SEMICONTINUITY OF MULTIFUNCTIONS CONNECTED WITH OPTIMIZATION WITH RESPECT TO CONES

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Abstract

This paper studies topological upper and lower semicontinuity of the minimal value multifunction and the solution multifunction for optimization problems, which are defined in terms of cones, subject to perturbations in constraints. It extends the results of Tanino and Sawaragi to finite dimensions and one of Berge to multiple objective optimization problems.

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1. Introduction

Throughout the paper we let Y be a real topological, Hausdorff, vector space.

DEFINITION 1.1. A set C in Y is a *cone* if $\lambda y \in C$ for all $y \in C$ and $\lambda \geqslant 0$. A *convex cone* C is one for which $\lambda_1 y_1 + \lambda_2 y_2 \in C$ for all $y_1, y_2 \in C$ and $\lambda_1, \lambda_2 \geqslant 0$. A *pointed cone* is one for which $C \cap (-C) = \{0\}$, where 0 denotes the zero element in Y.

DEFINITION 1.2. Let C be a pointed cone in Y. Then C induces a partial ordering $\leqslant$ in Y, namely, for $y_1, y_2 \in Y$ we say $y_1 \leqslant y_2$ if $y_2 - y_1 \in C$.
More details about ordered topological vector spaces are in [15].

DEFINITION 1.3 [9, 14]. Let B be a subset of Y. The point $y_0 \in B$ is a *vector minimal point* of B with respect to C, denoted $y_0 \in \min B$, if there exists no $y \in B$ for which $y \leqslant y_0$ and $y_0 \neq y$.

Let $B \subset Y$; $\overline{B}$ denotes closure in the topology of Y, and B^c denotes complementation.

DEFINITION 1.4. Let C be a cone in Y. A subset B is said to be *C-semicompact* if every open cover of B of the form $\{(\overline{C} + y_j)^c \colon y_j \in Y, j \in J\}$ has a finite subcover.

Let us note that this definition is slightly different to that in [6, Definition 2.5], where the elements y_i are assumed to be in B. We have decided to change the definition because, according to the previous one, no compact set in the space of real numbers with the usual ordering has such a kind of open cover.

Motivation for the notion of C-semicompactness is given in [18].

The following theorem is still valid with our definition of C-semicompactness; however, the proof given in [6] has a mistake which can easily be removed, and therefore we do not prove the theorem here.

THEOREM 1.1 [6]. *If C is a pointed, convex cone in Y, and if $B \subseteq Y$ is a non-empty $(-C)$-semicompact set, then $\min B$ is nonempty.*

LEMMA 1.1. *Let C and B be as in Theorem* 1.1. *Then for every $b_0 \in B$ there is $y \in \min B$ such that $y \leqslant b_0$.*

PROOF. Let us consider a nonempty set $B_0 = \{y \in B \colon y \leqslant b_0\}$. If $B_0 \subset \bigcup_{j \in J}(-\overline{C} + y_j)^c$ for some $y_j \in Y$, then $B \subset \bigcup_{j \in J}(-\overline{C} + y_j)^c \cup (-\overline{C} + b_0)^c$. B is $(-C)$-semicompact, so

$$B \subseteq \left(-\overline{C} + y_1\right)^c \cup \cdots \cup \left(-\overline{C} + y_p\right)^c \cup (-C + b_0)^c$$

for some $y_1, \ldots, y_p$ taken from the set $\{y_j\}_{j \in J}$. Since $B_0 \cap (-\overline{C} + b_0)^c = \varnothing$, we get that

$$B_0 \subset \bigcup_{n=1}^{p} \left(-\overline{C} + y_n\right)^c,$$

which implies that B_0 is $(-C)$-semicompact. By Theorem 1.1 there exists $y_0 \in \min B_0$, and it can be shown that $y_0 \in \min B$ as well.

Let T, X be topological Hausdorff spaces, $f \colon X \to Y$ a function and $\Gamma \colon T \to X$ a multifunction. Consider the family of minimization problems $\{P_t\}_{t \in T}$, P_t: $\min\{f(x) \colon x \in \Gamma(t)\}$, which can be considered as perturbations in constraints from P_{t_0} in some neighbourhood of t_0. Problem P_t can be studied from two sides:

(a) we can extend the notion of the minimal value function for f a real-valued function to the minimal value multifunction, namely, $M_f \colon T \to Y$ will be defined as $M_f(t) = \min f(\Gamma(t))$, for $t \in T$; and

(b) the solution multifunction will be defined as

$$S_f: T \to X, \qquad S_f(t) = \{ x \in \Gamma(t): f(x) \in M_f(t) \},$$

for $t \in T$, the set of all solutions to P_t.

If $f = $ id (the identity function on $X = Y$), then $S_{\mathrm{id}} = M_{\mathrm{id}}$, and this multifunction we shall call the *minimal* multifunction and denote it be M; so for a multifunction $\Gamma: T \to Y$, the minimal multifunction $M: T \to Y$ is defined by $M(t) = \min \Gamma(t)$ for $t \in T$.

Let us note that $x \in S_f(t)$ if and only if there exists no $\bar{x} \in \Gamma(t)$ for which $f(\bar{x}) \leqslant f(x)$ and $f(\bar{x}) \neq f(x)$. This means that $S_f(t)$ is the well-known set of Pareto optimal elements for the vector optimization problem P_t in Euclidean space.

Optimization with respect to cones and generalizations of Pareto optimality have been studied in a number of papers, [3, 5, 7, 8, 16] to name but a few. More references can be found in the above mentioned papers.

Different types of stability of S_f and M_f, when $Y = R$ and $C = R_+$ is the nonnegative half line, have been studied for years. An extensive bibliography on this subject is given in [1].

Let us recall the definitions of topological lower and upper semicontinuity.

DEFINITION 1.5. A multifunction $\Gamma: T \to X$ is said to be *lower semicontinuous* (*l.s.c.*) *at a point* $t_0 \in T$ if for every open set Q which meets $\Gamma(t_0)$ there exists a neighbourhood V of t_0 such that Q meets $\Gamma(t)$ for every $t \in V$.

A multifunction $\Gamma: T \to X$ is said to be *upper semicontinuous*(*u.s.c.*) *at* t_0 if for every open set Q which includes $\Gamma(t_0)$ there exists a neighbourhood V of t_0 such that Q includes $\Gamma(t)$ for every $t \in V$.

We say that Γ is *continuous at* t_0 if it is both l.s.c. and u.s.c. at t_0, Γ is *continuous* if it is continuous at every point $t \in T$.

2. Stability conditions for the minimal value multifunction and the solution multifunction

Denote by R^n the n-dimensional Euclidean space ordered by the nonnegative orthant R_+^n.

The well known Berge's theorem states: if X, T are topological spaces, and if $\Gamma: T \to X$ is a continuous multifunction such that for every $t \in T$, $\Gamma(t)$ is nonempty

and compact, then for every continuous, real-valued function $f: X \to R^1$
    (i) the minimal value function M_f is continuous,
    (ii) the solution multifunction S_f is u.s.c.
    The following simple example in R^2 shows that the solution multifunction S_{id} may not be u.s.c., when Γ is continuous and compact valued unless the pair $(Y, C) = (R, R_+)$, and therefore Berge's theorem ceases to be valid in the general case.

EXAMPLE 2.1. Let $\Gamma: [0, 1] \to R^2$ be defined as follows: $\Gamma(t) =$ the closed line segment between $(-1 + t, t)$ and $(0, 0)$ for $0 \leqslant t \leqslant 1$, so that Γ is compact valued and continuous. Then

$$S_{id}(t) = \min \Gamma(t) = \begin{cases} (-1, 0), & t = 0, \\ \Gamma(t), & 0 < t < 1, \\ (0, 0), & t = 1. \end{cases}$$

Hence S_{id} is not u.s.c. at 0 and at 1, and thus M_{id} is not continuous as well.
    But we can state the theorem for lower semicontinuity of the set of vector minimal elements for Γ as follows:

THEOREM 2.1. *Let T be a topological Hausdorff space, and let Y be a real topological Hausdorff vector space, ordered by a pointed, convex closed cone C. Let t_0 be a point in T and $\Gamma: T \to Y$ a multifunction continuous at t_0. Then the minimal multifunction*

$$M: T \to Y, \qquad M(t) = \min \Gamma(t) \quad for\ t \in T$$

is l.s.c. at t_0,
    (a) *if $\Gamma(t)$ is nonempty, $(-C)$-semicompact and closed in some neighbourhood of t_0, and $\Gamma(t_0)$ is compact, or* (b) *if T fulfills the first countability axiom at t_0, $\Gamma(t)$ is nonempty and $(-C)$-semicompact in some neighbourhood of t_0, and $\Gamma(t_0)$ is compact.*

PROOF . Let A be a linear ordered set, and let $\{V_\alpha\}_{\alpha \in A}$ be a nested base for the neighbourhood system of the point t_0. We can assume that Γ fulfills all assumptions of the theorem in every V_α.

    Suppose M is not l.s.c. at t_0. Then we can find $y_0 \in M(t_0)$ and its neighbourhood Q in Y, and a net $\{t_\alpha\}_{\alpha \in A}$ converging to t_0, such that

$$t_\alpha \in V_\alpha \quad \text{for every } \alpha \in A$$

and
$$M(t_\alpha) \cap Q = \varnothing.$$

Let $\mathscr{U}(0)$ be a local base in Y. Let us denote by $\{U'\}_{U \in \mathscr{U}(0)}$ a base of the neighbourhood system of the point y_0 of the kind
$$U' = (y_0 + U) \cap Q \quad \text{for } U \in \mathscr{U}(0).$$
Fix U'; since $y_0 \in \Gamma(t_0) \cap U'$ and Γ is l.s.c. at t_0, we can find elements $\alpha(U') \in A$ and $y_{\alpha(U')} \in \Gamma(t_{\alpha(U')}) \cap U'$. Let us note that the net $\{y_{\alpha(U')}\}_{U \in \mathscr{U}(0)}$ converges to y_0.

Define a set
$$\mathscr{U}(A) = \{\alpha(U'): U \in \mathscr{U}(0)\} \subseteq A.$$

The first case is when $\mathscr{U}(A)$ is a cofinal set in A. Let us note that for part (b) of Theorem 2.1 we can choose $\mathscr{U}(A)$ cofinal; and the closedness of $\Gamma(t)$ will not be necessary when $\mathscr{U}(A)$ is a cofinal set, so the first case establishes the part (b) of the theorem.

In this case $\{t_{\alpha(U')}\}_{U \in \mathscr{U}(0)}$ is a subnet of $\{t_\alpha\}_{\alpha \in A}$, so it converges to t_0 as well. By Lemma 1.1, one actually can find $z_{\alpha(U')} \in M(t_{\alpha(U')})$ such that $z_{\alpha(U')} \leqslant y_{\alpha(U')}$ for every $\alpha(U') \in \mathscr{U}(A)$.

Next, for every $U \in \mathscr{U}(0)$ let us define open sets
$$G_U = \Gamma(t_0) + U.$$
By upper semicontinuity of Γ at t_0, we obtain $\beta(U) \in A$ such that $\Gamma(t) \subseteq G_U$ for every $t \in V_{\beta(U)}$.

Now we will construct a new set $\{\gamma(U)\}_{U \in \mathscr{U}(0)}$. Take $U \in \mathscr{U}(0)$;

(i) if $\beta(U) \leqslant \alpha(U')$ we put $\gamma(U) = \alpha(U')$,

(ii) if $\beta(U) > \alpha(U')$ we can find $\alpha \in \mathscr{U}(A)$ such that $\alpha \geqslant \beta(U)$ because $\mathscr{U}(A)$ is a cofinal set in A, and then put $\gamma(U) = \alpha$. The set $\{\gamma(U)\}_{U \in \mathscr{U}(A)}$ is a cofinal in $\mathscr{U}(A)$. Indeed, if there exists $\alpha(U_0') \in \mathscr{U}(A)$, such that $\alpha(U_0') > \gamma(U)$ for every $U \in \mathscr{U}(0)$, then we consider $\gamma(U_0)$; we have $\alpha(U_0') > \gamma(U_0)$, and by the definition of the set $\{\gamma(U)\}_{U \in \mathscr{U}(0)}$ it must actually be true that $\beta(U_0) > \alpha(U_0')$, whence
$$\gamma(U_0) \geqslant \beta(U_0) > \alpha(U_0') > \gamma(U_0),$$
which is a contradiction. Since $\gamma(U) \geqslant \beta(U)$, so $V_{\gamma(U)} \subseteq V_{\beta(U)}$, and therefore $\Gamma(t) \subseteq G_U$ for every $t \in V_{\gamma(U)}$. Let us observe that $z_{\gamma(U)} \in M(t_{\gamma(U)})$ for every $U \in \mathscr{U}(0)$, because $\gamma(U) \in \mathscr{U}(A)$. Hence
$$z_{\gamma(U)} \in G_U = \Gamma(t_0) + U \quad \text{for every } U \in \mathscr{U}(0)$$
and $z_{\gamma(U)}$ can be written
$$z_\gamma(U) = x_U + u \quad \text{for some } x_U \in \Gamma(t_0) \text{ and } u \in U.$$
Since $\Gamma(t_0)$ is a compact set, we can choose a subnet of $\{x_U\}_{U \in \mathscr{U}(0)}$, which is convergent to $x_0 \in \Gamma(t_0)$, and consequently there exists a subnet of $\{z_{\gamma(U)}\}_{U \in \mathscr{U}(0)}$ which is convergent to x_0. By the definition, $z_{\gamma(U)} \leqslant y_{\gamma(U)}$, and C is closed, so

$x_0 \leqslant y_0$, whence $x_0 = y_0$. Therefore

$$M\bigl(t_{\gamma(U)}\bigr) \cap Q \neq \varnothing \quad \text{for some } U \in \mathscr{U}(0),$$

which is a contradiction.

The second case is when $\mathscr{U}(A)$ is not a cofinal set in A. The theorem is true if t_0 is an isolated point; suppose t_0 is not an isolated point. T is a Hausdorff space, so for every $\alpha \in A$ there exists $\gamma \in A$ such that $\gamma > \alpha$. Indeed, if there is a $\alpha \in A$ such that $\gamma \geqslant \alpha$ for any $\alpha \in A$, then we have $V_\gamma \subseteq V_\alpha$ for $\alpha \in A$. The point t_0 is not isolated, so we get $t_1 \in V_\gamma$ such that $t_1 \neq t_0$. T is a Hausdorff space, so there exist open sets U_0 and U_1 such that $t_0 \in U_0$, $t_1 \in U_1$ and $U_0 \cap U_1 = \varnothing$. $\{V_\alpha\}_{\alpha \in A}$ is a neighbourhood system of t_0, so we can find $\alpha \in A$ for which $V_\alpha \subseteq U_0$. Therefore $t_1 \in V_\gamma \subseteq U_0$, which is a contradiction.

Since $\mathscr{U}(A)$ is not a cofinal set now, there exists $\alpha_0 \in A$ such that $\alpha_0 > \alpha(U')$ for every $U \in \mathscr{U}(0)$. Then

$$V_\alpha \subseteq V_{\alpha_0} \subseteq V_{\alpha(U')} \quad \text{for every } U \in \mathscr{U}(0) \text{ and every } \alpha \geqslant \alpha_0$$

and

$$\Gamma(t) \cap U' \neq \varnothing \text{ for every } t \in V_\alpha, \quad \text{for every } \alpha \geqslant \alpha_0 \text{ and for every } U \in \mathscr{U}(0).$$

Hence, for every $\alpha \geqslant \alpha_0$ and for every $U \in \mathscr{U}(0)$ we can take $y_\alpha^U \in \Gamma(t_\alpha) \cap U'$. Fix $\alpha \geqslant \alpha_0$; the net $\{y_\alpha^U\}_{U \in \mathscr{U}(0)}$ is convergent to y_0, and $\Gamma(t_\alpha)$ is closed, so $y_0 \in \Gamma(t)$. By Lemma 1.1 we get $z_\alpha \in M(t_\alpha)$ such that $z_\alpha \leqslant y_0$. Consider the net $\{t_\alpha\}_{\alpha \geqslant \alpha_0}$. Exactly as in the first case, we find a subnet of $\{z_\alpha\}_{\alpha \geqslant \alpha_0}$ which is convergent to y_0.

This contradiction establishes lower semicontinuity of M.

COROLLARY 2.1. *Let T and Y be as in Theorem 2.1. Let $\Gamma: T \to Y$ be a continuous multifunction with $\Gamma(t)$ nonempty and compact for every $t \in T$. Then the minimal M is l.s.c. at every point $t \in T$.*

THEOREM 2.2. *Let T and Y be as in Theorem 2.1. Let a multifunction $\Gamma: T \to X$ be continuous at t_0 with $\Gamma(t)$ nonempty and compact in some neighbourhood of t_0. Then for every function $f: X \to Y$ which is continuous in some open set including $\Gamma(t_0)$, the minimal value multifunction M_f is l.s.c. at t_0.*

PROOF. Let f, Γ be as in the hypothesis: Then $f\Gamma: T \to Y$, defined by $(f\Gamma)(t) = f(\Gamma(t))$ for every $t \in T$, fulfills all of the assumptions of Theorem 2.1. Hence M_f, which is the minimal multifunction for $f\Gamma$, must be l.s.c. at t_0.

Theorem 2.2 extends part (i) of Berge's theorem to vector optimization, because for a real valued function f the multifunction M_f is actually a function and its lower semicontinuity as a multifunction is equivalent to the continuity.

THEOREM 2.3. *Let a multifunction* $\Gamma\colon T \to X$ *be continuous at* t_0 *and* $f\colon X \to Y$ *be a continuous and one-to-one function in some open set* G *including* $\Gamma(t_0)$. *Then the solution multifunction* $S_f\colon T \to X$ *is l.s.c. at* t_0

(a) *if* $\Gamma(t)$ *is nonempty and compact in some neighbourhood of* t_0, *and the function* f *is open* [13] *in* G, *or*

(b) *if* T *is first countable at* t_0, X *is a vector topological space with a countable local base,* $f(\Gamma(t))$ *is* $(-C)$*-semi-compact in some neighbourhood of* t_0, *and* $\Gamma(t_0)$ *is a compact set.*

PROOF. (a) Let Q be an open set in X, and let $x_0 \in S_f(t_0) \cap Q$. Since f is open in G, then $f(Q \cap G)$ is an open set and $f(x_0) \in M_f(t_0) \cap f(Q \cap G)$. By Theorem 2.2, M_f is l.s.c. at t_0, thus there is a neighbourhood V of t_0 such that $M_f(t) \cap f(Q \cap G) \neq \varnothing$ for every $t \in V$. Let $y_t \in M_f(t) \cap f(Q \cap G)$ for $t \in V$. Hence there exist elements $x_t \in \Gamma(t)$ and $q_t \in Q \cap G$ such that

$$f(x_t) = y_t = f(q_t) \quad \text{for } t \in V,$$

and one must actually have $x_t = q_t$ for some neighbourhood V_1 of t_0 contained in V because f is one-to-one in G and Γ is u.s.c. at t_0. Thus $x_t \in \Gamma(t) \cap f^{-1}(M_f(t)) \cap Q \cap G$, and therefore $S_f(t) \cap Q \neq \varnothing$ for every $t \in V_1$, which proves the lower semicontinuity of S_f at t_0.

(b) Suppose that S_f is not l.s.c. at t_0. Because of the assumptions imposed on the spaces T and X, this means that there exist a sequence $\{t_n\}_{n=1}^{\infty}$ converging to t_0 and a point $x_0 \in S_f(t_0)$ and its neighbourhood Q such that $S_f(t_n) \cap Q = \varnothing$ for $n = 1, 2, \ldots$. Since Γ is l.s.c. at t_0, we can find $\{x\}_{n=1}^{\infty}$ tending to x_0 such that $x_{n-1} \in \Gamma(t_n)$ for $n > N$. By Lemma 1.1, we get elements $a_n \in \Gamma(t_n)$ with $f(a_n) \in M_f(t_n)$ and $f(a_n) \leqslant f(x_n)$ for $n > N$.

Let $\{U_k\}_{k=1}^{\infty}$, $U_{k+1} \subset U_k$, a local base in X. Consider the open sets

$$G_k = \Gamma(t_0) + U_k \quad \text{for } k = 1, 2, \ldots.$$

By upper semicontinuity at t_0 of Γ we must actually have $\Gamma(t_{n_k}) \subset G_k$ for some subsequence $\{t_{n_k}\}_{n=1}^{\infty}$ of $\{t_n\}_{n=1}^{\infty}$. Thus $a_{n_k} \in C_k$ for $k = 1, 2, \ldots$, and since $\Gamma(t_0)$ is compact, a sequence $\{a_{n_k}\}_{k=1}^{\infty}$ possesses a convergent subsequence. Let $\{a_{n_k}\}_{k=1}^{\infty}$ itself tend to $a_0 \in \Gamma(t_0)$. Let us observe that $a_{n_k}, x_{n_k} \in G$ for $k > K$ because Γ is u.s.c. at t_0. The function f is continuous in G, $f(a_n) \leqslant f(x_n)$ for $n > N$, and so $f(a_0) \leqslant f(x_0)$. Since $f(x_0) \in M_f(t_0)$, one must actually have $f(a_0) = f(x_0)$. Both a_0 and x_0 belong to G, f is one-to-one in G, and so $a_0 = x_0$. Hence $a_{n_k} \in S_f(t_{n_k}) \cap Q$ for some k, which is a contradiction.

Let us note that the assumption given for X in part (b) of Theorem 2.3 is equivalent to the assumption that X is metrizable [12, page 49].

Theorems 2.1 and 2.3 are generalizations of results of Tanino and Sawaragi [17, Theorems 5.2 and 7.2, respectively] for infinite dimensions as well as an extension to a larger class of multifunctions. Tanino and Sawaragi have used slightly

different sequential definitions of semicontinuities of multifunctions which, however, imply our definitions under their assumptions about the spaces (T is such that the concept of convergence is defined in terms of sequences, and Y is a Euclidean space), and with the additional condition imposed on Γ that $\bigcup_{t \in V} \Gamma(t)$ be compact for some neighbourhood V of t_0. For details of the different kinds of semicontinuity of multifunctions see [2, 4, 10, 11, 13]. The following example shows a multifunction Γ with values in a Banach space which fulfills all of the assumptions of Theorem 2.1, but fails to satisfy one of Theorem 5.2 [17].

EXAMPLE 2.2. Let Y be the space of continuous real functions on the interval $[0, 1]$ with the sup norm, ordered by a closed, convex, pointed cone (even with nonempty interior)

$$C = \{ f : f(t) \geqslant 0 \text{ for every } t \in [0, 1] \}.$$

Define a multifunction $\Gamma : \bigcup_{n=1}^{\infty} \{1/n\} \cup \{0\} \to Y$ by

$$\Gamma(0) = \{0\}, \qquad \Gamma\left(\frac{1}{n}\right) = B\left(0, \frac{1}{n}\right) \cap C \quad \text{for } n = 1, 2, \ldots,$$

where $B(0, 1/n)$ denotes a closed ball centred at 0 with radius $1/n$. It is easy to see that Γ satisfies every assumption of Theorem 2.1 except that, for any natural number m, the set

$$\overline{\bigcup_{n \geqslant m} \Gamma\left(\frac{1}{n}\right)} = B\left(0, \frac{1}{m}\right) \cap C$$

is not a compact set, and thus we cannot apply either Theorem 5.2 or Theorem 7.2 of [17] to Γ.

Another remark on the paper of Tanino and Sawaragi [17] is that they consider variations in the domination structure D, which is a generalization of a notion of the order in a partially ordered space. Keeping the notation as in [17], we can extend the definition of $(-C)$-semicompactness to the case of the domination structure D by declaring that $B \subset Y$ is D-semicompact if every open cover of B of the form $\{(y_j + \overline{D[y_j]})^c : y_j \in Y, j \in J\}$ has a finite subcover.

Under the assumption about D in [17], and after replacing $(-C)$-semicompactness by D-semicompactness, Theorem 1.1 as well as our results in this paper remain true, and the proofs are in fact the same. However, this approach is not under consideration here, and for details of this subject the reader is referred to [17].

3. Examples

The following examples show that the assumptions of Theorem 2.1 cannot be weakened.

EXAMPLE 3.1. We define a continuous multifunction Γ with $\Gamma(t)$ closed and bounded for every t and $\Gamma(t_0)$ compact for some t_0, but such that M is not l.s.c. at t_0. Namely, let us take as Y the Banach space of all sequences of real numbers converging to 0, with the sup norm, and let us order it by the pointed, closed, convex cone C of all nonnegative sequences. Define $\Gamma: [0, 1] \to Y$ by

$$\Gamma(t) = \{ y: \|y\| \leqslant t \}.$$

Then

$$M(t) = \min \Gamma(t) = \varnothing \quad \text{for } t \neq 0 \text{ (for details see } [6]),$$

and

$$M(0) = \min \Gamma(0) = \{0\},$$

whence M is not l.s.c. at 0.

EXAMPLE 3.2. We cannot reject the assumption that Γ is u.s.c. at t_0 in Theorem 2.1. Indeed, let us define an l.s.c. multifunction

$$\Gamma: \{t: t \geqslant 0\} \to R^2 \text{ by}$$
$$\Gamma(t) = \begin{cases} I \cup \{(-1/t, 0)\}, & t > 0, \\ I, & t = 0, \end{cases}$$

where $I = \{(-x, x): 0 \leqslant x \leqslant 1\}$. Then Γ is compact valued but is not u.s.c. at 0, and so

$$M(t) = \begin{cases} I, & t = 0, \\ \{(-1/t, 0)\}, & 0 < t \leqslant 1, \\ \{(-x, x): 1 \leqslant x < 1/t\} \cup \{(-1/t, 0)\}, & t > 1, \end{cases}$$

is not l.s.c. at 0.

This example also shows that the minimal value function M_f may not be l.s.c. when f is continuous, and when Γ is l.s.c. and compact valued.

The next example proves that we cannot omit the assumption about $(-C)$-semicompactness for $\Gamma(t)$ in some neighbourhood of t_0 in Theorem 2.1(b).

EXAMPLE 3.3. Let $\Gamma: [0, 1] \to R^2$ be the continuous multifunction defined by

$$\Gamma(t) = \begin{cases} \{(x, y: 1 - x < y \leqslant 2 - x\} \cup \{(1, 0)\}, & 0 < t \leqslant 1, \\ \{(x, y: 1 - x \leqslant y \leqslant 2 - x\}, & t = 0. \end{cases}$$

Then

$$M(t) = \begin{cases} \{(1, 0)\}, & 0 < t \leqslant 1, \\ \{(x, 1 - x): 0 \leqslant x \leqslant 1\}, & t = 0, \end{cases}$$

is not l.s.c. at 0, although $\Gamma(0)$ is compact.

Now we show that in Theorem 2.1(b), M may not be l.s.c. at t_0 when $\Gamma(t_0)$ is not a compact set.

EXAMPLE 3.4. Let Γ: $[0,1] \to R^2$ be the continuous multifunction defined by

$$\Gamma(t) = \{(1, t)\} \cup \left\{\left(-\frac{1}{x}, x\right): t \leqslant x \leqslant 1\right\},$$

$$\Gamma(0) = \{(1, 0)\} \cup \left\{\left(-\frac{1}{x}, x\right): 0 < x \leqslant 1\right\}.$$

Then $\Gamma(t)$ is $(-R_+^2)$-semicompact for every $t \neq 0$, $\Gamma(0)$ is closed, and min $\Gamma(0) \neq \varnothing$, but

$$M(t) = \begin{cases} \{(1, 0)\}, & t = 0, \\ \left\{\left(-\frac{1}{t}, t\right)\right\}, & t \neq 0, \end{cases}$$

is not l.s.c. at 0.

As a final example, it will be recalled that we cannot obtain lower semicontinuity of S_f when f is a continuous function, even when f is linear and real valued, and Γ is convex and compact valued. Indeed, let us define a continuous multifunction Γ: $[0,1] \to R^2$ by

$$\Gamma(t) = \begin{cases} I \cap B(t), & t \neq 0, \\ I, & t = 0, \end{cases}$$

where $B(t)$ denotes the closed ball centred at $(0, 1/t)$ and with radius $1/t$ for $t \neq 0$, and where $I = \{(x, y): x, y \in [0, 1]\}$.

Let f: $R^2 \to R^1$ be defined by $f(x, y) = y$ for $(x, y) \in R^2$. Then $M_f(t) = 0$ for every t, and

$$S_f(t) = \{(x, y) \in \Gamma(t): y = 0\}$$

$$= \begin{cases} \{(0, 0)\}, & t \neq 0, \\ \{(x, 0): 0 \leqslant x \leqslant 1\}, & t = 0, \end{cases}$$

is not l.s.c. at $t = 0$.

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