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RANDOM AFFINE SIMPLEXES

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Abstract

For a fixed $k \in \{1, \dots, d\}$, consider arbitrary random vectors $X_0, \dots, X_k \in \mathbb{R}^d$ such that the $(k+1)$-tuples $(UX_0, \dots, UX_k)$ have the same distribution for any rotation U. Let A be any nonsingular $d \times d$ matrix. We show that the k-dimensional volume of the convex hull of affinely transformed X_i satisfies $|\operatorname{conv}(AX_0, \dots, AX_k)| \overset{\mathrm{D}}{=} (|P_\xi \mathcal{E}|/\kappa_k)\, |\operatorname{conv}(X_0, \dots, X_k)|$, where $\mathcal{E} := \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{x}^\top (A^\top A)^{-1} \boldsymbol{x} \le 1\}$ is an ellipsoid, P_ξ denotes the orthogonal projection to a uniformly chosen random k-dimensional linear subspace ξ independent of $X_0, \dots, X_k$, and κ_k is the volume of the unit k-dimensional ball. As an application, we derive the following integral geometry formula for ellipsoids: $c_{k,d,p} \int_{A_{d,k}} |\mathcal{E} \cap E|^{p+d+1} \, \mu_{d,k}(\mathrm{d}E) = |\mathcal{E}|^{k+1} \int_{G_{d,k}} |P_L \mathcal{E}|^p \, \nu_{d,k}(\mathrm{d}L)$, where $c_{k,d,p} = \big(\kappa_d^{k+1}/\kappa_k^{d+1}\big)(\kappa_{k(d+p)+k}/\kappa_{k(d+p)+d})$. Here $p > -1$ and $A_{d,k}$ and $G_{d,k}$ are the affine and the linear Grassmannians equipped with their respective Haar measures. The $p = 0$ case reduces to an affine version of the integral formula of Furstenberg and Tzkoni (1971).

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1. Main results

1.1. Basic notation

First we introduce some basic notion of integral geometry following [17]. The Euclidean space $\mathbb{R}^d$ is equipped with the Euclidean scalar product $\langle \cdot, \cdot \rangle$. The volume is denoted by $|\cdot|$. Some of the sets we consider have dimension less than d. In fact, we consider three classes: the convex hulls of $k+1$ points, orthogonal projections to k-dimensinal linear subspaces, and intersections with k-dimensional affine subspaces, where $k \in \{0, \dots, d\}$. In this case, $|\cdot|$ stands for the k-dimensional volume.

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The unit ball in $\mathbb{R}^k$ is denoted by $\mathbb{B}^k$. For $p > 0$, we write

$$\kappa_p := \frac{\pi^{p/2}}{\Gamma(p/2+1)}, \tag{1.1}$$

where, for an integer k, we have $\kappa_k = |\mathbb{B}^k|$.

For $k \in \{0, \ldots, d\}$, the linear (respectively affine) Grassmannian of k-dimensional linear (respectively affine) subspaces of $\mathbb{R}^d$ is denoted by $G_{d,k}$ (respectively $A_{d,k}$) and is equipped with a unique rotation invariant (respectively rigid motion invariant) Haar measure $\nu_{d,k}$ (respectively $\mu_{d,k}$), normalized by

$$\nu_{d,k}(G_{d,k}) = 1$$

and

$$\mu_{d,k}\left(\left\{E \in A_{d,k} : E \cap \mathbb{B}^d \neq \varnothing\right\}\right) = \kappa_{d-k},$$

respectively.

A compact convex subset K of $\mathbb{R}^d$ with nonempty interior is called a convex body. We define the intrinsic volumes of K by Kubota's formula,

$$V_k(K) = \binom{d}{k} \frac{\kappa_d}{\kappa_k \kappa_{d-k}} \int_{G_{d,k}} |P_L K| \nu_{d,k}(\mathrm{d}L), \tag{1.2}$$

where $P_L K$ denotes the image of K under the orthogonal projection to L.

For $L \in G_{d,k}$ (respectively $E \in A_{d,k}$), we denote by λ_L (respectively λ_E) the k-dimensional Lebesgue measures on L (respectively E).

1.2. Affine transformation of spherically symmetric distribution

For a fixed $k \in \{1, \ldots, d\}$, consider random vectors $X_0, \ldots, X_k \in \mathbb{R}^d$ (not necessarily independent and identically distributed (i.i.d.)) with an arbitrary *spherically symmetric* joint distribution. By this we mean that the $(k+1)$-tuple $(UX_0, \ldots, UX_k)$ has the same distribution for any orthogonal $d \times d$ matrix U. The convex hull

$$\operatorname{conv}(X_0, \ldots, X_k)$$

is a k-dimensional simplex (maybe degenerate) with well-defined k-dimensional volume

$$|\operatorname{conv}(X_0, \ldots, X_k)|. \tag{1.3}$$

How does the volume in (1.3) change under affine transformations? For $k = d$, the answer is obvious: it is multiplied by the determinant of the transformation. The $k < d$ case presents a more delicate problem.

Theorem 1.1. *Let A be any nonsingular $d \times d$ matrix, and let $\mathcal{E}$ be the ellipsoid defined by*

$$\mathcal{E} := \left\{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{x}^\top (A^\top A)^{-1} \boldsymbol{x} \leq 1\right\}. \tag{1.4}$$

Then we have

$$|\operatorname{conv}(AX_0, \ldots, AX_k)| \overset{\mathrm{D}}{=} \frac{|P_\xi \mathcal{E}|}{\kappa_k} |\operatorname{conv}(X_0, \ldots, X_k)|, \tag{1.5}$$

where P_ξ denotes the orthogonal projection to a uniformly chosen random k-dimensional linear subspace ξ independent of $X_0, \ldots, X_k$.

Due to Kubota's formula (see (1.2)), $\mathbb{E}\,|P_\xi \mathcal{E}|$ is proportional to $V_k(\mathcal{E})$. Thus, taking the expectation in (1.5) and using the formula

$$V_k(\mathbb{B}^d) = \binom{d}{k}\frac{\kappa_d}{\kappa_{d-k}}$$

readily implies the following corollary.

Corollary 1.1. *Under the assumptions of Theorem 1.1, we have*

$$\mathbb{E}\,|\operatorname{conv}(AX_0, \ldots, AX_k)| = \frac{V_k(\mathcal{E})}{V_k(\mathbb{B}^d)}\mathbb{E}\,|\operatorname{conv}(X_0, \ldots, X_k)|. \tag{1.6}$$

For a formula of $V_k(\mathcal{E})$, see [11]. Relation (1.6) can be generalized to higher moments using the notion of *generalized* intrinsic volumes introduced in [4], but we shall skip to describing details here.

The main ingredient of the proof of Theorem 1.1 is the following deterministic version of (1.5).

Proposition 1.1. *Let A and $\mathcal{E}$ be as in Theorem 1.1. Consider $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in \mathbb{R}^d$ and denote by L their span (linear hull). Then*

$$|\operatorname{conv}(0, A\boldsymbol{x}_1, \ldots, A\boldsymbol{x}_k)| = \frac{|P_L\mathcal{E}|}{\kappa_k}\,|\operatorname{conv}(0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_k)|. \tag{1.7}$$

Let us stress that here the origin is added to the convex hull.

Applying (1.7) to standard Gaussian vectors (details are in Section 2.3) leads to the following representation.

Corollary 1.2. *Under the assumptions of Theorem 1.1, we have*

$$\frac{|P_\xi\mathcal{E}|}{\kappa_k} \overset{\mathrm{D}}{=} \left(\frac{\det\left(G^\top A^\top AG\right)}{\det(G^\top G)}\right)^{1/2} \overset{\mathrm{D}}{=} \left(\frac{\det\left(G_\lambda^\top G_\lambda\right)}{\det\left(G^\top G\right)}\right)^{1/2}, \tag{1.8}$$

where G is a random $d \times k$ matrix with i.i.d. standard Gaussian entries N_{ij} and G_λ is a random $d \times k$ matrix with the entries $\lambda_i N_{ij}$, where $\lambda_1, \ldots, \lambda_d$ denote the singular values of A.

Thus, we obtain the following version of (1.5).

Corollary 1.3. *Under the assumptions of Theorem 1.1 and Corollary 1.2, we have*

$$\begin{aligned}|\operatorname{conv}(AX_0, \ldots, AX_k)| &\overset{\mathrm{D}}{=} \left(\frac{\det\left(G^\top A^\top AG\right)}{\det\left(G^\top G\right)}\right)^{1/2} |\operatorname{conv}(X_0, \ldots, X_k)| \\ &\overset{\mathrm{D}}{=} \left(\frac{\det\left(G_\lambda^\top G_\lambda\right)}{\det\left(G^\top G\right)}\right)^{1/2} |\operatorname{conv}(X_0, \ldots, X_k)|.\end{aligned}$$

The important special case $k = 1$ corresponds to the distance between two random points.

Corollary 1.4. *Under the assumptions of Theorem 1.1, we have*

$$|AX_0 - AX_1| \overset{\mathrm{D}}{=} \sqrt{\frac{\lambda_1^2 N_1^2 + \cdots + \lambda_d^2 N_d^2}{N_1^2 + \cdots + N_d^2}}\,|X_0 - X_1|,$$

where $N_1, \ldots, N_d$ are i.i.d. standard Gaussian variables and $\lambda_1, \ldots, \lambda_d$ denote the singular values of A.

1.3. Random points in ellipsoids

Now suppose that $X_0, \dots, X_k$ are independent and uniformly distributed in some convex body $K \subset \mathbb{R}^d$. A classical problem of stochastic geometry is to find the distribution of (1.3) starting with its moments

$$\mathbb{E}\,|\operatorname{conv}(X_0, \dots, X_k)|^p = \frac{1}{|K|^{k+1}} \int_{K^{k+1}} |\operatorname{conv}(\boldsymbol{x}_0, \dots, \boldsymbol{x}_k)|^p \, \mathrm{d}\boldsymbol{x}_0 \cdots \mathrm{d}\boldsymbol{x}_k. \tag{1.9}$$

The most studied case is $d = 2$, $k = p = 1$, when the problem reduces to calculating the mean distance between two uniformly chosen random points in a planar convex set (see [1], [2], [6], [13, Chapter 2], and [16, Chapter 4]).

For an arbitrary d and $k = 1$, there is an electromagnetic interpretation of (1.9) (see [8]): a transmitter X_0 and a receiver X_1 are placed uniformly at random in K. It is empirically known that the power received decreases with an inverse distance law of the form $1/|X_0 - X_1|^\alpha$, where α is the so-called path-loss exponent, which depends on the environment in which both are located (see [15]). Thus, with $k = 1$ and $p = -n\alpha$, (1.9) expresses the nth moment of the power received ($n < d/\alpha$).

The case of arbitrary k and d was studied only for K being a ball. In [14] it was shown (see also [17, Theorem 8.2.3]) that, for $X_0, \dots, X_k$ uniformly distributed in the unit ball $\mathbb{B}^d \subset \mathbb{R}^d$ and for an integer $p \geq 0$,

$$\mathbb{E}\,|\operatorname{conv}(X_0, \dots, X_k)|^p = \frac{1}{(k!)^p} \frac{\kappa_{d+p}^{k+1}}{\kappa_d^{k+1}} \frac{\kappa_{k(d+p)+d}}{\kappa_{(k+1)(d+p)}} \frac{b_{d,k}}{b_{d+p,k}}, \tag{1.10}$$

where κ_k is defined in (1.1) and for any real number $q > k - 1$ we write (see [17, Equation (7.8)])

$$b_{q,k} := \frac{\omega_{q-k+1} \cdots \omega_q}{\omega_1 \cdots \omega_k}, \tag{1.11}$$

with $\omega_p := p\kappa_p$ being equal to the area of the unit $(p-1)$-dimensional sphere when p is integer.

In [10, Proposition 2.8] this relation was extended to all real $p > -1$. It should be noted that Proposition 2.8 of [10] is formulated for real $p \geq 0$ only, but in the proof (see p. 23) it is argued that by analytic continuation, the formula holds for all real $p > -1$ as well. Theorem 1.1 implies (for details see Section 2.4) the following generalization of (1.10) for ellipsoids. Recall that P_ξ denotes the orthogonal projection to a uniformly chosen random k-dimensional linear subspace ξ independent of $X_0, \dots, X_k$.

Theorem 1.2. *For $X_0, \dots, X_k$ uniformly distributed in some nondegenerate ellipsoid $\mathcal{E} \subset \mathbb{R}^d$ and any real number $p > -1$, we have*

$$\mathbb{E}\,|\operatorname{conv}(X_0, \dots, X_k)|^p = \frac{1}{(k!)^p} \frac{\kappa_{d+p}^{k+1}}{\kappa_d^{k+1}} \frac{\kappa_{k(d+p)+d}}{\kappa_{(k+1)(d+p)}} \frac{b_{d,k}}{b_{d+p,k}} \frac{\mathbb{E}\,|P_\xi \mathcal{E}|^p}{\kappa_k^p}. \tag{1.12}$$

Note that (1.12) is indeed a generalization of (1.10) since $P_\xi \mathbb{B}^d = \mathbb{B}^k$ a.s. and $|\mathbb{B}^k|^p = \kappa_k^p$. For $k = 1$, (1.12) was recently obtained in [9].

For $p = 1$, the right-hand side of (1.12) is proportional to the kth intrinsic volume of $\mathcal{E}$ (see (1.2)), which implies the following result (for details, see Section 2.5).

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Corollary 1.5. *For $X_0, \dots, X_k$ uniformly distributed in some nondegenerate ellipsoid $\mathcal{E} \subset \mathbb{R}^d$, we have*

$$\mathbb{E}\,|\operatorname{conv}(X_0, \dots, X_k)| = \frac{1}{2^k} \frac{((d+1)!)^{k+1}}{((d+1)(k+1))!} \left(\frac{\kappa_{d+1}^{k+1}}{\kappa_{(d+1)(k+1)}} \right)^2 V_k(\mathcal{E}).$$

Very recently, for $X_0, \dots, X_k$ uniformly distributed in the unit ball $\mathbb{B}^d$, the formula for the distribution of $|\operatorname{conv}(X_0, \dots, X_k)|$ has been derived in [7]. For a random variable η and $\alpha_1, \alpha_2 > 0$, we write $\eta \sim B(\alpha_1, \alpha_2)$ to denote that η has a beta distribution with parameters α_1, α_2 and the density

$$\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} t^{\alpha_1-1} (1-t)^{\alpha_2-1}, \qquad t \in (0, 1).$$

It was shown in [7] that, for $X_0, \dots, X_k$ uniformly distributed in $\mathbb{B}^d$,

$$(k!)^2\, \eta(1-\eta)^k\, |\operatorname{conv}(X_0, \dots, X_k)|^2 \overset{\mathrm{D}}{=} (1-\eta')^k\, \eta_1 \cdots \eta_k, \tag{1.13}$$

where $\eta, \eta', \eta_1, \dots, \eta_k$ are independent random variables independent of $X_0, \dots, X_k$ such that

$$\eta, \eta' \sim B\left(\frac{d}{2}+1, \frac{kd}{2}\right), \qquad \eta_i \sim B\left(\frac{d-k+i}{2}, \frac{k-i}{2}+1\right).$$

Multiplying both sides of (1.13) by $|P_\xi \mathcal{E}|^2/\kappa_k^2$ and applying Theorem 1.1 and Corollary 1.2 (for details, see Section 2.4) leads to the following generalization of (1.13).

Theorem 1.3. *For $X_0, \dots, X_k$ uniformly distributed in some nondegenerate ellipsoid $\mathcal{E} \subset \mathbb{R}^d$, we have*

$$\begin{aligned}(k!)^2 \eta(1-\eta)^k |\operatorname{conv}(X_0, \dots, X_k)|^2 &\overset{\mathrm{D}}{=} \kappa_k^{-2}(1-\eta')^k \eta_1 \cdots \eta_k |P_\xi \mathcal{E}|^2 \\ &\overset{\mathrm{D}}{=} (1-\eta')^k \eta_1 \cdots \eta_k \left(\frac{\det(G_\lambda^\top G_\lambda)}{\det(G^\top G)} \right),\end{aligned}$$

where the matrices G and G_λ are defined in Corollary 1.2 and $\lambda_1, \dots, \lambda_d$ denote the length of semi-axes of $\mathcal{E}$.

Taking $k=1$ yields the distribution of the distance between two random points in $\mathcal{E}$.

Corollary 1.6. *Under the assumptions of Theorem 1.3, we have*

$$\eta(1-\eta)\,|X_0 - X_1|^2 \overset{\mathrm{D}}{=} (1-\eta')\eta_1 \left(\frac{\lambda_1^2 N_1^2 + \cdots + \lambda_d^2 N_d^2}{N_1^2 + \cdots + N_d^2} \right),$$

where $N_1, \dots, N_d$ are i.i.d. standard Gaussian variables.

1.4. Integral geometry formulae

Recall that $G_{d,k}$ and $A_{d,k}$ denote the linear and affine Grassmannians defined in Section 1.1.

For an arbitrary convex compact body K, $p > -d$, and $k = 1$, it is possible to express (1.9) in terms of the lengths of the one-dimensional sections of K (see [3] and [12]):

$$\int_{K^2} |\boldsymbol{x}_0 - \boldsymbol{x}_1|^p \,\mathrm{d}\boldsymbol{x}_0 \,\mathrm{d}\boldsymbol{x}_1 = \frac{2d\kappa_d}{(d+p)(d+p+1)} \int_{A_{d,1}} |K \cap E|^{p+d+1} \mu_{d,1}(\mathrm{d}E).$$

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This formula does not extend to $k > 1$. The next theorem shows that for ellipsoids this is possible.

Theorem 1.4. *For any nondegenerate ellipsoid* $\mathcal{E} \subset \mathbb{R}^d$, $k \in \{0, 1, \ldots, d\}$, *and any real number* $p > -d + k - 1$, *we have*

$$\int_{\mathcal{E}^{k+1}} |\operatorname{conv}(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_k)|^p \,\mathrm{d}\boldsymbol{x}_0 \ldots \mathrm{d}\boldsymbol{x}_k = \frac{1}{(k!)^p} \frac{\kappa_{d+p}^{k+1}}{\kappa_k^{p+d+1}} \frac{\kappa_{k(d+p)+k}}{\kappa_{(k+1)(d+p)}} \frac{b_{d,k}}{b_{d+p,k}} \int_{A_{d,k}} |\mathcal{E} \cap E|^{p+d+1} \mu_{d,k}(\mathrm{d}E). \tag{1.14}$$

The proof is given in Section 3.2.

Comparing this theorem with Theorem 1.2 readily gives the following connection between the volumes of k-dimensional cross-sections and projections of ellipsoids.

Theorem 1.5. *Under the assumptions of Theorem 1.4, we have*

$$\frac{\kappa_d^{k+1}}{\kappa_k^{d+1}} \frac{\kappa_{k(d+p)+k}}{\kappa_{k(d+p)+d}} \int_{A_{d,k}} |\mathcal{E} \cap E|^{p+d+1} \mu_{d,k}(\mathrm{d}E) = |\mathcal{E}|^{k+1} \int_{G_{d,k}} |P_L \mathcal{E}|^p \nu_{d,k}(\mathrm{d}L).$$

For $p = 0$, we obtain the following integral formula.

Corollary 1.7. *Under the assumptions of Theorem 1.4, we have*

$$\int_{A_{d,k}} |\mathcal{E} \cap E|^{d+1} \mu_{d,k}(\mathrm{d}E) = \frac{\kappa_k^{d+1}}{\kappa_d^{k+1}} \frac{\kappa_{d(k+1)}}{\kappa_{k(d+1)}} |\mathcal{E}|^{k+1}. \tag{1.15}$$

This result may be regarded as an affine version of the following integral formula of Furstenberg and Tzkoni (see [5]):

$$\int_{G_{d,k}} |\mathcal{E} \cap L|^d \nu_{d,k}(\mathrm{d}L) = \frac{\kappa_k^d}{\kappa_d^k} |\mathcal{E}|^k.$$

Our next theorem generalizes this formula in the same way as (1.14) generalizes (1.15).

Theorem 1.6. *For any nondegenerate ellipsoid* $\mathcal{E} \subset \mathbb{R}^d$, $k \in \{0, 1, \ldots, d\}$, *and any real number* $p > -d + k$, *we have*

$$\int_{\mathcal{E}^k} |\operatorname{conv}(0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_k)|^p \,\mathrm{d}\boldsymbol{x}_1 \ldots \mathrm{d}\boldsymbol{x}_k = \frac{1}{(k!)^p} \frac{\kappa_{d+p}^k}{\kappa_k^{p+d}} \frac{b_{d,k}}{b_{d+p,k}} \int_{G_{d,k}} |\mathcal{E} \cap L|^{p+d} \nu_{d,k}(\mathrm{d}L).$$

In probabilistic language it may be formulated as

$$|\mathbb{E}|^k \operatorname{conv}(0, X_1, \ldots, X_k)|^p = \frac{1}{(k!)^p} \frac{\kappa_{d+p}^k}{\kappa_k^{p+d}} \frac{b_{d,k}}{b_{d+p,k}} \mathbb{E}\, |\mathcal{E} \cap \xi|^{p+d},$$

where $X_1, \ldots, X_k$ are independent uniformly distributed random vectors in $\mathcal{E}$ and ξ is a uniformly chosen random k-dimensional linear subspace in $\mathbb{R}^d$.

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2. Proofs: part I

2.1. Proof of Theorem 1.1 assuming Proposition 1.1

First note that, with probability 1, the equation

$$\frac{|P_\xi \mathcal{E}|}{\kappa_k} \cdot |\operatorname{conv}(X_0, \ldots, X_k)| = 0$$

holds if and only if

$$|\operatorname{conv}(AX_0, \ldots, AX_k)| = 0,$$

which in turn is equivalent to

$$\dim \operatorname{conv}(X_0, \ldots, X_k) < k.$$

Therefore, to prove (1.5), it is enough to show that the conditional distributions of

$$|\operatorname{conv}(AX_0, \ldots, AX_k)| \quad \text{and} \quad \frac{|P_\xi \mathcal{E}|}{\kappa_k} |\operatorname{conv}(X_0, \ldots, X_k)|$$

given $\dim \operatorname{conv}(X_0, \ldots, X_k) = k$ are equal. Thus, without loss of generality, we can assume that the simplex $\operatorname{conv}(X_0, \ldots, X_k)$ is not degenerate with probability 1:

$$\dim \operatorname{conv}(X_0, \ldots, X_k) = k \quad \text{a.s.} \tag{2.1}$$

Our original proof was based on the Blaschke–Petkantschin formula and the characteristic function uniqueness theorem. (The original proof can be found in the first version of this paper available at https://arxiv.org/abs/1711.06578v1.) Later, Youri Davydov found a much simpler and nicer proof which also allows us to get rid of the assumption about the existence of the joint density of $X_0, \ldots, X_k$. Let us present this proof.

Since the joint distribution of $X_0, \ldots, X_k$ is spherically symmetric, we have for any orthogonal matrix U

$$\begin{aligned} |\operatorname{conv}(AX_0, \ldots, AX_k)| &= |\operatorname{conv}(0, A(X_1 - X_0), \ldots, A(X_k - X_0)| \\ &\stackrel{D}{=} |\operatorname{conv}(0, A(UX_1 - UX_0), \ldots, A(UX_k - UX_0)|. \end{aligned} \tag{2.2}$$

Now let Υ be a random orthogonal matrix chosen uniformly from $SO(n)$ with respect to the probabilistic Haar measure and independently of $X_0, \ldots, X_k$. By (2.1), with probability one the span of $X_1 - X_0, \ldots, X_k - X_0$ is a k-dimensional linear subspace of $\mathbb{R}^d$. Thus, the span

$$\xi := \operatorname{span}(\Upsilon X_1 - \Upsilon X_0, \ldots, \Upsilon X_k - \Upsilon X_0)$$

is a random uniformly chosen k-dimensional linear subspace in $\mathbb{R}^d$ independent of $X_0, \ldots, X_k$. Applying Proposition 1.1 to the vectors $\Upsilon X_1 - \Upsilon X_0, \ldots, \Upsilon X_k - \Upsilon X_0$, we obtain

$$\begin{aligned} &|\operatorname{conv}(0, A(\Upsilon X_1 - \Upsilon X_0), \ldots, A(\Upsilon X_k - \Upsilon X_0)| \\ &\quad = \frac{|P_\xi \mathcal{E}|}{\kappa_k} |\operatorname{conv}(0, \Upsilon X_1 - \Upsilon X_0, \ldots, \Upsilon X_k - \Upsilon X_0)| \\ &\quad = \frac{|P_\xi \mathcal{E}|}{\kappa_k} |\operatorname{conv}(\Upsilon X_0, \Upsilon X_1, \ldots, \Upsilon X_k)| \\ &\quad \stackrel{D}{=} \frac{|P_\xi \mathcal{E}|}{\kappa_k} |\operatorname{conv}(X_0, X_1, \ldots, X_k)|. \end{aligned}$$

Combining this with (2.2) for $U = \Upsilon$ completes the proof.

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2.2. Proof of Proposition 1.1

To avoid trivialities, we assume that $\dim L = k$, i.e. $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ are in general position. Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k \in \mathbb{R}^d$ be some orthonormal basis in L. Let O_L and X denote $d \times k$ matrices whose columns are $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k$ and $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$, respectively. It is easy to check that $O_L O_L^\top$ is a $d \times d$ matrix corresponding to the orthogonal projection operator P_L. Thus,

$$O_L O_L^\top X = X. \tag{2.3}$$

Recall that $\mathcal{E}$ is defined by (1.4). It is known (see, e.g. [18, Appendix H]) that the orthogonal projection $P_L\mathcal{E}$ is an ellipsoid in L and

$$|P_L\mathcal{E}| = \kappa_k \left[\det\left(O_L^\top H O_L\right)\right]^{1/2}, \tag{2.4}$$

where

$$H := A^\top A.$$

A well-known formula for the volume of a k-dimensional parallelepiped implies that, for any $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in \mathbb{R}^d$,

$$|\operatorname{conv}(0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_k)| = \frac{1}{k!}\left[\det\left(X^\top X\right)\right]^{1/2}. \tag{2.5}$$

Therefore,

$$k!\,|\operatorname{conv}(0, A\boldsymbol{x}_1, \ldots, A\boldsymbol{x}_k)| = \left[\det\left((AX)^\top AX\right)\right]^{1/2} = \left[\det\left(X^\top H X\right)\right]^{1/2}.$$

Applying (2.3) produces

$$\begin{aligned}\det(X^\top H X) &= \det\left(X^\top O_L O_L^\top H O_L O_L^\top X\right)\\ &= \det\left(O_L^\top H O_L\right)\det\left(X^\top O_L\right)\det\left(O_L^\top X\right)\\ &= \det\left(O_L^\top H O_L\right)\det\left(X^\top O_L O_L^\top X\right)\\ &= \det\left(O_L^\top H O_L\right)\det\left(X^\top X\right),\end{aligned}$$

which together with (2.4) and (2.5) completes the proof.

2.3. Proof of Corollary 1.2

Denote by $G_1, \ldots, G_k \in \mathbb{R}^d$ the columns of the matrix G. Hence, $AG_1, \ldots, AG_k \in \mathbb{R}^d$ are the columns of the matrix AG. Using Proposition 1.1 with $\boldsymbol{x}_i = G_i$ and applying (2.5) to G and AG gives

$$\left[\det\left(G^\top A^\top A G\right)\right]^{1/2} = \frac{|P_\eta \mathcal{E}|}{\kappa_k}\left[\det\left(G^\top G\right)\right]^{1/2},$$

or

$$\left(\frac{\det\left(G^\top A^\top A G\right)}{\det\left(G^\top G\right)}\right)^{1/2} = \frac{|P_\eta \mathcal{E}|}{\kappa_k},$$

where η is the span of $G_1, \ldots, G_k$. Since $G_1, \ldots, G_k$ are i.i.d. standard Gaussian vectors, η is uniformly distributed in $G_{d,k}$ with respect to $\nu_{d,k}$ (given $\dim \eta = k$ which holds a.s.), therefore, $\eta \overset{\mathrm{D}}{=} \xi$ and the corollary follows.

2.4. Proofs of Theorem 1.2 and Theorem 1.3

For any nondegenerate ellipsoid $\mathcal{E}$, there exists a unique *symmetric* positive-definite $d \times d$ matrix A such that

$$\mathcal{E} = A\mathbb{B}^d = \left\{ \boldsymbol{x} \in \mathbb{R}^d : \; \|A^{-1}\boldsymbol{x}\| \le 1 \right\} = \left\{ \boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{x}^\top A^{-2} \boldsymbol{x} \le 1 \right\}.$$

Since $X_0, \ldots, X_k$ are i.i.d. random vectors uniformly distributed in $\mathcal{E}$, then $A^{-1}X_0, \ldots, A^{-1}X_k$ are i.i.d. random vectors uniformly distributed in $\mathbb{B}^d$. It follows from Theorem 1.1 that

$$\begin{aligned} |\operatorname{conv}(X_0, \ldots, X_k)| &= \left| \operatorname{conv}\left(AA^{-1}X_0, \ldots, AA^{-1}X_k \right) \right| \\ &\overset{\mathrm{D}}{=} \left| \operatorname{conv}\left(A^{-1}X_0, \ldots, A^{-1}X_k \right) \right| \frac{|P_\xi \mathcal{E}|}{\kappa_k}. \end{aligned} \tag{2.6}$$

Taking the pth moment and applying (1.10) implies Theorem 1.2.

Now apply (1.13) to $A^{-1}X_0, \ldots, A^{-1}X_k$:

$$(k!)^2 \, \eta(1-\eta)^k \left| \operatorname{conv}\left(A^{-1}X_0, \ldots, A^{-1}X_k \right) \right|^2 \overset{\mathrm{D}}{=} (1-\eta')^k \, \eta_1 \cdots \eta_k.$$

Multiplying by $|P_\xi \mathcal{E}|/\kappa_k^p$ and applying (2.6) implies the first equation in Theorem 1.3. The second equation follows from (1.8).

2.5. Proof of Corollary 1.5

From Kubota's formula (see (1.2)) and Theorem 1.2, we have

$$\mathbb{E}\,|\operatorname{conv}(X_0, \ldots, X_k)| = \alpha_{d,k} V_k(\mathcal{E}),$$

where

$$\alpha_{d,k} := \frac{1}{k!} \frac{\kappa_{d+1}^{k+1}}{\kappa_d^{k+1}} \frac{\kappa_{k(d+1)+d}}{\kappa_{(k+1)(d+1)}} \frac{b_{d,k}}{b_{d+1,k}} \frac{\kappa_{d-k}}{\binom{d}{k}\kappa_d}.$$

From the definitions of $b_{d,k}$ (see (1.11)) and κ_p (see (1.1)), we obtain

$$\begin{aligned} \alpha_{d,k} &= \frac{\kappa_{d+1}^{k+1}}{\kappa_d^{k+1}} \frac{\kappa_{k(d+1)+d}}{\kappa_{(k+1)(d+1)}} \frac{(d+1-k)!\,\kappa_{d-k+1}}{(d+1)!\,\kappa_{d+1}} \frac{\kappa_{d-k}}{\kappa_d} \\ &= \frac{(d+1-k)!}{\pi^{k/2}(d+1)!} \left(\frac{\Gamma(d/2+1)}{\Gamma((d+1)/2+1)} \right)^{k+1} \\ &\quad \times \frac{\Gamma((k+1)(d+1)/2+1)}{\Gamma(((k+1)d+k)/2+1)} \frac{\Gamma((d+1)/2+1)}{\Gamma((d-k+1)/2+1)} \frac{\Gamma(d/2+1)}{\Gamma((d-k)/2+1)}. \end{aligned}$$

Using Legendre's duplication formula for the gamma function,

$$\Gamma(z)\,\Gamma\left(z+\tfrac{1}{2}\right) = 2^{1-2z}\pi^{1/2}\Gamma(2z),$$

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the recursion $\Gamma(1+z)=z\,\Gamma(z)$, and the fact that k, $d\in\mathbb{Z}$, we obtain

$$\begin{aligned}\alpha_{d,k} &= \frac{(d-k)!}{\pi^{k/2}d!}\frac{\Gamma\left((k+1)(d+1)/2+1\right)}{\Gamma\left(((k+1)d+k)/2+1\right)}\frac{\Gamma\left(d/2+1/2\right)\Gamma\left(d/2+1\right)}{\Gamma\left((d-k)/2+1/2\right)\Gamma\left((d-k)/2+1\right)}\\ &\quad\times\left(\frac{\Gamma\left(d/2+1\right)}{\Gamma\left((d+1)/2+1\right)}\right)^{k+1}\\ &= \frac{1}{(2\sqrt{\pi})^k}\frac{\Gamma\left((k+1)(d+1)/2+1\right)}{\Gamma\left(((k+1)d+k)/2+1\right)}\left(\frac{\Gamma\left(d/2+1\right)}{\Gamma\left((d+1)/2+1\right)}\right)^{k+1}\\ &= \frac{1}{(2\sqrt{\pi})^k}\frac{\left(\Gamma\left(d/2+1\right)\Gamma\left(d/2+1+1/2\right)\right)^{k+1}}{\Gamma\left((kd+d+k)/2+1\right)\Gamma\left((kd+k+d)/2+1+1/2\right)}\left(\frac{\kappa_{d+1}^{k+1}}{\kappa_{(d+1)(k+1)}}\right)^2\\ &= \frac{1}{2^k}\frac{((d+1)!)^{k+1}}{((d+1)(k+1))!}\left(\frac{\kappa_{d+1}^{k+1}}{\kappa_{(d+1)(k+1)}}\right)^2.\end{aligned}$$

3. Proofs: part II

3.1. Blaschke–Petkantschin formula

In our further calculations we will need to integrate some nonnegative measurable function h of k-tuples of points in $\mathbb{R}^d$. To this end, we integrate first over the k-tuples of points in a fixed k-dimensional linear subspace L with respect to the product measure λ_L^k and then integrate over $G_{d,k}$ with respect to $\nu_{d,k}$. The corresponding transformation formula is known as the linear Blaschke–Petkantschin formula (see [17, Theorem 7.2.1]):

$$\begin{aligned}&\int_{(\mathbb{R}^d)^k} h(\boldsymbol{x}_1,\dots,\boldsymbol{x}_k)\,\mathrm{d}\boldsymbol{x}_1\dots\mathrm{d}\boldsymbol{x}_k \\ &\quad= (k!)^{d-k}b_{d,k}\int_{G_{d,k}}\int_{L^k} h(\boldsymbol{x}_1,\dots,\boldsymbol{x}_k)|\operatorname{conv}(0,\boldsymbol{x}_1,\dots,\boldsymbol{x}_k)|^{d-k}\\ &\qquad\times\lambda_L(\mathrm{d}\boldsymbol{x}_1)\cdots\lambda_L(\mathrm{d}\boldsymbol{x}_k)\nu_{d,k}(\mathrm{d}L),\end{aligned} \tag{3.1}$$

where $b_{d,k}$ is defined in (1.11).

A similar affine version (see [17, Theorem 7.2.7]) may be stated as follows:

$$\begin{aligned}&\int_{(\mathbb{R}^d)^{k+1}} h(\boldsymbol{x}_0,\dots,\boldsymbol{x}_k)\,\mathrm{d}\boldsymbol{x}_0\dots\mathrm{d}\boldsymbol{x}_k \\ &\quad= (k!)^{d-k}b_{d,k}\int_{A_{d,k}}\int_{E^{k+1}} h(\boldsymbol{x}_0,\dots,\boldsymbol{x}_k)|\operatorname{conv}(\boldsymbol{x}_0,\dots,\boldsymbol{x}_k)|^{d-k}\\ &\qquad\times\lambda_E(\mathrm{d}\boldsymbol{x}_0)\cdots\lambda_E(\mathrm{d}\boldsymbol{x}_k)\mu_{d,k}(\mathrm{d}E).\end{aligned} \tag{3.2}$$

3.2. Proof of Theorem 1.4

Let

$$\begin{aligned}J &:= \int_{\mathcal{E}^{k+1}} |\operatorname{conv}(\boldsymbol{x}_0,\dots,\boldsymbol{x}_k)|^p\,\mathrm{d}\boldsymbol{x}_0\cdots\mathrm{d}\boldsymbol{x}_k\\ &= \int_{(\mathbb{R}^d)^{k+1}} |\operatorname{conv}(\boldsymbol{x}_0,\dots,\boldsymbol{x}_k)|^p\prod_{i=0}^{k}\mathbf{1}_{\mathcal{E}}(\boldsymbol{x}_i)\,\mathrm{d}\boldsymbol{x}_0\cdots\mathrm{d}\boldsymbol{x}_k.\end{aligned}$$

Using the affine Blaschke–Petkantschin formula (see (3.2)) with

$$h(\boldsymbol{x}_0, \dots, \boldsymbol{x}_k) := |\operatorname{conv}(\boldsymbol{x}_0, \dots, \boldsymbol{x}_k)|^p \prod_{i=0}^{k} \mathbf{1}_{\mathcal{E}}(\boldsymbol{x}_i)$$

yields

$$\begin{aligned} J &= (k!)^{d-k} b_{d,k} \int_{A_{d,k}} \int_{E^{k+1}} |\operatorname{conv}(\boldsymbol{x}_0, \dots, \boldsymbol{x}_k)|^{p+d-k} \\ &\quad \times \prod_{i=0}^{k} \mathbf{1}_{\mathcal{E}}(\boldsymbol{x}_i) \lambda_E(\mathrm{d}\boldsymbol{x}_0) \dots \lambda_E(\mathrm{d}\boldsymbol{x}_k) \mu_{d,k}(\mathrm{d}E) \\ &= (k!)^{d-k} b_{d,k} \int_{A_{d,k}} \int_{(E\cap\mathcal{E})^{k+1}} |\operatorname{conv}(\boldsymbol{x}_0, \dots, \boldsymbol{x}_k)|^{p+d-k} \lambda_E(\mathrm{d}\boldsymbol{x}_0) \cdots \lambda_E(\mathrm{d}\boldsymbol{x}_k) \mu_{d,k}(\mathrm{d}E). \end{aligned}$$

Now fix $E \in A_{d,k}$. Applying Theorem 1.2 to the ellipsoid $\mathcal{E} \cap E$ gives

$$\begin{aligned} &\frac{1}{|\mathcal{E}\cap E|^{k+1}} \int_{(E\cap\mathcal{E})^{k+1}} |\operatorname{conv}(\boldsymbol{x}_0, \dots, \boldsymbol{x}_k)|^{p+d-k} \lambda_E(\mathrm{d}\boldsymbol{x}_0) \cdots \lambda_E(\mathrm{d}\boldsymbol{x}_k) \\ &\qquad = \frac{1}{(k!)^{p+d-k}} \frac{\kappa_{d+p}^{k+1}}{\kappa_k^{k+1}} \frac{\kappa_{k(d+p)+k}}{\kappa_{(k+1)(d+p)}} \frac{b_{k,k}}{b_{d+p,k}} \frac{|\mathcal{E}\cap E|^{p+d-k}}{\kappa_k^{p+d-k}}, \end{aligned}$$

which leads to

$$J = \frac{1}{(k!)^p} \frac{\kappa_{d+p}^{k+1}}{\kappa_k^{p+d+1}} \frac{\kappa_{k(d+p)+k}}{\kappa_{(k+1)(d+p)}} \frac{b_{d,k}}{b_{d+p,k}} \int_{A_{d,k}} |\mathcal{E}\cap E|^{p+d+1} \mu_{d,k}(\mathrm{d}E).$$

3.3. Proof of Theorem 1.6

The proof is similar to the previous proof. Let

$$\begin{aligned} J &:= \int_{\mathcal{E}^k} |\operatorname{conv}(0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_k)|^p \, \mathrm{d}\boldsymbol{x}_1 \cdots \mathrm{d}\boldsymbol{x}_k \\ &= \int_{(\mathbb{R}^d)^k} |\operatorname{conv}(0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_k)|^p \prod_{i=1}^{k} \mathbf{1}_{\mathcal{E}}(\boldsymbol{x}_i) \, \mathrm{d}\boldsymbol{x}_1 \cdots \mathrm{d}\boldsymbol{x}_k. \end{aligned}$$

Using the linear Blaschke–Petkantschin formula (see (3.1)) with

$$h(\boldsymbol{x}_1, \dots, \boldsymbol{x}_k) := |\operatorname{conv}(0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_k)|^p \prod_{i=1}^{k} \mathbf{1}_{\mathcal{E}}(\boldsymbol{x}_i)$$

gives

$$\begin{aligned} J &= (k!)^{d-k} b_{d,k} \int_{G_{d,k}} \int_{L^k} |\operatorname{conv}(0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_k)|^{p+d-k} \\ &\quad \times \prod_{i=1}^{k} \mathbf{1}_{\mathcal{E}}(\boldsymbol{x}_i) \lambda_L(\mathrm{d}\boldsymbol{x}_1) \cdots \lambda_L(\mathrm{d}\boldsymbol{x}_k) \, \nu_{d,k}(\mathrm{d}L) \\ &= (k!)^{d-k} b_{d,k} \int_{G_{d,k}} \int_{(L\cap\mathcal{E})^k} |\operatorname{conv}(0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_k)|^{p+d-k} \lambda_L(\mathrm{d}\boldsymbol{x}_1) \cdots \lambda_L(\mathrm{d}\boldsymbol{x}_k) \nu_{d,k}(\mathrm{d}L). \quad (3.3) \end{aligned}$$

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Fix $L \in G_{d,k}$. Since $\mathcal{E} \cap L$ is an ellipsoid, there exists a linear transform $A_L : L \to \mathbb{R}^k$ such that $A_L(\mathcal{E} \cap L) = \mathbb{B}^k$. Applying the coordinate transform $\boldsymbol{x}_i = A_L \boldsymbol{y}_i,\ i = 1, 2, \ldots, k$, we get

$$\begin{aligned}\int_{(L\cap\mathcal{E})^k} &|\operatorname{conv}(0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_k)|^{p+d-k} \lambda_L(\mathrm{d}\boldsymbol{x}_1) \cdots \lambda_L(\mathrm{d}\boldsymbol{x}_k) \\ &= \frac{|\mathcal{E} \cap L|^{p+d}}{\kappa_k^{p+d}} \int_{(\mathbb{B}^k)^k} |\operatorname{conv}(0, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_k)|^{p+d-k}\, \mathrm{d}\boldsymbol{y}_1 \cdots \mathrm{d}\boldsymbol{y}_k. \end{aligned} \tag{3.4}$$

It is known (see, e.g. [17, Theorem 8.2.2]) that

$$\int_{(\mathbb{B}^k)^k} |\operatorname{conv}(0, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_k)|^{p+d-k}\, \mathrm{d}\boldsymbol{y}_1 \cdots \mathrm{d}\boldsymbol{y}_k = (k!)^{-p-d+k} \kappa_{d+p}^k \frac{b_{k,k}}{b_{d+p,k}}. \tag{3.5}$$

Substituting (3.5) and (3.4) into (3.3) completes the proof.

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