THE JOURNAL OF SYMBOLIC LOGIC
Volume 82, Number 1, March 2017

HANDMADE DENSITY SETS

GEMMA CAROTENUTO

Abstract. Given a metric space (X, d), equipped with a locally finite Borel measure, a measurable set $A \subseteq X$ is a *density set* if the points where A has density 1 are exactly the points of A. We study the topological complexity of the density sets of the real line with Lebesgue measure, with the tools—and from the point of view—of descriptive set theory. In this context a density set is always in $\mathbf{\Pi}^0_3$. We single out a family of true $\mathbf{\Pi}^0_3$ density sets, an example of true $\mathbf{\Sigma}^0_2$ density set and finally one of true $\mathbf{\Pi}^0_2$ density set.

§1. Introduction. A measure on a topological space is *locally finite* if every point has a neighborhood of finite measure. Fix a triple (X, d, μ), where μ is a locally finite Borel measure on the metric space (X, d). Given $A \subseteq X$, a measurable set, we say that A has *density* $r \in [0; 1]$ at a point $x \in X$ if

$$\mathscr{D}_A(x) \stackrel{\text{def}}{=} \lim_{\epsilon \to 0} \frac{\mu(A \cap B(x, \epsilon))}{\mu(B(x, \epsilon))} = r,$$

where $B(x, \epsilon)$ is the open ball of radius ϵ centered at x. The *density set of* A is the set

$$\Phi(A) = \{x \in X \mid A \text{ has density } 1 \text{ at } x\}.$$

We define a *density set* to be any set of the form $\Phi(B)$, for some B measurable subset of X. An exhaustive study of the topological complexity of density sets in the Cantor space, equipped with the usual metric and the coin-tossing measure, is carried out by A. Andretta and R. Camerlo in [1]. Unfortunately, since the notion of density heavily depends on the distance and the measure, there is no straightforward way to translate their results to other measure metric spaces.

Consider the real line with Lebesgue measure μ. It is well known that in this context the density sets are in $\mathbf{\Pi}^0_3$. In fact, for every measurable set $A \subseteq \mathbb{R}$,

$$\Phi(A) = \bigcap_{\substack{n \in \omega \\ n > 0}} \bigcup_{\substack{k \in \omega \\ k > 0}} \bigcap_{h \in (0; k^{-1}) \cap \mathbb{Q}} \left\{ x \in \mathbb{R} \mid \frac{\mu(A \cap (x - h; x + h))}{2h} \geq 1 - 1/n \right\}.$$

Thus, we have the first natural question:

QUESTION 1.1. *Does there exist a true* $\mathbf{\Pi}^0_3$ *density set*?

Received July 30, 2015.
2010 *Mathematics Subject Classification.* 03E15, 28A05.
Key words and phrases. Lebesgue measure, density.

0022-4812/17/8201-0012
DOI:10.1017/jsl.2016.53

In [6] the author considers a particular type of compact set of positive measure, which arises from some very regular Cantor constructions. If K is a compact set of this type, in [6] a combinatorial condition for a point x is determined so that K has density 1 at x (Theorem 3.2). Using this result, we show that $\Phi(K)$ is $\mathbf{\Pi}^0_3$-complete, giving a positive answer to Question 1.1 (Theorem 3.3).

QUESTION 1.2. *What about density sets of simpler topological complexities?*

Every open interval is trivially a density set, while C. Costantini showed that there is no true $\mathbf{\Pi}^0_1$ density set (personal communication). On the other hand, looking at the second level of the Borel hierarchy, we single out an example of true $\mathbf{\Sigma}^0_2$ density set (Proposition 4.3) and one of true $\mathbf{\Pi}^0_2$ density set (Proposition 5.5). The word *handmade* in the title of the paper refers to the fact that the density sets which we will illustrate here arise from *ad hoc* constructions, and in this sense it expresses the difference with a more systematic method of finding density sets of specific topological complexities, which will be followed in a forthcoming work on the same theme.

The paper is organized as follows. In Section 2, we will review some standard notations, and then we will introduce some terminology about Cantor constructions of the real line. In Section 3, we will prove that the density sets of an entire family of Cantor sets of positive measure are true $\mathbf{\Pi}^0_3$, through a combinatorial argumentation. In Section 4, we will give an example of a density set which is true $\mathbf{\Sigma}^0_2$, by means of a very *asymmetric* Cantor-construction. Finally, in Section 5, we will define a true $\mathbf{\Pi}^0_2$ density set, starting from the ternary Cantor set.

This work is part of the author's Ph.D. thesis ([3]), who would like to thank her advisor Alessandro Andretta for all the useful suggestions and stimulating discussions on this argument. The anonymous referee should be also thanked for thoroughly reading the manuscript of this paper.

§2. Notation and preliminaries.

2.1. Basics. For the basic concepts of descriptive set theory and measure theory, the reader is referred to [4] and [5], respectively. The notation of this paper is standard, but for the reader's benefit we summarize it below.

2.1.1. Sequences. The set of all natural numbers is denoted by $\omega = \{0, 1, 2, \dots\}$. A *sequence* s is a function from an ordinal to a set: its domain is called *length* and is denoted by $\mathrm{lh}(s)$. We denote by ${}^nX$, ${}^{<\omega}X$, and ${}^{\omega}X$ the sets of the sequences of elements of X of length n, of finite length and of length ω, respectively. We set ${}^{\leq\omega}X = {}^{<\omega}X \cup {}^{\omega}X$. Notice that we allow the case $n = 0$ and by definition $X^0 = \{\emptyset\}$, where $\emptyset$ denotes here the *empty sequence*. A sequence s of length n is denoted by $\langle s(0), s(1), \dots, s(n-1)\rangle$ and a sequence z of length ω by $\langle z(0), z(1), \dots\rangle$. If $s \in {}^{\leq\omega}X$ is constantly equal to some $x \in X$, we simply indicate s by $x^{(n)}$, if $\mathrm{lh}(s) = n < \omega$, or by $x^{(\infty)}$, if $\mathrm{lh}(s) = \omega$. If $m < \mathrm{lh}(s)$, we indicate by $s \restriction m$ the restriction of s to the domain m. Given $s \in {}^{<\omega}X$ and $z \in {}^{\leq\omega}X$, we put $s \subseteq z$, if $s = z \restriction \mathrm{dom}(s)$. Two sequences $s, t \in {}^{\leq\omega}X$ are *incompatible*, in symbols $s \perp t$, if there is some $n < \mathrm{lh}(s), \mathrm{lh}(t)$ such that $s(n) \neq t(n)$; otherwise they are *compatible*, that is s extends t or t extends s. For $s \in {}^{<\omega}X$ and $z \in {}^{\leq\omega}X$, the concatenation of s with z is denoted by $s^\frown z$, or even by sz, if there is no danger of confusion. If

https://doi.org/10.1017/jsl.2016.53 Published online by Cambridge University Press

$z = \langle x \rangle$, we often write $s^\frown x$, instead of $s^\frown\langle x \rangle$. If X is a linear order and $s, t \in {}^n 2$ for some $n > 0$, by $s <_{\text{lex}} t$ we mean that s is smaller than t in the lexicographic order.

2.1.2. Sets, functions and metric spaces. If $A \subseteq X$ and the set X is clear from the context, $\complement A$ is the set theoretic complement of A with respect to X, i.e., $\complement A = X \setminus A$. If f is a function, we denote its domain and its range by $\operatorname{dom}(f)$ and $\operatorname{ran}(f)$, respectively. If $f : X \to Y$ and $A \subseteq X$, then $f[A]$ denotes the image of A under f and $f^{-1}[B]$ denotes the preimage of B under f, where $B \subseteq Y$.

Let (X, d) be a metric space. For $x \in X$ and $r > 0$ the *open ball* with center x and radius r is the set $\mathrm{B}(x; r) = \{y \in X \mid d(x, y) < r\}$. Given $\emptyset \neq A, B \subseteq X$, we set $d(x, A) = \inf\{d(x, y) \mid y \in A\}$ and $d(A, B) = \inf\{d(x, y) \mid x \in A, y \in B\}$, and we call *$r$-ball around A* the set $\mathrm{B}(A; r) = \{x \in X \mid d(x, A) < r\}$.

We follow the modern logician notation for the classes of the Borel hierarchy: by $\mathbf{\Sigma}^0_1$ we mean the family of the open sets, by $\mathbf{\Sigma}^0_{n+1}$ the family of the countable unions of $\mathbf{\Pi}^0_n$ sets, where $\mathbf{\Pi}^0_n$ is the family of the complements of $\mathbf{\Sigma}^0_n$ sets. Therefore, in particular, $\mathbf{\Sigma}^0_2 = F_\sigma$, $\mathbf{\Pi}^0_2 = G_\delta$, and $\mathbf{\Pi}^0_3 = F_{\sigma\delta}$.

Given any class $\mathbf{\Gamma}(X)$ of subsets of X, let $\check{\mathbf{\Gamma}}(X)$ be its *dual* class, i.e., $\check{\mathbf{\Gamma}}(X) = \{X \setminus A \mid A \in \mathbf{\Gamma}(X)\}$. We say that a set $A \subseteq X$ is *true* $\mathbf{\Gamma}$, if $A \in \mathbf{\Gamma}(X) \setminus \check{\mathbf{\Gamma}}(X)$.

2.1.3. Wadge reducibility. Let X and Y be topological spaces and $A \subseteq X$, $B \subseteq Y$. We say that A is *Wadge reducible* to B, and we write $(X, A) \leq_W (Y, B)$, if there is a continuous function $f : X \to Y$ such that $A = f^{-1}[B]$, i.e., $x \in A \Leftrightarrow f(x) \in B$. If $X = Y$ and the space X is clear from the context, then $(X, A) \leq_W (Y, B)$ is usually abbreviated by $A \leq_W B$.

Let $\mathbf{\Gamma}$ be a class of sets in Polish spaces. If Y is a Polish space and $A \in \mathbf{\Gamma}(Y)$, we say that A is $\mathbf{\Gamma}$-*complete* if $B \leq_W A$, for any $B \in \mathbf{\Gamma}(X)$, where X is a zero-dimensional Polish space. Note that if A is $\mathbf{\Gamma}$-*complete*, then $X \setminus A$ is $\check{\mathbf{\Gamma}}$-*complete*. Moreover, given a $\mathbf{\Gamma}$-*complete* set B and $A \in \mathbf{\Gamma}$, if $B \leq_W A$, then A is $\mathbf{\Gamma}$-*complete*. As a consequence of the Wadge's Lemma (see, e.g., [4, p. 156]), we have the following result:

THEOREM 2.1 (Wadge). *Let X be a zero-dimensional Polish space and $\mathbf{\Gamma}(X)$ a class of Borel subsets of X, closed under continuous preimages. If Γ is not closed under relative complementation, then for every $A \subseteq X$*

$$A \text{ is } \mathbf{\Gamma}\text{-complete} \Leftrightarrow A \text{ is true } \mathbf{\Gamma}.$$

2.2. Cantor constructions. We exclusively consider the Lebesgue measure on $\mathbb{R}$, denoted by the symbol μ. For the sake of simplicity, the length (i.e., the Lebesgue measure) of an interval I is denoted by $|I|$.

The Cantor ternary set, usually denoted by $E_{1/3}$, is the subset of $\mathbb{R}$ created by repeatedly deleting the open middle third of a set of line segments, starting by $[0; 1]$ (Figure 1). This definition is generalized to a wide class of subsets of $\mathbb{R}$

FIGURE 1. A Cantor construction.

https://doi.org/10.1017/jsl.2016.53 Published online by Cambridge University Press

as follows.[1] We say that an interval is *nondegenerate* if it is nonempty and not a singleton. Moreover, if X and Y are subsets of the real line, by $X < Y$ we mean that

$$\forall x \in X\, \forall y \in Y (x < y),$$

where $<$ is the usual order on $\mathbb{R}$. Let $K_\emptyset$ be a closed nondegenerate interval. If $I_\emptyset$ is an open interval completely contained in $K_\emptyset$,[2] we can extract $I_\emptyset$ from $K_\emptyset$, creating two closed nondegenerate subintervals $K_0 < K_1$ of $K_\emptyset$. If we infinitely repeat this operation we obtain a closed nondegenerate interval K_s and an open interval I_s completely contained in K_s for every $s \in {}^{<\omega}2$, such that $K_s \setminus I_s$ is the union of the intervals $K_{s^\frown 0}$ and $K_{s^\frown 1}$.

Notice that if $s \subseteq t$, then $K_s \supseteq K_t$; while if $s, t \in {}^n2$ and $s <_{\text{lex}} t$, then $K_s < K_t$. We set

$$K = K_\emptyset \setminus \bigcup_{s \in {}^{<\omega}2} I_s = \bigcap_{n \in \omega} \bigcup_{s \in {}^n 2} K_s.$$

It is evident that $K \neq \emptyset$ and that K is compact. Moreover, if

$$\forall x \in {}^\omega 2 \big(\lim_{n \to \infty} |K_{x \restriction n}| = 0 \big),$$

then K has empty interior and the map

$$H_K : {}^\omega 2 \to K, \qquad H_K(x) = \text{the unique element of } \bigcap_n K_{x \restriction n}$$

is an homeomorphism. In this case $\langle K_s \mid s \in {}^{<\omega}2 \rangle$ is called *Cantor construction*, K is the resulting *Cantor set* and H_K is the *canonical homeomorphism associated to* the Cantor construction. It is easy to see that different Cantor constructions can give rise to the same resulting Cantor set and thus, in general, H_K depends on the Cantor construction, and not simply on K. We indicate by a_s and b_s the left extremity and the right extremity of K_s, respectively: namely, for every $s \in {}^{<\omega}2$

$$K_s = [a_s; b_s], \quad \text{and so } I_s = (b_{s^\frown 0}; a_{s^\frown 1}).$$

DEFINITION 2.2. A Cantor construction $\langle K_s \mid s \in {}^{<\omega}2 \rangle$ and its resulting Cantor set K are said to be

- *centered*, if the I_s's are centered in the K_s's, i.e., $|K_{s^\frown 0}| = |K_{s^\frown 1}|$,
- *uniform*, if $\text{lh}(s) = \text{lh}(t) \Rightarrow |I_s| = |I_t|$,
- *symmetric*, if they are centered and uniform.

In Lebesgue measure theory, $E_{1/3}$ is a classical example of a set which is uncountable and has zero measure. On the other hand, it is possible to define Cantor set of positive measure, as we will do in Section 3.

Now we will focus on the reciprocal positions of some elements of a generic Cantor construction. As usual, we say that two intervals I and J are *contiguous*, or that the one is *contiguous to* the other one, if

$$\sup I = \inf J \quad \text{or} \quad \sup J = \inf I.$$

[1]The notation below follows [2].

[2]We say that $(a; b)$ is *completely contained* in $[c; d]$ if $c < a$ and $b < d$.

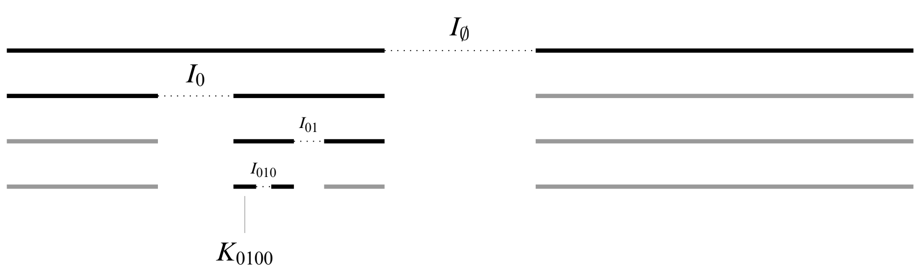
https://doi.org/10.1017/jsl.2016.53 Published online by Cambridge University Press

Given a Cantor construction $\langle K_s \mid s \in {}^{<\omega}2\rangle$, it is evident that for every $s \in {}^{<\omega}2 \setminus \{\emptyset\}$ the compact K_s is contiguous to the open interval $I_{s\restriction(\mathrm{lh}(s)-1)}$, which is the extracted interval K_s arises from. If s is of the form $0^{(n)}$ or $1^{(n)}$ for some $n > 0$, then $I_{s\restriction(\mathrm{lh}(s)-1)}$ is the only extracted interval to which K_s is contiguous. When s is not constant, K_s is contiguous to another extracted interval: the following definition, introduced in [6], will permit us to individuate it easily.

Definition 2.3. For every $s \in {}^{<\omega}2 \setminus \{\emptyset\}$, s not constant, let $\mathbf{m}(s) < \mathrm{lh}(s) - 1$ be greatest such that

$$s(\mathbf{m}(s)) \neq s(\mathrm{lh}(s) - 1).$$

For example if $s = 0100$, then $\mathbf{m}(s) = 1$. It is easy to see that for every nonconstant $s \in {}^{<\omega}2 \setminus \{\emptyset\}$ the interval $I_{s\restriction\mathbf{m}(s)}$ is contiguous to K_s. Obviously, $\mathbf{m}(s) \leq \mathrm{lh}(s) - 2$, and so $I_{s\restriction\mathbf{m}(s)} \neq I_{s\restriction(\mathrm{lh}(s)-1)}$. For example, for $s = 0100$ we have that both the interval $I_{s\restriction(\mathrm{lh}(s)-1)} = I_{010}$ and the interval $I_{s\restriction\mathbf{m}(s)} = I_0$ are contiguous to K_s:



§3. **A family of true $\mathbf{\Pi}^0_3$ density sets.** In this section we will achieve the following result.

Theorem 3.1. *If K is a symmetric Cantor set, with $\mu(K) \neq 0$, then $\Phi(K)$ is a true $\mathbf{\Pi}^0_3$ set.*

First of all we will present some properties of the symmetric Cantor sets. As shown in [2], if $K \subseteq \mathbb{R}$ is a symmetric Cantor set, then it is possible to recover from K the unique symmetric Cantor construction that yields K. Therefore, if K is a symmetric Cantor set, then there exists a canonical homeomorphism associated to it: the canonical homeomorphism associated to its unique uniform Cantor construction. Moreover, a symmetric Cantor set is uniquely determined by the initial compact $K_\emptyset$, which from now on we suppose to be equal to $[0; 1]$, and by a sequence of positive reals $(u_n)_n$ with $\sum_{n=0}^{\infty} u_n \leq 1$, describing the total length of the intervals removed at each step of the construction process. That is, $|I_\emptyset| = u_0$, $|I_0| = |I_1| = 2^{-1} \cdot u_1$, and more in general, for every $x \in {}^{\omega}2$ and $n \in \omega$

$$|I_{x\restriction n}| = 2^{-n} \cdot u_n. \tag{1}$$

In Definition 2.3 we have associated an index $\mathbf{m}(s)$ to every $s \in {}^{<\omega}2 \setminus \{\emptyset\}$ which is not constant. For technical reasons, we set $\mathbf{m}(s) = -1$ for every s constant and we put $u_{-1} = 1$. Moreover, for every $s \in {}^{<\omega}2 \setminus \{\emptyset\}$ let

$$\mathbf{p}(s) = \mathrm{lh}(s) - 1 - \mathbf{m}(s).$$

https://doi.org/10.1017/jsl.2016.53 Published online by Cambridge University Press

Thus, if s is constant, then $\mathbf{p}(s) = \mathrm{lh}(s)$; otherwise, $\mathbf{p}(s)$ is the length of the constant tail of s. Observe that $\mathbf{p}(s) \geq 1$, since $\mathbf{m}(s) \leq \mathrm{lh}(s) - 2$.

THEOREM 3.2 (Tacchi, [6]). *Let K be a symmetric Cantor set of positive measure and H the canonical homeomorphism associated with K. For every $a \in K$, $a = H(x)$ for some $x \in {}^{\omega}2$,*

$$a \in \Phi(K) \Leftrightarrow \lim_{n\to\infty} 2^{\mathbf{p}(x\restriction n)} \cdot u_{\mathbf{m}(x\restriction n)} = 0,$$

where $(u_i)_{i<\omega}$ is the sequence which characterizes K.

Each Cantor set K is closed, thus $\Phi(K) \subseteq K$. Thanks to Theorem 3.2, in order to obtain a proof of Theorem 3.1 it will be enough to prove the following result:

THEOREM 3.3. *For every $(u_i)_{i\in\omega}$ sequence of positive real numbers, with $\sum_{i=0}^{\infty} u_i < 1$, the set*

$$A = \{x \in {}^{\omega}2 \mid \lim_{n\to\infty} 2^{\mathbf{p}(x\restriction n)} \cdot u_{\mathbf{m}(x\restriction n)} = 0\}$$

is $\mathbf{\Pi}^0_3$-complete.

Fix a sequence $(u_i)_{i\in\omega}$ as in Theorem 3.3. Consider the function

$$r : {}^{<\omega}2 \setminus \{\emptyset\} \to \mathbb{R}^+, \quad r(s) = 2^{\mathbf{p}(s)} \cdot u_{\mathbf{m}(s)}$$

and for every $s \in {}^{<\omega}2 \setminus \{\emptyset\}$ such that $r(s) \in (0;1]$, set

$$\rho(s) = n \Leftrightarrow r(s) \in (2^{-(n+1)}; 2^{-n}].$$

Note that the intervals $(2^{-(n+1)}; 2^{-n}]$, for $n \in \omega$, form a partition of $(0;1]$ and that if A is defined as in Theorem 3.3, then

$$x \in A \Leftrightarrow \lim_{n\to\infty} r(x \restriction n) = 0 \Leftrightarrow \lim_{n\to\infty} \rho(x \restriction n) = \infty. \tag{2}$$

LEMMA 3.4. *Let $s \in {}^{<\omega}2$ with $s(\mathrm{lh}(s)-1) = 1$, $\rho(s) = n$ and length of s such that*

$$\forall N \geq \mathrm{lh}(s) - 1\big(u_N < 2^{-(n+1)}\big).$$

Then for every $j < \omega$ there exists $t \supsetneq s$, with $t(\mathrm{lh}(t)-1) = 1$, such that

- $\rho(t) = j$,
- $\forall z\big(s \subseteq z \subset t \Rightarrow \rho(z) \geq \min\{n, j\}\big)$,
- $\forall N \geq \mathrm{lh}(t) - 1\big(u_N < \min\{2^{-(n+1)}, 2^{-(j+1)}\}\big)$.

In order to simplify the proof of Lemma 3.4, and later its application, we will distinguish the following three cases: $n < j$, $j < n$ and $j = n$, which will be separately examined in Lemmas 3.5, 3.6, and 3.7, respectively.

LEMMA 3.5. *Let $s \in {}^{<\omega}2$ with $s(\mathrm{lh}(s)-1) = 1$, $\rho(s) = n$ and length of s such that*

$$\forall N \geq \mathrm{lh}(s) - 1\big(u_N < 2^{-(n+1)}\big).$$

Then for every $j > n$ there exists $t \supsetneq s$, with $t(\mathrm{lh}(t)-1) = 1$, such that:

- $\rho(t) = j$,
- $\forall z\big(s \subseteq z \subset t \Rightarrow \rho(z) \geq n\big)$,
- $\forall N \geq \mathrm{lh}(t) - 1\big(u_N < 2^{-(j+1)}\big)$.

https://doi.org/10.1017/jsl.2016.53 Published online by Cambridge University Press

Proof. Since the sequence $(u_n)_n$ approaches zero, there exists $v > \mathrm{lh}(s) - 1$ such that $v' = v - (\mathrm{lh}(s) - 1)$ is even and

$$\forall N \geq v\big(u_N < 2^{-(j+1)}\big).$$

Put $k = v'/2$. Let M be least such that $2^M \cdot u_{v+1} \in (2^{-(j+1)}; 2^{-j}]$. We claim that the following extension of s

$$t = s^\frown(01)^{(k)\frown}0^\frown 1^{(M)}$$

satisfies our requirements.

Observe that t effectively ends with 1, i.e., $M \neq 0$, since $u_v < 2^{-(j+1)}$.

Now we will prove that $\rho(t) = j$. Notice that $\mathbf{m}(t) = \mathrm{lh}(s) + v' = v + 1$ and $\mathbf{p}(t) = M$, so we have that $r(t) = 2^M \cdot u_{v+1}$ and then, by construction, $r(t) \in (2^{-(j+1)}, 2^{-j}]$, hence $\rho(t) = j$.

Let $s \subset z \subset t$. If $s \subset z \subseteq s^\frown(01)^{(k)\frown}0^\frown 1$, then $\mathbf{m}(z) \geq \mathrm{lh}(s) - 1$, by recalling that s ends with 1. Then $u_{\mathbf{m}(z)} < 2^{-(n+1)}$ by hypothesis and thus, since $\mathbf{p}(z) = 1$, we obtain $r(z) = 2 \cdot u_{\mathbf{m}(z)} < 2^{-n}$. Otherwise, if $s^\frown(01)^{(k)\frown}0^\frown 1 \subset z \subset t$, then $\mathbf{m}(z) = \mathbf{m}(t) = v+1$ and $\mathbf{p}(z) < \mathbf{p}(t) = M$. So $r(z) = 2^{\mathbf{p}(z)} \cdot u_{v+1}$ and the minimality of M implies that $r(z) \leq 2^{-(j+1)}$. We conclude that in both cases $\rho(z) \geq n$.

We finally observe that by construction $\mathrm{lh}(t) = \mathrm{lh}(s) + v' + 1 + M = v + M$, so $\mathrm{lh}(t) - 1 \geq v$. Then $\forall N \geq \mathrm{lh}(t) - 1\big(u_N < 2^{-(j+1)}\big)$. ⊣

Lemma 3.6. *Let $s \in {}^{<\omega}2$ with $s(\mathrm{lh}(s) - 1) = 1$ and $\rho(s) = n$. Then for every $j < n$ there exists $t \supsetneq s$, with $t(\mathrm{lh}(t) - 1) = 1$, such that*

- $\rho(t) = j$,
- $\forall z\big(s \subseteq z \subset t \Rightarrow \rho(z) > j\big)$.

Proof. Let $M = n - j$. We consider the following extension of s

$$t = s^\frown 1^{(M)}.$$

Notice that, since s ends with 1, then $\mathbf{m}(t) = \mathbf{m}(s)$ and $\mathbf{p}(t) = \mathbf{p}(s) + M$, thus

$$r(t) = 2^M \cdot r(s) \in (2^{-(j+1)}; 2^{-j}],$$

i.e., $\rho(t) = j$. It is evident that if $s \subset z \subset t$, then $\rho(z) > j$. ⊣

Lemma 3.7. *Let $s \in {}^{<\omega}2$ with $s(\mathrm{lh}(s) - 1) = 1$ and length of s such that*

$$\forall N \geq \mathrm{lh}(s) - 1\big(u_N < 2^{-(n+1)}\big).$$

Then there exists $t \supsetneq s$, with $t(\mathrm{lh}(t) - 1) = 1$, such that

- $\rho(t) = \rho(s)$,
- $\forall z\big(s \subseteq z \subset t \Rightarrow \rho(z) \geq \rho(s)\big)$.

Proof. It is enough to apply Lemma 3.5 for $j = \rho(s) + 1$ and then Lemma 3.6 for $j = \rho(s)$ to the resulting extension. ⊣

Proof of Theorem 3.3. It is clear that $A \in \mathbf{\Pi}^0_3$. Consider now the set

$$P_3 = \{z \in {}^{\omega\times\omega}2 \mid \forall n \forall^\infty m(z(n,m) = 0)\}.$$

Since P_3 is a $\mathbf{\Pi}^0_3$-complete set (see [4, p. 179]), it will be enough to show that $P_3 \leq_W A$. For every 0-1 matrix $a = \langle a(i,j) \mid i,j < n\rangle$ of order n, we will construct a sequence $\varphi(a) \in {}^{<\omega}2$ such that

$$\text{if } a \subset b \text{ then } \varphi(a) \subset \varphi(b).$$

https://doi.org/10.1017/jsl.2016.53 Published online by Cambridge University Press

Therefore the function

$$f : {}^{\omega\times\omega}2 \to {}^{\omega}2, \qquad f(z) = \bigcup_n \varphi(z \restriction n \times n)$$

will be continuous. Since $(u_n)_n$ approaches zero, then there exists $v > 0$, v even such that $\forall N > v - 2(u_N < 1/2)$. Let $k = v/2$ and M be such that $2^M \cdot u_{v-2} \in (1/2; 1]$. Let

$$t_0 = (01)^{(k)\frown}1^{(M)}.$$

It is easy to see that $\rho(t_0) = 0$ and $\forall N > \mathrm{lh}(t_0) - 1(u_N < 1/2)$. We set $\varphi(\emptyset) = t_0$. If $a = \langle a(i,j) \mid i,j \leq n\rangle$ is a $(n+1)$-order matrix, and $\varphi(a \restriction n \times n)$ is already defined, let

$$\varphi(a) = t,$$

where t is defined as follows:

CASE 1: $\forall i \leq n(a(i,n) = 0)$. Let t be the extension of $\varphi(a \restriction n \times n)$ given by Lemma 3.5 for $j = n+1$. Observe that in particular we have that

$$\forall z\big(\varphi(a \restriction n \times n) \subseteq z \subset t \Rightarrow \rho(z) \geq \rho(\varphi(a \restriction n \times n))\big).$$

CASE 2: $\exists i \leq n(a(i,n) = 1)$. Let i_0 be the least such i. Let t be the extension of $\varphi(a \restriction n \times n)$ such that $\rho(t) = i_0$, defined as in Lemma 3.4. Note that

$$\forall z\big(\varphi(a \restriction n \times n) \subseteq z \subset t \Rightarrow \rho(z) \geq \min\{\rho(\varphi(a \restriction n \times n)), i_0\}\big).$$

Let $z \in P_3$. For every $i \in \omega$, let m_i be such that $\forall m \geq m_i(z(i,m) = 0)$. Given $k \in \omega$, let $v_k = \max\{m_0, \dots, m_k, k\}$. Notice that for every $n \geq v_k$ the least $i_0 \leq n$ such that $z(i_0, n) = 0$, if it exists, is larger than k and hence $\rho(\varphi(z \restriction n \times n)) \geq k$. Therefore $\lim_{i\to\infty} \rho(f(z) \restriction i) = \infty$ and thus $f(z) \in A$, by (2).

On the other hand, let $z \notin P_3$. Let n_0 be the least n such that $\exists^\infty m(z(n_0, m) = 1)$. Then for arbitrarily large n, $\varphi(z \restriction n \times n)$ is computed as in Case 2, hence $\rho(f(z) \restriction i) = n_0$ for infinitely many i's. Therefore $f(z) \notin A$, by (2). ⊣

§4. An example of true $\boldsymbol{\Sigma}^0_2$ density set. For $A \subseteq \mathbb{R}$ a measurable set and $x \in \mathbb{R}$, the *right density of A at x* is defined as

$$\mathscr{D}^+_A(x) = \lim_{\epsilon\to 0} \frac{\mu(A \cap (x; x+\epsilon))}{\epsilon}.$$

The *left density* is defined similarly. If $\mathscr{D}^+_A(x)$ and $\mathscr{D}^-_A(x)$ both exist, then $\mathscr{D}_A(x)$ exists and in this case

$$\mathscr{D}_A(x) = \frac{\mathscr{D}^+_A(x) + \mathscr{D}^-_A(x)}{2}.$$

In this section we will define an open set U of $\mathbb{R}$ such that $\Phi(U) \in \boldsymbol{\Sigma}^0_2 \setminus \boldsymbol{\Pi}^0_2$. The definition of U goes through the Cantor construction $\langle K_s \mid s \in {}^{<\omega}2\rangle$, where $K_\emptyset = [0; 1]$ and for all $s \in {}^{<\omega}2$

$$|K_{s\frown 1}| = 2^{-1} \cdot |K_s| \quad \text{and} \quad |K_{s\frown 0}| = 2^{-(\mathrm{lh}(s)+2)} \cdot |K_s|.$$

Thus it results that $|I_s| = (2^{-1} - 2^{-(\mathrm{lh}(s)+2)}) \cdot |K_s|$.

For example

$$K_0 = [0; 1/4] \quad \text{and} \quad K_1 = [1/2; 1],$$

$$K_{00} = [0; 1/32],\ K_{01} = [1/8; 1/4],\ K_{11} = [1/2; 1/2 + 1/16], \text{and } K_{11} = [3/4; 1],$$

$$\vdots$$

We indicate by K the associated Cantor set and by H the associated homeomorphism. Obviously, K is neither centered nor uniform. Moreover, the relative length of I_s with respect to $|K_s|$, i.e., the fraction $|I_s|/|K_s|$, is directly proportional to $\mathrm{lh}(s)$; so in particular K has no ratio.

For every $s \in {}^{<\omega}2$, we define an open interval $U_s \subset I_s = (b_{s^\frown 0}; a_{s^\frown 1})$ by setting

$$U_s = (b_{s^\frown 0}; \mathbf{c}(s)),$$

where

$$\frac{\mathbf{c}(s) - b_{s^\frown 0}}{a_{s^\frown 1} - b_{s^\frown 0}} = 1 - 2^{-N(s)}, \tag{3}$$

and

$$N(s) = \text{ number of final consecutive 1's in } s,$$

that is, $N(s)$ is the largest k such that $s = t^\frown 1^{(k)}$ for some t.

So U_s and I_s have the same left extremity; while $\mathbf{c}(s) < a_{s^\frown 1}$, and the more the final sequence of 1's in s is large, the closer $\mathbf{c}(s)$ is to $a_{s^\frown 1}$. On the other hand, if $s = \emptyset$ or s ends in 0, then $\mathbf{c}(s)$ coincides with $b_{s^\frown 0}$, i.e., $U_s = \emptyset$.

We set

$$U = \bigcup_{s \in {}^{<\omega}2} U_s.$$

Since U is open, then $U \subseteq \Phi(U)$; moreover, by construction the other density points of U will all belong to K.

Proposition 4.1. *For every $a \in K$*

$$a \in \Phi(U) \quad \Leftrightarrow \quad a = b_{s^\frown 0} \text{ for some } s \in {}^{<\omega}2,\ s \text{ ends by } 1.$$

In other words, the only points of K which are points of density 1 in U are the left extremities of the U_s's with $U_s \neq \emptyset$. Therefore

$$\Phi(U) = U \cup \{b_{s^\frown 0} \mid s \in {}^{<\omega}2 \ \&\ s(\mathrm{lh}(s) - 1) = 1\}.$$

Proof. ($\Rightarrow$) Let $a = H(x)$, for some $x \in {}^{\omega}2$. Suppose that $a \neq b_{s^\frown 0}$ for any s ending by 1. We distinguish two cases:

Case 1: $\exists^\infty n(x(n) = 0)$. Then $a \in K_{s^\frown 0}$ for infinitely many s. Observe that $U_{s^\frown 0} = \emptyset$, and so $I_{s^\frown 0} \cap K_{s^\frown 0} \subset \complement U$ for every s. Therefore for infinitely many s

$$\begin{aligned}\frac{\mu(U \cap B(a; |K_{s^\frown 0}|))}{2 \cdot |K_{s^\frown 0}|} &< \frac{2 \cdot |K_{s^\frown 0}| - |I_{s^\frown 0}|}{2 \cdot |K_{s^\frown 0}|} \\ &= \frac{2 \cdot |K_{s^\frown 0}| - (2^{-1} - 2^{-(\mathrm{lh}(s)+2)}) \cdot |K_{s^\frown 0}|}{2 \cdot |K_{s^\frown 0}|} \\ &= 1 - 2^{-2} + 2^{-(\mathrm{lh}(s)+3)} < 7/8,\end{aligned}$$

and thus $a \notin \Phi(U)$.

CASE 2: $\forall^\infty n(x(n) = 1)$. Let $a = b_t$ with $t \subset x$ minimal such that $x = t^\frown 1^{(\infty)}$. Since by assumption

$$H(t^\frown 1^{(\infty)}) = a \neq b_{s^\frown 0} = H(s^\frown 0^\frown 1^{(\infty)})$$

for any s ending by 1, then by the minimality of t

$$t = \emptyset \quad \text{or} \quad t = 0 \quad \text{or} \quad t \text{ ends by } 00.$$

Suppose $t = \emptyset$, and so $a = 1$. Since $U \subset [0;1]$, it follows that $\mathscr{D}_U^+(a) = 0$. Also if $t = 0$, that is, $a = b_0$, we have that $\mathscr{D}_U^+(a) = 0$, since $(b_0; a_1) = I_\emptyset \subset \complement U$. Finally, consider the case $t = u^\frown 00$, for some $u \in {}^{<\omega}2$. Then $a = b_{u^\frown 00}$ is the left extremity of $I_{u^\frown 0}$. Since $I_{u^\frown 0} \subset \complement U$, we also have $\mathscr{D}_U^+(a) = 0$.

($\Leftarrow$) Now we will show the converse. Let $s \in {}^{<\omega}2$, with s ending by 1. Trivially, since $\emptyset \neq U_s = (b_{s^\frown 0}; \mathbf{c}(s)) \subset U$, it follows that $\mathscr{D}_U^+(b_{s^\frown 0}) = 1$. It remains to show that $\mathscr{D}_U^-(b_{s^\frown 0}) = 1$. Consider the following subset of $U \cap K_{s^\frown 01}$, which intersects U in a left neighborhood of $b_{s^\frown 0}$,

$$\tilde{U} = \tilde{U}(s) := \bigcup_{n>0} U_{s^\frown 01^{(n)}}$$

(see Figure 2) and the function

$$f : K_{s^\frown 01} \setminus \{b_{s^\frown 0}\} \to [0;1], \qquad f(z) = \frac{\lambda(\tilde{U} \cap (z; b_{s^\frown 0}))}{b_{s^\frown 0} - z}.$$

Trivially, since $\tilde{U} \subseteq U$, we have that

$$f(z) < \frac{\mu(U \cap (z; b_{s^\frown 0}))}{b_{s^\frown 0} - z}, \text{ for all } z \in K_{s^\frown 01} \setminus \{b_{s^\frown 0}\}. \tag{4}$$

Notice that f alternates decreasing phases with increasing ones. For example, f is decreasing on $(b_{s^\frown 010}; \mathbf{c}(s^\frown 01))$, because the whole interval is included in $\tilde{U}$, while f is increasing on $(\mathbf{c}(s^\frown 01); b_{s^\frown 0110})$, since $\tilde{U} \cap (\mathbf{c}(s^\frown 01); b_{s^\frown 0110}) = \emptyset$. In general, it is easy to see that f is decreasing on $(b_{s^\frown 01^{(k)}0}; \mathbf{c}(s^\frown 01^{(k)}))$ and is increasing on $(\mathbf{c}(s^\frown 01^{(k)}); b_{s^\frown 01^{(k+1)}0})$ for every $k > 0$. Then f has a local minimum point at each $\mathbf{c}(s^\frown 01^{(k)})$ and only there. So by the continuity of f, in order to show that $f(z)$ converges to 1 as z approaches $b_{s^\frown 0}$, it will be enough to prove that $f(\mathbf{c}(s^\frown 01^{(k)})) \to 1$, as $k \to \infty$, i.e.,

$$\lim_{k\to\infty} \frac{\mu(\tilde{U} \cap (\mathbf{c}(s^\frown 01^{(k)}); b_{s^\frown 0}))}{b_{s^\frown 0} - \mathbf{c}(s^\frown 01^{(k)})} = 1,$$

or equivalently

$$\lim_{k\to\infty} \frac{\mu(\complement\tilde{U} \cap (\mathbf{c}(s^\frown 01^{(k)}); b_{s^\frown 0}))}{b_{s^\frown 0} - \mathbf{c}(s^\frown 01^{(k)})} = 0. \tag{5}$$

Let us verify the limit relation (5). Set

$$Q_k = \mu(\complement\tilde{U} \cap (\mathbf{c}(s^\frown 01^{(k)}); b_{s^\frown 0}))/(b_{s^\frown 0} - \mathbf{c}(s^\frown 01^{(k)})$$

and notice that

$$\mu(\complement\tilde{U} \cap (\mathbf{c}(s^\frown 01^{(k)}); b_{s^\frown 0})) = \sum_{n=0}^{\infty} (a_{s^\frown 01^{(k+n)}1} - \mathbf{c}(s^\frown 01^{(k+n)})) + \sum_{n=1}^{\infty} |K_{s^\frown 01^{(k+n)}0}|. \tag{6}$$

The following easy facts about our Cantor construction and the set U will permit us to get an upper bound to the value Q_k.

LEMMA 4.2. *For every $s \in {}^{<\omega}2$ and for every $n, k \in \omega$,*

(a) $|K_{s^\frown 01^{(n)}}| = 2^{-n} \cdot |K_{s^\frown 0}|$,
(b) $a_{s^\frown 01^{(k+n)}1} - \mathbf{c}(s^\frown 01^{(k+n)}) < 2^{-(2k+2n+1)} \cdot |K_{s^\frown 0}|$,
(c) $|K_{s^\frown 01^{(k+n)}0}| = 2^{-(\mathrm{lh}(s)+2k+2n+3)} \cdot |K_{s^\frown 0}|$,
(d) $b_{s^\frown 0} - \mathbf{c}(s^\frown 01^{(k)}) > 2^{-(k+1)} \cdot |K_{s^\frown 0}|$.

PROOF. The equalities in (a) and (c) hold by construction. Notice that we have that

$$a_{s^\frown 01^{(k+n)}1} - \mathbf{c}(s^\frown 01^{(k+n)}) = 2^{-(k+n)} \cdot |I_{s^\frown 01^{(k+n)}}|$$

and so we obtain (b), considering that $|I_{s^\frown 01^{(k+n)}}| < 2^{-1} \cdot |K_{s^\frown 01^{(k+n)}}|$ and then using (a). Finally, we obtain (d) by observing that

$$b_{s^\frown 0} - \mathbf{c}(s^\frown 01^{(k)}) > b_{s^\frown 0} - a_{s^\frown 01^{(k)}1} = |K_{s^\frown 01^{(k)}1}|$$

and again applying (a). ⊣

From the equality (6) and the relations (b), (c), and (d) we have that

$$Q_k < \frac{\sum_{n=0}^{\infty} 2^{-(2k+2n+1)} \cdot |K_{s^\frown 0}| + \sum_{n=1}^{\infty} 2^{-(\mathrm{lh}(s)+2k+2n+3)} \cdot |K_{s^\frown 0}|}{2^{-(k+1)} \cdot |K_{s^\frown 0}|},$$

thus

$$\begin{aligned} Q_k &< \frac{2^{-(2k+1)} \cdot \sum_{n=0}^{\infty} 2^{-2n} + 2^{-(\mathrm{lh}(s)+2k+3)} \cdot \sum_{n=1}^{\infty} 2^{-2n}}{2^{-(k+1)}} \\ &= 2^{-k} \cdot (4/3 + 2^{-\mathrm{lh}(s)-2} \cdot 1/3). \end{aligned}$$

Then Q_k converges to 0 as k tends to ∞, i.e., the limit relation (5) is proved. By (4) we have that $\mathscr{D}_U^-(b_{s^\frown 0}) = 1$. ⊣

PROPOSITION 4.3. *The density set of U is a true $\mathbf{\Sigma}_2^0$.*

PROOF. By Proposition 4.1

$$\Phi(U) = U \cup \{b_{s^\frown 0} \mid s \in {}^{<\omega}2 \ \&\ s(\mathrm{lh}(s) - 1) = 1\}, \tag{7}$$

so it is clear that $\Phi(U) \in \mathbf{\Sigma}_2^0$. Set $A = H^{-1}[\Phi(U)]$. Since $U \subseteq \complement K$, then by (7) we obtain

$$A = H^{-1}(\{b_{s^\frown 0} \mid s \in {}^{<\omega}2 \ \&\ s(\mathrm{lh}(s) - 1) = 1\}).$$

Observe that $b_{s^\frown 0} = H(s^\frown 0^\frown 1^{(\infty)})$ for every $s \in {}^{<\omega}2$, and thus

$$A = \{x \in {}^{\omega}2 \mid x = t^\frown 10^\frown 1^{(\infty)},\ \text{for some } t \in {}^{<\omega}2\}.$$

Notice that A is countable, therefore F_σ; moreover, A is dense and codense in ${}^{\omega}2$. Therefore by the Baire Category Theorem A is not G_δ, and so $\Phi(U) \in \mathbf{\Sigma}_2^0 \setminus \mathbf{\Pi}_2^0$. ⊣

§5. An example of true $\mathbf{\Pi}_2^0$ density set. From now on by $\langle K_s \mid s \in {}^{<\omega}2\rangle$ we mean the usual construction which gives rise to the Cantor ternary set $E_{1/3}$, that is $K_\emptyset = [0; 1]$, $K_0 = [0; 1/3]$, $K_1 = [2/3; 1]$, and so on. To simplify the notation, we put for every $n \in \omega$

$$\epsilon_n = 3^{-n}.$$

https://doi.org/10.1017/jsl.2016.53 Published online by Cambridge University Press

https://doi.org/10.1017/jsl.2016.53 Published online by Cambridge University Press

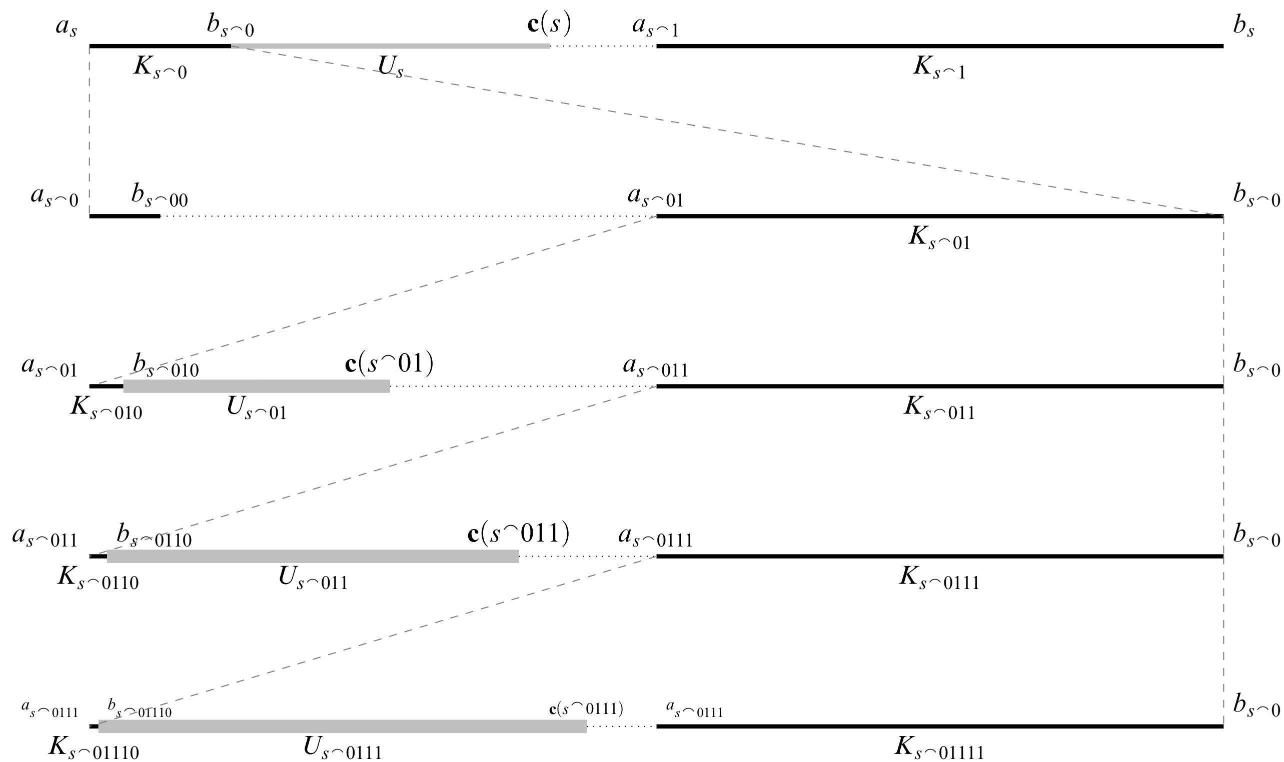


FIGURE 2. The union of the thick grey lines is a finite approximation of $\tilde{U}(s)$, for some s ending by 1.

Observe that $|K_s| = \epsilon_{\mathrm{lh}(s)}$, for every $s \in {}^{<\omega}2$. For the following Remark see, e.g., [7, p. 680], or [2] for a more general version.

REMARK 5.1. Given $X \subseteq \mathbb{R}$ measurable, X has density 1 at $a \in \mathbb{R}$ if and only if there exists a sequence of positive reals $(h_n)_n$ converging to 0 such that

$$\liminf_{n\to\infty} \frac{h_{n+1}}{h_n} > 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\mu(X \cap (a - h_n; a + h_n))}{2 \cdot h_n} = 1.$$

Note that the sequence $(\epsilon_n)_n$ satisfies the conditions above.

DEFINITION 5.2. For every $s \in {}^{<\omega}2$, the *round* of K_s is the open interval

$$\mathcal{R}(K_s) = \mathrm{B}(K_s; \epsilon_{\mathrm{lh}(s)+1}),$$

i.e., $\mathcal{R}(K_s)$ is the $3^{-(\mathrm{lh}(s)+1)}$-ball around K_s.

For example, $\mathcal{R}(K_\emptyset) = (-1/3; 4/3)$, $\mathcal{R}(K_0) = (-1/9; 4/9)$, $\mathcal{R}(K_1) = (5/9; 10/9)$, and so on. The following properties are straightforward. For every $s, t \in {}^{<\omega}2$,

- $|\mathcal{R}(K_s)| = \epsilon_{\mathrm{lh}(s)} + 2 \cdot \epsilon_{\mathrm{lh}(s)+1} = 5 \cdot \epsilon_{\mathrm{lh}(s)+1}$;
- if $s \subseteq t$, then $\mathcal{R}(K_t) \subseteq \mathcal{R}(K_s)$.

we are interested in characterizing the points of $E_{1/3}$ which are density points of some measurable $X \subseteq \mathbb{R}$. Since $E_{1/3} \subseteq [0; 1]$, we can consider only the sets X which are contained in $(-1/3; 4/3)$. The next result shows how to detect the elements of $\Phi(X) \cap E_{1/3}$ in terms of rounds.

LEMMA 5.3. *Let $X \subseteq (-1/3; 4/3)$, X measurable. For every $a \in E_{1/3}$, $a = H(x)$ for some $x \in {}^{\omega}2$,*

$$a \in \Phi(X) \Leftrightarrow \lim_{n\to\infty} \frac{\mu(X \cap \mathcal{R}(K_{x\restriction n}))}{|\mathcal{R}(K_{x\restriction n})|} = 1.$$

PROOF. $(\Leftarrow)$ It is easy to see that for every $n \in \omega$

$$(a - \epsilon_{n+1}; a + \epsilon_{n+1}) \subseteq \mathcal{R}(K_{x\restriction n}),$$

being $a \in K_{x\restriction n}$, and thus

$$\mathcal{R}(K_{x\restriction n}) = (a - \epsilon_{n+1}; a + \epsilon_{n+1}) \cup (\mathcal{R}(K_{x\restriction n}) \setminus (a - \epsilon_{n+1}; a + \epsilon_{n+1})). \tag{8}$$

Observe that

$$\mu(\mathcal{R}(K_{x\restriction n}) \setminus (a - \epsilon_{n+1}; a + \epsilon_{n+1})) = \epsilon_n + 2 \cdot \epsilon_{n+1} - 2 \cdot \epsilon_{n+1} = \epsilon_n,$$

so by (8) we obtain

$$\begin{aligned}\frac{\mu(X \cap (a - \epsilon_{n+1}; a + \epsilon_{n+1}))}{2 \cdot \epsilon_{n+1}} &\geq \frac{\mu(X \cap \mathcal{R}(K_{x\restriction n}))}{2 \cdot \epsilon_{n+1}} - \frac{\epsilon_n}{2 \cdot \epsilon_{n+1}} \\ &= \frac{\mu(X \cap \mathcal{R}(K_{x\restriction n}))}{5 \cdot \epsilon_{n+1}} \cdot \frac{5 \cdot \epsilon_{n+1}}{2 \cdot \epsilon_{n+1}} - \frac{3 \cdot \epsilon_{n+1}}{2 \cdot \epsilon_{n+1}} \\ &= 5/2 \cdot \frac{\mu(X \cap \mathcal{R}(K_{x\restriction n}))}{|\mathcal{R}(K_{x\restriction n})|} - 3/2.\end{aligned}$$

Therefore, if $\lim_n \mu(X \cap \mathcal{R}(K_{x\restriction n}))/|\mathcal{R}(K_{x\restriction n})| = 1$, then

$$\lim_{n\to\infty} \frac{\mu(X \cap (a - \epsilon_{n+1}; a + \epsilon_{n+1}))}{2 \cdot \epsilon_{n+1}} = 1,$$

and thus $a \in \Phi(X)$, thanks to Remark 5.1.

($\Rightarrow$) Now suppose that

$$\liminf_{n\to\infty} \frac{\mu(X \cap \mathcal{R}(K_{x\restriction n}))}{|\mathcal{R}(K_{x\restriction n})|} = \delta,$$

for some $0 \leq \delta < 1$, and so

$$\forall n \exists m \geq n (\mu(X \cap \mathcal{R}(K_{x\restriction m})) \leq \delta \cdot 5 \cdot \epsilon_{m+1}). \tag{9}$$

Claim 1. $\mathcal{R}(K_{x\restriction(n+1)}) \subseteq (a - \epsilon_n; a + \epsilon_n)$.

Proof. Note that[3]

$$\begin{aligned}\mathcal{R}(K_{x\restriction(n+1)}) &= (a_{x\restriction(n+1)} - \epsilon_{n+2} \,;\, b_{x\restriction(n+1)} + \epsilon_{n+2}) \\ &= (a_{x\restriction(n+1)} - \epsilon_{n+2} \,;\, a_{x\restriction(n+1)} + 4 \cdot \epsilon_{n+2}).\end{aligned}$$

Since $a \in K_{x\restriction(n+1)}$, then

$$a_{x\restriction(n+1)} \leq a \leq a_{x\restriction(n+1)} + \epsilon_{n+1}. \tag{10}$$

By the second relation in (10), it follows that

$$a - \epsilon_n \leq a_{x\restriction(n+1)} + \epsilon_{n+1} - \epsilon_n = a_{x\restriction(n+1)} - 2 \cdot \epsilon_{n+1} = a_{x\restriction(n+1)} - 6 \cdot \epsilon_{n+2},$$

and so

$$a - \epsilon_n < a_{x\restriction(n+1)} - \epsilon_{n+2}.$$

Similarly, it is possible to show that $a_{x\restriction(n+1)} + 4 \cdot \epsilon_{n+2} < a + \epsilon_n$; hence the Claim is proved. ⊣

By Claim 1, we have that

$$\begin{aligned}&\mu(X \cap (a - \epsilon_n; a + \epsilon_n)) \\ &\quad = \mu(X \cap \mathcal{R}(K_{x\restriction(n+1)})) + \mu(X \cap ((a - \epsilon_n; a + \epsilon_n) \setminus \mathcal{R}(K_{x\restriction(n+1)}))),\end{aligned} \tag{11}$$

and so (9) implies that for every n there exists $m \geq n$ such that

$$\mu(X \cap (a - \epsilon_m; a + \epsilon_m)) \leq \delta \cdot 5 \cdot \epsilon_{m+1} + 2 \cdot \epsilon_m - 5 \cdot \epsilon_{m+1} = (5 \cdot \delta + 1) \cdot \epsilon_{m+1},$$

thus

$$\frac{\mu(X \cap (a - \epsilon_m; a + \epsilon_m))}{2 \cdot \epsilon_m} \leq \frac{(5 \cdot \delta + 1) \cdot \epsilon_{m+1}}{6 \cdot \epsilon_{m+1}} < 1, \quad \text{being } \delta < 1.$$

We conclude that $a \notin \Phi(X)$. ⊣

Now we will define a subset V of $\mathbb{R}$ such that $\Phi(V) \in \boldsymbol{\Pi}^0_2 \setminus \boldsymbol{\Sigma}^0_2$, by considering particular open sets inside the I_s's, the extracted intervals of our Cantor construction. Since V will be open, then $V \subseteq \Phi(V)$; moreover, we will show that $a \in E_{1/3} \setminus \Phi(V)$ if and only if $x = H^{-1}(a)$ is eventually equal to 1 (Proposition 5.4).

[3]Recall that by a_s and b_s we denote the left and the right extremity of K_s, respectively.

https://doi.org/10.1017/jsl.2016.53 Published online by Cambridge University Press

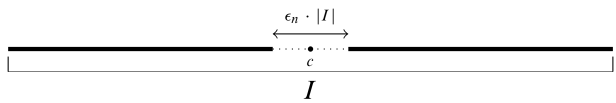


FIGURE 3. $\mathrm{Fill}_n(I)$ in black.

Given $n > 0$ and an open interval I with central point c, the *n-filling* of I is the open set (Figure 3)

$$\mathrm{Fill}_n(I) = I \setminus \mathrm{Cl}(\mathrm{B}(c; \epsilon_n \cdot |I|/2)).$$

For example,

$$\mathrm{Fill}_1((0;1)) = (0;1/3) \cup (2/3;1) \quad \text{and} \quad \mathrm{Fill}_2((0;1)) = (0;4/9) \cup (5/9;1).$$

Note that the larger is n, the closer $\mathrm{Fill}_n(I)$ is to I. More precisely,

$$\frac{\mu(\mathrm{Fill}_n(I))}{|I|} = 1 - \epsilon_n. \tag{12}$$

For every $s \in {}^{<\omega}2$, let

$$V_s = \bigcup \{\mathrm{Fill}_{\mathrm{lh}(s)}(I_t) \mid t \in {}^{<\omega}2, t \supseteq s\},$$

and put

$$V = \bigcup \{V_s \mid s \in {}^{<\omega}2, s(\mathrm{lh}(s) - 1) = 0\} \cup \bigcup \{\mathrm{Fill}_1(I_s) \mid s \in {}^{<\omega}2\} \cup (-1/3;0).$$

PROPOSITION 5.4. *For every $a \in E_{1/3}$, with $x = H^{-1}(a) \in {}^{\omega}2$,*

$$a \in \Phi(V) \Leftrightarrow \exists^\infty n(x(n) = 0).$$

PROOF. ($\Rightarrow$) Let $x \in {}^{\omega}2$ and suppose that there exists N least such that $\forall n \geq N(x(n) = 1)$. By (12) for every $n \geq N$,

$$\frac{\mu(V \cap K_{x\restriction n})}{|K_{x\restriction n}|} = \begin{cases} 2/3, & \text{if } N = 0 \text{ or } N = 1, \\ 1 - \epsilon_N, & \text{otherwise.} \end{cases}$$

Thus $\limsup_n \mu(V \cap K_{x\restriction n})/|K_{x\restriction n}| \neq 1$. It follows that

$$\lim_{n\to\infty} \frac{\mu(V \cap \mathcal{R}(K_{x\restriction n}))}{|\mathcal{R}(K_{x\restriction n})|} \neq 1,$$

being $K_{x\restriction n} \subsetneq \mathcal{R}(K_{x\restriction n})$ and $|K_{x\restriction n}|/|\mathcal{R}(K_{x\restriction n})| = 3/5$ for every n. Thanks to Lemma 5.3 we conclude that $a = H(x) \notin \Phi(V)$.

($\Leftarrow$) First notice that by construction for all $s \in {}^{<\omega}2$, s not constantly equals to 1,

$$\mathcal{R}(K_s) \setminus K_s \subseteq \bigcup \{\mathrm{Fill}_1(I_s) \mid s \in {}^{<\omega}2\} \cup (-1/3;0), \tag{13}$$

and so $\mathcal{R}(K_s) \setminus K_s$ is entirely contained in V. Let $x \in {}^{\omega}2$ such that $\exists^\infty n(x(n) = 0)$. By (12) we have that for all large enough n

$$\frac{\mu(V \cap K_{x\restriction n})}{|K_{x\restriction n}|} \geq 1 - \epsilon_{i(n)+1},$$

https://doi.org/10.1017/jsl.2016.53 Published online by Cambridge University Press

where $i(n) = i(x, n)$ is the greatest $i < n$ such that $x(i) = 0$. Therefore

$$\lim_{n\to\infty} \frac{\mu(V \cap K_{x\restriction n})}{|K_{x\restriction n}|} = 1, \tag{14}$$

since $i(n) \to \infty$, and thus $\epsilon_{i(n)+1} \to 0$, as $n \to \infty$. By (13) and (14) we obtain

$$\lim_{n\to\infty} \frac{\mu(V \cap \mathcal{R}(K_{x\restriction n}))}{|\mathcal{R}(K_{x\restriction n})|} = 1,$$

and so $a = H(x) \in \Phi(V)$, by Lemma 5.3 ⊣

Proposition 5.5. *The density set of V is a true $\boldsymbol{\Pi}^0_2$.*

Proof. First note that $V \subseteq \Phi(V)$, being V an open set. Moreover, by construction $\Phi(V) \setminus V \subseteq E_{1/3}$. Therefore, thanks to Proposition 5.4 we get

$$\Phi(V) = V \cup H[Z], \quad \text{where } Z = \{x \in {}^{\omega}2 \mid \exists^{\infty} n(x(n) = 0)\}.$$

It is well known that $Z \in \boldsymbol{\Pi}^0_2 \setminus \boldsymbol{\Sigma}^0_2$, see, e.g., [4, p. 179]. Thus $\Phi(V) \in \boldsymbol{\Pi}^0_2$, and $\Phi(V) \notin \boldsymbol{\Sigma}^0_2$, being $Z = H^{-1}[\Phi(V)]$. ⊣

REFERENCES

[1] A. Andretta and R. Camerlo, *The descriptive set theory of the Lebesgue Density Theorem*. ***Advances in Mathematics***, vol. 234 (2013), pp. 1–42.

[2] A. Andretta, R. Camerlo, and C. Costantini, *Lebesgue density and exceptional points*, `http://arxiv.org/abs/1510.04193`, 2015.

[3] G. Carotenuto, *On the topological complexity of the density sets of the real line*, Ph.D. thesis, Università di Salerno, `http://elea.unisa.it:8080/jspui/bitstream/10556/1972/1/tesi_G.Carotenuto.pdf`, 2015.

[4] A. S. Kechris, *Classical descriptive set theory*. ***Advances in Mathematics***, vol. 234 (1995), pp. 1–42.

[5] J. C. Oxtoby, ***Measure and Category: A Survey of the Analogies Between Topological and Measure Spaces***, Graduate Texts in Mathematics, Springer-Verlag, New York, 1980.

[6] J. P. Tacchi, *Points of density of Cantor sets*. ***European Journal of Combinatorics***, vol. 16 (1995), no. 6, pp. 645–653 (In French).

[7] W. Wilczyński, *Density topologies*, ***Handbook of Measure Theory*** (E. Pap, editor), North-Holland, Amsterdam, 2002, pp. 675–702.

DIPARTIMENTO DI MATEMATICA
UNIVERSITA' DI SALERNO
SALERNO, ITALY
E-mail: gcarotenuto@unisa.it