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HARMONIC AND EQUIANHARMONIC EQUATIONS IN THE GROTHENDIECK–TEICHMÜLLER GROUP. III

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Abstract We study behaviours of the 'equianharmonic' parameter of the Grothendieck–Teichmüller group introduced by Lochak and Schneps. Using geometric construction of a certain one-parameter family of quartics, we realize the Galois action on the fundamental group of a punctured Mordell elliptic curve in the standard Galois action on a specific subgroup of the braid group $\hat{B}_4$. A consequence is to represent a matrix specialization of the 'equianharmonic' parameter in terms of special values of the adelic beta function introduced and studied by Anderson and Ihara.

Keywords: Grothendieck–Teichmüller group; Galois actions on profinite braid groups; adelic beta function

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1. Introduction

In this paper, we continue our studies [**19**, **20**] on the Galois representations in profinite fundamental groups. Due to Belyi's fundamental theorem [**2**], the absolute Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ can be embedded in the automorphism group $\mathrm{Aut}(\hat{F}_2)$ of the free profinite group $\hat{F}_2$ regarded as $\pi_1(\boldsymbol{P}^1_{\bar{\mathbb{Q}}} - \{0, 1, \infty\}, \overrightarrow{01})$ with standard generators x, y, z (x, y, z: loops around 0, 1, ∞ satisfying $xyz = 1$). The elements $\sigma \in G_{\mathbb{Q}}$ are parametrized by pairs $(\lambda_\sigma, f_\sigma) \in \hat{\mathbb{Z}}^\times \times \hat{F}_2'$, where $\lambda : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}^\times$ is the cyclotomic character and f_σ is a uniquely determined element (for each σ) in the commutator subgroup $\hat{F}_2'$ of $\hat{F}_2$ so that $\sigma \in G_{\mathbb{Q}}$ acts as $\sigma(x) = x^{\lambda_\sigma}$, $\sigma(y) = f_\sigma^{-1} y^{\lambda_\sigma} f_\sigma$. Studies of behaviours of the mysterious parameter f_σ on $G_{\mathbb{Q}}$ lead to various versions of the Grothendieck–Teichmüller group $\mathrm{GT} \subset \mathrm{Aut}(\hat{F}_2)$ defined by functional equations in (λ, f) found satisfied by the image of $G_{\mathbb{Q}}$ (Drinfeld, Ihara, others: see, for example, [**3**, **8**, **18**]).

Grothendieck [**5**] proposed to consider geometric constraints of the image of $G_{\mathbb{Q}}$ in GT coming from 'lego' structures of a tower of the moduli spaces of curves. As a first step

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toward Galois–Teichmüller lego through 'special loci' *inside* the moduli spaces of curves, Lochak and Schneps [10] introduced certain accessory parameters g and $h : \mathrm{GT} \to \hat{F}_2$ so as to decompose (uniquely) the main parameter $f : \mathrm{GT} \to \hat{F}_2$ as follows:

$$f(x,y) = g(y,x)^{-1}g(x,y) = \begin{cases} y^{-(\lambda-1)/2}h(y,z)^{-1}h(x,y) & (\lambda \equiv 1 \bmod 6), \\ y^{-(\lambda-1)/2}h(y,z)^{-1}y^{-1}h(x,y) & (\lambda \equiv -1 \bmod 6). \end{cases} \quad (1.1)$$

On $G_{\mathbb{Q}}(\subset \mathrm{GT})$, these parameters g and h represent Galois transformations of certain paths on $\boldsymbol{P}^1 - \{0,1,\infty\}$ connecting the infinity loci $\{0,1,\infty\}$ to the special loci $\{-1,\frac{1}{2},2\}$ and $\{\rho,\rho^{-1}\}$ (where $\rho := \mathrm{e}^{2\pi\mathrm{i}/6}$). Using this interpretation, in [19], we showed that the parameters g and h can actually be directly written by (λ, f) on the image of $G_{\mathbb{Q}}$ in GT, and presented several new-type equations satisfied by the Galois image.

In [20], we switched our view of $\boldsymbol{P}^1 - \{0,1,\infty\}$ to its interpretation as the (coarse) moduli space of elliptic curves of level 2 structures. Then, the special loci $\{-1,\frac{1}{2},2\}$ and $\{\rho,\rho^{-1}\}$ represent respectively moduli of the lemniscate elliptic curve $Y^2 = X^3 - X$ and of the Mordell elliptic curve $Y^2 = X^4 - X$, so naturally the arithmetic features of elliptic curves came into our scope for the study of the Grothendieck–Teichmüller parameters.

The purpose of this paper is to study the Mordell curve case in details and to develop a parallel theory to the lemniscate curve case investigated in [20].

One aspect of the above phenomenon may be sharply featured in the matrix specialization of these parameters f, g, h, when $x, y \in \hat{F}_2$ are specialized to the generator matrices of the level 2 modular group $\Gamma(2) \subset \mathrm{SL}_2(\mathbb{Z})$. Noting that the profinite completion $\widehat{\mathrm{SL}_2(\mathbb{Z})}$ has still a big kernel towards $\mathrm{SL}_2(\hat{\mathbb{Z}})$, we may consider specializations in the latter group as a first approximation of information of f_σ, g_σ, h_σ ($\sigma \in G_{\mathbb{Q}}$). In [14, Corollary 4.13], using the Tate elliptic curve (that deforms a degenerated pointed curve), we explicitly computed the matrix $f_\sigma((\begin{smallmatrix}1&2\\0&1\end{smallmatrix}),(\begin{smallmatrix}1&0\\-2&1\end{smallmatrix})) \in \mathrm{SL}_2(\hat{\mathbb{Z}})$, which turned out later in [18, Remark 2.7] to be decomposed as the following 'intriguing' form:

$$f_\sigma\left(\begin{pmatrix}1&2\\0&1\end{pmatrix},\begin{pmatrix}1&0\\-2&1\end{pmatrix}\right) = (-1)^{(\lambda_\sigma-1)/2}\begin{pmatrix}1&0\\-8\rho_2(\sigma)&1\end{pmatrix}\begin{pmatrix}\lambda_\sigma^{-1}&0\\0&\lambda_\sigma\end{pmatrix}\begin{pmatrix}1&-8\rho_2(\sigma)\\0&1\end{pmatrix} \quad (\sigma \in G_{\mathbb{Q}}). \quad (1.2)$$

Here, $\rho_2 : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}$ designates the Kummer 1-cocycle along the positive roots of 2 (see also [15,16]). The matrix specialization $g_\sigma((\begin{smallmatrix}1&2\\0&1\end{smallmatrix}),(\begin{smallmatrix}1&0\\-2&1\end{smallmatrix}))$, $h_\sigma((\begin{smallmatrix}1&2\\0&1\end{smallmatrix}),(\begin{smallmatrix}1&0\\-2&1\end{smallmatrix})) \in \mathrm{SL}_2(\hat{\mathbb{Z}})$ should then decompose the right-hand side of the above formula (1.2) according to their defining properties (1.1). It turns out that those decomposing factors are explicitly given in the language of the Anderson–Ihara adelic beta function [1,6,7] specialized to the 'adelic periods' of the lemniscate and Mordell elliptic curves (Theorem 4.16; see [20] for the lemniscate case).

As another direction of application, we obtain formulae of $G_{\mathbb{Q}}$-actions on some specific braids in 'power conjugate forms' (Proposition 4.13). Unknown is whether they hold for all elements of GT, i.e. if they could be constraints for the famous question $G_{\mathbb{Q}} \subsetneqq \mathrm{GT}$.

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The contents of the present paper are as follows. In § 2, after preparing basic notions on braid configuration spaces $\boldsymbol{A}^n \setminus D$ and standard tangential base points on them, we focus on Cardano–Ferrari connection of those with four and three strings and on Mordell transformation to certain moduli space of elliptic curves. In this context, we find two *canonical* special loci—the *lemniscate* and *Mordell* loci—in $\boldsymbol{A}^4 \setminus D$, the braid configuration space with four strings. In § 3, we introduce specific paths between standard tangential base points on $\boldsymbol{A}^4 \setminus D$ and on the Mordell locus, and look at related loops in terms of braid generators. In § 4, we analyse the Galois actions on the fundamental group of a punctured Mordell elliptic curve realized as a 6-cyclic cover over the projective line with three-point ramification. Then, we finally fit this picture in the frame of Galois-braid group, i.e. the arithmetic fundamental group of $\boldsymbol{A}^4 \setminus D$ and obtain our main results.

2. Braid configuration space and Mordell locus

2.1. We start with regarding the affine n-space $\boldsymbol{A}^n_u$ (over $\mathbb{Q}$) to be the moduli space of the monic polynomials $f_u(X) = X^n + u_1 X^{n-1} + \cdots + u_n$ with coordinates $u = (u_1, \dots, u_n)$. Our main concern is then the subspace $\boldsymbol{A}^n \setminus D$ of $\boldsymbol{A}^n$ consisting of points of those $f_u(X)$ with multiplicity free zeros. To introduce a standard base point for its fundamental group, we follow the construction by Ihara and Matsumoto [**8**]. First take the S_n-cover $\boldsymbol{A}^n_v \setminus \Delta$ of $\boldsymbol{A}^n_u \setminus D$ with the coordinates $v = (v_1, \dots, v_n)$ representing the ordered zeros of $f_u(X) \in \boldsymbol{A}^n_u \setminus D$. Then, putting $v = (t_1 t_2 \cdots t_n, t_2 \cdots t_n, \dots, t_n)$ gives a homomorphism of the structure ring of $\boldsymbol{A}^n_v \setminus \Delta$ into the Puiseux power series ring

$$\mathbb{Q}\Big[v_1, \dots, v_n, \prod_{i<j} (v_i - v_j)^{-1}\Big] \to \bar{\mathbb{Q}}\{\{t_1, \dots, t_n\}\},$$

which defines a tangential base point $\vec{v}$ on $\boldsymbol{A}^n_v \setminus \Delta$. We define the standard base point $\vec{b}$ on $\boldsymbol{A}^n_u \setminus D$ to be the image of $\vec{v}$. In the above definition of $\vec{v}$, we may (and sometimes do) assume $t_1 = 0$. This gives an equivalent base point to $\vec{v}$, as the parallel transformation of $\boldsymbol{A}^1$ by $-(t_1 \cdots t_n)$ moves the above v to $(0, t_2 \cdots t_n(1 - t_1), \dots, t_n(1 - t_1 \cdots t_{n-1}))$ that defines a Galois equivalent tangential base point defined by $(0, t_2 t_3 \cdots t_n, t_3 \cdots t_n, \dots, t_n)$. (See, for example, [**17**, 5.9] for the definition of Galois equivalence of tangential base points: a version of Deligne's notion of 'toroidal transformations'.)

2.2. The geometric part of the fundamental group $\pi_1(\boldsymbol{A}^n_u \setminus D, \vec{b})$ is then naturally identified with the profinite completion of the Artin braid group B_n with standard generators $\tau_1, \dots, \tau_{n-1}$ with relations: $\tau_i \tau_j = \tau_j \tau_i$, $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$ $(i = 1, \dots, n-1,\ i+1 < j)$. We have the exact sequence of profinite groups

$$1 \to \hat{B}_n \to \pi_1(\boldsymbol{A}^n_u \setminus D, \vec{b}) \to G_{\mathbb{Q}} \to 1.$$

The geometric part of $\pi_1(\boldsymbol{A}^n_v \setminus D, \vec{v})$ is then the profinite completion of the pure braid group $P_n \subset B_n$.

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2.3. We shall also consider the 'modulo $\boldsymbol{G}_m$ versions' of the above discussed spaces, namely consider $\mathcal{A}_v^n := (\boldsymbol{A}_v^n \setminus \Delta)/\boldsymbol{G}_m$ and $\mathcal{A}_u^n := (\boldsymbol{A}_u^n \setminus D)/\boldsymbol{G}_m$ (as stacks), where the actions of $\boldsymbol{G}_m$ are given by those induced from (simple) scalar multiplications of the coordinates of v. This corresponds to neglecting the effect of the coordinate t_n in the definition of $\vec{v}$ (or normalizing, say, $t_n = 1$). For abuse of notations, we shall write the images of $\vec{v}$, $\vec{b}$ in $\mathcal{A}_v^n$, $\mathcal{A}_u^n$ by the same symbols respectively. The geometric part of the fundamental group $\pi_1(\mathcal{A}_u^n, \vec{b})$ can be identified with $\hat{B}_n$ divided by the centre which is pro-cyclic subgroup generated by $\omega_n := (\tau_1 \cdots \tau_{n-1})^n$.

2.4. It is sometimes useful to consider the deformation retract $(\boldsymbol{A}_u^n \setminus D)_0$ of the space $(\boldsymbol{A}_u^n \setminus D)$ by killing the second coefficient u_1 of the monic $f_u(X)$ via Tschirnhausen transformation replacing X by $X - u_1/n$. This process does not affect on homotopy invariants and is obviously defined over $\mathbb{Q}$. Noting also that this retraction is compatible with the above $\boldsymbol{G}_m$-action, we may form the quotient stack $(\mathcal{A}_u^n)_0 := (\boldsymbol{A}_u^n \setminus D)_0/\boldsymbol{G}_m$. For simplicity, we shall write those images of the tangential base point $\vec{b}$ on $(\boldsymbol{A}_u^n \setminus D)_0$ and on $(\mathcal{A}_u^n)_0$ by the same symbol as long as no troubles occur from this convention.

Ferrari morphism

2.5. The classical formula (due to Ferrari) tells us how to obtain zeros of any quartic $f(T) = T^4 + aT^3 + bT^2 + cT + d = 0$. After replacing T by $T - a/4$, we may assume $a = 0$. The fundamental observation is that, if T_1, T_2, T_3, T_4 are the zeros of $f(T) = 0$ (where we are now assuming $T_1 + T_2 + T_3 + T_4 = 0$), then, the resolvents

$$\left.\begin{aligned} X_1 &= (\tfrac{1}{2}(T_1 - T_2 - T_3 + T_4))^2 = -(T_1 + T_4)(T_2 + T_3), \\ X_2 &= (\tfrac{1}{2}(T_1 - T_2 + T_3 - T_4))^2 = -(T_1 + T_3)(T_2 + T_4), \\ X_3 &= (\tfrac{1}{2}(T_1 + T_2 - T_3 - T_4))^2 = -(T_1 + T_2)(T_3 + T_4) \end{aligned}\right\} \tag{2.6}$$

are the zeros of the cubic

$$(\mathcal{F}f)(X) := X^3 + 2bX^2 + (b^2 - 4d)X - c^2. \tag{2.7}$$

The solutions $T_1, \dots, T_4$ are then obtained as linear combinations of square roots $\sqrt{X_i}$ $(i = 1, 2, 3)$ satisfying $\sqrt{X_1}\sqrt{X_2}\sqrt{X_3} = -c$.

2.8. We shall consider the above mapping $f \mapsto \mathcal{F}f$ as a $\mathbb{Q}$-morphism $\boldsymbol{A}_u^4 \setminus D \to \boldsymbol{A}_u^3 \setminus D$ and call it the *Ferrari morphism*. The standard tangential base point $\vec{b}$ on $\boldsymbol{A}_u^4 \setminus D$ is equivalent to the image of one on $\boldsymbol{A}_v^4 \setminus \Delta$ induced from $(v_1, v_2, v_3, v_4) = (0, t^3, t^2, t)$. Its image by $\mathcal{F}$ is then the image of a tangential base point from $(v_1, v_2, v_3) = (t^5, t^4, t^3)$ on $\boldsymbol{A}_v^3 \setminus \Delta$ which is Galois equivalent to $\vec{b}$ on $\boldsymbol{A}_u^3 \setminus D$. Thus, we may consider the Ferrari morphism $\mathcal{F}$ gives a $G_{\mathbb{Q}}$-compatible homomorphism

$$\pi_1(\mathcal{F}) : \pi_1(\boldsymbol{A}_u^4 \setminus D, \vec{b}) \to \pi_1(\boldsymbol{A}_u^3 \setminus D, \vec{b}).$$

By simple path tracing, we find on the geometric part of the fundamental groups, $\pi_1(\mathcal{F})$ maps the generators τ_1, τ_2, τ_3 of B_4 respectively to τ_1, τ_2, τ_1 of B_3. This surjection

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$B_4 \to B_3$ was studied closely in Gorin and Lin [**4**], in which its kernel is shown to be isomorphic to a free group of rank 2 at discrete level. At the profinite level, it is not difficult to see that the kernel of $\pi_1(\mathcal{F})$ is a free profinite group of rank 2. In this article, we prefer to call it the *Ferrari kernel* $\mathfrak{F}_2$ and to choose its standard generators as

$$\left.\begin{aligned} \xi_1 &:= \tau_3\tau_1^{-1}, \\ \xi_2 &:= \tau_1\tau_2\tau_1\tau_3^{-1}\tau_2^{-1}\tau_1^{-1}, \end{aligned}\right\} \tag{2.9}$$

so that a relation $[\xi_1, \xi_2] = (\tau_1\tau_2\tau_3)^4(\tau_1\tau_2)^{-6}$ holds as in [**20**, (4.2.2)] (dating also back to [**14**, § 4] and [**16**]). It is also useful to consider the reduced form

$$\mathcal{F}_0 : (\boldsymbol{A}_u^4 \setminus D)_0 \to (\boldsymbol{A}_u^3 \setminus D)_0$$

of $\mathcal{F}$ obtained by taking obvious restriction at source space and composition with retraction at target space. By simple calculation, one finds

$$\mathcal{F}_0(T^4 + bT^2 + cT + d) = X^3 - (\tfrac{1}{3}b^2 + 4d)X - (\tfrac{2}{27}b^3 - \tfrac{8}{3}bd + c^2). \tag{2.10}$$

In view of $\boldsymbol{G}_m$-actions, the scalar multiples of zeros of quartics give rise to doubly scalar multiples of cubic resolvents. So the induced morphism $(\mathcal{A}_u^4)_0 \to (\mathcal{A}_u^3)_0$ factors through

$$\bar{\mathcal{F}}_0 : (\mathcal{A}_u^4)_0 \to (\tilde{\mathcal{A}}_u^3)_0 := (\boldsymbol{A}_u^3 \setminus D)_0/\boldsymbol{G}_m^2.$$

Finally, we remark that the geometric fundamental group of $(\tilde{\mathcal{A}}_u^3)_0$ based at (the image of) $\vec{b}$ is isomorphic to $\hat{B}_3/\langle\omega_3^2\rangle \cong \widehat{\mathrm{SL}_2(\mathbb{Z})}$. The reduced Ferrari morphism $\bar{\mathcal{F}}_0$ induces a surjection from $\hat{B}_4/\langle\omega_4\rangle$ on it, and its kernel is naturally isomorphic to $\mathfrak{F}_2$. So we do identify the Ferrari kernel $\mathfrak{F}_2$ with its image in $\hat{B}_4/\langle\omega_4\rangle \subset \pi_1(\mathcal{A}_u^4, \vec{b})$. In summary, we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathfrak{F}_2 & \longrightarrow & \hat{B}_4 & \longrightarrow & \hat{B}_3 & \longrightarrow & 1 \\ & & \Vert & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathfrak{F}_2 & \longrightarrow & \hat{B}_4/\langle\omega_4\rangle & \longrightarrow & \hat{B}_3/\langle\omega_3^2\rangle \cong \widehat{\mathrm{SL}_2(\mathbb{Z})} & \longrightarrow & 1. \end{array}$$

Mordell transformation

2.11. From the above formula of $\mathcal{F}_0$, one sees that, for any quartic $f(T) = T^4 + bT^2 + cT + d$, the elliptic curve

$$E_f : Y^2 = -(\mathcal{F}f)_0(-X) = X^3 - (\tfrac{1}{3}b^2 + 4d)X - (-\tfrac{2}{27}b^3 + \tfrac{8}{3}bd - c^2) \tag{2.12}$$

has always a specific rational point $P_f := (X, Y) = (-\frac{2}{3}b, c)$. We shall call this correspondence from $f \mapsto (E_f, P_f)$ the Mordell transformation $\mathcal{M}$. In fact, one can easily verify that the classical birational transformation of [**13**, p. 77] maps the curve

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$y^2 = f(x) = x^4 + bx^2 + cx + d$ with two infinity points (∞_+, ∞_-), where $\infty_\pm$ corresponds respectively to $(\xi, \eta) = (0, \pm 1)$ after the change of variables $\xi = x^{-1}$, $\eta = yx^{-2}$, to the elliptic curve $E_f : Y^2 = -(\mathcal{F}f)_0(-X)$ with two rational points (P_f, O). In our notation, this transformation is given explicitly by

$$\left.\begin{aligned} x &= \frac{-3Y - 3c}{6X + 4b}, & X &= 2x^2 - 2y + \tfrac{1}{3}b, \\ y &= -\tfrac{1}{2}X + \tfrac{1}{6}b + x^2, & Y &= 4x(y - x^2 - \tfrac{1}{2}b) - c. \end{aligned}\right\} \qquad (2.13)$$

It is probably worth mentioning here that, when $(\xi, \eta) \to (0, 1)$, the quantities $(y - x^2)$ and $x(y - x^2 - \frac{1}{2}b)$ converge to $\frac{1}{2}b$, $\frac{1}{2}c$ respectively.

Conversely, given an elliptic curve $E : Y^2 = X^3 - \gamma_2 X - \gamma_3$ with $4\gamma_2^3 - 27\gamma_3^2 \neq 0$ and a point $P = (X_0, Y_0)$ on it, we can form a quartic $f(T) = \mathcal{M}^{-1}(E, P)$ to be

$$f(T) = T^4 - \tfrac{3}{2}X_0 T^2 + Y_0 T + \tfrac{1}{4}(\gamma_2 - \tfrac{3}{4}X_0^2). \qquad (2.14)$$

We shall call this converse mapping $\mathcal{M}^{-1}$ the inverse Mordell transformation.

2.15. These correspondences $\mathcal{M}$, $\mathcal{M}^{-1}$ can be formulated more conceptually to be related with $\mathcal{F}_0$ by introducing the moduli scheme $M_{1,1}^{\omega}$ of the pairs (E, ω) of elliptic curves E with nowhere vanishing invariant differentials ω on E and the total space $M_{1,2}^{\omega}$ as the universal (once punctured) elliptic curve over $M_{1,1}^{\omega}$. (See, for example, [**9**] for $M_{1,1}^{\omega}$.) In this article, we do not pursue the most sophisticated treatment of these spaces, but do note the existence of a natural commutative diagram:

$$\begin{array}{ccc} (\boldsymbol{A}_u^4 \setminus D)_0 & \xrightarrow{\ \mathcal{M}\ } & M_{1,2}^{\omega} \\ {\scriptstyle \mathcal{F}_0}\downarrow & & \downarrow{\scriptstyle \mathrm{proj}} \\ (\boldsymbol{A}_u^3 \setminus D)_0 & \xrightarrow{\ \iota\ } & M_{1,1}^{\omega} \end{array}$$

where horizontal arrows give isomorphisms of schemes.

Lemniscate and Mordell loci

2.16. We are now ready to introduce our main objects to study in this paper. The one-parameter family of lemniscate elliptic curves $Y^2 = X^3 - (s^2 - s)X$ with rational points (s, s) has discriminant $4s^3(s-1)^3$, hence its inverse Mordell transformation should give a $\mathbb{Q}$-embedding $f^{\mathrm{lem}} : \boldsymbol{P}_s^1 - \{0, 1, \infty\} \to \boldsymbol{A}_u^4 \setminus D$. It reads

$$f^{\mathrm{lem}}(s) = \mathcal{M}^{-1}([Y^2 = X^3 - (s^2 - s)X], (s, s)) = T^4 - \tfrac{3}{2}sT^2 + sT + \tfrac{1}{16}s^2 - \tfrac{1}{4}s. \quad (2.17)$$

In a parallel way, the one-parameter family of Mordell elliptic curves $Y^2 = X^3 - (s^3 - s^2)$ with rational points (s, s) has discriminant $-27s^4(s-1)^2$, hence produces a $\mathbb{Q}$-embedding $f^{\mathrm{mor}} : \boldsymbol{P}_s^1 - \{0, 1, \infty\} \to \boldsymbol{A}_u^4 \setminus D$ according to

$$f^{\mathrm{mor}}(s) = \mathcal{M}^{-1}([Y^2 = X^3 - (s^3 - s^2)], (s, s)) = T^4 - \tfrac{3}{2}sT^2 + sT - \tfrac{3}{16}s^2. \qquad (2.18)$$

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These special families are actually 'canonical' in the sense that they have the following characterization. Before stating it, recall from [**20**, (3.2), (3.5)] that the triangle group $\Delta(\infty, 2, 4)$ (respectively $\Delta(\infty, 3, 6)$) has a natural normal subgroup of index 4 (respectively 6) corresponding to the fundamental group of lemniscate (respectively Mordell) elliptic curve minus origin realized as a 4-cyclic (respectively 6-cyclic) cover over $\boldsymbol{P}^1_{01\infty}$. We denote by K_4^{lem}, K_6^{mor} these normal subgroups respectively.

Proposition 2.19. *The $\mathbb{Q}$-embedding f^{lem} (respectively f^{mor}) : $\boldsymbol{P}^1 - \{0, 1, \infty\} \to \boldsymbol{A}_u^4 \setminus D$, together with any geometric path $\vec{b} \rightsquigarrow f^{\mathrm{lem}}(\overrightarrow{01})$ (respectively $\vec{b} \rightsquigarrow f^{\mathrm{mor}}(\overrightarrow{01})$), induces a commutative diagram of homomorphisms of profinite groups*

$$\begin{array}{ccc} \pi_1(\boldsymbol{P}^1 - \{0,1,\infty\}, \overrightarrow{01}) & \longrightarrow & \pi_1(\mathcal{A}_u^4, \vec{b}) \\ & \searrow \quad \swarrow & \\ & G_{\mathbb{Q}} & \end{array}$$

whose geometric part

(i) *exactly factors through the triangle group $\hat{\Delta}(\infty, 2, 4)$ (respectively $\hat{\Delta}(\infty, 3, 6)$),*

(ii) *has the image containing the Ferrari kernel $\mathfrak{F}_2$ that exactly coincides with the profinite closures of K_4^{lem} (respectively K_6^{mor}).*

These properties (i), (ii) characterize the embeddings f^{lem}, f^{mor} up to equivalence under the $\boldsymbol{G}_m$-action on $\boldsymbol{A}_u^4 \setminus D$.

Proof. We analyse the loci

$$\begin{aligned} L^{\mathrm{lem}} &= \{T^4 + bT^2 + cT + d \in \boldsymbol{A}_u^4 \setminus D \mid \tfrac{2}{27}b^3 - \tfrac{8}{3}bd + c^2 = 0\}, \\ L^{\mathrm{mor}} &= \{T^4 + bT^2 + cT + d \in \boldsymbol{A}_u^4 \setminus D \mid \tfrac{1}{3}b^2 + 4d = 0\} \end{aligned}$$

under the $\boldsymbol{G}_m$-action on $\boldsymbol{A}_u^4 \setminus D$, and find, after simple observation, orbifold isomorphisms $L^{\mathrm{lem}}/\boldsymbol{G}_m \cong (\boldsymbol{P}^1 - \{0,1,\infty\}) \cup (\cdot/\mu_2) \cup (\cdot/\mu_4)$ and $L^{\mathrm{mor}}/\boldsymbol{G}_m \cong (\boldsymbol{P}^1 - \{0,1,\infty\}) \cup (\cdot/\mu_2) \cup (\cdot/\mu_3)$. It also follows that the projection images of f^{lem} and f^{mor} in $\mathcal{A}_u^4$ respectively detect the above loci $L^{\mathrm{lem}}/\boldsymbol{G}_m$, $L^{\mathrm{mor}}/\boldsymbol{G}_m$. This proves (i). For (ii), we shift our view on these loci to those images by $\mathcal{M}$ and by forgetting the ω-structure corresponding to division by the $\boldsymbol{G}_m$-action. The image of f^{lem} (respectively f^{mor}) in $M_{1,2}$ is then same as that of L^{lem} (respectively L^{mor}) that is the fibre over the lemniscate modulus $[\cdot/\mu_4]$ (respectively Mordell modulus $[\cdot/\mu_6]$) on $M_{1,1}$. This observation verifies the property (ii) for our f^{lem}, f^{mor}. To deduce the characterization of these embeddings up to $\boldsymbol{G}_m$-equivalence, we note that the properties (i), (ii) and $G_{\mathbb{Q}}$-compatibility of the induced homomorphisms on fundamental groups force that the arithmetic fundamental groups of the $\mathbb{Q}$-images of $\mathcal{M} \circ f^{\mathrm{lem}}$, $\mathcal{M} \circ f^{\mathrm{mor}}$ in $M_{1,2}^{\omega}$ should isomorphic to those of the lemniscate and Mordell elliptic curves respectively. The fundamental conjecture of anabelian geometry solved by Tamagawa and Mochizuki (cf. [**12, 21**]; the affine case needed here was settled first by [**21**]) insures that those images must surely lie over the prescribed loci at geometric level. □

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In [**20**], we considered a certain one-parameter family of quartics over $\boldsymbol{P}^1_t - \{0,1,\infty\}$ given as $f^{\mathrm{lem}}(1/(1-t))(-T+\frac{1}{2})$ from our $f^{\mathrm{lem}}(s)(T)$ here in the above. In the next section, we will investigate the family $f^{\mathrm{mor}}(1/(1-t))(T)$ over $\boldsymbol{P}^1_t - \{0,1,\infty\}$ in a parallel way to [**20**].

3. Monodromy of paths between tangential base points

3.1. Now we shall analyse the monodromy properties of zeros of the quartic family

$$f^{\mathrm{mor}}_t(T)\biggl(=f^{\mathrm{mor}}\biggl(\frac{1}{1-t}\biggr)(T)\biggr) := T^4 - \frac{3}{2}\frac{1}{1-t}T^2 + \frac{1}{1-t}T - \frac{3}{16}\frac{1}{(1-t)^2}$$

explicitly by using Cardano–Ferrari formula. The above quartics for $t \notin \{0,1,\infty\}$ have discriminant $-27t^2/(1-t)^8$ and the cubic resolvents are given by

$$X_1 = \frac{1}{1-t} - \frac{\sqrt[3]{t}}{1-t}\omega^2, \qquad X_2 = \frac{1}{1-t} - \frac{\sqrt[3]{t}}{1-t}\omega, \qquad X_3 = \frac{1}{1-t} - \frac{\sqrt[3]{t}}{1-t}. \tag{3.2}$$

We shall take the branches $\sqrt{X_i}$ $(i=1,2,3)$ so that, for small positive t, they approximately take values

$$\sqrt{X_1} \approx -1 + \tfrac{1}{2}\omega^2\sqrt[3]{t}, \qquad \sqrt{X_2} \approx 1 - \tfrac{1}{2}\omega\sqrt[3]{t}, \qquad \sqrt{X_3} \approx 1 - \tfrac{1}{2}\sqrt[3]{t}. \tag{3.3}$$

Using these, we specify the four zeros of $f^{\mathrm{mor}}_t(T)=0$ as

$$\left.\begin{aligned} T_1 &:= \tfrac{1}{2}(\sqrt{X_3} - \sqrt{X_2} - \sqrt{X_1}) \approx \tfrac{1}{2}(1+\omega^2\sqrt[3]{t}),\\ T_2 &:= \tfrac{1}{2}(\sqrt{X_1} + \sqrt{X_2} + \sqrt{X_3}) \approx \tfrac{1}{2}(1+\omega\sqrt[3]{t}),\\ T_3 &:= \tfrac{1}{2}(\sqrt{X_1} - \sqrt{X_2} - \sqrt{X_3}) \approx \tfrac{1}{2}(1+\sqrt[3]{t}),\\ T_4 &:= \tfrac{1}{2}(\sqrt{X_2} - \sqrt{X_1} - \sqrt{X_3}) \approx -\tfrac{3}{2}. \end{aligned}\right\} \tag{3.4}$$

Below we basically consider T_i $(i=1,\dots,4)$ as single-valued functions analytically extended from small positive t to the upper half sphere $\boldsymbol{P}^1_t(\mathbb{C})_{\Im(t)>0}$, and then, to multi-valued functions on $\boldsymbol{P}^1_t(\mathbb{C}) - \{0,1,\infty\}$. We shall trace the image of the ordered triple $(T_1,T_2,T_3,T_4) \in \boldsymbol{A}^4_v$ along moves of $t \in \boldsymbol{P}^1(\mathbb{C}) - \{0,1,\infty\}$ starting from $\overrightarrow{01}_t$ (whose image in $\boldsymbol{A}^4_u$ is determined by $f^{\mathrm{mor}}_t(T)=0$ as a set $\{T_1,T_2,T_3,T_4\}$).

3.5. Let us first look at the tangential base point $\vec{t} = (T_1,T_2,T_3,T_4)$ on $\boldsymbol{A}^4_v$ lifting $f^{\mathrm{mor}}(\overrightarrow{01}_t)$ on $\boldsymbol{A}^4_u$. It is equivalent to

$$\begin{aligned}\vec{t} = (T_1,T_2,T_3,T_4) &\sim (0, T_2-T_1, T_3-T_1, T_4-T_1)\\ &\sim \biggl(0, \frac{\omega-\omega^2}{2}\sqrt[3]{t}, \frac{1-\omega^2}{2}\sqrt[3]{t}, -2\biggr)\\ &\sim (0, \boldsymbol{t}_1\boldsymbol{t}_2\boldsymbol{t}_3, \boldsymbol{t}_2\boldsymbol{t}_3, \boldsymbol{t}_3)\end{aligned}$$

with $\boldsymbol{t}_1 = \rho$, $\boldsymbol{t}_2 = -(\sqrt{3}/4)\zeta_{12}\sqrt[3]{t}$, $\boldsymbol{t}_3 = -2$, where $\zeta_{12} = \mathrm{e}^{2\pi\mathrm{i}/12}$. Set $K = -(\sqrt{3}/4)\zeta_{12}$. Observing $\sqrt{3}\zeta_{12} = \frac{1}{2}(3+\sqrt{-3})$ and expansions of the T_i, we find that $\vec{t}$ is defined over

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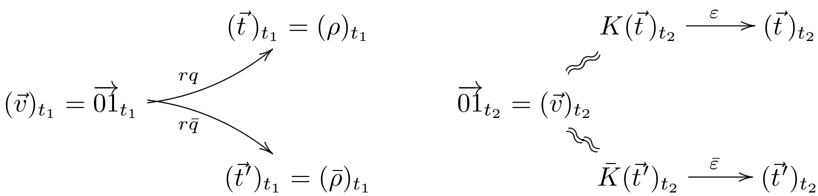
$\mathbb{Q}(\sqrt{-3})$. The conjugate $\vec{t'}$ of $\vec{t}$ over $\mathbb{Q}$ is then easily seen to be the tangential base point defined by (T_2, T_1, T_3, T_4) as

$$\begin{aligned}\vec{t'} = (T_2, T_1, T_3, T_4) &\sim (0, T_1 - T_2, T_3 - T_2, T_4 - T_2)\\ &\sim \left(0, \frac{\omega^2 - \omega}{2}\sqrt[3]{t}, \frac{1-\omega}{2}\sqrt[3]{t}, -2\right)\\ &\sim (0, \boldsymbol{t}_1\boldsymbol{t}_2\boldsymbol{t}_3, \boldsymbol{t}_2\boldsymbol{t}_3, \boldsymbol{t}_3)\end{aligned}$$

with $\boldsymbol{t}_1 = \rho^{-1} = \bar{\rho}$, $\boldsymbol{t}_2 = -(\sqrt{3}/4)\zeta_{12}^{-1}\sqrt[3]{t} = \bar{K}\sqrt[3]{t}$, $\boldsymbol{t}_3 = -2$ (we denote by $\bar{K}$ the complex conjugation of K).

3.6. Now, we shall connect by specific paths these $\vec{t}$, $\vec{t'}$ with the Ihara–Matsumoto tangential base point $\vec{v}$ on $\boldsymbol{A}_v^4$ defined by $v = (0, t_1t_2t_3, t_2t_3, t_3)$ valued in the Puiseux ring $\bar{\mathbb{Q}}\{\{t_1, t_2, t_3\}\}$. Recall here that we introduced in [**19**] certain standard paths $r : \overrightarrow{01} \rightsquigarrow \frac{1}{2}$ and $q : \frac{1}{2} \rightsquigarrow \rho$ on $\boldsymbol{P}^1(\mathbb{C}) - \{0, 1, \infty\}$, and introduced $\bar{q} : \frac{1}{2} \rightsquigarrow \bar{\rho}$ to be the complex conjugation of q. On the t_1-line, we use the compositions $(rq)_{t_1}$, $(r\bar{q})_{t_1}$ to move from $\overrightarrow{01}_{t_1}$ to $t_1 = \rho, \bar{\rho}$ respectively:



On the t_2-line, the base point $\vec{v}$ is defined by the power root system $\{\sqrt[n]{t_2}\}_n$, which must be regarded as $\{(K\sqrt[3]{t})^{1/n}\}_n$ for $(\vec{t})_{t_2}$ and as $\{(\bar{K}\sqrt[3]{t})^{1/n}\}_n$ for $(\vec{t'})_{t_2}$. Define the path ε (respectively $\bar{\varepsilon}$) on the t_2-line as the shortest path (i.e. the most economical homothety around 0) from $\overrightarrow{01}_{t_2}$ to $(\vec{t})_{t_2}$ (respectively $(\vec{t'})_{t_2}$).

3.7. We may consider the composition of the above paths rq (respectively $r\bar{q}$) and ε (respectively $\bar{\varepsilon}$) to be a path on $\boldsymbol{A}_u^4 \setminus D$ from $\vec{b}$ to $f^{\mathrm{mor}}(\overrightarrow{01}_t)$. Figure 1 illustrates how the four zeros of $f_t^{\mathrm{mor}}(T) = 0$ move along these paths.

3.8. Now, f_t^{mor} embeds $\boldsymbol{P}_t^1 - \{0, 1, \infty\}$ into $\boldsymbol{A}_u^4 \setminus D$, so every loop on the former space based at $\overrightarrow{01}_t$ has an interpretation as a loop on the latter space based at $\vec{b}$ after conjugated by $rq\varepsilon$ or $r\bar{q}\bar{\varepsilon}$. To understand braids representing those loops, particularly useful is first to observe how the four zeros of $f_t^{\mathrm{mor}}(T)$ behave along the boundary of the upper half sphere of $\boldsymbol{P}_t^1(\mathbb{C}) - \{0, 1, \infty\}$. This is illustrated in Figure 2. Note that around $t = 1$, the four zeros rotate around $T = \infty$ because of the denominator of $f_t^{\mathrm{mor}}(T)$.

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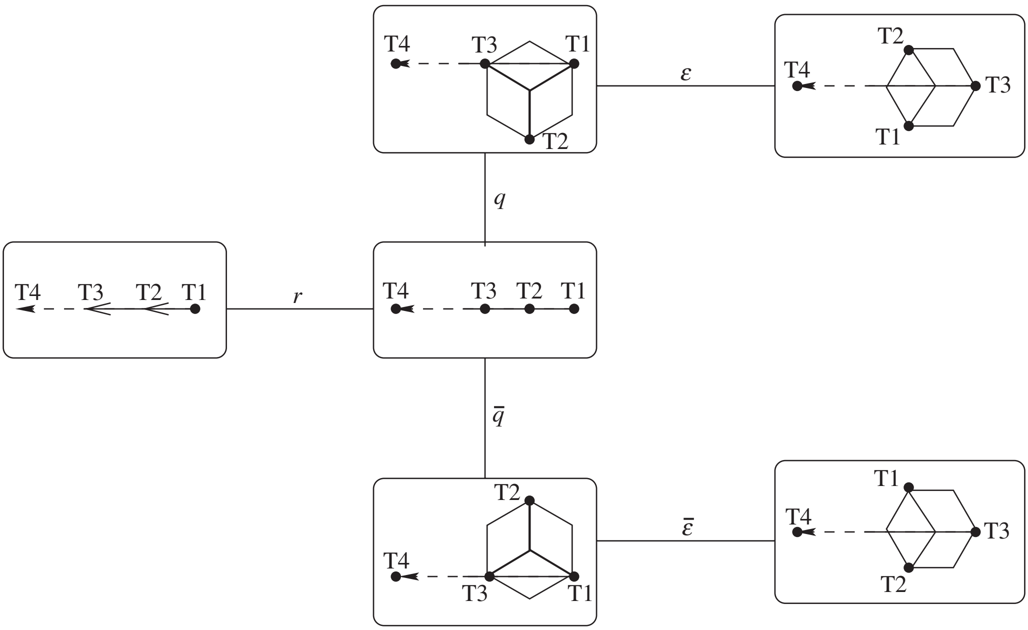

Figure 1. Motion of zeros along paths.

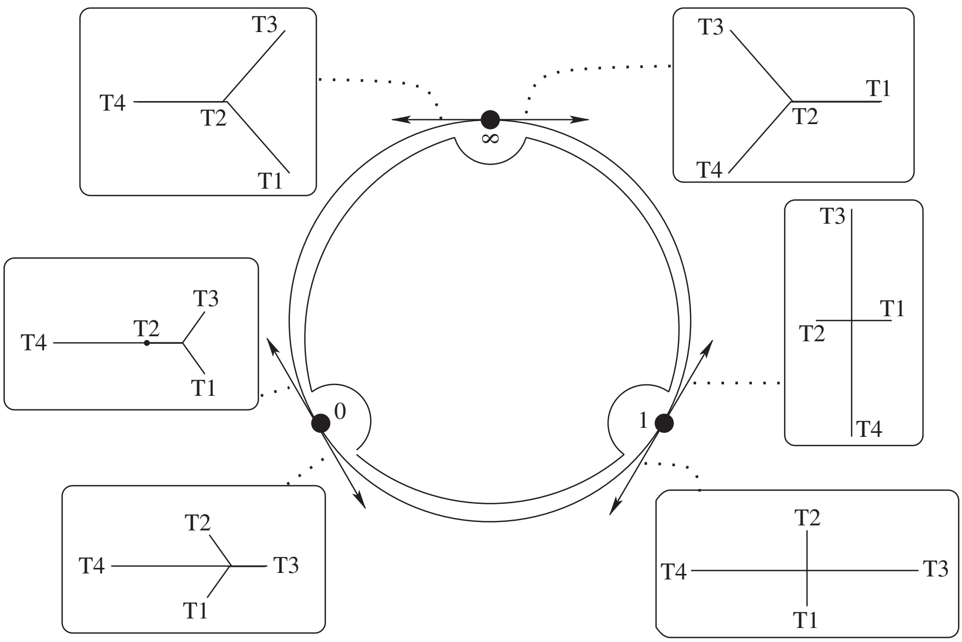

Figure 2. Motion of zeros on the upper hemisphere.

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Notation 3.9. Let us here introduce notations for several specific braids that will play important roles in the subsequent discussions:

$$\xi_+ := \tau_1\tau_2, \qquad \xi_{+2} := \tau_2^{-1}\tau_1^{-1}\tau_2^{-1}\tau_1^{-1}\tau_2^{-1}\tau_3^{-1}, \qquad \xi_{+3} := \tau_3\tau_2\tau_1\tau_2,$$
$$\xi_- := \tau_2\tau_1, \qquad \xi_{-2} := \tau_1^{-1}\tau_3^{-1}\tau_2^{-1}\tau_1^{-1}\tau_1^{-1}\tau_2^{-1}, \qquad \xi_{-3} := \tau_2(\tau_1\tau_1\tau_2\tau_3)\tau_2^{-1},$$
$$\eta := \tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2,$$

We shall also frequently use the same notations to denote their images in quotients of braid groups.

Note that $\omega_3 = \xi_+^3 = \xi_-^3 = \eta^2$ generates the centre of B_3. It is also easy to see that $\xi_{+2}^2 = \xi_{-2}^2 = \omega_4^{-1}$ and $\xi_{+3}^3 = \xi_{-3}^3 = \omega_4$, both generate the centre of B_4. In particular, $\langle \xi_+, \xi_{+2}, \xi_{+3}\rangle$ generates a group isomorphic to the triangle group $\Delta(\infty, 2, 3)$ in $B_4/\langle\omega_4\rangle$. This was actually motivating observation at our early stage of subsequent studies of [**19**].

Lemma 3.10. *In $\pi_1(\boldsymbol{A}_u^4 \setminus D, \vec{b})$, we have*

(i) $(rq\varepsilon)x(rq\varepsilon)^{-1} = \xi_+$, $(rq\varepsilon)y(rq\varepsilon)^{-1} = \xi_{+2}$, $(rq\varepsilon)z(rq\varepsilon)^{-1} = \xi_{+3}$;

(ii) $(r\bar{q}\bar{\varepsilon})x(r\bar{q}\bar{\varepsilon})^{-1} = \xi_-$, $(r\bar{q}\bar{\varepsilon})y(r\bar{q}\bar{\varepsilon})^{-1} = \xi_{-2}$, $(r\bar{q}\bar{\varepsilon})z(r\bar{q}\bar{\varepsilon})^{-1} = \xi_{-3}$;

(iii) $r\bar{q}\bar{\varepsilon} \cdot (rq\varepsilon)^{-1} = \tau_2^2\tau_1\tau_2^2$.

Proof. These follow from observation of behaviours of four zeros of $f_t(T)$ when t moves along the issued loops. The observation can be done in a way based on Figure 2 of those four zeros along the boundary segments of the upper half sphere. □

3.11. Finally, taking modulo the action of $\boldsymbol{G}_m$ on the quadruples (T_1, T_2, T_3, T_4), we could work on the spaces $\mathcal{A}_v^4 = (\boldsymbol{A}_v^4 - \Delta)/\boldsymbol{G}_m$ and $\mathcal{A}_u^4 = (\boldsymbol{A}_u^4 - D)/\boldsymbol{G}_m$. In this case, the effect of t_3-line on homotopy invariants disappears. We write the images of composed paths rq and $r\bar{q}$ on $\mathcal{A}_u^4$ by the same symbols for simplicity.

Conclusion of this section

The morphism f_t^{mor} and the path $rq\varepsilon$ geometrically realizes an embedding of the triangle group $\Delta(\infty, 2, 3)$ onto $\langle \xi_+, \xi_{+2}, \xi_{+3}\rangle/\text{centre} \subset \pi_1(\mathcal{A}_u^4, \vec{b})$.

4. Galois actions in terms of GT-parameters

Mordell curve as a 6-cyclic cover

4.1. In [**20**, (3.4)], we considered a projective model of Mordell elliptic curve defined by the equation

$$E^{\mathrm{mor}} : Y^2 = X^4 - X.$$

This curve can be realized as a 6-cyclic cover of $\boldsymbol{P}_t^1$ unramified outside $t = 0, 1, \infty$ by $X = \sqrt[3]{t}$ and $Y = \sqrt[6]{t}\sqrt{t-1}$. Let the unique point over $t = 0$ to be the origin O of E^{mor}, and fix once and for all any lift of the tangential base point $\overrightarrow{01}_t$ on $E^{\mathrm{mor}} \setminus O$ to

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consider its fundamental group. Then, in particular, the geometric fundamental group $\pi_1^{\mathrm{geom}}(E^{\mathrm{mor}} \setminus O)$ is a normal subgroup of

$$\pi_1^{\mathrm{geom}}(\boldsymbol{P}_t^1 - \{0,1,\infty\}; e_1|2, e_\infty|3; \overrightarrow{01}) = \hat{\Delta}(\infty,2,3) = \langle x,y,z \mid xyz = y^2 = z^3 = 1\rangle$$

of index 6. The Galois group action on it at the base point $\overrightarrow{01}$ can be extended to the action of GT: each element $\sigma = (\lambda, f) \in \mathrm{GT}$ acts on the generators x, y, z by the following formula:

$$\left.\begin{aligned} &x \mapsto x^\lambda, \\ &y \mapsto f(y,x)y^\lambda f(x,y), \\ &z \mapsto x^{(\lambda-1)/2} f(z,x) z^\lambda f(x,z) x^{(1-\lambda)/2}. \end{aligned}\right\} \tag{4.2}$$

4.3. We here repeat the discussion in [**20**, (3.4)] for the sake of notation normalization: we wish to use $x_1 := zx^{-2}$, $x_2 := x^4 z$ for generators of $\pi_1^{\mathrm{geom}}(E^{\mathrm{mor}} \setminus O)$ which are related with those x_1', x_2' in [**20**] by $x_1 = x^2 x_1' x^{-2}$, $x_2 = x x_2' x^{-1}$. Having a generator $z_0 := x^6$ for the inertia group over $O \in E^{\mathrm{mor}}$, we keep the standard presentation

$$\pi_1^{\mathrm{geom}}(E^{\mathrm{mor}} \setminus O) = \langle x_1, x_2, z_0 \mid [x_1, x_2] z_0 = 1\rangle.$$

The Tate module $T_f(E^{\mathrm{mor}})$ appears as the abelianization of $\pi_1^{\mathrm{geom}}(E^{\mathrm{mor}} \setminus O)$. It is isomorphic to a subgroup of index 6 of the quotient triangle group $\hat{\Delta}(6,2,3)$ of $\hat{\Delta}(\infty,2,3)$. They fit in the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1^{\mathrm{geom}}(E^{\mathrm{mor}} - \{O\}) & \longrightarrow & \hat{\Delta}(\infty,2,3) & \longrightarrow & \mathbb{Z}/6\mathbb{Z} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & T_f(E^{\mathrm{mor}}) & \longrightarrow & \hat{\Delta}(6,2,3) & \longrightarrow & \mathbb{Z}/6\mathbb{Z} & \longrightarrow & 1. \end{array}$$

The inner automorphism by x induces an action on $T_f(E^{\mathrm{mor}})$ of order 6, given by $\zeta_6 : \bar{x}_1 \mapsto -\bar{x}_2$, $\bar{x}_2 \mapsto \bar{x}_1 + \bar{x}_2$. We extend this action naturally to the action of the ring $\hat{\mathbb{Z}}[\zeta_6]$ on $T_f(E^{\mathrm{mor}})$. Under these notations, [**20**, Proposition 3.5] may be rephrased as the following proposition.

Proposition 4.4. *Let $\bar{x}_1$, $\bar{x}_2$ be the basis of the Tate module $T_f(E^{\mathrm{mor}})$ which are the images of x_1, x_2 respectively. Then, each element $\sigma = (\lambda, f) \in \mathrm{GT}$ acts on $T_f(E^{\mathrm{mor}})$ by*

$$\begin{aligned} \bar{x}_1 &\mapsto \mathbb{B}_\sigma(\zeta_6, \zeta_6^2) \cdot \bar{x}_1, \\ \bar{x}_2 &\mapsto \zeta_6^{\lambda-1} \mathbb{B}_\sigma(\zeta_6, \zeta_6^2) \cdot \bar{x}_2. \end{aligned}$$

Proof. By using (4.2), one computes

$$\begin{aligned} \sigma(x_1) = \sigma(zx^{-2}) &= x^{(\lambda-1)/2} f(z,x) z^\lambda f(x,z) x^{(1-5\lambda)/2} \\ &= (x^{(\lambda-1)/2} f(z,x) x^{(1-\lambda)/2})(x^{(\lambda-1)/2} z^{(1-\lambda)/4}) \\ &\qquad \times (z^{(5\lambda-1)/4} x^{(1-5\lambda)/2})(x^{(5\lambda-1)/2} f(x,z) x^{(1-5\lambda)/2}). \end{aligned}$$

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Here, note that $z^{1/4}$ makes sense as $z^3 = 1$. Denoting by '$\equiv$' the congruence modulo the commutator subgroup of the kernel of $\hat{\Delta}(\infty, 2, 3) \to \hat{\Delta}(6, 2, 3)$, we have $f(x, z) \equiv \mathbb{A}_\sigma(x, z) \cdot [x, z]$ (cf. [**20**, (FA)]). Now, $[x, z] = xzx^{-1}z^{-1} = x^{-3}x_2x^3x_1^{-1} \equiv -\bar{x}_2 - \bar{x}_1$, and by induction, one easily verifies that

$$x^{2a}z^{-a} \equiv \frac{1-\zeta_6^{2a}}{1-\zeta_6^2}(-\bar{x}_1).$$

Putting these together into the above computation, we obtain

$$\sigma(\bar{x}_1) \equiv (1 + \mathbb{A}_\sigma(\zeta_6, \zeta_6^2)(\zeta_6 - 1)(\zeta_6^2 - 1))\frac{1-\zeta_6^{2\lambda}}{1-\zeta_6^2}\zeta_6^{(\lambda-1)/2}\bar{x}_1 \equiv \mathbb{B}_\sigma(\zeta_6, \zeta_6^2)\bar{x}_1.$$

It follows from $x_2 = x^6x^{-2}x_1x^2$ that $\sigma(x_2) \equiv \zeta_6^{-2\lambda+2}\mathbb{B}_\sigma(\zeta_6, \zeta_6^2)\bar{x}_2$. We then obtain the desired formula after noting that $\zeta_6^{3(\lambda-1)} = 1$. □

Galois actions in braid configuration space

4.5. Now, we shall make the objects discussed in the previous subsection fit in the braid configuration space. Let $\bar{f}^{\mathrm{mor}} : \boldsymbol{P}^1 - \{0, 1, \infty\} \to \mathcal{A}_u^4$ be the morphism induced from the one-parameter family $f^{\mathrm{mor}}(1/(1-t))$ of quartics considered in § 3, and let $rq\varepsilon$ be the path from $\vec{b}$ to $f^{\mathrm{mor}}(\overrightarrow{01}_t)$ introduced in § 2. By Proposition 2.19, $\bar{f}^{\mathrm{mor}}$ gives an embedding of $\pi_1(\boldsymbol{P}_t^1 - \{0, 1, \infty\}; e_1|2, e_\infty|3; \overrightarrow{01})$ into $\pi_1(\mathcal{A}_u^4, \bar{f}^{\mathrm{mor}}(\overrightarrow{01}))$. Conjugation by the path $rq\varepsilon$ maps isomorphically this latter fundamental group based at $\bar{f}^{\mathrm{mor}}(\overrightarrow{01})$ to that based at $\vec{b}$, and it sends $\pi_1^{\mathrm{geom}}(E^{\mathrm{mor}} \setminus O)$ (based at any lift of $\overrightarrow{01}_t$) to K_6^{mor}. We first relate the generators of it with those of $\pi_1^{\mathrm{geom}}(\mathcal{A}_u^4, \vec{b})$ as follows.

Lemma 4.6. *In $\pi_1^{\mathrm{geom}}(\mathcal{A}_u^4, \vec{b})$, we have:*

(i) $(rq\varepsilon)x_1(rq\varepsilon)^{-1} = \tau_3\tau_1^{-1}(= \xi_1)$;

(ii) $(rq\varepsilon)x_2(rq\varepsilon)^{-1} = \tau_1\tau_2\tau_1\tau_3^{-1}\tau_2^{-1}\tau_1^{-1}(= \xi_2)$;

(iii) $(rq\varepsilon)z_0(rq\varepsilon)^{-1} = (\tau_1\tau_2\tau_1)^4$.

Namely, these loops correspond to the generators of the Ferrari kernel $\mathfrak{F}_2$ given in (2.9).

Proof. These equations follow immediately from Lemma 3.10 and the definition of x_1, x_2, z_0 in § 4.3. □

Through Lemmas 3.10 and 4.6, if $G_{\mathbb{Q}}$-action on $rq\varepsilon$ is known, then, $G_{\mathbb{Q}}$-actions at $\bar{f}^{\mathrm{mor}}(\overrightarrow{01}_t)$ on the triangle group $\hat{\Delta}(\infty, 2, 3)$ and $T_f(E^{\mathrm{mor}})$ described in § 4.1, (4.2), § 4.3 and Proposition 4.4 can be transferred into the $G_{\mathbb{Q}}$-actions of those images in $\pi_1(\mathcal{A}_u^4, \vec{b})$. To investigate $\sigma(rq\varepsilon)$ for $\sigma \in G_{\mathbb{Q}}$, we introduce a Kummer character for roots of the coefficient appearing in the t_2-component of $f^{\mathrm{mor}}(\overrightarrow{01}_t)$ in § 3.

Definition 4.7. Let $\sqrt[n]{K} = 3^{1/2n}4^{-1/n}\zeta_{12n}^{-5}$ and $\sqrt[n]{\bar{K}} = 3^{1/2n}4^{-1/n}\zeta_{12n}^{5}$. Define the Kummer character $\rho_K : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}$ with respect to the roots of K by

$$\zeta_n^{\rho_K(\sigma)} = \sigma(\sqrt[n]{K})/\sqrt[n]{\sigma(K)} \quad (\sigma \in G_{\mathbb{Q}}).$$

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One can easily obtain an explicit description of ρ_K:

$$\rho_K(\sigma) = \frac{\rho_3(\sigma)}{2} - 2\rho_2(\sigma) + \frac{-5\lambda_\sigma \pm 5}{12} \quad (\sigma \in G_{\mathbb{Q}}), \tag{4.8}$$

where $\pm$ varies according as $\lambda_\sigma \equiv \pm 1 \bmod 6$.

Lemma 4.9. *For each* $\sigma \in G_{\mathbb{Q}}$,

$$\sigma(\varepsilon) = \varepsilon^{\pm}(\tau_1\tau_2)^{3\rho_K(\sigma)}.$$

Here, $\varepsilon^{\pm}$ *denotes* ε *or* $\bar{\varepsilon}$ *according as* $\lambda_\sigma \equiv \pm 1 \bmod 6$ *respectively.*

Proof. This follows directly from definitions. □

4.10. As for Galois actions on the path rq, we have from [**10**, **19**] the following description:

$$\sigma(rq) = \begin{cases} h_\sigma(\tau_1^2, \tau_2^2)^{-1} rq & (\lambda_\sigma \equiv 1 \bmod 6), \\ h_\sigma(\tau_1^2, \tau_2^2)^{-1} r\bar{q} & (\lambda_\sigma \equiv -1 \bmod 6). \end{cases}$$

Combining Lemma 4.9 and the above information, we obtain the following proposition.

Proposition 4.11. *For every* $\sigma \in G_{\mathbb{Q}}$, *the loop*

$$u_\sigma := \sigma(rq\varepsilon) \cdot (rq\varepsilon)^{-1}$$

in $\pi_1(\mathcal{A}_u^4)$ *is given explicitly by*

$$\begin{aligned} u_\sigma &= \begin{cases} h_\sigma(\tau_1^2, \tau_2^2)^{-1}(\tau_1\tau_2)^{3\rho_K(\sigma)} & (\lambda_\sigma \equiv 1 \bmod 6), \\ h_\sigma(\tau_1^2, \tau_2^2)^{-1}(\tau_1\tau_2)^{3\rho_K(\sigma)}(\tau_2^2\tau_1\tau_2^2) & (\lambda_\sigma \equiv -1 \bmod 6), \end{cases} \\ &= \tau_1^{((\lambda_\sigma-1)/2)-3\rho_3(\sigma)} f_\sigma(\tau_1\tau_2, \tau_1^2)(\tau_1\tau_2)^{3(\rho_3(\sigma)-2\rho_2(\sigma)-((\lambda_\sigma-1)/2))}. \end{aligned}$$

In the above proposition, it is rather amazing to observe that u_σ has a uniform expression by f_σ unconcerned with congruence classes of λ_σ modulo 6.

Proof. Lemma 4.9, § 4.10 and Lemma 3.10 (iii) imply that the path $rq\varepsilon$ is transformed by $\sigma \in G_{\mathbb{Q}}$ as:

$$\sigma(rq\varepsilon) = \begin{cases} h_\sigma(\tau_1^2, \tau_2^2)^{-1}(\tau_1\tau_2)^{3\rho_K(\sigma)} rq\varepsilon & (\lambda_\sigma \equiv 1 \bmod 6), \\ h_\sigma(\tau_1^2, \tau_2^2)^{-1}(\tau_1\tau_2)^{3\rho_K(\sigma)}(\tau_2^2\tau_1\tau_2^2) rq\varepsilon & (\lambda_\sigma \equiv -1 \bmod 6). \end{cases}$$

This gives the above first expression of u_σ. For the second, we employ the formula of [**19**, Theorem A (HF_1)] which reads:

$$h(\tau_1^2, \tau_2^2) = (\xi_\pm)^{(\lambda \mp 1 - 6\rho_3)/4} f(\tau_1^2, \xi_\pm)\tau_1^{3\rho_3 - ((\lambda \mp 1)/2)}.$$

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Here, ξ_+, ξ_- denote $\tau_1\tau_2$, $\tau_2\tau_1$ respectively, and the sign $\mp$ is taken according as $\lambda\ (:= \lambda_\sigma) \equiv \pm 1 \bmod 6$ respectively. Using this and noting $\tau_1\xi_-\tau_1^{-1} = \xi_+$ and the fact that $\xi_+^3 = \eta^2$ commutes with τ_1 for the case $\lambda \equiv -1 \bmod 6$, we find

$$u_\sigma = \tau_1^{(\lambda-1)/2-3\rho_3} f(\xi_+, \tau_1^2) \cdot \begin{cases} \xi_+^{(6\rho_3-\lambda-1)/4+3\rho_K} & (\lambda \equiv 1 \bmod 6), \\ \xi_+^{(6\rho_3-\lambda-1)/4+3\rho_K+3} & (\lambda \equiv -1 \bmod 6). \end{cases}$$

Substituting ρ_K by the explicit expression (4.8), we find that the exponent of ξ_+ in the last factor coincides with $3(\rho_3 - 2\rho_2 - \frac{1}{2}(\lambda - 1))$ in both cases of $\lambda \equiv \pm 1 \bmod 6$. This concludes our second expression of u_σ. □

4.12. Now we are ready to give a new description of $G_{\mathbb{Q}}$-actions on the braids ξ_+, ξ_{+2}, ξ_{+3} under the standard action on $\hat{B}_4$ given by Drinfeld's formula:

$$\begin{aligned} \tau_1 &\mapsto \tau_1^\lambda, \\ \tau_2 &\mapsto f(\tau_1^2, \tau_2^2)^{-1} \tau_2^\lambda f(\tau_1^2, \tau_2^2), \\ \tau_3 &\mapsto f(\eta^2, \tau_3^2)^{-1} \tau_3^\lambda f(\eta^2, \tau_3^2). \end{aligned}$$

Proposition 4.13. *Under the standard action of $G_{\mathbb{Q}}$ on $\hat{B}_4$, the elements ξ_+, ξ_{+2}, ξ_{+3} are transformed in the following 'power-conjugate' form:*

(i) $\sigma(\xi_+) = u_\sigma \xi_+^{\lambda_\sigma} u_\sigma^{-1}$;

(ii) $\sigma(\xi_{+2}) = u_\sigma f_\sigma(\xi_{+2}, \xi_+) \xi_{+2}^{\lambda_\sigma} f_\sigma(\xi_+, \xi_{+2}) u_\sigma^{-1}$;

(iii) $\sigma(\xi_{+3}) = u_\sigma \xi_+^{(\lambda_\sigma-1)/2} f_\sigma(\xi_{+3}, \xi_+) \xi_{+3}^{\lambda_\sigma} f_\sigma(\xi_+, \xi_{+3}) \xi_+^{(1-\lambda_\sigma)/2} u_\sigma^{-1}$;

for $\sigma \in G_{\mathbb{Q}}$. Here u_σ is the loop given in Proposition 4.11.

Proof. According to the work of Ihara and Matsumoto [**8**], on $G_{\mathbb{Q}}$, Drinfeld's action coincides with that given from the tangential base point $\vec{b}$. The above proposition and § 4.1 ensure that the above (i), (ii), (iii) hold in $\pi_1^{\mathrm{geom}}(\mathcal{A}_u^4, \vec{b}) \cong \hat{B}_4/\langle\omega_4\rangle$. So they hold in $\hat{B}_4$ up to the power of ω_4 under the standard $G_{\mathbb{Q}}$-action. The ambiguity of modulo centre can be killed by comparing both sides in the image of abelianization of $\hat{B}_4$ isomorphic to $\hat{\mathbb{Z}}$. □

4.14. Finally, we shall compute a matrix specialization of the parameter h_σ for $\sigma \in G_{\mathbb{Q}}$. Identify $\pi_1^{\mathrm{geom}}(\boldsymbol{P}^1 - \{0, 1, \infty\}, e_1|2, e_\infty|3; \overrightarrow{01}_t)$ as a subgroup of $\pi_1(\mathcal{A}_u^4, \vec{b})$ by conjugating it by the path $rq\varepsilon : \vec{b} \rightsquigarrow f^{\mathrm{mor}}(\overrightarrow{01}_t)$, and the abelian subquotient corresponding to $T_f(E^{\mathrm{mor}})$ with $\hat{\mathbb{Z}}^2$ by $\bar{\xi}_1 \mapsto \binom{1}{0}$, $\bar{\xi}_2 \mapsto \binom{0}{1}$. Recall that the conjugate actions of τ_1, τ_2 on $\langle \xi_1, \xi_2 \rangle$, the kernel of Ferrari morphism is given by

$$\mathrm{Int}(\tau_1) : \begin{cases} \xi_1 \mapsto \xi_1, \\ \xi_2 \mapsto \xi_2\xi_1, \end{cases} \qquad \mathrm{Int}(\tau_2) : \begin{cases} \xi_1 \mapsto \xi_1\xi_2^{-1}, \\ \xi_2 \mapsto \xi_2 \end{cases} \tag{4.15}$$

(cf. [**14**, (4.9.1)] and [**20**, (5.4.1)], but the latter had a typo in $\mathrm{Int}(\tau_2)(\xi_1)$). Therefore, their actions on $T_f(E^{\mathrm{mor}})$ are represented by the matrices $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$ respectively.

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Theorem 4.16. *For each $\sigma \in G_{\mathbb{Q}}$, we have*

$$h_\sigma\left(\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}\right) = (-1)^{\rho_K(\sigma)}\begin{pmatrix}\pm 1 & 0\\ 0 & 1\end{pmatrix}\mathbb{B}_\sigma\left(\begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}^{\pm 1}, \begin{pmatrix}-1 & 1\\ -1 & 0\end{pmatrix}^{\pm 1}\right)\cdot\begin{pmatrix}\lambda_\sigma^{-1} & -8\rho_2(\sigma)\lambda_\sigma^{-1}\\ 0 & 1\end{pmatrix}$$

in $\mathrm{GL}_2(\hat{\mathbb{Z}})$, *where* $\pm$ *takes the sign simultaneously according as* $\lambda_\sigma \equiv \pm 1 \bmod 6$.

Proof. Let $\sigma \in G_{\mathbb{Q}}$ act on the both sides of Lemma 4.6 (i), (ii). Then, we obtain

$$u_\sigma(rq\varepsilon)\sigma(x_i)(rq\varepsilon)^{-1}u_\sigma^{-1} = \sigma(\xi_i) \quad (i = 1, 2)$$

and view it in $\hat{\mathbb{Z}}^2$. For the right-hand side, by [**20**, Proposition 3.3], the matrix representation of $G_{\mathbb{Q}}$-action on the basis $(\bar{\xi}_1, \bar{\xi}_2)$ is given by

$$\begin{pmatrix}\lambda_\sigma & 8\rho_2(\sigma)\\ 0 & 1\end{pmatrix}.$$

For the left-hand side, noting that the action of $\mathrm{Int}(x)$ is given by left multiplication by $R = \left(\begin{smallmatrix}0 & 1\\ -1 & 1\end{smallmatrix}\right)$, we see that the pair $(\sigma(\bar{x}_1), \sigma(\bar{x}_2))$ conjugated by $rq\varepsilon$ may be given by

$$\left(\mathbb{B}_\sigma(R, R^2)\begin{pmatrix}1\\ 0\end{pmatrix}, R^{\lambda-1}\mathbb{B}_\sigma(R, R^2)\begin{pmatrix}1\\ 0\end{pmatrix}\right) = \mathbb{B}_\sigma(R, R^2)J_\lambda,$$

where $J_\lambda = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}1 & -1\\ 0 & -1\end{smallmatrix}\right)$ according as $\lambda (:= \lambda_\sigma) \equiv \pm 1 \bmod 6$. Combining these with the expression of u_σ, we obtain

$$(-1)^{\rho_K(\sigma)}h_\sigma\left(\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}\right)^{-1}\cdot V_\lambda\cdot\mathbb{B}_\sigma(R, R^2)\cdot J_\lambda = \begin{pmatrix}\lambda & 8\rho_2(\sigma)\\ 0 & 1\end{pmatrix},$$

where $V_\lambda = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}-1 & 1\\ 0 & -1\end{smallmatrix}\right)$ according as $\lambda \equiv \pm 1 \bmod 6$. When $\lambda \equiv -1 \bmod 6$, since $J_\lambda R J_\lambda = R^{-1}$, $J_\lambda^2 = 1$ and $V_\lambda J_\lambda = \left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$, the left-hand side above turns out to be

$$(-1)^{\rho_K(\sigma)}h_\sigma\left(\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}\right)^{-1}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\cdot\mathbb{B}_\sigma(R^{-1}, R^{-2}).$$

This completes the proof of our theorem. □

5. Miscellany

We add some corrections of previous articles [**20**] and [**11**].

Nakamura and Tsunogai [20]

Page 200: $\prod_{nc=0}$ should read $\prod_{c=0}^{n-1}$.
(5.1) (1): the position of ‘,’ should move to the end of line.
(5.4.1): $\mathrm{Int}(\tau_2) : \xi_1 \mapsto \xi_1\xi_2^{-1}$.

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Lochak, Nakamura and Schneps [11]

Page 70, (6.5.2): the sign of $\frac{1}{2}(\lambda - 1)$ in the exponents of τ_1 should be altered as follows:

$$\xi \mapsto \tau_1^{(\lambda-1)/2-3\rho_3} f(\xi, \tau_1^2) \cdot \xi^\lambda \cdot f(\tau_1^2, \xi)\tau_1^{3\rho_3-((\lambda-1)/2)} \quad (\xi = \tau_1\tau_2).$$

This, of course, is nothing but Proposition 4.13 (i), but an alternative simpler proof can be given as follows. Regard $\hat{B}_3/\langle\omega_3\rangle$ as the geometric fundamental group of the orbifold $(\boldsymbol{P}^1 - \{0, 1, \infty\})/S_3$ based at the image of $\overrightarrow{01}$ so that a lift of τ_1 (respectively τ_2) in the groupoid $\pi_1(\boldsymbol{P}^1 - \{0, 1, \infty\}, \mathfrak{B})$ are given respectively as $\tau_1 = [0_1^\infty]$ (respectively $\tau_2 = \langle 0, 1\rangle[1_0^\infty]\langle 1, \infty\rangle$) in the notation of [**14**, § 2.6]. Then, $\tau_1\tau_2 = [0_1^\infty]\langle 0, \infty\rangle[\infty_0^1]\langle\infty, 1\rangle = \langle 0, 1\rangle[1_\infty^0]^{-1}$ The standard action of $\sigma \in G_{\mathbb{Q}}$ on it is given as $\sigma(\tau_1\tau_2) = f_\sigma(x, y)^{-1}y^{(1-\lambda)/2}(\tau_1\tau_2)$, where $x = \tau_1^2$, $y = \tau_2^2$. To this first factor, we apply the equianharmonic equation

$$f(\tau_1^2, \tau_2^2) = \tau_2^{-3\rho_3-((\lambda-1)/2)} f(\tau_2^2, \tau_1\tau_2)^{-1}(\tau_1\tau_2)^{(\lambda-1)/2} f(\tau_1^2, \tau_1\tau_2)\tau_1^{3\rho_3-((\lambda-1)/2)} \qquad (\mathrm{II}')$$

from [**19**]. This immediately leads to the desired result.

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