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Regularity and propagation of moments in some nonlinear Vlasov systems

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We introduce a new variant to prove the regularity of solutions to transport equations of the Vlasov type. Our approach is mainly based on the proof of propagation of velocity moments, as in a previous paper by Lions and Perthame. We combine it with moment lemmas which assert that, locally in space, velocity moments can be gained from the kinetic equation itself. We apply our theory to two cases. First, to the Vlasov–Poisson system, and we solve a long-standing conjecture, namely the propagation of any moment larger than two. Next, to the Vlasov–Stokes system, where we prove the same result for fairly singular kernels.

1. Introduction

We consider the regularity of solutions to Vlasov systems. These are nonlinear transport equations arising as the *mean field* limits of many-particle systems and are classical models arising, for instance, in plasma physics, astrophysics, fluid dynamics, etc. Due to the nonlinearity, which arises because the force field acting on the particles depends on the density repartition of the particles themselves, these models exhibit a rather complex behaviour. A particular example of this complexity is the difficulty to prove the regularity of solutions with smooth initial data.

We will describe our method on two examples of such systems. The first example is the famous Vlasov–Poisson (VP) system. It describes the evolution of a density $f(x,v,t)$ of particles which, at time $t \geqslant 0$ and position $x \in \mathbb{R}^3$, move with velocity $v \in \mathbb{R}^3$ and interact through self-consistent Coulombic or Newtonian forces. It reads

$$\left.\begin{aligned} \frac{\partial}{\partial t} f + v \cdot \nabla_x f + \operatorname{div}_v(Ff) = 0, \\ f(x,v,0) = f^0(x,v) \geqslant 0, \end{aligned}\right\} \tag{1.1}$$

with the force field

$$F(t,x) = \pm \frac{x}{|x|^3} \star \rho(t,x), \tag{1.2}$$

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and, as usual, from the microscopic density f, we compute the macroscopic density ρ and the current j with the formulae

$$\rho(x,t) = \int_{\mathbb{R}^3} f(x,v,t)\,\mathrm{d}v, \qquad j(x,t) = \int_{\mathbb{R}^3} v f(x,v,t)\,\mathrm{d}v. \tag{1.3}$$

The second example is the Vlasov–Stokes (VS) system, which describes the evolution of particles interacting through a fluid described by a Stokes flow (see [10] for another VS system and [14] for the derivation of the system below from an interacting system of particles),

$$\left.\begin{aligned} \frac{\partial}{\partial t} f + v \cdot \nabla_x f + \mathrm{div}_v[(F - v) f] = 0, \\ f(x,v,0) = f^0(x,v) \geqslant 0, \end{aligned}\right\} \tag{1.4}$$

$$F(x,t) = A(x) \star j(x,t). \tag{1.5}$$

Here, the matrix $A \in C^\infty(\mathbb{R}^3 \setminus 0)$ is assumed to satisfy two properties. The first property gives a limitation on the possible singularity at the origin and the second expresses the dissipation of the kinetic energy of the system (a natural condition since it is realized for the particle system)

$$|A(x)| \leqslant \frac{C}{|x|^\beta}, \quad 0 < \beta < 2, \tag{1.6}$$

$$\int_{\mathbb{R}^3} j(x) \cdot A(x) \star j(x)\,\mathrm{d}x \leqslant 0 \quad \forall\, j \in (\mathcal{D}(\mathbb{R}^3))^3. \tag{1.7}$$

For these two models, we are interested in the propagation of v-moments

$$M_k(t) = \sup_{0 \leqslant s \leqslant t} \int_{\mathbb{R}^6} |v|^k f(x,v,s)\,\mathrm{d}v\,\mathrm{d}x. \tag{1.8}$$

Classical energy bounds (see [8, 13]) show that the second moment is *a priori* bounded,

$$M_2(t) \leqslant C(\|f^0\|_\infty, M_2(0)), \tag{1.9}$$

where we denote by $\|u(\cdot)\|_p$ the L^p norm of the function u in its arguments x or (x,v), depending on the context.

Here, we prove the propagation of v-moments for k larger than two. As it is well known, this is a definitive step towards regularity of solutions because of the classical interpolation inequalities

$$\left.\begin{aligned} \|\rho(\cdot,t)\|_{(k+3)/3} \leqslant C \|f(\cdot,\cdot,t)\|_\infty^{k/(3+k)} M_k(t)^{3/(3+k)}, \\ \|j(\cdot,t)\|_{(k+3)/4} \leqslant C \|f(\cdot,\cdot,t)\|_\infty^{(k-1)/(3+k)} M_k(t)^{4/(3+k)}. \end{aligned}\right\} \tag{1.10}$$

These inequalities, combined with the Young (or generalized Young) inequalities, furnish regularity for the force fields F,

$$\left.\begin{aligned}\left\|\frac{x}{|x|^3}\star\rho\right\|_r &\leqslant C\|f(t)\|_\infty^{k/(k+3)}M_k(t)^{3/(3+k)}, & r&=3\left(\frac{3+k}{6-k}\right),\\ \|A\star j\|_r &\leqslant C\|f(t)\|_\infty^{(k-1)/(k+3)}M_k(t)^{4/(3+k)}, & \frac{1}{r}&=\tfrac{1}{3}\beta+\frac{1-k}{k+3}.\end{aligned}\right\}\tag{1.11}$$

For k large enough ($k>6$ for the VP case, $k>3(\beta+1)/(3-\beta)$ for the VS case), a control of M_k therefore yields an L^∞ bound on F and thus allows us to prove the propagation of the (x,v) support of f, or of its derivatives, and thus to deduce its regularity.

Let us recall that the issue of the regularity of large solutions to nonlinear transport equations is a classical question still unresolved for several three-dimensional models (e.g. Vlasov–Maxwell, Boltzmann). Several theories have been proposed for understanding the mechanisms that provide regularity. For the VP system, a method based on directly proving regularity through characteristics has been proposed by Pfaffelmoser [20], Batt [1–3], Batt and Rein [6], Schaeffer [21] and Horst [11,12]. The case of two-dimensional (or two dimensions and a half) Vlasov–Maxwell systems is treated in [9]. For the BGK model of the Boltzmann equation, existence of smooth solutions follows from the control of propagation of the $L^\infty(\mathbb{R}^6)$ norms of $|v|f(\cdot,\cdot,t)$ (see [19]). For the Vlasov–Poisson Fokker–Planck system, still another theory has been developed by Bouchut [4,5].

Here, we will follow an approach based on proving the propagation of the velocity moments M_k, as in a previous paper by Lions and Perthame [15]. We combine it with moment lemmas which assert that, locally in space, velocity moments can be gained from the kinetic equation itself (see [7,16,17] or lemma 2.2 below). This induces a difficulty in getting back global regularity in space despite the local aspect of the moment lemmas. We solve it using, indirectly, the propagation of x-moments of f. This method allows us to simplify the method of [15] and to improve the results in the sense that we can prove the propagation of lower moments, since we show it on the VP system, and also to handle stronger singularities in the nonlinearity, since we illustrate it on the VS system. Namely, for the VP system, we prove the following result.

Theorem 1.1 (VP system). *We assume that $f^0\in L^\infty(\mathbb{R}^6)$ and that, for some $k_0>2$, we have*

$$\int_{\mathbb{R}^6}(1+|v|^{k_0}+|x|^{1/3+0})f^0(x,v)\,\mathrm{d}v\,\mathrm{d}x<+\infty,\tag{1.12}$$

$$\rho^0(x,t):=\int_{\mathbb{R}^3}f^0(x-vt,v)\,\mathrm{d}v\in L^1_{\mathrm{loc}}(0,+\infty;L^{3(k_0+3)/(k_0+6)}(\mathbb{R}^3)).\tag{1.13}$$

Then there exists a weak solution to (1.1) which satisfies, for all $t,T>0$,

$$f(x,v,t)\geqslant 0,\qquad \|f(\cdot,\cdot,t)\|_\infty\leqslant\|f^0(\cdot,\cdot)\|_\infty,\tag{1.14}$$

$$\int_{\mathbb{R}^6}(1+|v|^{k_0}+|x|^{1/3+0})f(x,v,t)\,\mathrm{d}v\,\mathrm{d}x\in L^\infty(0,T),\tag{1.15}$$

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$$F \in L^1(0,T; L^{k_0+4-0}(\mathbb{R}^3)), \tag{1.16}$$

$$\rho \in L^\infty(0,T; L^{(k_0+3)/3}(\mathbb{R}^3)). \tag{1.17}$$

REMARK 1.2.

(1) Throughout this paper, when we use notation like $u \in L^{p+0}$ we mean that there exists an $\varepsilon > 0$ such that $u \in L^{p+\varepsilon}$.

(2) Notice that in the above theorem, we solve a question asked in [15]. Namely, to prove the propagation of a v-moment of order larger than two, while in [15] it is fundamental to control initially moments larger than three.

(3) Also, the regularity of the force field can be completed as follows

$$F \in L^\infty(0,T; L^r(\mathbb{R}^3)),$$

$$r = 3\left(\frac{k+3}{6-k}\right) \quad \text{for } 2 < k < 6, \quad r = \infty \quad \text{for } k > 6.$$

(4) An improvement is still possible. A careful application of the same proof shows that the assumption $f^0 \in L^\infty$ can be relaxed to some L^p.

Turning now to the VS system, we prove the following result. It is the first regularity result for this system. The difficulty here comes from the lower L^p regularity available on j compared to ρ, and thus on the corresponding force F.

THEOREM 1.3 (VS system). *We assume that* $0 < \beta < \frac{8}{5}$, $f^0 \in L^\infty(\mathbb{R}^6)$ *and that, for some* $k_0 > 2$, *we have*

$$\int_{\mathbb{R}^6} (1 + |v|^{k_0} + |x|^2) f^0(x,v) \, \mathrm{d}v \, \mathrm{d}x < +\infty, \tag{1.18}$$

$$J^0(x,t) := \int_{\mathbb{R}^3} |v| f^0(x - vt, v) \, \mathrm{d}v \in L^1_{\mathrm{loc}}(0,+\infty; L^p(\mathbb{R}^3)), \quad \frac{1}{p} \leqslant \frac{k_0+5}{k_0+4} - \tfrac{1}{3}\beta. \tag{1.19}$$

Then there exists a weak solution to (1.4) which satisfies, for all $t, T > 0$,

$$f(x,v,t) \geqslant 0, \qquad \|f(\cdot,\cdot,t)\|_\infty \leqslant \mathrm{e}^{3t} \|f^0(\cdot,\cdot)\|_\infty, \tag{1.20}$$

$$\int_{\mathbb{R}^6} (1 + |v|^{k_0} + |x|^2) f(x,v,t) \, \mathrm{d}v \, \mathrm{d}x \in L^\infty(0,T), \tag{1.21}$$

$$F \in L^1(0,T; L^{k_0+4-0}(\mathbb{R}^3)), \tag{1.22}$$

$$j \in L^\infty(0,T; L^{(k_0+3)/4}(\mathbb{R}^3)). \tag{1.23}$$

REMARK 1.4.

(1) Improving the possible singularity of the matrix A, i.e. the upper value of β, is an open question.

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(2) Again, the regularity of the force field can be completed as follows

$$F \in L^\infty(0,T;L^r(\mathbb{R}^3)), \qquad \begin{cases} \dfrac{1}{r} = \tfrac{1}{3}\beta + \dfrac{1-k}{k+3} & \text{for } 2 < k < 3\left(\dfrac{\beta+1}{3-\beta}\right), \\ r = \infty \quad \text{for } k > 3\left(\dfrac{\beta+1}{3-\beta}\right). \end{cases} \tag{1.24}$$

The end of this paper explains the proof of these results. In a second section, we give the main lemmas and show the strategy of proof. The most fundamental estimate is specific to each case and its proof is detailed in separate sections.

2. Proofs of the main theorems

In this section we are concerned with the proofs of the main theorems 1.1 and 1.3. Before going to the new ingredients, we recall the general method and some necessary preliminary lemmas valid for both the VP and VS systems.

First of all, as is usual to prove these theorems, we consider a sequence of classical solutions to a regularized system (with regularized positive convolution operators which define the forces, this is possible in truncating for high frequencies), with regularized and compactly supported initial data. It is enough to prove the estimates of the theorems for these solutions and then to pass to the limit on the regularization. Secondly, for such solutions, the positivity and L^∞ bounds stated in the theorems are true thanks to the maximum principle, as well as the kinetic energy bounds (see [8] or [13]), which can be kept by appropriate regularizations of the force kernel. The only difficult point is then to prove the propagation of moments higher than two. This proof follows the same lines for the two systems. But the form of the VS system makes it longer due to the friction term which, however, does not add any specific difficulty. Therefore, we restrict our proof to the simplified system where we neglect the friction term, i.e. we only consider (1.1) with the two cases of forces F.

In the following, we set

$$K_\infty = \|f^0\|_\infty, \tag{2.1}$$

and we recall some technical lemmas.

2.1. Preliminary lemmas

The first lemma concerns the propagation of moments for solutions to (1.1).

LEMMA 2.1. *Let $k > 0$. Then, for $0 \leqslant t \leqslant T$, the moments $M_k(t)$ defined in (1.8) satisfy*

$$M_k(t) \leqslant C(T,K_\infty)\left(M_k(0) + \left(\int_0^t \|F(s)\|_{k+3}\,\mathrm{d}s\right)^{k+3}\right). \tag{2.2}$$

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This lemma is well known and can be proven easily using, explicitly, the Vlasov equation and the inequality

$$\int_{\mathbb{R}^3} |v|^{k-1} f(x,v,t)\,\mathrm{d}v \leqslant C(K_\infty)\biggl(\int_{\mathbb{R}^3} |v|^k f(x,v,t)\,\mathrm{d}v\biggr)^{(k+2)/(k+3)}, \tag{2.3}$$

which generalizes (1.10).

The second result is a so-called moment lemma about the gain of velocity moments by integration in time. It was first used in [17] to solve the BGK model. A more direct and systematic approach was devised in [16]. The possibility to use it in order to control macroscopic quantities was proved in [7].

LEMMA 2.2. *Let $\alpha > 0$, $k \geqslant 1$ and $t > 0$ and let f be a smooth solution to the Vlasov equations (1.1), $\|F(\cdot)\|_{k+3} \in L^1(0,T)$ and $M_k(t) < \infty$. Then the inequality*

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^6} \frac{|v|^{k+1}}{1+|x|^{1+\alpha}} f(x,v,s)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s \\ \leqslant C(K_\infty)\biggl[M_k(t) + \int_0^t \|F(s)\|_{k+3}\,\mathrm{d}s\, M_k(t)^{(k+2)/(k+3)}\biggr]\end{aligned} \tag{2.4}$$

holds for some constant which also depends upon t, k and α.

Proof. We multiply the Vlasov equation by

$$|v|^{k-1}\frac{x\cdot v}{(1+|x|^\alpha)^{1/\alpha}}, \quad \alpha > 0, \quad k \geqslant 1, \tag{2.5}$$

and integrate over $\mathbb{R}^3_x \times \mathbb{R}^3_v \times (0,t)$. After integration by parts and using (2.3), this yields

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^6} \frac{|v|^{k+1}}{(1+|x|^\alpha)^{1/\alpha}}\biggl(1 - \frac{|x|^{\alpha-2}(x\cdot v)^2}{(1+|x|^\alpha)|v|^2}\biggr) f(x,v,s)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s \\ \leqslant \int_{\mathbb{R}^6} |v|^k (f(x,v,t) + f(x,v,0))\,\mathrm{d}v\,\mathrm{d}x \\ + k\int_0^t \int_{\mathbb{R}^3_x} |F(x,s)| \biggl(\int_{\mathbb{R}^3_v} |v|^{k-1} f(x,v,s)\,\mathrm{d}v\biggr)\mathrm{d}x\,\mathrm{d}s \\ \leqslant 2\biggl(M_k(t) + C(K_\infty,k)\int_0^t \|F(s)\|_{k+3}\,\mathrm{d}s\, M_k(t)^{(k+2)/(k+3)}\biggr).\end{aligned} \tag{2.6}$$

Finally, we remark that

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^6} \frac{|v|^{k+1}}{1+|x|^{1+\alpha}} f(x,v,s)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s \\ \leqslant C(\alpha)\int_0^t \int_{\mathbb{R}^6} \frac{|v|^{k+1}}{(1+|x|^\alpha)^{1/\alpha}}\biggl(1 - \frac{|x|^{\alpha-2}(x\cdot v)^2}{(1+|x|^\alpha)|v|^2}\biggr) f(x,v,s)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s,\end{aligned}$$

which concludes the proof. □

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2.2. Another formula for the force fields

The following result is the main new ingredient in the proofs of the theorems. It improves the method introduced in the analysis of the VP system by [15] in order to use the moment lemma 2.2. Since the exponents coming in for the two systems are quite different, we state the result separately.

LEMMA 2.3 (VP system). *Smooth solutions to the regularized VP system (1.1) (see above) satisfy*

$$\int_0^T \|F(t)\|_r \,\mathrm{d}t \leqslant C(T) \int_0^T \|\rho^0(t)\|_p \,\mathrm{d}t + C_1\bigg(\int_0^T \int_{\mathbb{R}^6} \frac{|v|^{k+1}}{1+|x|^{1+0}} f(x,v,t)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t\bigg)^{1/r} \tag{2.7}$$

for all $3 < r < k+4$ *and with* $1/r = 1/p - \frac{1}{3}$ *and where* C_1 *also depends on the parameters* k, r *and* K_∞, *the initial energy* $M_2(0)$ *and* $\int_{\mathbb{R}^6} |x|^{1/3+0} f^0(x,v)\,\mathrm{d}x\,\mathrm{d}v$.

LEMMA 2.4 (VS system). *Smooth solutions to the regularized VS system (1.4) (see above) satisfy*

$$\int_0^T \|F(t)\|_r \,\mathrm{d}t \leqslant C(T) \int_0^T \|J^0(t)\|_p \,\mathrm{d}t + C_2 \int_0^T \int_{\mathbb{R}^6} \bigg(\frac{|v|^{k+1}}{1+|x|^{1+0}} f(x,v,t)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t\bigg)^{1/r} \tag{2.8}$$

for all r *such that*

$$\frac{1}{k+1} + \tfrac{1}{3}(\beta-2) < \frac{1}{r} < \tfrac{1}{3}(\beta-1)$$

and with $1/r > 1/p + \frac{1}{3}(\beta-3)$. *Here,* C_2 *also depends on the parameters* β, p, k, r *and* K_∞, *the initial kinetic energy* $M_2(0)$ *and*

$$\int_{\mathbb{R}^6} |x|^2 f^0(x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

The proofs of these lemmas are given in the next section. With these three types of lemmas, we are now able to prove our main theorem.

2.3. Concluding the proofs of the main theorems

2.3.1. *The Vlasov–Poisson case*

We combine lemmas 2.3 and 2.2 so as to get

$$\int_0^T \|F(t)\|_r \,\mathrm{d}t \leqslant C \int_0^T \|\rho^0(t)\|_p \,\mathrm{d}t + C\bigg(M_k(T) + M_k(t)^{(k+2)/(k+3)} \int_0^T \|F(t)\|_{k+3} \,\mathrm{d}t\bigg)^{1/r}, \tag{2.9}$$

with $3 < r < k+4$ and $1/r = 1/p - \frac{1}{3}$.

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The assumption on ρ^0 in theorem 1.1 allows us to control the integral of $\|\rho^0\|_p$ for all corresponding r between 3 and k_0+3 included. We already know that the kinetic energy M_2 is bounded in time and thus we can apply the result (2.9) for $k=2$ (recall we already control F in $L^1_t L^p_x$ for all $p<5$ thanks to (2.7)), thus obtaining that the integral in time of the L^r norm of F is bounded for any $3<r<6$.

As a consequence, using lemma 2.1, we immediately propagate every moment $2<k<3$, thus concluding the theorem for $k_0<3$. Then, for $k_0 \geqslant 3$, we repeat the above argument using any $k<3$, which allows, with (2.9), to reach $r<7$ and therefore, using lemma 2.1 again, to propagate any moments up to $k<4$, thus concluding the theorem for $k_0<4$. One can easily see that each repetition of these two steps allows to gain one unit on k for the propagation of moments, and we are thus able to reach any value k_0.

2.3.2. *The Vlasov–Stokes case*

The proof for VS system follows the same lines. We use lemmas 2.4 and 2.2 to obtain

$$\int_0^T \|F(t)\|_r \,\mathrm{d}t \leqslant C\int_0^T \|J^0\|_p \,\mathrm{d}t + C\biggl(M_k(T) + M_k(t)^{(k+2)/(k+3)}\int_0^T \|F(t)\|_{k+3}\,\mathrm{d}t\biggr)^{1/r}, \tag{2.10}$$

with

$$\frac{3}{\beta-1} < r < \biggl(\frac{1}{k+1} - \tfrac{1}{3}(2-\beta)\biggr)^{-1} \quad \text{and} \quad \frac{1}{p} < \frac{1}{r} + \tfrac{1}{3}(3-\beta).$$

As with the VP case, the assumption on J^0 in the theorem is enough to upper bound the J^0 term in (2.10), and this for all for $3/(\beta-1) < r \leqslant k_0+3$. First, we use, from the energy, that M_2 is bounded, and from (2.8), we deduce that F is bounded in $L^1_t L^p_x$ for all $p<5$. And thus, from the inequality (2.10), we get that F belongs to $L^1_t L^r_x$, r being given by the above formula, and, using lemma 2.1, we propagate every moment of order k' less than $r-3$, i.e.

$$\frac{1}{k'+3} > \frac{1}{k+1} - \tfrac{1}{3}(2-\beta). \tag{2.11}$$

For $k=2$, this allows us to control moments with $k'>2$ (this explains the limitation $\beta<\frac{8}{5}$). Moreover, iterating the argument, for $k \geqslant 2$, k' can be chosen strictly larger (with a uniform gap) than k because the L^{k+3}_x bound on F is then automatic. And we can repeat this procedure until we propagate all the desired moments thus concluding the proof.

3. Estimates on the force fields

In this section, we prove the fundamental lemmas 2.3 and 2.4 stated in the previous section, which allow us to obtain a better L^r estimate on the force field, working in L^1 in time rather than L^∞, and using some kind of localization in space. The exponents arising in the proof depend on the specific form of the force fields and therefore differ somewhat in the two cases of Poisson and Stokes flows. The proofs are thus presented in two subsections.

They both use the following common expression for the Vlasov equation (recall that we neglect the friction term in the Stokes case to simplify the proofs)

$$\frac{\partial}{\partial t} f + v \cdot \nabla_x f + \mathrm{div}_v(F_{\mathrm{L}} f) = \mathrm{div}_v(F_{\mathrm{S}} f), \tag{3.1}$$

where the long-range part of the force is given by $F_{\mathrm{L}} = F_{\mathrm{S}} + F$ and the short-range part of the force is defined for the VP system as $F_{\mathrm{S}} := \pm \rho \star (\chi_R x/|x|^3)$ and for the VS system as $F_{\mathrm{S}} := j \star (\chi_R A)$, where χ_R is a smooth cut-off function of the ball of radius R, which vanishes for $|x| > 2R$, and such that $\chi_R(x) = 1$ for $|x| \leqslant R$. We also use a representation of the solution related to the well-defined characteristics

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} X(t) &= V(t), & X(0) &= x, \\ \frac{\mathrm{d}}{\mathrm{d}t} V(t) &= F_{\mathrm{L}}(t, X(t)), & V(0) &= v. \end{aligned}\right\} \tag{3.2}$$

We now choose the truncation parameter R large enough compared to the final time T, so that these characteristics and their partial Jacobians behave like $X(t) = x - vt$, $V(t) = v$, which are obtained for $F_{\mathrm{L}} = 0$, i.e. the limit as $R \to \infty$. For the sake of simplicity, the proofs below are written for these limiting characteristics, but the arguments hold for those given by (3.2) as it was checked in [15], the numerous changes of variables only require to control Jacobians of $\partial X/\partial x$, $\partial X/\partial v$, etc.

With this simplification, we have

$$\begin{aligned} f(x, v, t) &= f^0(x - vt, v) + \int_0^t (\mathrm{div}_v\, F_{\mathrm{S}} f)(x - vs, v, t - s)\, \mathrm{d}s \\ &= f^0(x - vt, v) + \int_0^t \mathrm{div}_v(F_{\mathrm{S}}) f(x - vs, v, t - s)\, \mathrm{d}s \\ &\qquad\qquad + \int_0^t \mathrm{div}_x(F_{\mathrm{S}}) f(x - vs, v, t - s) s\, \mathrm{d}s. \end{aligned} \tag{3.3}$$

3.1. The Vlasov–Poisson system

Now we restrict our attention to the VP case. From the above formula, we deduce

$$\rho(x, t) = \int_{\mathbb{R}^3} f^0(x - vt, v)\, \mathrm{d}v + \int_0^t \int_{\mathbb{R}^3} \mathrm{div}_x(F_{\mathrm{S}} f)(x - vs, v, t - s) s\, \mathrm{d}s\, \mathrm{d}v.$$

Using the fact that F wins a full derivative compared to ρ in L^p norms, we deduce, from the Calderon–Zygmung theory (see [22]), with $1/r = 1/p - \frac{1}{3}$ and recalling the definition of ρ^0 in theorem 1.1,

$$\|F(\cdot, t)\|_r \leqslant \|\rho^0(\cdot, t)\|_p + \int_0^t \left\| \int_{\mathbb{R}^3} (F_{\mathrm{S}} f)(x - vs, v, t - s)\, \mathrm{d}v \right\|_r s\, \mathrm{d}s. \tag{3.4}$$

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We now treat the second $L^r(\mathbb{R}^3_x)$ norm term in a new way. We write, x being fixed, using Holder's inequality in $\mathrm{d}v$,

$$
\begin{aligned}
\biggl|\int (F_{\mathrm{S}} f)(x-vs,v,t-s)\,\mathrm{d}v\biggr|
&\leqslant \|F_{\mathrm{S}}(x-vs,t-s)|(1+|x-vs|^{1/3+0})\|_{3/2+0}\\
&\qquad\times \|f(x-vs,v,t-s)(1+|x-vs|^{1/3+0})^{-1}\|_{3-0}\\
&\leqslant K_\infty^{2/3-0}\frac{1}{s^{2-0}}\|F_{\mathrm{S}}(\cdot,t-s)|(1+|\cdot|^{1/3+0})\|_{3/2+0}\\
&\qquad\times\biggl(\int_{\mathbb{R}^3} f(x-vs,v,t-s)\times(1+|x-vs|^{1+0})^{-1}\,\mathrm{d}v\biggr)^{1/3-0}\\
&\leqslant K_\infty^{2/3+(k+1)/(k+4)-0}\frac{1}{s^{2-0}}\|F_{\mathrm{S}}(\cdot,t-s)(1+|\cdot|^{1/3+0})\|_{3/2+0}\\
&\qquad\times\biggl(\int_{\mathbb{R}^3}|v|^{k+1}f(x-vs,v,t-s)\times(1+|x-vs|^{1+0})^{-1}\,\mathrm{d}v\biggr)^{1/4+k-0}.
\end{aligned}
$$

Here, we have used a variant of the general interpolation inequality (2.3). We now conclude, using $r=4+k-0$, that

$$
\begin{aligned}
\|F(\cdot,t)\|_r \leqslant \|\rho^0(\cdot,t)\|_p + C(K_\infty)\int_0^t \|F_{\mathrm{S}}(\cdot,t-s)(1+|\cdot|^{1/3+0})\|_{3/2+0}&\\
\times\biggl(\int_{\mathbb{R}^6}|v|^{k+1}f(x-vs,v,t-s)(1+|x-vs|^{1+0})^{-1}\,\mathrm{d}v\,\mathrm{d}x\biggr)^{1/r}\frac{\mathrm{d}s}{s^{1-0}}.&
\end{aligned}
\tag{3.5}
$$

After integrating in time and changing the variable $\mathrm{d}x=\mathrm{d}(x-vs)$, we obtain lemma 2.3. Indeed, the mass and energy propagations imply that

$$
\begin{aligned}
|F_{\mathrm{S}}(x,\sigma)|(1+|x|^\alpha)| &\leqslant C\int_{|x-y|\leqslant R}\frac{\rho(y,\sigma)}{|x-y|^2}(1+|x-y|^\alpha+|y|^\alpha)\\
&\leqslant C\rho\star\frac{1}{|x|^2}+C\rho\star\frac{1}{|x|^{2-\alpha}}+C\biggl(|x|^\alpha\rho\star\frac{1}{|x|^2}\biggr).
\end{aligned}
$$

For $0\leqslant\alpha<2$, each of these terms is bounded in $L^1\cap L^s$ for some $s>\frac{3}{2}$ thanks to the *a priori* estimates $\rho\in L^\infty(0,\infty;L^1)$ (mass conservation), $\rho\in L^\infty(0,\infty;L^{5/3})$ (energy conservation and (1.10)), and $|x|^\alpha\rho\in L^\infty(0,\infty;L^1\cap L^s)$ (propagation of x-moments of order less than two and interpolation with $\rho\in L^\infty(0,\infty;L^{5/3})$).

3.2. The Vlasov–Stokes system

The proof of lemma 2.4 for the VS system follows the same ideas as for the VP system. First of all, in place of (3.2) we use the characteristics defined by the regular force term, which is the sum of the friction term and the long-range part of

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the force $F(t,x)$, to find the estimate (see the beginning of § 3 for notation)

$$\int_0^T \|F(\cdot,t)\|_r \,\mathrm{d}t \leqslant \int_0^T \|J^0(t)\|_p \,\mathrm{d}t + \int_0^T \bigg\| \int_0^t \int_{\mathbb{R}^3} F_{\mathrm{S}} f(x-vs,v,t-s)\,\mathrm{d}v\,\mathrm{d}s \bigg\|_a \mathrm{d}t \\ + \int_0^T \bigg\| \int_0^t s \int_{\mathbb{R}^3} |v||F_{\mathrm{S}}| f(x-vs,v,t-s)\,\mathrm{d}v\,\mathrm{d}s \bigg\|_b \mathrm{d}t, \tag{3.6}$$

with the relations

$$\frac{1}{r} = \frac{1}{a} + \tfrac{1}{3}\beta - 1 = \frac{1}{b} + \tfrac{1}{3}(\beta - 2). \tag{3.7}$$

We denote

$$\left.\begin{aligned} \mathrm{I} &= \int_0^T \bigg\| \int_0^t \int_{\mathbb{R}^3} F_{\mathrm{S}} f(x-vs,v,t-s)\,\mathrm{d}v\,\mathrm{d}s \bigg\|_a \mathrm{d}t, \\ \mathrm{II} &= \int_0^T \bigg\| \int_0^t s \int_{\mathbb{R}^3} |v||F_{\mathrm{S}}| f(x-vs,v,t-s)\,\mathrm{d}v\,\mathrm{d}s \bigg\|_b \mathrm{d}t. \end{aligned}\right\} \tag{3.8}$$

In the next two subsections, we explain how we can upper bound these two terms.

3.2.1. *Bound on the term* I *in the inequality (3.6)*

We write

$$\mathrm{I} \leqslant \int_0^T \bigg\| \int_0^t \|F_{\mathrm{S}}(\cdot\cdot)\|_{L_v^c} \times \|f(\cdot\cdot)\|_{L_v^{c^\star}}\,\mathrm{d}s \bigg\|_{L_x^a} \mathrm{d}t, \tag{3.9}$$

where $c^\star$ is the conjugate exponent of c and with

$$\|F_{\mathrm{S}}(x-vs,t-s)\|_{L_v^c} = s^{-3/c}\|F_{\mathrm{S}}(\cdot,t-s)\|_{L_x^c} \leqslant Cs^{-3/c} \quad \text{for } c > 3. \tag{3.10}$$

Indeed, we have that the conservation of the kinetic energy implies that j belongs to $L^1 \cap L^{5/4}(\mathbb{R}^3)$ and hence we already know that F_{S} belongs to $L^{3/\beta} \cap L^{15/(5\beta-3)}$. For $1 \leqslant \beta < \frac{8}{5}$, we have $\frac{15}{8} < 3/\beta \leqslant 3$ and $3 < 15/(5\beta-3) \leqslant \frac{15}{2}$. Therefore, the force term F_{S} always belongs to $L^\infty([0,T], L^c(\mathbb{R}^3))$ for some $c > 3$.

Next we recall that

$$\int_{\mathbb{R}^3} f(x-vs,v,t-s)\,\mathrm{d}v \leqslant CK_\infty \bigg(\int |v|^\delta f(x-vs,v,t-s)\,\mathrm{d}v \bigg)^{3/(3+\delta)}. \tag{3.11}$$

Combining the last two inequalities, for any $\frac{3}{2} < a < \frac{5}{2}$,

$$\begin{aligned} \mathrm{I} &\leqslant C \int_0^T \bigg\| \int_0^t s^{-3/c} \|f(x-vs,v,t-s)\|_{L_v^{c^\star}}\,\mathrm{d}s \bigg\|_{L_x^a} \mathrm{d}t \\ &\leqslant C \int_0^T \int_0^t s^{-3/c} \bigg\| \bigg(\int f(x-vs,v,t-s)\,\mathrm{d}v \bigg)^{1/c^\star} \bigg\|_{L_x^a} \mathrm{d}s\,\mathrm{d}t \\ &\leqslant C \int_0^T \int_0^t s^{-3/c} \bigg(\int |v|^\delta f\,\mathrm{d}v\,\mathrm{d}x \bigg)^{1/a} \mathrm{d}s\,\mathrm{d}t \\ &\leqslant C, \end{aligned} \tag{3.12}$$

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since we have $\frac{1}{3}(3+\delta)=a/c$, and so δ is less than two, for a is less than $\frac{5}{2}$ and $c^\star$ is less than $\frac{3}{2}$ but as close to $\frac{3}{2}$ as we want.

Since the system conserves mass and kinetic energy, it also conserves all the moments between zero and two in velocity. Eventually, we have proved that

$$\int_0^T \Big\| \int_0^t \int_{\mathbb{R}^3} |F_{\mathrm{S}}| f(x-vs,v,t-s)\,\mathrm{d}v\,\mathrm{d}s\Big\|_a \,\mathrm{d}t$$

is bounded for all a between $\frac{3}{2}$ and $\frac{5}{2}$ included. Using relation (3.7), this can be used for any r between $3/(\beta-1)$ and $+\infty$, since β is less than $\frac{9}{5}$ (we work with β less than $\frac{8}{5}$).

3.2.2. *Bound on the term* II *in the estimate (3.6)*

First, we perform the same manipulation as in the previous section,

$$\begin{aligned}\mathrm{II} \leqslant \int_0^T \bigg\| \int_0^t \bigg\| &(1+|x-vs|)F_{\mathrm{S}}(x-vs,t-s)\|_{L^d_v} \\ &\times \bigg\| \frac{vf(x-vs,v,t-s)}{1+|x-vs|}\bigg\|_{L^{d^\star}_v} \mathrm{d}s\bigg\|_{L^b_x} \mathrm{d}t. \end{aligned} \tag{3.13}$$

We choose for d a number slightly larger than $\frac{3}{2}$ and we bound the term with F_{S} by $s^{-3/d}\|(1+|x|)F_{\mathrm{S}}(x,t-s)\|_d$ and decompose this last term as

$$\begin{aligned} |(1+|x|)&F_{\mathrm{S}}(x,t-s)| \\ &\leqslant \int (1+|x|)\frac{\chi_R(x-y)}{|x-y|^\beta}|j(y)|\,\mathrm{d}y \\ &\leqslant \int \frac{\chi_R(x-y)}{|x-y|^\beta} j(y)\,\mathrm{d}y + C\int \frac{\chi_R(x-y)}{|x-y|^{\beta-1}} j(y)\,\mathrm{d}y + C\int \frac{\chi_R(x-y)}{|x-y|^\beta}|y| j(y)\,\mathrm{d}y. \end{aligned} \tag{3.14}$$

For $\beta<2$, the first two terms on the right-hand side are obviously in L^d for some d larger than $\frac{3}{2}$ but as close as we wish. To bound the last term in L^d, we need an L^1 estimate on $|y|j(y)$, which is given by the following lemma.

Lemma 3.1. *If the kinetic energy is bounded, then (1.4) conserves all the moments in*

$$\int |x|^\delta \rho(x,t)\,\mathrm{d}x$$

for δ between zero and two.

Proof. This lemma is a straightforward consequence of the simple relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} (1+|x|)^\delta \rho(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^6} \delta(1+|x|)^{\delta-1}\frac{x}{|x|}\cdot vf(x,v,t)\,\mathrm{d}x\,\mathrm{d}v. \tag{3.15}$$

□

Thanks to this lemma, we know that

$$\int |x|^2 \rho(x,t)\,\mathrm{d}x$$

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belongs to $L^\infty([0,T])$ and thus

$$\int |x|\cdot|j(x,t)|\,\mathrm{d}x$$

also because of the inequality

$$\int_{\mathbb{R}^3} |x|\cdot|j(x,t)|\,\mathrm{d}x \leqslant \frac{1}{2}\int_{\mathbb{R}^6} |v|^2 f\,\mathrm{d}x\,\mathrm{d}v + \frac{1}{2}\int_{\mathbb{R}^3} |x|^2\rho(x,t)\,\mathrm{d}x. \tag{3.16}$$

As a consequence, for any d greater than $\frac{3}{2}$ but close enough, we have

$$\|(1+|x-vs|)F_{\mathrm{S}}(\cdot\cdot)\|_{L^d_v} \in L^\infty([0,T]\times\mathbb{R}^3). \tag{3.17}$$

We immediately deduce that

$$\begin{aligned}\mathrm{II} &\leqslant C\int_0^T\int_0^t s^{1-3/d}\left\|\frac{vf(x-vs,v,t-s)}{1+|x-vs|}\right\|_{L^b_x(L^{d^\star}_v)}\mathrm{d}s\,\mathrm{d}t\\ &\leqslant C\int_0^T\int_0^t s^{1-3/d}\left\|\left(\int\frac{|v|^{d^\star}f(x-vs,v,t-s)}{(1+|x-vs|)^{d^\star}}\,\mathrm{d}v\right)^{1/d^\star}\right\|_b \mathrm{d}s\,\mathrm{d}t,\end{aligned} \tag{3.18}$$

using the inequality

$$\int_{\mathbb{R}^3}\frac{|v|^\alpha f(x-vs,v,t-s)}{(1+|x-vs|)^\delta}\,\mathrm{d}v \leqslant C\left(\int_{\mathbb{R}^3}\frac{|v|^\gamma f(\cdot\cdot)}{(1+|x-vs|)^\delta}\,\mathrm{d}v\right)^{(3+\alpha)/(3+\gamma)}. \tag{3.19}$$

Then recalling that $d=\frac{3}{2}+0$, if b is greater than $d^\star$, we find

$$\begin{aligned}\mathrm{II} &\leqslant C\int_0^T\int_0^t s^{1-3/d}\left(\int_{\mathbb{R}^6}\frac{|v|^\delta f(x-vs,v,t-s)}{(1+|x-vs|)^{d^\star}}\,\mathrm{d}v\,\mathrm{d}x\right)^{1/b}\mathrm{d}s\,\mathrm{d}t\\ &\leqslant C\int_0^T\int_0^t s^{1-3/d}\left(\int_{\mathbb{R}^6}\frac{|v|^\delta f(x,v,t-s)}{(1+|x|)^{d^\star}}\,\mathrm{d}v\,\mathrm{d}x\right)^{1/b}\mathrm{d}s\,\mathrm{d}t\\ &\leqslant C\int_0^T\int_s^T s^{1-3/d}\left(\int_{\mathbb{R}^6}\frac{|v|^\delta f(x,v,t-s)}{(1+|x|)^{d^\star}}\,\mathrm{d}v\,\mathrm{d}x\right)^{1/b}\mathrm{d}t\,\mathrm{d}s\\ &\leqslant \tilde{C}\int_0^T\left(\int_{\mathbb{R}^6}\frac{|v|^\delta f(x,v,t)}{(1+|x|)^{d^\star}}\,\mathrm{d}v\,\mathrm{d}x\right)^{1/b}\mathrm{d}t\\ &\leqslant C'\left(\int_0^T\int_{\mathbb{R}^6}\frac{|v|^\delta f(x,v,t)}{(1+|x|)^{d^\star}}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{1/b},\end{aligned} \tag{3.20}$$

with the relation $d^\star/b=(3+d^\star)/(3+\delta)$. Since $d^\star=3-0$, if we denote $k=\delta-1$, this implies

$$k=b-1+0. \tag{3.21}$$

We can now conclude the proof. The result from § 3.2.1 is valid for r between $3/(\beta-1)$ and $+\infty$, and the result from § 3.2.2 for b larger than $3-0$, which means r larger than $3/(\beta-1)$ thanks to relation (3.7). Hence for r larger than $3/(\beta-1)$,

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we can put together these two results and get

$$\int_0^T \|F(\cdot,t)\|_r \,\mathrm{d}t \leqslant \int_0^T \|J^0\|_p \,\mathrm{d}t + C\biggl(1 + \biggl(\int_0^T \int_{\mathbb{R}^6} \frac{|v|^{k+1} f(x,v,t)}{(1+|x|)^{d^\star}} \,\mathrm{d}v \,\mathrm{d}x \,\mathrm{d}t\biggr)^{1/b}\biggr), \tag{3.22}$$

with

$$\frac{1}{b} = \frac{1}{r} + \tfrac{1}{3}(2-\beta), \quad k = b - 1 + 0, \tag{3.23}$$

which is exactly lemma 2.4.

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