \in \mathfrak{q}(n)\}.$$

Let $\rho : \mathfrak{q}(n) \to \mathrm{End}_{\mathbb{C}}(V^{\otimes k})$ be the superalgebra homomorphism induced from the action of $\mathfrak{q}(n)$ on $V^{\otimes k}$. In [16], Sergeev proved the following double centralizer theorem along the lines of the Schur–Weyl duality for the queer superalgebra.

Theorem 2.1 *Let $\rho : \mathfrak{q}(n) \to \mathrm{End}_{\mathbb{C}}(V^{\otimes k})$ and $\Psi_k : \mathrm{Ser}_k \to \mathrm{End}_{\mathbb{C}}(V^{\otimes k})$ be the maps defined as above. Let A be the image of Ψ_k and let B be the subalgebra of $\mathrm{End}_{\mathbb{C}}(V^{\otimes k})$ spanned by the image of ρ. Then A and B are centralizers of each other.*

Remark 2.2 It may be noted that in the statement of the above theorem we consider the left action of the Sergeev algebra whereas Sergeev considers the action from the right and uses Ser_k^{op} instead.

3 Loop superalgebras

3.1

Let $\mathfrak{g}$ be either a simple Lie superalgebra or $\mathfrak{gl}(m|n)$, or any Lie superalgebra among $\mathfrak{q}(n)$ or $\mathfrak{p}(n)$. Let A be an associative, commutative and finitely generated $\mathbb{C}$-algebra with unity. Then $\mathfrak{g} \otimes A$ has a natural structure of Lie superalgebra:

$$[x \otimes a, y \otimes b] = [xy] \otimes ab, \quad \text{for } x, y \in \mathfrak{g}, a, b \in A.$$

With this the even part of $\mathfrak{g} \otimes A$ is $(\mathfrak{g})_0 \otimes A$ and the odd part is $(\mathfrak{g})_1 \otimes A$. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the standard triangular decomposition of $\mathfrak{g}$. Then $\mathfrak{g} \otimes A = (\mathfrak{n}^- \otimes A) \oplus (\mathfrak{h} \otimes A) \oplus (\mathfrak{n}^- \otimes A)$ is a triangular decomposition of $\mathfrak{g} \otimes A$.

3.2

Let $T(\mathfrak{g}\otimes A)=\bigoplus_{k=0}^{\infty}T^k(\mathfrak{g}\otimes A)$, where $T^k(\mathfrak{g}\otimes A)=(\mathfrak{g}\otimes A)^{\otimes k}$, be the tensor algebra of $\mathfrak{g}\otimes A$ and let $U(\mathfrak{g}\otimes A)$ be the enveloping algebra of $\mathfrak{g}\otimes A$. The universal enveloping algebra of $\mathfrak{g}\otimes A$ is a quotient of the tensor algebra $T(\mathfrak{g}\otimes A)$ modulo the two-sided ideal I generated in $T(\mathfrak{g}\otimes A)$ by all elements of the form $x\otimes y-(-1)^{|x||y|}y\otimes x-[x,y]$, where $x,y\in\mathfrak{g}\otimes A$. Note that $T(\mathfrak{g}\otimes A)$ has a $\mathbb{Z}_2$-grading extending that on $\mathfrak{g}\otimes A$. Since the ideal I is homogeneous, $U(\mathfrak{g}\otimes A)$ is $\mathbb{Z}_2$-graded.

We have a natural surjective algebra homomorphism:

$$\pi: T(\mathfrak{g}\otimes A)\to U(\mathfrak{g}\otimes A).$$

Let

$$U(\mathfrak{g}\otimes A)^{\mathfrak{g}}=\{\boldsymbol{y}\in U(\mathfrak{g}\otimes A):[x,\boldsymbol{y}]=0,\quad \text{for all } x\in\mathfrak{g}\}$$

be the subalgebra of $\mathfrak{g}$-invariants. Note that it is naturally $\mathbb{Z}_2$-graded.

If we take $A=\mathbb{C}$, then $U(\mathfrak{g}\otimes A)^{\mathfrak{g}}$ reduces to $U(\mathfrak{g})^{\mathfrak{g}}$. Using a PBW basis, we see that if an element of $U(\mathfrak{g})$ supercommutes with all elements of $\mathfrak{g}$, it supercommutes with all elements of the PBW basis. So $U(\mathfrak{g})^{\mathfrak{g}}$ is the center of $U(\mathfrak{g})$, which we denote by $Z(\mathfrak{g})$.

Let $\mathfrak{h}=\mathfrak{h}_0+\mathfrak{h}_1$ be a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+$ be a triangular decomposition of $\mathfrak{g}$. We note that for basic Lie superalgebras $\mathfrak{h}=\mathfrak{h}_0$, whereas for $\mathfrak{g}=\mathfrak{q}(n)$ and $\mathfrak{p}(n)$, $\mathfrak{h}_1\neq 0$. Let $\{x_1,x_2,\dots,x_m\}$, $\{h_1,h_2,\dots,h_n\}$, and $\{y_1,y_2,\dots,y_p\}$ be homogeneous bases for $\mathfrak{n}^-$, $\mathfrak{h}$ and $\mathfrak{n}^+$ respectively. Then the monomials $x_1^{r_1}\dots x_m^{r_m}h_1^{s_1}\dots h_n^{s_n}y_1^{t_1}\dots y_p^{t_p}$, where the exponents of the even elements runs through the set of all positive integers, and the exponents of odd elements ranges over $\{0,1\}$, form a PBW basis of $U(\mathfrak{g})$. The monomials where $r_1=\dots=r_m=t_1=\dots=t_p=0$ form a basis of $U(\mathfrak{h})$, while the set of all remaining monomials is a basis of $(\mathfrak{n}^-U(\mathfrak{g})+U(\mathfrak{g})\mathfrak{n}^+)$. This yields a decomposition of $U(\mathfrak{g})$ as $U(\mathfrak{g})=U(\mathfrak{h})\oplus(\mathfrak{n}^-U(\mathfrak{g})+U(\mathfrak{g})\mathfrak{n}^+)$. The Harish Chandra homomorphism HC is the restriction to the center $Z(\mathfrak{g})$ of the projection map $\pi: U(\mathfrak{g})\to U(\mathfrak{h})$ with kernel $(\mathfrak{n}^-U(\mathfrak{g})+U(\mathfrak{g})\mathfrak{n}^+)$. With the identification of $U(\mathfrak{h})$ to the symmetric algebra $S(\mathfrak{h})$, it is well known that for all basic Lie superalgebras and $\mathfrak{q}(n)$, the image of the Harish Chandra homomorphism is contained in the commutative algebra $S(\mathfrak{h}_0)$ (*see* [3, Sections 2.2 and 2.3]). Since the Harish Chandra homomorphism is injective, it follows that the center $Z(\mathfrak{g})=U(\mathfrak{g})^{\mathfrak{g}}$ comprises only even elements, hence $U(\mathfrak{g}\otimes A)^{\mathfrak{g}}$ also contains only even elements. For an alternative proof of this fact, we refer to [20, 3.2, Proposition]. It is known that the center of $U(\mathfrak{p}(n))=\mathbb{C}$ (*see* [7]). We give an alternate proof of this fact in Section 6. Therefore, it now follows that the center of $U(\mathfrak{g}\otimes A)$ contains only even elements for all basic Lie superalgebras, $\mathfrak{q}(n)$ and $\mathfrak{p}(n)$.

3.3

For any $\theta\in[\mathfrak{g}^{\otimes k}]^{\mathfrak{g}}$, $\theta=\sum_i x_1^i\otimes x_2^i\otimes\dots\otimes x_k^i$ and for any $a_1,a_2,\dots,a_k\in A$, we define

$$\theta(a_1,a_2,\dots,a_k)=\sum_i(x_1^i\otimes a_1)(x_2^i\otimes a_2)\dots(x_k^i\otimes a_k)\in U(\mathfrak{g}\otimes A).$$

The element $\hat{\theta}(a_1, a_2, \ldots, a_k) := \sum_i (x_1^i \otimes a_1) \otimes (x_2^i \otimes a_2) \otimes \cdots \otimes (x_k^i \otimes a_k) \in T(\mathfrak{g} \otimes A)$ maps to $\theta(a_1, a_2, \ldots, a_k)$ under the surjective map π and since $[x, \hat{\theta}(a_1, a_2, \ldots, a_k)] = 0$ for all $x \in \mathfrak{g}$, we get that $[x, \theta(a_1, a_2, \ldots, a_k)] = 0$ for all $x \in \mathfrak{g}$. Hence, we have

$$\theta(a_1, a_2, \ldots, a_k) \in U(\mathfrak{g} \otimes A)^{\mathfrak{g}}.$$

We then have the following proposition.

Proposition 3.1 *Let $\mathfrak{g}$ be a simple Lie superalgebra or $\mathfrak{gl}(m|n)$, or any Lie superalgebra among $\mathfrak{q}(n)$ or $\mathfrak{p}(n)$. Then the subalgebra $U(\mathfrak{g} \otimes A)^{\mathfrak{g}}$ is spanned by $\theta(a_1, a_2, \ldots, a_k)$, $k \geq 0$ and $a_i \in A$ for $i = 1, 2, \ldots, k$ where θ runs over a homogeneous basis of $T(\mathfrak{g})^{\mathfrak{g}}$.*

Further, the subalgebra $U(\mathfrak{g} \otimes A)^{\mathfrak{g}}$ is generated as an algebra by $\theta(a_1, a_2, \ldots, a_k)$, $k \geq 0$ and $a_1, a_2, \ldots, a_k \in A$, where θ runs over a set of homogeneous algebra generators of $T(\mathfrak{g})^{\mathfrak{g}}$.

Proof The map

$$\pi : T(\mathfrak{g} \otimes A) \to U(\mathfrak{g} \otimes A).$$

is surjective. By using the PBW basis for $U(\mathfrak{g} \otimes A)$, we see that π splits. So we have a surjective degree preserving algebra homomorphism

$$T(\mathfrak{g} \otimes A)^{\mathfrak{g}} \to U(\mathfrak{g} \otimes A)^{\mathfrak{g}}.$$

Note that $T(\mathfrak{g} \otimes A)^{\mathfrak{g}}$ is a graded algebra and the grading comes from the grading of $T(\mathfrak{g})^{\mathfrak{g}}$. Since $\mathfrak{g} \otimes \mathbb{C}a_1 \cong \mathfrak{g}$ as $\mathfrak{g}$-modules, $(\mathfrak{g} \otimes \mathbb{C}a_1) \otimes (\mathfrak{g} \otimes \mathbb{C}a_2) \otimes \cdots \otimes (\mathfrak{g} \otimes \mathbb{C}a_k)$ as a $\mathfrak{g}$-module is isomorphic to $\mathfrak{g} \otimes \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$. Then the spanning (resp. algebra generating) set of $T(\mathfrak{g})^{\mathfrak{g}}$ maps to a spanning (resp. algebra generating) set of $U(\mathfrak{g} \otimes A)^{\mathfrak{g}}$. ■

4 Tensor FFT for Queer Lie superalgebra

In this section after listing out some isomorphisms that we require in this note, we prove the tensor version of the first fundamental theorem of the queer Lie superalgebra.

4.1 Isomorphisms

Let V be a super vector space over $\mathbb{C}$.

1. We have $V \otimes V^* \cong \mathrm{End}_{\mathbb{C}}(V)$ defined by $(v, \phi) \mapsto T_{(v,\phi)}$ where $T_{(v,\phi)}(w) = \phi(w)v$ for $v, w \in V$.

This implies that $\mathrm{End}_{\mathbb{C}}(V)^{\otimes k} \cong (V \otimes V^*)^{\otimes k}$ for $k \geq 1$.

2. For two super vector spaces V and W, we have an isomorphism $V \otimes W \to W \otimes V$ defined by $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$, where v and w are homogeneous elements of V and W, respectively.

3. Let V and W be two super vector spaces. For simple homogeneous tensors $\boldsymbol{v} = v_1 \otimes v_2 \otimes \cdots \otimes v_k \in V^{\otimes k}$ and $\boldsymbol{w} = w_1 \otimes w_2 \otimes \cdots \otimes w_k \in W^{\otimes k}$, we define $p(\boldsymbol{v}, \boldsymbol{w}) := \sum_{j=1}^{k-1} d_j(\boldsymbol{v}, \boldsymbol{w})$ where $d_j(\boldsymbol{v}, \boldsymbol{w}) := |w_j|(|v_{j+1}| + \cdots + |v_k|)$, for $1 \leq j \leq k-1$. We extend this definition by linearity to arbitrary tensors.

We have an evaluation map $(V^*)^{\otimes k} \times V^{\otimes k} \to \mathbb{C}$ defined by:

$$(f_1 \otimes f_2 \otimes \cdots \otimes f_k, v_1 \otimes v_2 \otimes \cdots \otimes v_k) \mapsto (-1)^{p(\boldsymbol{f},\boldsymbol{v})} f_1(v_1)\dots f_k(v_k),$$

where $\boldsymbol{f} = f_1 \otimes f_2 \otimes \cdots \otimes f_k$, and $\boldsymbol{v} = v_1 \otimes v_2 \otimes \cdots \otimes v_k$ for homogeneous elements $f_i \in V^*$ and $v_i \in V$. This evaluation map extends to a non-degenerate bilinear form and hence we get an isomorphism between $(V^*)^{\otimes k}$ and $(V^{\otimes k})^*$.

4. Let τ be the permutation which takes $(1, 2, \dots, k, k+1, \dots, 2k)$ to $(\tau(1), \tau(2), \dots, \tau(k), \tau(k+1), \dots, \tau(2k)) = (1, 3, 5, \dots, 2k-1, 2, 4, 6, \dots, 2k)$. By applying the permutation τ to $(V \otimes V^*)^{\otimes k} = (V \otimes V^*) \otimes (V \otimes V^*) \otimes \cdots \otimes (V \otimes V^*)$ we get a $\mathrm{End}_{\mathbb{C}}(V)$-module isomorphism between $(V \otimes V^*)^{\otimes k}$ and $V^{\otimes k} \otimes (V^*)^{\otimes k}$. We then have

$$\mathrm{End}_{\mathbb{C}}(V)^{\otimes k} \cong (V \otimes V^*)^{\otimes k} \cong V^{\otimes k} \otimes V^{*\otimes k}.$$

Let $\boldsymbol{v} = v_1 \otimes v_2 \otimes \cdots \otimes v_k$ and $\boldsymbol{f} = f_1 \otimes f_2 \otimes \cdots \otimes f_k$. Then the map in the reverse direction is defined by $\boldsymbol{v} \otimes \boldsymbol{f} \mapsto (-1)^{p(\boldsymbol{v},\boldsymbol{f})}(v_1 \otimes f_1) \otimes (v_2 \otimes f_2) \otimes \cdots \otimes (v_k \otimes f_k)$.

Remark: All the isomorphisms mentioned above are in particular $\mathfrak{q}(n)$-equivariant.

4.2 Tensor FFT

The tensor version of the first fundamental theorem (FFT) of invariant theory (see [6]) for $\mathfrak{q}(n)$ describes a spanning set for $(V^{\otimes k} \otimes V^{*\otimes k})^{\mathfrak{q}(n)}$ which can be derived from the Shur–Weyl–Sergeev duality described above.

For $n \geq 1$, let $V = \mathbb{C}^n \oplus \mathbb{C}^n$. Then $\mathrm{End}_{\mathbb{C}}(V) = \mathfrak{gl}(n|n)$. We take the standard basis of V as $\{e_{-1}, \dots, e_{-n}, e_1, \dots, e_n\}$ and we denote the index set for this basis by B. Put $|i| = 0$ if $i > 0$ and 1 otherwise. The $\mathbb{Z}_2$-gradation of V is defined by setting $|e_i| = |i|$. The standard basis of $\mathrm{End}_{\mathbb{C}}(V)$ consists of the matrix units E_{ij} where $E_{ij}e_k = \delta_{jk}e_i$. Note that $\mathrm{End}_{\mathbb{C}}(V)$ is also $\mathbb{Z}_2$-graded by $|E_{ij}| = |i| + |j|$.

To every basis element of the Sergeev algebra Ser_k, we wish to associate an element of $V^{\otimes k} \otimes V^{*\otimes k}$. In order to avoid cumbersome notations, we let $\boldsymbol{c}$ denote the element $c_1^{\varepsilon_1} \dots c_k^{\varepsilon_k}$. Let $\{i_1, \dots, i_k\}$ be a multi-subset of B of cardinality k. The basis element of $V^{\otimes k}$ which corresponds to I is $\boldsymbol{e_I} \coloneqq e_{i_1} \otimes \cdots \otimes e_{i_k}$. We simply write $\gamma(I, \sigma)$ for $\gamma(\boldsymbol{e_I}, \sigma)$. For any two length-$k$ multi-subsets of indices $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_k\}$ of B, we define $p(I, J) \coloneqq p(\boldsymbol{e_I}, \boldsymbol{e_J})$.

Now we assign the element $\sigma^{-1} \otimes \boldsymbol{c} \in \mathrm{Ser}_k$ to the following element:

$$\xi^{(k)}_{\sigma^{-1},\boldsymbol{c}} = \sum_I \varsigma(\boldsymbol{c}, I, \sigma)(P^{\varepsilon_{\sigma(1)}} e_{i_{\sigma(1)}} \otimes \cdots \otimes P^{\varepsilon_{\sigma(k)}} e_{\sigma(i_k)}) \otimes (e^*_{i_1} \otimes \cdots \otimes e^*_{i_k}), \tag{4.1}$$

where $I = \{i_1, \dots, i_k\}$ runs over all multi-subsets of B of cardinality k and

$$\varsigma(\boldsymbol{c}, I, \sigma) = (-1)^{\gamma(I,\sigma) + \sum_{s>t} \varepsilon_s |e_{i_t}| + p(I,I)}.$$

With the notations just introduced, the first fundamental theorem for $\mathfrak{q}(n)$ is stated below.

Theorem 4.1 *For all $\sigma \in S_k$, $\boldsymbol{c} \in \mathbf{Cl}_k$, we have that $\xi^{(k)}_{\sigma^{-1},\boldsymbol{c}} \in (V^{\otimes k} \otimes V^{*\otimes k})^{\mathfrak{q}(n)}$, and the space $(V^{\otimes k} \otimes V^{*\otimes k})^{\mathfrak{q}(n)}$ is spanned by the set $\{\xi^{(k)}_{\sigma^{-1},\boldsymbol{c}} : \boldsymbol{c} \in \mathbf{Cl}_k,\ \sigma \in S_k\}$.*

Proof By the isomorphisms (1) and (3) in the previous subsection, we have an isomorphism $\Theta: V^{\otimes k} \otimes V^{*\otimes k} \to \mathrm{End}_{\mathbb{C}}(V^{\otimes k})$.

Since the tensor product action of $\mathfrak{q}(n)$ on $V \otimes V^*$ corresponds to the adjoint action on $\mathfrak{q}(n)$, Θ induces an isomorphism $\tilde{\Theta}$ to the subspace $\mathfrak{q}(n)$-invariants:

$$\tilde{\Theta}: (V^{\otimes k} \otimes V^{*\otimes k})^{\mathfrak{q}(n)} \to \mathrm{End}_{\mathbb{C}}(V^{\otimes k})^{\mathfrak{q}(n)} = \mathrm{End}_{\mathfrak{q}(n)}(V^{\otimes k}). \tag{4.2}$$

By Schur–Weyl–Sergeev duality (Theorem 2.1), we have $\Psi_k(\mathrm{Ser}_k) = \mathrm{End}_{\mathfrak{q}(n)}(V^{\otimes k})$. We claim that Θ maps $\xi^{(k)}_{\sigma^{-1},\boldsymbol{c}}$ to $\Psi_k(\sigma^{-1} \otimes \boldsymbol{c})$. To show this, we take $J = \{j_1, j_2, \dots, j_k\}$ and then we have

$$\begin{aligned}
\Theta(\xi^{(k)}_{\sigma^{-1},\boldsymbol{c}})(e_{j_1} \otimes \cdots \otimes e_{j_k}) &= \sum_I \varsigma(\boldsymbol{c}, I, \sigma)(P^{\varepsilon_{\sigma(1)}} e_{i_{\sigma(1)}} \otimes \cdots \otimes P^{\varepsilon_{\sigma(k)}} e_{i_{\sigma(k)}}) \\
&\qquad \otimes (e^*_{i_1} \otimes \cdots \otimes e^*_{i_k})(e_{j_1} \otimes \cdots \otimes e_{j_k}) \\
&= \sum_I (-1)^{p(I,J)} \varsigma(\boldsymbol{c}, I, \sigma)[e^*_{i_1}(e_{j_1}) \dots e^*_{i_k}(e_{j_k})] \\
&= (-1)^{\gamma(J,\sigma) + \sum_{s>t} \varepsilon_s |e_{j_t}|}(P^{\varepsilon_{\sigma(1)}} e_{j_{\sigma(1)}} \otimes \cdots \otimes P^{\varepsilon_{\sigma(k)}} e_{j_{\sigma(k)}}) \\
&= \Psi_k(\sigma^{-1} \otimes \boldsymbol{c}).
\end{aligned}$$

Since the isomorphism Θ is $\mathfrak{q}(n)$-equivariant, for all $x \in \mathfrak{q}(n)$, it follows that

$$\Theta(x \cdot \xi^{(k)}_{\sigma^{-1},\boldsymbol{c}}) = x \cdot \Theta(\xi^{(k)}_{\sigma^{-1},\boldsymbol{c}}) = [x, \Psi_k(\sigma^{-1} \otimes \boldsymbol{c})] = 0.$$

The last equality is valid as $\Psi_k(\sigma^{-1} \otimes \boldsymbol{c}) \in \mathrm{End}_{\mathfrak{q}(n)}(V^{\otimes k})$. Since Θ is injective, we conclude that $\xi^{(k)}_{\sigma^{-1},\boldsymbol{c}} \in (V^{\otimes k} \otimes V^{*\otimes k})^{\mathfrak{q}(n)}$, for all $\sigma \in S_k$, $\boldsymbol{c} \in \mathbf{Cl}_k$. Since $\tilde{\Theta}$ is an isomorphism and the elements $\sigma^{-1} \otimes \boldsymbol{c}$ span the Sergeev superalgebra Ser_k, it follows that the set $\{\xi^{(k)}_{\sigma^{-1},\boldsymbol{c}} : \boldsymbol{c} \in \mathbf{Cl}_k,\ \sigma \in S_k\}$ spans $(V^{\otimes k} \otimes V^{*\otimes k})^{\mathfrak{q}(n)}$. ∎

5 The $\mathfrak{q}(n)$ invariants of the universal enveloping algebra

Let V be the vector superspace $\mathbb{C}^n \oplus \mathbb{C}^n$. Then $\mathrm{End}_{\mathbb{C}}(V) = \mathfrak{gl}(n|n)$. We have a $\mathfrak{q}(n)$-equivariant epimorphism $\phi: \mathfrak{gl}(n|n) \to \mathfrak{q}(n)$ given by $E_{i,j} \mapsto F_{i,j}$. As $\mathrm{End}_{\mathbb{C}}(V) \cong V \otimes V^*$ ($E_{i,j}$ identifies to $e_i \otimes e_j^*$), ϕ induces a surjective map (also denoted by ϕ) $(V \otimes V^*)^{\otimes k} \to \mathfrak{q}(n)^{\otimes k}$.

Let A be an associative, commutative, and finitely generated $\mathbb{C}$-algebra with unity. For $k \in \mathbb{N}$ and $a_1, a_2, \dots, a_k \in A$, we define

$$\begin{aligned}
&C^{(k)}_n(a_1, \dots, a_k) \\
&\quad := \sum_I (-1)^{|i_2| + \cdots + |i_k|}(F_{i_1,i_2} \otimes a_1)(F_{i_1,i_2} \otimes a_1) \dots (F_{i_{k-1},i_k} \otimes a_{k-1})(F_{i_k,i_1} \otimes a_k),
\end{aligned}$$

where the sum is over all the multi-subsets $I = \{i_1, i_2, \dots, i_k\}$ of B of cardinality k. Note that $C^{(k)}_n(a_1, \dots, a_k) \in U(\mathfrak{q}(n) \otimes A)$.

It is easy to see that $[x, C_n^{(k)}(a_1,\ldots,a_k)] = 0$ for all $x \in \mathfrak{q}(n)$ and $a_1, a_2, \ldots, a_k \in A$ and so $C_n^{(k)}(a_1,\ldots,a_k) \in U(\mathfrak{q}(n)\otimes A)^{\mathfrak{q}(n)}$. These central operators generalize the classical Gelfand invariants for the general linear Lie algebra. We call them the "generalized casimir operators" for $\mathfrak{q}(n)$. In fact, as in the classical case, in this section we show that these central operators generate $U(\mathfrak{q}(n)\otimes A)^{\mathfrak{q}(n)}$ as an algebra. We proceed to describe a spanning set of $U(\mathfrak{q}(n))^{\mathfrak{q}(n)}$. First we introduce few notations.

- For a k-length multi-subset $I = (i_1,\ldots,i_k)$ of B, we put $J_I = (i_2, i_3, \ldots, i_k, i_1)$.
- For $\sigma \in S_k$ and $\boldsymbol{c} \in \mathbf{C}l_k$, we set
$$J_I{}^{\sigma\otimes\boldsymbol{c}} = ((-1)^{\varepsilon_{\sigma(k)}} i_{\sigma(k)}, (-1)^{\varepsilon_{\sigma(1)}} i_{\sigma(1)}, \ldots, (-1)^{\varepsilon_{\sigma(k-1)}} i_{\sigma(k-1)}).$$
- Set $\nu(\boldsymbol{c}, I, \sigma) := \varsigma(\boldsymbol{c}, I, \sigma)(-1)^{p(J_I{}^{\sigma\otimes\boldsymbol{c}}, J_I)+\sum \varepsilon_t |e_{i_t}|}$, where $\varsigma(\boldsymbol{c}, I, \sigma)$ is defined in Equation (4.1).
- Let $\zeta^{(k)}_{\sigma^{-1},\boldsymbol{c}}$ denote the following element in $U(\mathfrak{q}(n))$:

$$\zeta^{(k)}_{\sigma^{-1},\boldsymbol{c}} := \sum_I \nu(\boldsymbol{c}, I, \sigma) F_{(-1)^{\varepsilon_{\sigma(k)}} i_{\sigma(k)}, i_2} F_{(-1)^{\varepsilon_{\sigma(1)}} i_{\sigma(1)}, i_3} \ldots F_{(-1)^{\varepsilon_{\sigma(k-2)}} i_{\sigma(k-2)}, i_k} F_{(-1)^{\varepsilon_{\sigma(k-1)}} i_{\sigma(k-1)}, i_1}, \tag{5.1}$$

where the sum is over all the multi-subsets $I = \{i_1, i_2, \ldots, i_k\}$ of B of cardinality k.

Theorem 5.1 *The space $U(\mathfrak{q}(n))^{\mathfrak{q}(n)}$ is spanned by the set described below:*

$$\bigcup_{k\geq 0}\left\{\zeta^{(k)}_{\sigma^{-1},\boldsymbol{c}} : \boldsymbol{c} \in \mathbf{C}l_k,\ \sigma \in S_k\right\}.$$

Proof We have a canonical projection map $T(\mathfrak{q}(n)) \to U(\mathfrak{q}(n))$ which respects the gradings of both the algebras. Since the grading of $T(\mathfrak{q}(n))^{\mathfrak{q}(n)}$ is induced from that of $T(\mathfrak{q}(n))$, we obtain a degree preserving algebra epimorphism $T(\mathfrak{q}(n))^{\mathfrak{q}(n)} \to U(\mathfrak{q}(n))^{\mathfrak{q}(n)}$. We claim that the kth-graded component of $T(\mathfrak{q}(n))^{\mathfrak{q}(n)}$, i.e., $(\mathfrak{q}(n)^{\otimes k})^{\mathfrak{q}(n)}$ is spanned by the set $\{\eta^{(k)}_{\sigma^{-1},\boldsymbol{c}} : \boldsymbol{c} \in \mathbf{C}l_k,\ \sigma \in S_k\}$ with

$$\eta^{(k)}_{\sigma^{-1},\boldsymbol{c}} = \sum_I \nu(\boldsymbol{c}, I, \sigma) F_{(-1)^{\varepsilon_{\sigma(k)}} i_{\sigma(k)}, i_2} \otimes F_{(-1)^{\varepsilon_{\sigma(1)}} i_{\sigma(1)}, i_3} \otimes \cdots \otimes F_{(-1)^{\varepsilon_{\sigma(k-2)}} i_{\sigma(k-2)}, i_k} \otimes F_{(-1)^{\varepsilon_{\sigma(k-1)}} i_{\sigma(k-1)}, i_1}.$$

To prove the claim we note that by Theorem 4.1, the space $(V^{\otimes k} \otimes V^{*\otimes k})^{\mathfrak{q}(n)}$ is spanned by the set $\{\xi^{(k)}_{\sigma^{-1},\boldsymbol{c}} : \boldsymbol{c} \in \mathbf{C}l_k,\ \sigma \in S_k\}$, where

$$\xi^{(k)}_{\sigma^{-1},\boldsymbol{c}} = \sum_I \varsigma(\boldsymbol{c}, I, \sigma)(P^{\varepsilon_{\sigma(1)}} e_{i_{\sigma(1)}} \otimes \cdots \otimes P^{\varepsilon_{\sigma(k)}} e_{\sigma(i_k)}) \otimes (e^*_{i_1} \otimes \cdots \otimes e^*_{i_k}),$$

and the sum is over all the multi-subsets $I = \{i_1, i_2, \ldots, i_k\}$ of B of cardinality k. Let $\tau \in S_{2k}$ be the permutation defined by:

$$(\tau(1), \tau(2), \ldots, \tau(k), \tau(k+1), \tau(k+2), \ldots, \tau(2k)) = (3, 5, \ldots, 2k-1, 1, 2k, 2\ldots, 2k-2).$$

By applying τ to $(V \otimes V^*)^{\otimes k} = (V \otimes V^*) \otimes (V \otimes V^*) \otimes \cdots \otimes (V \otimes V^*)$, we get a $\mathfrak{q}(n)$-module isomorphism between $(V \otimes V^*)^{\otimes k}$ and $V^{\otimes k} \otimes V^{*\otimes k}$. So we get $((V \otimes V^*)^{\otimes k})^{\mathfrak{q}(n)} \cong (V^{\otimes k} \otimes V^{*\otimes k})^{\mathfrak{q}(n)}$. Under this isomorphism the element $\xi^{(k)}_{\sigma^{-1},\boldsymbol{c}} \in (V^{\otimes k} \otimes V^{*\otimes k})^{\mathfrak{q}(n)}$ maps to $\theta^{(k)}_{\sigma^{-1},\boldsymbol{c}}$ in $(\mathrm{End}_{\mathbb{C}}(V)^{\otimes k})^{\mathfrak{q}(n)}$, where

$$\theta^{(k)}_{\sigma^{-1},\boldsymbol{c}} = \sum_I \varsigma(\boldsymbol{c}, I, \sigma)(-1)^{p(J_I{}^{\sigma\otimes\boldsymbol{c}}, J_I)}(P^{\varepsilon_{\sigma(k)}} e_{i_{\sigma(k)}} \otimes e^*_{i_2}) \otimes (P^{\varepsilon_{\sigma(1)}} e_{i_{\sigma(1)}} \otimes e^*_{i_3}) \otimes \cdots \otimes (P^{\varepsilon_{\sigma(k-1)}} e_{\sigma(i_{k-1})} \otimes e^*_{i_1}).$$

Using the fact that $P^\varepsilon e_i = (-1)^{\varepsilon|e_i|} e_{-i}$ for each i and the identification $E_{i,j} = e_i \otimes e^*_j$, we get that

(5.2)
$$\theta^{(k)}_{\sigma^{-1},\boldsymbol{c}} = \sum_I \nu(\boldsymbol{c}, I, \sigma) E_{(-1)^{\varepsilon_{\sigma(k)}} i_{\sigma(k)}, i_2} \otimes E_{(-1)^{\varepsilon_{\sigma(1)}} i_{\sigma(1)}, i_3} \otimes \cdots \otimes E_{(-1)^{\varepsilon_{\sigma(k-2)}} i_{\sigma(k-2)}, i_k} \otimes E_{(-1)^{\varepsilon_{\sigma(k-1)}} i_{\sigma(k-1)}, i_1}.$$

Under ϕ the image of $\theta^{(k)}_{\sigma^{-1},\boldsymbol{c}}$ in $(\mathfrak{q}(n)^{\otimes k})^{\mathfrak{q}(n)}$ is the following element:

(5.3)
$$\eta^{(k)}_{\sigma^{-1},\boldsymbol{c}} = \sum_I \nu(\boldsymbol{c}, I, \sigma) F_{(-1)^{\varepsilon_{\sigma(k)}} i_{\sigma(k)}, i_2} \otimes F_{(-1)^{\varepsilon_{\sigma(1)}} i_{\sigma(1)}, i_3} \otimes \cdots \otimes F_{(-1)^{\varepsilon_{\sigma(k-2)}} i_{\sigma(k-2)}, i_k} \otimes F_{(-1)^{\varepsilon_{\sigma(k-1)}} i_{\sigma(k-1)}, i_1}.$$

Since ϕ surjective, we get that $\{\eta^{(k)}_{\sigma^{-1},\boldsymbol{c}} : \boldsymbol{c} \in \mathbf{Cl}_k,\ \sigma \in S_k\}$ spans $(\mathfrak{q}(n)^{\otimes k})^{\mathfrak{q}(n)}$. This means that the images of these elements under the canonical map $T(\mathfrak{q}(n))^{\mathfrak{q}(n)} \to U(\mathfrak{q}(n))^{\mathfrak{q}(n)}$ span the kth-graded component of $U(\mathfrak{q}(n))^{\mathfrak{q}(n)}$. By definition, the image of $\eta^{(k)}_{\sigma^{-1},\boldsymbol{c}}$ is the element $\zeta^{(k)}_{\sigma^{-1},\boldsymbol{c}}$. Consequently, the result follows. ∎

Let A be a finitely generated associative commutative $\mathbb{C}$-algebra with identity. For $a_1, \ldots, a_k \in A$, we define:

(5.4)
$$\zeta^{(k)}_{\sigma^{-1},\boldsymbol{c}}(a_1, \ldots, a_k) := \sum_{I:|I|=k} \nu(\boldsymbol{c}, I, \sigma)(F_{(-1)^{\varepsilon_{\sigma(k)}} i_{\sigma(k)}, i_2} \otimes a_1)(F_{(-1)^{\varepsilon_{\sigma(1)}} i_{\sigma(1)}, i_3} \otimes a_2) \ldots (F_{(-1)^{\varepsilon_{\sigma(k-2)}} i_{\sigma(k-2)}, i_k} \otimes a_{k-1})(F_{(-1)^{\varepsilon_{\sigma(k-1)}} i_{\sigma(k-1)}, i_1} \otimes a_k),$$

where the sum is over all the multi-subsets $I = \{i_1, i_2, \ldots, i_k\}$ of B of cardinality k.

With this notation, we have the following corollary.

Corollary 5.2 *Let A be a finitely generated associative commutative $\mathbb{C}$-algebra with identity, and $\boldsymbol{a} := (a_1, \ldots, a_k)$ be an arbitrary k-tuple. Then $U(\mathfrak{q}(n) \otimes A)^{\mathfrak{q}(n)}$ is spanned by the following set:*

$$\bigcup_{k\geq 0} \bigcup_{\boldsymbol{a}\in A} \left\{\zeta^{(k)}_{\sigma^{-1},\boldsymbol{c}}(a_1, \ldots, a_k) : \boldsymbol{c} \in \mathbf{Cl}_k,\ \sigma \in S_k\right\}.$$

Proof By the above Theorem, we see that $U(\mathfrak{q}(n))^{\mathfrak{q}(n)}$ is spanned by the set described in formula (5.1). Therefore, it follows from Proposition 3.1 that $U(\mathfrak{q}(n)\otimes A)^{\mathfrak{q}(n)}$ is spanned by the elements given above. ∎

5.1 Center of $\mathfrak{q}(n)$

In this subsection, we describe a set of algebra generators of $Z(\mathfrak{q}(n)) = U(\mathfrak{q}(n))^{\mathfrak{q}(n)}$. In what follows we denote the k-cycle $(12\dots k)$ by σ_k and $|e_i|$ by just $|i|$.

Theorem 5.3 *The centre $Z(\mathfrak{q}(n))$ is generated as an algebra by the following elements:*

$$C_n^{(k)} \coloneqq \sum_I (-1)^{|i_2|+\cdots+|i_k|} F_{i_1,i_2} F_{i_1,i_2} \dots F_{i_{k-1},i_k} F_{i_k,i_1},$$

where k runs over the set of all odd positive integers and I over all multi-subsets of B of cardinality k.

Proof Choose an arbitrary basis element $\boldsymbol{c} = c_1^{\varepsilon_1}\dots c_k^{\varepsilon_k}$ of $\mathbf{C}l_k$. Then formula (5.3) shows that the invariant associated with $\sigma_k^{-1}\otimes\boldsymbol{c}$ in $(\mathfrak{q}(n)^{\otimes k})^{\mathfrak{q}(n)}$ is given by

$$\eta^{(k)}_{\sigma_k^{-1},\boldsymbol{c}} = \sum_I \nu(\boldsymbol{c}, I, \sigma_k) F_{(-1)^{\varepsilon_1} i_1, i_2} \otimes \cdots \otimes F_{(-1)^{\varepsilon_k} i_k, i_1} \tag{5.5}$$

We begin by calculating explicitly the element in Equation (5.6). In this case, the sign $\nu(\boldsymbol{c}, I, \sigma_k)$ is given by

$$\nu(\boldsymbol{c}, I, \sigma_k) = (-1)^{\gamma(I,\sigma_k)+p(I,I)+p(J_I^{\sigma\otimes\boldsymbol{c}},J_I)+\sum_t \varepsilon_t|e_{i_t}|+\sum_{s>t}\varepsilon_s|e_{i_t}|}. \tag{5.6}$$

Since $\mathrm{Inv}(\sigma_k) = \{(1,k),(2,k),\dots,(k-1,k)\}$, it follows that $\gamma(I,\sigma_k) = |i_1|(|i_2| + \cdots |i_k|)$.

By definition

$$p(I,I) = |i_1|(|i_2| + \cdots |i_k|) + |i_2|(|i_3| + \cdots |i_k|) + \cdots + |i_{k-1}||i_k|.$$

We also have

$$\sum_{s>t} \varepsilon_s |e_{i_t}| + \sum_t \varepsilon_t |e_{i_t}| = |i_1|(\varepsilon_1 + \cdots + \varepsilon_k) + |i_2|(\varepsilon_2 + \cdots + \varepsilon_k) + \cdots + |i_k|\varepsilon_k.$$

Recall that $J_I{}^{\sigma\otimes\boldsymbol{c}} = ((-1)^{\varepsilon_{\sigma(k)}} i_{\sigma(k)}, (-1)^{\varepsilon_{\sigma(1)}} i_{\sigma(1)}, \dots, (-1)^{\varepsilon_{\sigma(k-1)}} i_{\sigma(k-1)})$ for $\sigma \in S_k$ where $J_I = (i_2,\dots,i_k,i_1)$. So for $\sigma = \sigma_k$, we get that

$$p(J_I{}^{\sigma_k\otimes\boldsymbol{c}}, J_I) = |i_2|(|(-1)^{\varepsilon_2} i_2| + \cdots + |(-1)^{\varepsilon_k} i_k|) + |i_3|(|(-1)^{\varepsilon_3} i_3| + \cdots + |(-1)^{\varepsilon_k} i_k|) + \cdots + |i_k||(-1)^{\varepsilon_k} i_k|.$$

Since $|(-1)^{\varepsilon} i| = \varepsilon + |i|$ and $|i|^2 = |i|$, we get that

$$p(J_I{}^{\sigma_k\otimes\boldsymbol{c}}, J_I) = (|i_2| + |i_3| + \cdots + |i_k|) + |i_2|(\varepsilon_2 + \cdots + \varepsilon_k) + \cdots + |i_{k-1}|(\varepsilon_{k-1} + \varepsilon_k) + |i_k|\varepsilon_k + p(I,I) - \gamma(I,\sigma_k). \tag{5.7}$$

$$= (|i_2| + |i_3| + \cdots + |i_k|) + \sum_{s>t} \varepsilon_s |e_{i_t}| + \sum_t \varepsilon_t |e_{i_t}| - (\varepsilon_1 + \cdots + \varepsilon_k)|i_1| + p(I,I) - \gamma(I,\sigma_k).$$

So $\nu(\boldsymbol{c}, I, \sigma_k) = (-1)^{(\varepsilon_1+\cdots+\varepsilon_k)|i_1| + (|i_2|+\cdots+|i_k|)}$.

In particular, if $\boldsymbol{c} = c_{t_1} \dots c_{t_m}$ with $0 \neq t_1 > \cdots > t_m$, then $\nu(\boldsymbol{c}, I, \sigma_k) = (-1)^{m|i_1|+|i_2|+\cdots+|i_k|}$. Using this information we can rewrite Equation (5.5) corresponding to $\sigma = \sigma_k$, and $\boldsymbol{c} = c_{t_1} \dots c_{t_m}$ as

(5.5a) $$\sum_I (-1)^{m|i_1|+|i_2|+\cdots+|i_k|} F_{i_1,i_2} \otimes \cdots \otimes F_{i_{t_1-1},i_{t_1}} \otimes F_{-i_{t_1},i_{t_1+1}} \otimes \cdots \otimes F_{i_{t_m-1},i_{t_m}} \otimes F_{-i_{t_m},i_{t_m+1}} \otimes \cdots \otimes F_{i_k,i_1}.$$

In the sum (5.5a) we first replace i_{t_1} by $-i_{t_1}$, and then using the relation $F_{i,j} = F_{-i,-j}$, we replace the factor $F_{i_{t_1-1},i_{t_1}} \otimes F_{-i_{t_1},i_{t_1+1}}$ in (5.5a) by $F_{-i_{t_1-1},i_{t_1}} \otimes F_{i_{t_1},i_{t_1+1}}$. Since for any $1 \leq q \leq k$, we have $(-1)^{|i_q|} = -(-1)^{|-i_q|}$, this replacement will have the sole effect of multiplication by -1. Performing the same operation inductively on i_{t_1-1} and then up to on i_2, we can bring the negative sign to i_1 while keeping the signs of all intermediate subscripts between i_2 and i_{t_1} positive. This implies that the sum (5.5a) is same as

(5.5b)
$$(-1)^{t_1-1} \sum_I (-1)^{m|i_1|+|i_2|+\cdots+|i_k|} F_{-i_1,i_2} \otimes \cdots \otimes F_{i_{t_1-1},i_{t_1}} \otimes F_{i_{t_1},i_{t_1+1}} \otimes \cdots \otimes F_{i_{t_m-1},i_{t_m}} \otimes F_{-i_{t_m},i_{t_m+1}} \otimes \cdots \otimes F_{i_k,i_1}.$$

Now, we repeat the same procedure in the sum (5.5b) with i_{t_2} and then inductively up to i_{t_m} to obtain that the sum (5.5a) is same as:

(5.5c) $$(-1)^{t_1+\cdots+t_m-m} \sum_I (-1)^{m|i_1|+|i_2|+\cdots+|i_k|} F_{(-1)^m i_1,i_2} \otimes F_{i_2,i_3} \otimes \cdots \otimes F_{i_k,i_1}.$$

We denote by f the sum in (5.5c) without the sign $(-1)^{t_1+\cdots+t_m-m}$. When m is an even integer, we have $f = C_n^{(k)}$. If m is odd we show that $f = 0$.

When m is odd, we have

$$f = \sum_I (-1)^{|i_1|+|i_2|+\cdots+|i_k|} F_{-i_1,i_2} \otimes F_{i_2,i_3} \otimes \cdots \otimes F_{i_k,i_1}.$$

By changing the signs of all the subscripts and using $F_{i,j} = F_{-i,-j}$, we get that $f = (-1)^k f$. Therefore, $f = 0$ when k is odd. We have the following identity:

$$|F_{-i_1,i_2}| = |-i_1| + |i_2| = 1 + |i_1| + |i_2| = |F_{i_2,i_3} \otimes \cdots \otimes F_{i_k,i_1}| + 1.$$

So, f can also be written as

$$f = \sum_I (-1)^{|i_1|+|i_2|+\cdots+|i_k|} F_{i_2,i_3} \otimes \cdots \otimes F_{i_k,i_1} \otimes F_{-i_1,i_2}.$$

Now we replace $i_2, \dots, i_k$ by their negatives to obtain $f = (-1)^{k-1} f$. This implies that $f = 0$ when k is even too.

To sum up, we have obtained that for any positive integer k, the element in (5.5) corresponding to $(\sigma_k^{-1} \otimes c_{t_1} \dots c_{t_m})$ is 0 when m is odd; and when m is even, it equals either $C_n^{(k)}$ or $-C_n^{(k)}$. Therefore, for each $k \geq 1$, we associate to it the element $C_n^{(k)}$.

Since the symmetric group S_k is generated by the cycles $(12) = \sigma_2$, and $(12 \dots k) = \sigma_k$, it follows that any element of $\cup_{k \geq 1} S_k$ can be written as a product of various σ_k's where $k \in \mathbb{N}$. The multiplication for the Sergeev algebra enables us to write $(\sigma_1\sigma_2 \otimes$

$\boldsymbol{c}) = (\sigma_1 \otimes 1) \cdot (\sigma_2 \otimes \boldsymbol{c})$, for any $\sigma_1, \sigma_2 \in S_k$. As Ψ_k is an algebra homomorphism, we get that $\eta^{(k)}_{\sigma,\boldsymbol{c}} = \eta^{(k)}_{\sigma_1,\boldsymbol{1}}\eta^{(k)}_{\sigma_2,\boldsymbol{c}}$. From here we conclude that the elements $\{C^k_n : k \in \mathbb{N}\}$ generate $Z(\mathfrak{q}(n))$ as an algebra. ∎

Corollary 5.4 *Let A be a finitely generated associative commutative $\mathbb{C}$-algebra with identity. Then $U(\mathfrak{q}(n) \otimes A)^{\mathfrak{q}(n)}$ is generated as an algebra by the following elements:*

$$C^{(k)}_n(a_1, \dots, a_k) = \sum_{I:|I|=k} (-1)^{|i_2|+\cdots+|i_k|}(F_{i_1,i_2} \otimes a_1) \otimes \cdots \otimes (F_{i_k,i_1} \otimes a_k),$$

where k runs over the set of all odd positive integers, $a_1, \dots, a_k \in A$ are arbitrary, and I runs over all multi-subsets of B of cardinality k.

Proof Follows directly from Proposition 3.1. ∎

Remark: Let $V(\lambda_1), \dots, V(\lambda_m)$ be finite-dimensional irreducible modules for $\mathfrak{q}(n)$. Let $A = \mathbb{C}[t, t^{-1}]$ be the algebra of Laurent polynomials in one variable, and $d_1, \dots, d_m$ be m distinct nonzero complex numbers. We denote the m-tuples $(\lambda_1, \dots, \lambda_m)$, and $(d_1, \dots, d_m)$, respectively by $\boldsymbol{\lambda}$ and $\boldsymbol{d}$. Then $V(\boldsymbol{\lambda}, \boldsymbol{d}) \coloneqq V(\lambda_1) \otimes \cdots \otimes V(\lambda_m)$ is a $\mathfrak{q}(n) \otimes A$-module, where the action is defined by

$$(x \otimes f(t)) \cdot (v_1 \otimes \cdots \otimes v_m) = f(d_1)(x \cdot v_1 \otimes \cdots \otimes v_m) + (-1)^{|x||v_1|} f(d_2)(v_1 \otimes x \cdot v_2 \otimes \cdots \otimes v_m)$$
$$+ \cdots + (-1)^{|x|(|v_1|+\cdots+|v_{m-1}|} f(d_m)(v_1 \otimes \cdots \otimes x \cdot v_m) \text{ for } x \otimes f(t) \in \mathfrak{q}(n) \otimes A \text{ and } v_i \in V(\lambda_i).$$

Then $V(\boldsymbol{\lambda}, \boldsymbol{d})$ with the above action is called an evaluation module. A proof similar to the proof given in [14, Section IV, Part C] shows that the $\mathfrak{q}(n) \otimes A$-module $V(\boldsymbol{\lambda}, \boldsymbol{d})$ is irreducible. Moreover, if $V(\boldsymbol{\lambda}, \boldsymbol{d}) = \bigoplus_{\mu \in \mathfrak{h}^*} V(\boldsymbol{\lambda}, \boldsymbol{d})_\mu$ is the weight space decomposition, where $V(\boldsymbol{\lambda}, \boldsymbol{d})_\mu = \{v \in V(\boldsymbol{\lambda}, \boldsymbol{d}) : h \cdot v = \mu(h)v, \ \forall \ h \in \mathfrak{h}\}$, then we have that

$$C^{(k)}_n(p_{i_1}(t), \dots, p_{i_k}(t)) \cdot V(\boldsymbol{\lambda}, \boldsymbol{d})^+_\mu \subseteq V(\boldsymbol{\lambda}, \boldsymbol{d})^+_\mu,$$

where $V(\boldsymbol{\lambda}, \boldsymbol{d})^+_\mu = \{v \in V(\boldsymbol{\lambda}, \boldsymbol{d})_\mu : \mathfrak{q}(n)^+ v = 0\}$, $\mathfrak{q}(n)^+$ is the sum of all root spaces, p_i is the Lagrange's polynomial defined by $p_i(t) = \prod_{k \neq i}(t - d_k)/(d_i - d_k)$, for $1 \le i \le m$, $i_1, \dots, i_k \in \{1, \dots, m\}$, and $C^{(k)}_n(p_1, \dots, p_k) = \sum_I (-1)^{|i_2|+\cdots+|i_k|} F_{i_1,i_2}(p_1) \otimes \cdots \otimes F_{i_k,i_1}(p_k)$, with $F_{p,q}(p_i) = F_{p,q} \otimes p_i$. This means that the central operators $C^{(k)}_n(p_{i_1}(t), \dots, p_{i_k}(t))$ send a highest weight vector to another highest weight vector. This is an effective method of producing new highest weight vectors from a given one. For details, we refer to [14, Section IV, Part D].

6 Invariants of the periplectic Lie superalgebra

6.1 Periplectic Lie superalgebra

The periplectic Lie superalgebra is the Lie superalgebra preserving an odd non-degenerate symmetric or skew-symmetric bilinear form. It is thus a superanalog of the orthogonal or symplectic Lie algebra.

For $n \in \mathbb{N}$, let $V = \mathbb{C}^n \oplus \mathbb{C}^n$ equipped with a non-degenerate odd symmetric bilinear form

$$\beta : V \otimes V \to \mathbb{C}, \beta(v,w) = \beta(w,v) \text{ and } \beta(v,w) = 0 \text{ for } |v| = |w|.$$

Then $\mathfrak{p}(V)$ is the Lie sub-superalgebra of $\mathrm{End}_{\mathbb{C}}(V)$ preserving β, i.e., satisfying

$$\beta(Xv,w) + (-1)^{|X||v|}\beta(v,Xw) = 0.$$

It can also be defined as the fixed point Lie sub-superalgebra of the involution $\sigma : \mathfrak{gl}(n|n) \to \mathfrak{gl}(n|n)$ by $E_{i,j} \mapsto -(-1)^{|i||j|+|i|}E_{n+j,n+i}$.

We note that $\mathfrak{p}(V)$ acts on V by matrix multiplication and on the r-fold tensor product $V^{\otimes r}$ by the rule given in Equation (2.1).

Since the bilinear form is nondegenerate on V, we can choose bases $v_1, v_2, \ldots, v_n$ for V_0 and $v_{n+1}, v_{n+2}, \ldots, v_{2n}$ for V_1 such that $\beta(v_{n+i}, v_j) = \beta(v_j, v_{n+i}) = \delta_{ij}$ and $\beta(v_i, v_j) = \beta(v_{n+i}, v_{n+j}) = 0$ for $i, j = 1, 2, \ldots, n$. The basis elements v_i and v_{n+i} are dual to each other with respect to the bilinear form. This form enables us to identify V and V^*. An element $f \in V^*$ is identified with $v_f \in V$ such that $f(u) = \beta(u, v_f)$ for every $u \in V$. The matrix of β with respect to this basis is

$$\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

We use the notation $v^*_{i+n} := v_i$ and $v^*_i := v_{i+n}$.

With respect to the above basis an element $X \in \mathfrak{p}(V)$ can be represented in matrix form as

$$X = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix},$$

where A, B, C are $n \times n$ matrices such that $B = B^T$ and $C = -C^T$. We identify $\mathfrak{p}(V)$ with the space $\mathfrak{p}(n)$ of matrices in the above form. The elements $E_{i,j} - E_{j+n,i+n}, E_{i,j+n} + E_{j,i+n}, E_{i+n,j} - E_{j+n,i}$, for $1 \le i, j \le n$ form a basis for $\mathfrak{p}(n)$.

There is a grading $\mathfrak{p}(n) = \mathfrak{p}(n)_{-1} \oplus \mathfrak{p}(n)_0 \oplus \mathfrak{p}(n)_1$, where $\mathfrak{p}(n)_0 \cong \mathfrak{gl}_n$, $\mathfrak{p}(n)_{-1} \cong \wedge^2((\mathbb{C}^n)^*)$ and $\mathfrak{p}(n)_1 \cong Sym^2(\mathbb{C}^n)$ as $\mathfrak{p}(n)_0$-modules. It is well known that the derived superalgebra $\mathfrak{sp}(n) = \mathfrak{p}(n) \cap \mathfrak{sl}(n,n) = [\mathfrak{p}(n), \mathfrak{p}(n)]$ is simple for $n \ge 3$. We have $\mathfrak{p}(n) = \mathbb{C}I' \oplus \mathfrak{sp}(n)$, where $I' = \left(\begin{smallmatrix} I_n & 0 \\ 0 & -I_n \end{smallmatrix}\right)$.

6.2

We note that the following element

$$c = \sum_{i=1}^{2n} (-1)^{|i|} v_i \otimes v^*_i,$$

where $v^*_i = v_{i+n}$ for $1 \le i \le n$, and $v^*_i = v_{i-n}$ when $n < i \le 2n$, is $\mathfrak{p}(V)$-invariant [9]. Then we have that

$$c^{\otimes k} = \sum_I (-1)^{|i_1|+|i_3|+\cdots+|i_{2k-1}|}(v_{i_1} \otimes v_{i_2}) \otimes (v_{i_3} \otimes v_{i_4}) \otimes \cdots \otimes (v_{i_{2k-1}} \otimes v_{i_{2k}}), \tag{6.1}$$

where the sum runs over all $2k$-multi-subsets $I = (i_1, i_2, \ldots, i_{2k})$ of the index set $\{1, 2, \ldots, 2n\}$ with the property that $|i_1 - i_2| = |i_3 - i_4| = \cdots = |i_{2k-1} - i_{2k}| = n$. As the action of $\mathfrak{p}(V)$ on $V^{\otimes 2k}$ commutes with the action of S_{2k}, all the $2k$-tensors $\theta_\sigma := \sigma \circ c^{\otimes k}$ are also $\mathfrak{p}(V)$ invariants for every $\sigma \in S_{2k}$. We have that

$$\theta_{\sigma^{-1}} = \sum_I (-1)^{\gamma(I,\sigma)+(|i_1|+|i_3|+\cdots+|i_{2k-1}|)} (v_{i_{\sigma(1)}} \otimes v_{i_{\sigma(2)}}) \otimes (v_{i_{\sigma(3)}} \otimes v_{i_{\sigma(4)}}) \otimes \cdots \otimes (v_{i_{\sigma(2k-1)}} \otimes v_{i_{\sigma(2k)}}),$$

where I is as described above, and $\gamma(I, \sigma)$ is the sign resulting from the action of σ^{-1} on $v_I = v_{i_1} \otimes \cdots \otimes v_{i_{2k}}$. It is known that the space $(V^{\otimes 2k})^{\mathfrak{p}(V)}$ is spanned by the set $\{\theta_\sigma : \sigma \in S_{2k}\}$ (see [4],[19]).

We have a $\mathfrak{p}(V)$-module isomorphism $\phi_1 : V \otimes V \cong \mathfrak{gl}(V)$ which sends $v_i \otimes v_{i+n}$ to $(-1)^{|i|} E_{i,i}$ and $v_{i+n} \otimes v_i$ to $(-1)^{|i+n|} E_{i+n,i+n}$ for $1 \le i \le n$. There is a surjective homomorphism $\phi_2 : \mathfrak{gl}(V) \to \mathfrak{p}(V)$, that sends a basis vector $E_{i,j}$ of $\mathfrak{gl}(V)$ to a certain basis vector of $\mathfrak{p}(V)$ by the following rule:

$$\phi_2(E_{i,j}) = \begin{cases} E_{i,j} - E_{j+n,i+n} & \text{for} \quad 1 \le i, j \le n \\ E_{i,j} - E_{j-n,i-n} & \text{for} \quad n < i, j \le 2n \\ E_{i,j} + E_{j-n,i+n} & \text{for} \quad 1 \le i \le n \quad \text{and} \quad n < j \le 2n \\ E_{i,j} - E_{j+n,i-n} & \text{for} \quad n < i \le 2n \quad \text{and} \quad 1 \le j \le n. \end{cases} \tag{6.2}$$

We denote the composition $\phi_2 \circ \phi_1 : V \otimes V \to \mathfrak{p}(V)$ by ϕ. Then,

$$\phi(v_i \otimes v_j) = \begin{cases} (-1)^{|i|} E_{i,j+n} + E_{j,i+n} & \text{for} \quad 1 \le i, j \le n \\ (-1)^{|i|} E_{i,j-n} - E_{j,i-n} & \text{for} \quad n < i, j \le 2n \\ (-1)^{|i|} E_{i,j-n} - E_{j,i+n} & \text{for} \quad 1 \le i \le n \quad \text{and} \quad n < j \le 2n \\ (-1)^{|i|} E_{i,j+n} - E_{j,i-n} & \text{for} \quad n < i \le 2n \quad \text{and} \quad 1 \le j \le n. \end{cases} \tag{6.3}$$

The map ϕ induces an epimorphism $\phi^{\otimes k} : (V \otimes V)^{\otimes k} = V^{\otimes 2k} \to \mathfrak{p}(V)^{\otimes k}$ defined by

$$\phi^{\otimes k}((v_1 \otimes v_2) \otimes \cdots \otimes (v_{2k-1} \otimes v_{2k})) = \phi(v_1 \otimes v_2) \otimes \cdots \otimes \phi(v_{2k-1} \otimes v_{2k}),$$

which further induces a surjective map η on the $\mathfrak{p}(V)$-invariants:

$$\eta : (V^{\otimes 2k})^{\mathfrak{p}(V)} \to (\mathfrak{p}(V)^{\otimes k})^{\mathfrak{p}(V)}.$$

We now calculate the image of the element $c \in V \otimes V$ under ϕ. Using formula (6.3), we find that

$$\phi(c) = \phi\Big(\sum_{i=1}^n (-1)^{|i|} v_i \otimes v_{i+n}\Big) + \phi\Big(\sum_{j=1}^n (-1)^{|j+n|} v_{j+n} \otimes v_j\Big)$$
$$= \sum_{i=1}^n (E_{i,i} - E_{n+i,n+i}) + \sum_{j=1}^n (E_{n+j,n+j} - E_{j,j}) = 0.$$

Therefore, $\eta(c^{\otimes k}) = \phi^{\otimes k}(c^{\otimes k}) = 0$. Since the action of $\mathfrak{p}(V)$ is compatible with the action of the symmetric group S_{2k} on $V^{\otimes 2k}$ we conclude that $\eta(\theta_{\sigma^{-1}}) = 0$ for every $\sigma \in S_{2k}$. This means $(\mathfrak{p}(V)^{\otimes k})^{\mathfrak{p}(V)} = 0$, for every positive integer k, and consequently, $T(\mathfrak{p}(V))^{\mathfrak{p}(V)} = \mathbb{C}$. Thus by Proposition 3.1, we obtain the following.

Proposition 6.1 *Let A be an associative commutative finitely generated $\mathbb{C}$-algebra with identity. Then $U(\mathfrak{p}(V)\otimes A)^{\mathfrak{p}(V)} = \mathbb{C}$. In particular, for $A = \mathbb{C}$, we get that the center of $U(\mathfrak{p}(V))$ consists only of scalars.*

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