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BIFURCATION PROPERTIES FOR A CLASS OF CHOQUARD EQUATION IN WHOLE $\mathbb{R}^3$†

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Abstract. This paper concerns the study of some bifurcation properties for the following class of Choquard-type equations:

$$\begin{cases} -\Delta u = \lambda f(x)\left[u + (I_\alpha * f(\cdot)H(u))\,h(u)\right], & \text{in } \mathbb{R}^3, \\ \lim_{|x|\to\infty} u(x) = 0, \quad u(x) > 0, \quad x \in \mathbb{R}^3, \quad u \in D^{1,2}(\mathbb{R}^3), \end{cases} \tag{P}$$

where $I_\alpha(x) = 1/|x|^\alpha$, $\alpha \in (0, 3)$, $\lambda > 0$, $f : \mathbb{R}^3 \to \mathbb{R}$ is a positive continuous function and $h : \mathbb{R} \to \mathbb{R}$ is a bounded Hölder continuous function. The main tools used are Leray–Schauder degree theory and a global bifurcation result due to Rabinowitz.

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1. Introduction and main result. The main goal of this paper is to study some bifurcation properties for the following class of Choquard-type equations:

$$\begin{cases} -\Delta u = \lambda f(x)\left[u + (I_\alpha * f \cdot H(u))\,h(u)\right], & \text{in } \mathbb{R}^3, \\ \lim_{|x|\to\infty} u(x) = 0, \quad u(x) > 0, \quad x \in \mathbb{R}^3, \quad u \in D^{1,2}(\mathbb{R}^3), \end{cases} \tag{P}$$

where $I_\alpha(x) = 1/|x|^\alpha$, $\lambda \in \mathbb{R}$ and $\alpha \in (0, 3)$.

The Choquard equation

$$-\Delta u + u = (I_2 * |u|^2)u, \quad \text{in} \quad \mathbb{R}^3, \tag{1.1}$$

frequently appears in the context of various physical models. It seems to originate from Fröhlich and Pekar's model of the polaron, where free electrons in an ionic lattice interact with phonons associated with deformations of the lattice or with the polarisation that it creates on the medium (interaction of an electron with its own hole); see [**7**, **8**, **21**]. The Choquard equation was also introduced by Ph. Choquard in 1976 in the modelling of a one-component plasma [**12**].

In recent years, various papers related to this equation and some variants have been published. In the autonomous Choquard equation case

$$-\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u, \quad \text{in} \quad \mathbb{R}^N, \tag{1.2}$$

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where $\alpha \in (0, N)$ and $p > 1$, the reader can find some results in Alves and Yang [5], Efinger [6], Genev and Venkok [9], Ghergu and Taliaferro [10], Ma and Zhao [14], Moroz and Van Schaftingen [16, 17], Stuart [24] and their references. In the non-autonomous Choquard equation case

$$-\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u, \quad \text{in} \quad \mathbb{R}^N, \tag{1.3}$$

where V is a non-constant potential, we cite Alves, Figueiredo and Yang [1], Alves, Nóbrega and Yang [2], Mercuri, Moroz and Van Schaftingen [15], Moroz and Van Schaftingen [19], Sun and Zhang [25], Yang and Wei [26] and their references.

For a more complete list of what was done with the Choquard equations, see the paper of Moroz and Van Schaftingen a bibliography review [18].

After a bibliography review, we observe that most of the results related to the Choquard equations were made using variational methods, and that there are few results using non-variational methods for Choquard-type equation in whole $\mathbb{R}^N$, we found only the paper due to Küpper, Zhang and Xia [11], where a few of bifurcation theories have been used to prove the main result.

In this paper, we proved a result related to Choquard equation via non-variational methods, precisely via Leray–Schauder degree theory and the global bifurcation result due to Rabinowitz [22]. We would like to point out that we focus on the case $N = 3$ since such a problem stems from physics, although the argument can be carried out for the space with dimension $N \geq 4$ by supposing $\alpha \in (0, \min\{4, N\})$.

Before stating our main results, we need to fix the assumptions on the functions f and h. In the sequel, $f : \mathbb{R}^3 \to \mathbb{R}$ is a positive continuous function that verifies the following conditions:

(f_1) There exists a bounded function $P \in C(\mathbb{R}^+, \mathbb{R}^+) \cap L^1(\mathbb{R}^N)$ such that $0 < f(x) \leq P(|x|)$, for all $x \in \mathbb{R}^3$.

Moreover, P verifies

(P_1) $P(|\,.\,|) \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$.

Related to the function $h : \mathbb{R} \to \mathbb{R}^+$, we assume that it is a bounded Hölder continuous function satisfying the following conditions:

(h_1) $h(t) \geq 0$, for all $t \in \mathbb{R}$;

(h_2) $\displaystyle\lim_{t\to 0} \frac{h(t)}{t} = a > 0$;

(h_3) $\displaystyle\limsup_{t\to +\infty} \frac{|h(t)|}{t} < +\infty$.

Our main results are the following.

THEOREM 1.1. *Assume that* (f_1), (P_1) *and* $(h_1)-(h_3)$ *hold. Then,* λ_1 *is a bifurcation point from the trivial solutions for* (P), *where* λ_1 *is the first eigenvalue of the linear problem*

$$\begin{cases} -\Delta u = \lambda f(x)u, & \textit{in} \quad \mathbb{R}^3, \\ \displaystyle\lim_{|x|\to+\infty} u(x) = 0. \end{cases} \tag{AP}$$

More precisely, if Σ *denotes the closure of the positive solutions of* (P), *then there exists an unbounded component* $\Sigma_1 \subset \Sigma$ *of solutions of* (P) *emanating from* $(\lambda_1, 0)$. *Furthermore,* $(\lambda_1, 0)$ *is the unique point with this property.*

The paper is organised as follows: in Section 2, we study the existence of eigenvalue for a special linear eigenvalue problem. In Section 3 is studied the regularity of the solutions, because it is crucial to get some estimates, while in Section 4 we prove our main result.

Notations

- ω_3 is the volume of the unit ball in $\mathbb{R}^3$.
- Γ is the fundamental solution of Laplace equation in $\mathbb{R}^3$.
- χ_B is the characteristic function of B.
- $B_r(x)$ denotes the ball centred at the x with radius $r > 0$ in $\mathbb{R}^3$.
- $L^s(\mathbb{R}^3)$, for $1 \leq s \leq \infty$, denotes the Lebesgue space with usual norm denoted by $|u|_s$.
- $L^2_H(\mathbb{R}^3)$ denotes the class of real-valued Lebesgue measurable functions u such that

$$\int_{\mathbb{R}^3} H(x)|u(x)|^2 dx < \infty.$$

  $L^2_H(\mathbb{R}^3)$ is a Hilbert space endowed with the inner product

$$(u, v)_{2,H} = \int_{\mathbb{R}^3} H(x)u(x)v(x)dx, \quad \forall u, v \in L^2_H(\mathbb{R}^3).$$

  The norm associated with this inner product will be denoted by $|\cdot|_{2,H}$.
- $D^{1,2}(\mathbb{R}^3)$ denotes the Sobolev space endowed with inner product

$$(u, v)_{1,2} = \int_{\mathbb{R}^3} \nabla u \nabla v dx, \quad u, v \in D^{1,2}(\mathbb{R}^3).$$

  The norm associated with this inner product will be denoted by $\|\cdot\|_{1,2}$.
- We denote by E the Banach space given by

$$E := \{u \in C(\mathbb{R}^3); \quad \sup_{x\in\mathbb{R}^3} |u(x)| < \infty\},$$

  endowed with the norm $|\cdot|_\infty$.
- If u is a measurable function, we denote by u^+ and u^- the positive and negative parts of u, respectively, which are given by

$$u^+ = \max\{u, 0\} \quad \text{and} \quad u^- = \max\{-u, 0\}.$$

- $C, C_1, \ldots, C_n$ are real constants.

2. A linear solution operator. In this section, we study the existence and properties of an important operator, which will be used to prove some bifurcation properties for problem (P).

Initially, due to its importance, we show that the embedding $D^s(\mathbb{R}^3) \hookrightarrow L^2_f(\mathbb{R}^3)$ is compact.

LEMMA 2.1. *Assume $(f_1) - (P_1)$. Then, the embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^2_f(\mathbb{R}^3)$ is compact.*

Proof. Let $\{u_n\}$ be a sequence in $D^{1,2}(\mathbb{R}^3)$ with $u_n \rightharpoonup 0$ in $D^{1,2}(\mathbb{R}^3)$ and $u_n(x) \to 0$ $a.e.$ in $\mathbb{R}^3$. As the space $D^{1,2}(\mathbb{R}^3)$ is naturally embedding in $L^6(\mathbb{R}^3)$, we have $\{u_n^2\}$ is a bounded sequence in $L^3(\mathbb{R}^3)$. Thus, up to a subsequence if necessary,

$$u_n^2 \rightharpoonup 0 \text{ in } L^3(\mathbb{R}^3),$$

or equivalently,

$$\int_{\mathbb{R}^3} u_n^2 \varphi dx \to 0, \quad \forall \varphi \in L^{\frac{3}{2}}(\mathbb{R}^3).$$

Since $(f_1)-(P_1)$ imply that $f \in L^r(\mathbb{R}^3)$ for all $r \geq 1$, it follows that

$$\int_{\mathbb{R}^3} f(x) u_n^2 dx \to 0.$$

This shows that $u_n \to 0$ in $L^2_f(\mathbb{R}^3)$. □

As a by-product of the last lemma, we can apply Riesz's Theorem to deduce that for each $v \in L^2_P(\mathbb{R}^3)$, there is an unique solution $u \in D^{1,2}(\mathbb{R}^3)$ of the problem

$$\begin{cases} -\Delta u = f(x) v \quad \text{in} \quad \mathbb{R}^3, \\ \lim\limits_{|x| \to +\infty} u(x) = 0. \end{cases} \tag{$(WLP)_v$}$$

From this, we can define a *solution operator* $S : L^2_P(\mathbb{R}^3) \to L^2_P(\mathbb{R}^3)$ such that $S(v) = u$, where u is the unique solution of the above weight linear problem $(WLP)_v$. By Lemma 2.1, S is a compact self-adjoint operator, then by spectral theory there exists a complete orthonormal basis $\{u_n\}$ of $L^2_P(\mathbb{R}^3)$ and a corresponding sequence of positive real numbers $\{\lambda_n\}$ with $\lambda_n \to \infty$, when $n \to \infty$, such that

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$$

and

$$-\Delta u_n = \lambda_n f(x) u_n, \quad \text{in} \quad \mathbb{R}^3.$$

Moreover, using Lagrange multiplier, we have the following characterisation for λ_1:

$$\lambda_1 = \inf_{v \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |\nabla v|^2 dx}{\int_{\mathbb{R}^3} f(x) |v(x)|^2 dx}.$$

The above identity is crucial to show that λ_1 is a simple eigenvalue and that a corresponding eigenfunction φ_1 can be chosen positive in $\mathbb{R}^3$.

Arguing as in Alves, de Lima and Souto [3], we have that $S(E) \subset E_0$, where

$$E_0 = \{u \in E; \quad \sup_{x \in \mathbb{R}^3} \{|x| \cdot |u(x)|\} < \infty\}. \tag{2.1}$$

Since $E_0 \subset E$, the operator $T := S|_E : E \to E$ is well defined, and it is a linear compact operator. Moreover, it is clear that $\lim_{|x| \to \infty} S(u)(x) = 0$, for all $u \in E$.

3. Regularity results. The main goal of this section is to prove some regularity results for problem (P). Have this in mind, we consider the following class of problem:

$$\begin{cases} -\Delta u = \lambda f(x) g(x, u), \ \text{in } \mathbb{R}^3, \\ \lim_{|x| \to \infty} u(x) = 0, \end{cases} \tag{P'}$$

where, for each $u \in D^{1,2}(\mathbb{R}^3)$, we set

$$g(x, u) = u^+(x) + (I_\alpha * f(\cdot) H(u(\cdot)))\, h(u^+(x)). \tag{3.1}$$

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We claim that (P') is equivalent to (P), because all solutions of (P') are positive. Indeed, given $u \in D^{1,2}(\mathbb{R}^3)$, a nonzero solution to (P'), we have

$$\int_{\mathbb{R}^3} \nabla u \nabla \varphi dx = \lambda \int_{\mathbb{R}^3} f(x) g(x, u) \varphi \, dx, \ \ \forall \varphi \in D^{1,2}(\mathbb{R}^3). \tag{3.2}$$

Fixing $\varphi = u^-$, we get

$$\|u^-\|_{1,2}^2 = \int_{\mathbb{R}^3} \nabla u \nabla u^- dx = \lambda \int_{\mathbb{R}^3} f(x) g(x, u) u^-(x) \, dx = 0. \tag{3.3}$$

Thus, $u^- = 0$ and, consequently, $u = u^+ \geq 0$. Since u is nonzero and $g(x, u) \geq 0$, the maximum principle ensures that $u > 0$.

The next step is to prove the equivalence between (P') and the functional equation

$$u = \lambda T(\phi(u)), \qquad \text{in} \qquad E, \tag{Q}$$

where $\phi(u) := g(\cdot, u(\cdot))$ for all $u \in E$.

In our opinion, it is not clear that a solution u of (P') is also a solution of (Q), because we need to show some results of regularity to guarantee that u belongs to E. To achieve our goals, we need to establish the following lemma.

LEMMA 3.1. *For all $v \in E$, we have that $\phi(v) \in E$, that is, $\phi(E) \subset E$.*

Proof. By definition of g and the hypotheses on h, it is enough to show that function

$$v(x) := \int_{\mathbb{R}^3} \frac{f(y) H(u(y))}{|x-y|^\alpha} dy, \qquad x \in \mathbb{R}^3, \tag{3.4}$$

belongs to E, for all $u \in E$.

First of all, we will prove that v is a continuous function in $\mathbb{R}^3$. Fixed $x_0 \in \mathbb{R}^3$, for all $x \in \mathbb{R}^3$ it holds

$$|v(x) - v(x_0)| = \left| \int_{\mathbb{R}^3} \left(\frac{f(y) H(u(y))}{|x-y|^\alpha} - \frac{f(y) H(u(y))}{|x_0 - y|^\alpha} \right) dy \right|.$$

Given $\delta_1 > 0$ and $x \in B_{\delta_1/2}(x_0)$, we have $B_{\delta_1}(x_0) \subset B_{2\delta_1}(x)$ and

$$\begin{aligned}
\left| \int_{B_{\delta_1}(x_0)} \left(\frac{f(y)}{|x-y|^\alpha} - \frac{f(y)}{|x_0-y|^\alpha} \right) dy \right| &\leq \int_{B_{\delta_1}(x_0)} \frac{P(y)}{|x-y|^\alpha} dy + \int_{B_{\delta_1}(x_0)} \frac{P(y)}{|x_0-y|^\alpha} dy \\
&\leq \int_{B_{\delta_1}(x_0)} \frac{P(y)}{|x_0-y|^\alpha} dy \\
&\quad + \int_{B_{2\delta_1}(x)} \frac{P(y)}{|x-y|^\alpha} dy \\
&\leq C_1 |P|_\infty \delta_1^{3-\alpha} + C_2 |P|_\infty (2\delta_1)^{3-\alpha}.
\end{aligned}$$

From this, given $\epsilon > 0$, let us fix $\delta_1 > 0$ verifying

$$C_1 |P|_\infty \delta_1^{3-\alpha} + C_2 |P|_\infty (2\delta_1)^{3-\alpha} < \epsilon/2,$$

consequently

$$\left| \int_{B_{\delta_1}(x_0)} \left(\frac{f(y)}{|x-y|^\alpha} - \frac{f(y)}{|x_0-y|^\alpha} \right) dy \right| < \frac{\epsilon}{2}. \tag{3.5}$$

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On the other hand, note that

$$\left|\int_{\mathbb{R}^3\setminus B_{\delta_1}(x_0)}\left(\frac{f(y)}{|x-y|^\alpha}-\frac{f(y)}{|x_0-y|^\alpha}\right)dy\right|\le\int_{\mathbb{R}^3\setminus B_{\delta_1}(x_0)}\left(\frac{P(y)}{|x-y|^\alpha}+\frac{P(y)}{|x_0-y|^\alpha}\right)dy.$$

As $x\in B_{\delta_1/2}(x_0)$ and $|y-x_0|>\delta_1$, we derive

$$|x-y|\ge|y-x_0|-|x_0-x|\ge\frac{\delta_1}{2},$$

then

$$\frac{1}{|x-y|^\alpha}\le\left(\frac{2}{\delta_1}\right)^\alpha,$$

and

$$\frac{1}{|x_0-y|^\alpha}\le\left(\frac{1}{\delta_1}\right)^\alpha.$$

Hence,

$$\frac{P(y)}{|x-y|^\alpha}+\frac{P(y)}{|x_0-y|^\alpha}\le CP(y),\ \forall y\in\mathbb{R}^3\setminus B_{\delta_1}(x_0)\text{ and }\forall x\in B_{\delta_1/2}(x_0).$$

By Lebesgue's Theorem

$$\lim_{x\to x_0}\int_{\mathbb{R}^3\setminus B_{\delta_1}(x_0)}\left|\frac{f(y)}{|x-y|^\alpha}-\frac{f(y)}{|x_0-y|^\alpha}\right|dy=0.$$

The last limit together with (3.5) implies that there exists $\delta\in(0,\delta_1/2)$ such that

$$\int_{\mathbb{R}^3}\left|\frac{f(y)}{|x-y|^\alpha}-\frac{f(y)}{|x_0-y|^\alpha}\right|dy<\frac{\epsilon}{2},\quad\text{for}\quad|x-x_0|<\delta,$$

showing that v is continuous at x_0. As x_0 is arbitrary, v is continuous in $\mathbb{R}^3$. In order to conclude that $v\in E$, firstly we would like to point out that the above previous arguments ensure that the functions

$$\Gamma(x)=\int_{\mathbb{R}^3}\frac{P(y)}{|x-y|^\alpha}dy\quad\text{and}\quad\beta(x)=\int_{\mathbb{R}^3}\frac{P(y)}{|x-y|}dy,$$

belong to $C(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$. This information permits to conclude that

$$|v(x)|\le C\int_{\mathbb{R}^3}\frac{f(y)|u(y)|^2}{|x-y|^\alpha}dy\le C\|u\|_\infty^2\Gamma(x),\qquad\forall x\in\mathbb{R}^N,$$

and so,

$$|v(x)|\le C|u|_\infty^2,\qquad\forall x\in\mathbb{R}^3,$$

for some $C>0$, proving the desired result. □

In the next theorem, we will show that any solution of (P') belongs to $L^\infty(\mathbb{R}^3)$. However, a key point in the proof is the following inequality.

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PROPOSITION 3.1. [13] [Hardy–Littlewood–Sobolev inequality]:
Let $s, r > 1$ and $0 < \alpha < N$ with $1/s + \alpha/N + 1/r = 2$. If $U \in L^s(\mathbb{R}^N)$ and $V \in L^r(\mathbb{R}^N)$, then there exists a sharp constant $C(s, N, \mu, r)$, independent of U and V such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{U(x)V(y)}{|x-y|^\alpha}\,dxdy \le C(s, N, \alpha, r)|U|_s|V|_r.$$

By using the above inequality, if $U, V \in L^{\frac{6}{6-\alpha}}(\mathbb{R}^3)$, we have that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{U(x)V(y)}{|x-y|^\alpha}\,dxdy \le C(N, \mu)|U|_{\frac{6}{6-\alpha}}|V|_{\frac{6}{6-\alpha}},$$

and so, for each $U \in L^{\frac{6}{6-\alpha}}(\mathbb{R}^3)$, the linear operator $J : L^{\frac{6}{6-\alpha}}(\mathbb{R}^3) \to \mathbb{R}$ given by

$$J(V) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{U(x)V(y)}{|x-y|^\alpha}\,dxdy,$$

is continuous. Thereby, by Riesz's representation theorem, we have that $I_\alpha * U \in L^{\frac{6}{\alpha}}(\mathbb{R}^3)$.

We would like to point out that by (f_1), (P_1), (h_2) and (h_3), if $u \in D^{1,2}(\mathbb{R}^3)$, we have that $f(\cdot)H(u(\cdot)) \in L^{\frac{6}{6-\alpha}}(\mathbb{R}^3)$, and so, the above remarks can be applied for the function $U(x) = f(x)H(u(x))$ for $x \in \mathbb{R}^3$.

THEOREM 3.1. *Let $v \in D^{1,2}(\mathbb{R}^3)$ a weak solution of problem*

$$-\Delta v = \lambda f(x)g(x, v), \quad in \quad \mathbb{R}^3. \tag{3.6}$$

Then, there exists $M > 0$ such that

$$|v|_\infty \le M|v|_6.$$

Proof. Let $u \in D^{1,2}(\mathbb{R}^3)$ be a solution of (3.6), then u satisfies the equation

$$-\Delta u = \lambda a(x)u, \quad x \in \mathbb{R}^3, \tag{3.7}$$

where

$$a(x) = \begin{cases} f(x) + (I_\alpha * f \cdot H(u)) \cdot f(x)\frac{h(u)}{u}, & u(x) \neq 0 \\ 0, \quad u(x) = 0. \end{cases}$$

By the Hardy–Littlewood–Sobolev inequality, we know that $(I_\alpha * f \cdot H(u)) \in L^{6/\alpha}(\mathbb{R}^3)$. On the other hand, by conditions on f and h, we have $f(x) \cdot \frac{h(u)}{u} \in L^{6/(4-\alpha)}(\mathbb{R}^3)$. Consequently, $a(x) \in L^{3/2}(\mathbb{R}^3)$. Indeed, since

$$|a(x)|^{3/2} \le C\left(|f(x)|^{3/2} + |I_\alpha * f \cdot H(u)|^{3/2} \cdot \left|f(x) \cdot \frac{h(u)}{u}\right|^{\frac{3}{2}}\right),$$

and

$$|I_\alpha * f \cdot H(u)|^{3/2} \in L^{4/\alpha}(\mathbb{R}^3), \qquad \left|f(x) \cdot \frac{h(u)}{u}\right|^{3/2} \in L^{4/(4-\alpha)}(\mathbb{R}^3),$$

we have $|a|^{3/2} \in L^1(\mathbb{R}^3)$, that is, $a(x) \in L^{3/2}(\mathbb{R}^3)$. Now, arguing as in Struwe [23],

$$u \in L^p(\mathbb{R}^3), \quad \forall p \in [6, \infty) \quad \text{and} \quad u \in L^p_{loc}(\mathbb{R}^3), \quad \forall p \in [1, \infty).$$

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If $p \geq 6$, for all $q \in (1, 4/\alpha)$, we have

$$|I_\alpha * f \cdot H(u)|^{3q/2} \in L^t(\mathbb{R}^3) \quad \text{and} \quad \left| f(x) \cdot \frac{h(u)}{u} \right|^{3q/2} \in L^{t'}(\mathbb{R}^3),$$

where $t = \frac{4}{q\alpha}$ and $t' = \frac{t}{t-1} = \frac{4}{4-q\alpha}$, and so, $a(x) \in L^{3q/2}(\mathbb{R}^3)$. Therefore, by Alves and Souto [4, Proposition 2.6], there is $M = M(q, |a|_q) > 0$ such that

$$|v|_\infty \leq M|v|_6.$$

□

Now, the equivalence between (P') and (Q) follows by the following result.

Corollary 3.1. *If $v \in D^{1,2}(\mathbb{R}^3)$ is a weak solution of (3.6), then $v \in C(\mathbb{R}^3)$.*

Proof. By Theorem 3.1, $v \in L^\infty(\mathbb{R}^3)$. Then, as v can be rewritten of the form

$$v(x) = \int_{\mathbb{R}^3} \frac{\lambda f(x) g(x, v)}{|x-y|} dy, \qquad \text{(see [3])}, \tag{3.8}$$

the continuity of v follows as in the proof of Lemma 3.1. □

To conclude this section, we have the following lemma that will be very important in the next section.

Lemma 3.2. *The operator $\phi : E \to E$, given by $\phi(u) := g(\cdot, u(\cdot))$, is continuous.*

Proof. Let $(u_n) \subset E$ and $u \in E$, such that $u_n \to u$, when $n \to \infty$. We need prove that $\phi(u_n) \to \phi(u)$, when $n \to \infty$. By definition of g and hypotheses on h, it is enough to show that

$$\phi_{u_n}(x) := \int_{\mathbb{R}^3} \frac{f(y) H(u_n(y))}{|x-y|^\alpha} dy \to \phi_u(x) := \int_{\mathbb{R}^3} \frac{f(y) H(u(y))}{|x-y|^\alpha} dy \quad \text{in} \quad L^\infty(\mathbb{R}^3), \tag{3.9}$$

where $\phi_{u_n} := \phi(u_n)$ and $\phi_u := \phi(u)$. Indeed, note that

$$\begin{aligned} |\phi_{u_n}(x) - \phi_u(x)| &= \left| \int_{\mathbb{R}^3} \frac{f(y)H(u_n(y)) - f(y)H(u(y))}{|x-y|^\alpha} dy \right| \\ &\leq \int_{\mathbb{R}^3} \frac{f(y)|H(u_n(y)) - H(u(y))|}{|x-y|^\alpha} dy \\ &\leq C\|u_n - u\|_\infty \int_{\mathbb{R}^3} \frac{P(y)}{|x-y|^\alpha} dy \\ &\leq C_1 \|u_n - u\|_\infty. \end{aligned}$$

Consequently,

$$\phi_{u_n} \to \phi_u \quad \text{in} \quad L^\infty(\mathbb{R}^3) \quad \text{when} \quad u_n \to u_n \quad \text{in} \quad L^\infty(\mathbb{R}^3), \tag{3.10}$$

showing the continuity of ϕ. □

4. Bifurcation result. In this section, based in the Global Bifurcation theorem due Rabinowitz [22], we will obtain some results involving the existence of solutions for the non-linear problem (P), by considering bifurcation of solutions from the trivial solutions.

4.1. Bifurcation from the trivial solutions. First of all, note that $T \circ \phi : E \to E$ is a compact operator, because $\phi : E \to E$ is continuous (see Lemma 3.2) and T is compact (see Section 2). In what follows, we set $F : [0, \infty) \times E \to E$ by

$$F(\lambda, u) := F_\lambda(u) := u - \lambda T[\phi(u)], \text{ for all } u \in E \text{ and } \lambda \geq 0. \tag{4.1}$$

We recall that λ is said a bifurcation point of $F(t, u) = 0$, whenever

$$(\lambda, 0) \in \Sigma = \overline{\{(t, u) \in \mathbb{R} \times E : F(t, u) = 0, u \neq 0\}}.$$

The reader is invited to see that $\Sigma \subset \mathbb{R}^+ \times E$ is the closure of the set formed by nontrivial solutions of (Q).

In the sequel, $ind(F_\lambda, 0)$ denotes the Leray–Schauder index of F_λ at 0 given by

$$ind(F_\lambda, 0) = \lim_{\varepsilon \to 0} deg(F_\lambda, B_\varepsilon(0), 0).$$

Our first goal is to show that the unique possible bifurcation point of $F(t, u)$ is λ_1. To this end, we need to prove some preliminary results.

PROPOSITION 4.1. *If u is a positive solution of the problem (P), then $\lambda < \lambda_1$.*

Proof. Let u be a positive solution to problem (P), then

$$\int_{\mathbb{R}^3} \nabla u \nabla \varphi_1 dx = \lambda \int_{\mathbb{R}^3} f(x) u \varphi_1 dx + \lambda \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f(x) f(y) H(u(y)) h(u(x)) \varphi_1(x)}{|x-y|^\alpha} dx dy.$$

Since φ_1 is a eigenfunction associated with λ_1 and $h(t) > 0$ for $t > 0$, it follows that

$$\lambda_1 \int_{\mathbb{R}^3} f(x) u \varphi_1 dx > \lambda \int_{\mathbb{R}^3} f(x) u \varphi_1 dx.$$

Therefore, $\lambda_1 > \lambda$. □

In the next two lemmas, we are going to prove that $ind(F_\lambda, 0) = 1$, if $\lambda < \lambda_1$ and $ind(F_\lambda, 0) = 0$, if $\lambda > \lambda_1$. These two facts jointly Global Bifurcation Theorem due to Rabinowitz ensure that λ_1 is a bifurcation point to branch of trivial solutions.

LEMMA 4.1. *If $\lambda < \lambda_1$, then $ind(F_\lambda, 0) = 1$, where*

$$F_\lambda(u) = u - \lambda T[\phi(u)].$$

Proof. We will show that there is $\varepsilon_0 > 0$ such that

$$F_{t\lambda}(u) \neq 0, \forall t \in [0, 1] \text{ and } 0 < \|u\| < \varepsilon < \varepsilon_0. \tag{4.2}$$

On the otherwise, there is $\{u_k\} \subset E$ and $t_k \in [0, 1]$

$$u_k \to 0 \text{ in } E \quad \text{ and } \quad F_{t_k \lambda}(u_k) = 0.$$

Then,

$$u_k = \lambda_k T[\phi(u_k)], \text{ where } \lambda_k = t_k \lambda. \tag{4.3}$$

Since $u_k > 0$, we can to define

$$v_k = \frac{u_k}{\|u_k\|},$$

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and so, from (4.3),

$$v_k = \lambda_k T\left[\frac{\phi(u_k)}{\|u_k\|}\right].$$

Arguing as in the proof of Lemma 3.2, we have that

$$I_\alpha * f(\cdot)H(u_k(.)) \to 0, \quad \text{in} \quad E. \tag{4.4}$$

This limit together with $(h_1)-(h_2)$ gives

$$\left\|\frac{\phi(u_k)}{\|u_k\|}\right\| = \left\|\frac{u_k + (I_\alpha * f(\cdot)H(u_k(\cdot)))\, h(u_k(x))}{\|u_k\|}\right\| \le C\|v_k\|, \tag{4.5}$$

from where it follows that $\left\{\frac{\phi(u_k)}{\|u_k\|}\right\}$ is bounded in E. As T is compact, there is $\{u_{k_j}\} \subset \{u_k\}$ and $w \in E$ such that

$$T\left(\frac{\phi(u_k)}{\|u_k\|}\right) \to w \ \text{ in } \ E.$$

Thus, there is $v \in E \setminus \{0\}$ such that

$$v_{k_j} \to v \ \text{ in } \ E \quad \text{and} \quad v(x) \ge 0, \quad \forall x \in \mathbb{R}^3.$$

Then, using (4.4) and the hypothesis of h, we have

$$\frac{\phi(u_{k_j})}{\|u_{k_j}\|} = \frac{\phi(u_{k_j})}{u_{k_j}}\frac{u_{k_j}}{\|u_{k_j}\|} = \frac{\phi(u_{k_j})}{u_{k_j}} v_{k_j} \to v \quad \text{in } E.$$

Supposing that $\lambda_{k_j} \to \overline{\lambda}$ and taking to limit with $k_j \to +\infty$ in (4.3), we find that

$$v = \overline{\lambda} T v, \qquad \text{where} \qquad \|v\| = 1 \text{ and } v(x) \ge 0, \quad \forall x \in \mathbb{R}^3.$$

Therefore, $\overline{\lambda}$ is an eigenvalue to the problem

$$-\Delta v = \overline{\lambda} f(x) v, \ \text{ in } \mathbb{R}^3.$$

From Maximum Principle, we have $v(x) > 0$ for all $x \in \mathbb{R}^3$, and so, we must have $\overline{\lambda} = \lambda_1$. However,

$$\overline{\lambda} = \lim_{k \to +\infty} t_k \lambda = t_0 \lambda \le \lambda < \lambda_1,$$

which is a contradiction. This proves the lemma. □

LEMMA 4.2. *If $\lambda > \lambda_1$, then ind$(F_\lambda, 0) = 0$.*

Proof. Let φ_1 be the first positive eigenfunction associated to eigenvalue λ_1, that is,

$$-\Delta \varphi_1 = \lambda_1 f(x) \varphi_1, \ \text{ in } \mathbb{R}^3,$$

or equivalently,

$$\varphi_1 = \lambda_1 T \varphi_1.$$

CLAIM 4.1. There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and $c \ge 0$,

$$F_\lambda(u) \ne c\varphi_1, \ \forall u \in E \text{ with } 0 < \|u\| \le \varepsilon.$$

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Suppose by contradiction that the claim is false. Then, there are $\{u_k\} \subset E \setminus \{0\}$ and $\{c_k\} \subset [0, \infty)$ with

$$u_k \to 0 \text{ in } E \quad \text{and} \quad F_\lambda(u_k) = c_k \varphi_1, \ \forall k \in \mathbb{N}.$$

Thus, for each $k \in \mathbb{N}$

$$\begin{cases} -\Delta u_k = \lambda f(x)\phi(u_k) + \lambda_1 c_k f(x)\varphi_1, \ \text{in } \mathbb{R}^3 \\ u_k \in D^{1,2}(\mathbb{R}^3) \cap E, \end{cases}$$

or equivalently

$$\int_{\mathbb{R}^3} \nabla u_k \nabla v dx = \lambda \int_{\mathbb{R}^3} f(x)\phi(u_k) v dx + \lambda_1 c_k \int_{\mathbb{R}^3} f(x)\varphi_1(x) v dx, \ \forall v \in D^{1,2}(\mathbb{R}^3).$$

Fixing $v = \varphi_1$, we find

$$\int_{\mathbb{R}^3} \nabla u_k \nabla \varphi_1 dx = \lambda \int_{\mathbb{R}^3} f(x)\phi(u_k)\varphi_1 dx + \lambda_1 c_k \int_{\mathbb{R}^3} f(x)\varphi_1^2 dx.$$

Recalling that

$$\int_{\mathbb{R}^3} \nabla \varphi_1 \nabla w dx = \lambda_1 \int_{\mathbb{R}^3} f(x)\varphi_1 w dx, \ \forall w \in D^{1,2}(\mathbb{R}^3),$$

it follows that

$$\int_{\mathbb{R}^3} \nabla \varphi_1 \nabla u_k dx = \lambda_1 \int_{\mathbb{R}^3} f(x)\varphi_1 u_k dx.$$

Therefore,

$$\begin{aligned} \lambda_1 \int_{\mathbb{R}^3} f(x)\varphi_1 u_k dx &= \lambda \int_{\mathbb{R}^3} f(x)\phi(u_k)\varphi_1 dx + \lambda_1 c_k \int_{\mathbb{R}^3} f(x)\varphi_1^2 dx \\ &\geq \lambda \int_{\mathbb{R}^3} f(x)\phi(u_k)\varphi_1 dx. \end{aligned} \tag{4.6}$$

By (4.4),

$$\frac{\phi(u_k)}{u_k} \to 1 \quad \text{in} \quad E,$$

then there is $k_0 \in \mathbb{N}$ such that

$$\phi(u_k) > \frac{\lambda_1}{\lambda} u_k, \quad \text{for all } \lambda > \lambda_1 > 0 \text{ and } k \geq k_0. \tag{4.7}$$

Here we have used the fact that $u_k(x) > 0$ for all $x \in \mathbb{R}^N$. From (4.6) and (4.7),

$$\lambda_1 \int_{\mathbb{R}^3} f(x)\varphi_1 u_k dx > \lambda \int_{\mathbb{R}^3} f(x) \frac{\lambda_1}{\lambda} u_k \varphi_1 dx = \lambda_1 \int_{\mathbb{R}^3} f(x)\varphi_1 u_k dx.$$

which is a contradiction, proving the claim.

Now, let $0 < \varepsilon \leq \varepsilon_0$ be, as F_λ is bounded in $\overline{B_{\varepsilon_0}(0)} \subset E$, then, there exists $c > 0$ large sufficiently so that

$$F_\lambda(u) \neq c\varphi_1, \ \forall u \in \overline{B_{\varepsilon_0}(0)}.$$

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By Claim 4.1,

$$F_\lambda(u) \neq tc\varphi_1, \text{ for } 0 < \|u\| \leq \varepsilon \text{ and } t \in [0, 1].$$

Hence the homotopy invariance yields

$$deg(F_\lambda, B_\varepsilon(0), 0) = deg(F_\lambda - c\varphi_1, B_\varepsilon(0), 0) = 0,$$

finishing the proof. □

4.2. Proof of Theorem 1.1.

Proof of Theorem 1.1. From Lemmas 4.1 and 4.2, we can apply the Global Bifurcation Theorem due Rabinowitz [22] to establish the existence of Σ_1. Indeed, note that $ind(F_\lambda, 0) = 1$ for all $0 < \lambda \leq \lambda_1$ by Lemma 4.1. Now, suppose by contradiction that $(\lambda_1, 0)$ is not a bifurcation point of (Q). Then,

$$F_\lambda(u) \neq 0, \quad \forall \lambda \in [\lambda_1 - \varepsilon, \lambda_1 + \varepsilon] \text{ and } 0 < \|u\| < \varepsilon,$$

for some $\varepsilon > 0$. Thus, there exist $\tilde{\lambda}$ and $\hat{\lambda}$ such that

$$\lambda_1 - \varepsilon < \tilde{\lambda} < \lambda_1 < \hat{\lambda} < \lambda_1 + \varepsilon,$$

and

$$deg(F_{\tilde{\lambda}}, B_\varepsilon(0), 0) = deg(F_{\hat{\lambda}}, B_\varepsilon(0), 0),$$

consequently,

$$ind(F_{\tilde{\lambda}}, 0) = ind(F_{\hat{\lambda}}, 0),$$

which contradicts Lemma 4.2. Therefore, λ_1 is a bifurcation point. The uniqueness of bifurcation point is proved with the same argument employed in Lemma 4.1. The proof that Σ_1 is unbounded follows as in [22]. □

REFERENCES

1. C. O. Alves, G. M. Figueiredo and M. Yang, Multiple semiclassical solutions for a nonlinear Choquard equation with magnetic field, *Asymptot. Anal.* **96**(2), 135–159 (2016).

2. C. O. Alves, A. B. Nóbrega and M. Yang, Multi-bump solutions for Choquard equation with deepening potential well, *Calc. Var. Partial Differ. Eq.* **55**(3), 48 (2016).

3. C. O. Alves, R. N. de Lima and M. A. S. Souto, Existence of a solution for a non-local problem in $\mathbb{R}^N$ via bifurcation theory, *Proc. Edin. Math. Soc.* **61**, 825–845 (2018).

4. C. O. Alves and M. A. S. Souto, Existence of solutions for a class of elliptic equations in $\mathbb{R}^N$ with vanishing potentials, *J. Differ. Equ.* **252**, 5555–5568 (2012).

5. C. O. Alves and J. Yang, Existence and regularity of solutions for a Choquard equation with zero mass, *To appear in Milan J. Math.*

6. H. J. Efinger, On the theory of certain nonlinear Schrödinger equations with nonlocal interaction, *Nuovo Cimento B* **80**(2), 260–278 (1984).

7. H. Fröhlich, Theory of electrical breakdown in ionic crystal, *Proc. R. Soc. Ser. A* **160**, 230–241 (1937).

8. H. Fröhlich, Electrons in lattice fields, *Adv. Phys.* **3**(11) (1954).

9. H. Genev and G. Venkov, Soliton and blow-up solutions to the time-dependent, *Discrete Contin. Dyn. Syst. Ser. S* **5**(5), 903–923 (2012).

10. M. Ghergu and S. D. Taliaferro, Pointwise bounds and blow-up for Choquard–Pekar inequalities at an isolated singularity, *J. Differ. Equ.* **261**(1), 189–217 (2016).

https://doi.org/10.1017/S0017089519000260 Published online by Cambridge University Press

11. T. Küpper, Z. Zhang and H. Xia, Multiple positive solutions and bifurcation for an equation related to Choquard's equation, *Proc. Edin. Math. Soc.* **46**, 597–607 (2003).

12. E. H. Lieb, Existence and uniqueness of the minimizing solution of Choquard nonlinear equation, *Stud. Appl. Math.* **57**(2), 93–105 (1976/77).

13. E. Lieb and M. Loss, *Analysis*, Gradute Studies in Mathematics (AMS, Providence, Rhode Island, 2001).

14. L. Ma and L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation, *Arch. Ration. Mech. Anal.* **195**(2), 455–467 (2010).

15. C. Mercuri, V. Moroz and J. Van Schaftingen, Groundstates and radial solutions to nonlinear Schrodinger–Poisson–Slater equations at the critical frequency, *Calc. Var. Partial Differ. Equ.* **55**, 146 (2016). https://doi.org/10.1007/s00526-016-1079-3

16. V. Moroz and J. Van Schaftingen, Groundstates of nonlinear Choquard equations:existence, qualitative properties and decay asymptotics, *J. Funct. Anal*. **265**, 153–184 (2013).

17. V. Moroz and J. Van Schaftingen, Existence of groundstates for a class of nonlinear Choquard equations. *Trans. Amer. Math. Soc.* **367**, 6557–6579 (2015).

18. V. Moroz and J. Van Schaftingen, A guide to the Choquard equation, *J. Fixed Point Theory Appl.* **19**, 773–813 (2017).

19. V. Moroz and J. Van Schaftingen, Groundstates of nonlinear Choquard equation: Hardy–Littlewood–Sobolev critical exponent, *To appear in Commun. Contemp. Math*. arXiv:1403.7414v1

20. I. M. Moroz, R. Penrose and P. Tod, Spherically-symmetric solution of the Schrödinger-Newton equation, *Classical Quantum Gravity* **15**, 2733–2742 (1998).

21. S. Pekar, *Untersuchung über die Elektronentheorie der Kristalle* (Akademie Verlag, Berlin, 1954), p. 2.

22. P. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Funct. Anal.* **7**, 487–513 (1971).

23. M. Struwe, *Variational methods: applications to nonlinear partial differential equations and Hamiltonian systems* (Springer, Berlin, 1990).

24. C. A. Stuart, Bifurcation for variational problems when the linearisation has no eigenvalues, *J. Funct. Anal.* **38**(2), 169–187 (1980).

25. X. Sun and Y. Zhang, Multi-peak solution for nonlinear magnetic Choquard type equation, *J. Math. Phys.* **55**(3), 031508 (2014).

26. M. Yang and Y. Wei, Existence and multiplicity of solutions for nonlinear Schrödinger equations with magnetic field and Hartree type nonlinearities, *J. Math. Anal. Appl.* **403**(2), 680–694 (2013).