Almost Simplicial Polytopes: The Lower and Upper Bound Theorems

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Abstract. We study *n*-vertex *d*-dimensional polytopes with at most one nonsimplex facet with, say, $d + s$ vertices, called *almost simplicial polytopes*. We provide tight lower and upper bound theorems for these polytopes as functions of *d*, *n*, and *s*, thus generalizing the classical Lower Bound Theorem by Barnette and the Upper Bound Theorem by McMullen, which treat the case where $s = 0$. We characterize the minimizers and provide examples of maximizers for any *d*. Our construction of maximizers is a generalization of cyclic polytopes, based on a suitable variation of the moment curve, and is of independent interest.

1 Introduction

In 1970, McMullen [19] proved the Upper Bound Theorem (UBT) for *simplicial polytopes*, polytopes with each facet being a simplex, while between 1971 and 1973, Barnette [4, 5] proved the Lower Bound Theorem (LBT) for the same polytopes. Both results are major achievements in the combinatorial theory of polytopes; see, for example, the books [13, 27] for further details and discussion.

These results can be phrased as follows: let $C(d, n)$ (resp. $S(d, n)$) denote a cyclic (resp. stacked) *d*-polytope on *n* vertices, and, for a polytope *P*, let $f_i(P)$ denote the number of its *i*-dimensional faces. Then the classical LBT and UBT read as follows.

Theorem 1.1 (Classical LBT and UBT) *For any simplicial d-polytope P on n vertices, and any* $0 \le i \le d - 1$,

$$f_i\big(S(d, n)\big) \le f_i(P) \le f_i\big(C(d, n)\big).$$

The numbers $f_i(S(d, n))$ and $f_i(C(d, n))$ are explicit known functions of (d, n, i), to be discussed later.

We generalize the UBT and LBT to the following context: consider a pair (P, F) where *P* is a polytope, *F* is a facet of *P*, and all facets of *P* are different from *F* are simplices. We call such a polytope *P* an *almost simplicial polytope* (ASP) and a pair (P, F) an *ASP-pair*. Since every ridge of the facet *F* is shared with another facet of *P*,

the facet F is necessarily simplicial. We will be interested only in the combinatorics of P.

Let $\mathcal{P}(d, n, s)$ denote the family of d-polytopes P on n-vertices such that (P, F) is an ASP-pair, where F has $d + s$ vertices ($s \geq 0$). Note that $\mathcal{P}(d, n, 0)$ consists of the simplicial d-polytopes on n vertices. In this paper, we define certain polytopes $C(d, n, s), S(d, n, s) \in \mathcal{P}(d, n, s)$, explicitly compute their face numbers, and prove the following theorem.

Theorem 1.2 (LBT and UBT for ASP)   *For any d, n, s, any polytope $P \in \mathcal{P}(d, n, s)$, and any $0 \leq i \leq d - 1$,*

$$f_i\big(S(d, n, s)\big) \leq f_i(P) \leq f_i\big(C(d, n, s)\big).$$

Further, for $d \geq 4$, the polytopes $P \in \mathcal{P}(d, n, s)$ with $f_i(P) = f_i(S(d, n, s))$ for some $1 \leq i \leq d - 1$ are characterized combinatorially and satisfy the above equality for all $0 \leq i \leq d - 1$.

The characterization of the equality case above generalizes Kalai's result [14] that for $d \geq 4$ equality in the classical LBT holds for some $1 \leq i \leq d - 1$ if and only if P is stacked. The polytopes $C(d, n, s)$ form an ASP analog of cyclic polytopes and satisfy a combinatorial Gale-evenness type description of their facets.

The combinatorics of P could also be understood by looking at certain triangulations of F. Consider an ASP-pair (P, F). Projectively transform P into a combinatorially equivalent polytope such that the orthogonal projection of $\mathbb{R}^d$ onto the hyperplane spanned by the facet F maps $P \setminus F$ into the relative interior of F; see [27, Ex. 2.18]. Think of F as sitting in $(\mathbb{R}^{d-1}, 0)$. Under this setting, the facet F admits a regular triangulation $\mathcal{C}$ that is obtained by a lifting of the vertices of $\mathcal{C}$ that leaves the vertices of F fixed; see [12, Sec. 17.3]. Then the polytope P becomes the convex hull of the lifted vertices of $\mathcal{C}$. Consequently, specifying the ASP-pair (P, F) amounts to specifying the aforementioned triangulation of F.

We pay special attention to simplicial balls of the form $P' := \partial P \setminus \{F\}$, which are a subfamily of certain balls considered by Billera and Lee [7] in their study of polytope pairs. In particular, their results give tight upper and lower bound theorems for the face numbers of simplicial $(d - 1)$-dimensional balls of the *polytope-antistar* form, that is, balls of the form $\partial Q \setminus \{v\}$, or $\partial Q \setminus v$ for simplicity, where Q is a simplicial d-polytope and v is a vertex of Q that is deleted. These bounds are given as functions of $d, f_0(\partial Q \setminus v), f_0(Q/v)$, where Q/v denotes the vertex figure of v in Q. For an ASP-pair (P, F), let Q be obtained from P by stacking a pyramid over F with a new vertex v; the stacking operation is defined in detail in Section 3. Then $F \cong Q/v$ and $P' = \partial P \setminus \{F\} = \partial Q \setminus v$. Thus, our balls P' form a subfamily of the balls $\partial Q \setminus v$ considered in [7]. The bounds we obtain in Theorem 1.2 are strictly stronger than those of [7], which apply to all polytope-antistar balls.

Let $f(P) = (1, f_0(P), f_1(P), \ldots, f_{d-1}(P))$ denote the *f-vector* of P, a vector recording the face numbers of P. The following problem naturally arises.

Problem 1.3   *For ASP-pairs (P, F), characterize the pairs of f-vectors $(f(P), f(F))$.*

A solution to the problem above would generalize the well-known g-theorem characterizing the face numbers of simplicial polytopes, conjectured by McMullen [20] and proved by Billera–Lee [6] (sufficiency) and Stanley [24] (necessity); the g-theorem solves the case $s = 0$ and provides some restrictions when $s > 0$. We leave this general problem to a future study. We remark that for the corresponding problem for the larger family of polytope pairs [7], there is currently no conjectured characterization, after Kolins' [18] counterexamples to the characterization conjectured by Billera and Lee [7].

The proof of the LBT for ASP and the characterization of the equality cases are based on framework-rigidity arguments (*cf.* Kalai [14]) and on an adaptation of the well-known McMullen–Perles–Walkup reduction (MPW reduction) [14, Sec. 5] to ASP; see Section 3.

The numerical bounds obtained in the UBT for ASP are a special case of a recent result of Adiprasito and Sanyal [1, Thm. 3.9], who proved the bounds for homology balls whose boundary is an induced subcomplex. While their proof relies on machinery from commutative algebra, our proof is elementary and is based on a suitable shelling of P. Further, our construction of maximizers $C(d, n, s)$ is a generalization of cyclic polytopes, based on a suitable variation of the moment curve, and is of independent interest; see Section 4.

Our proof techniques are likely to extend beyond ASP to polytopes P where all non-simplex proper faces $F_1, \ldots, F_n$ are facets, but probably not beyond that, as a key fact that we use in this paper, and probably will need in the extended setting, is that the f-vector of P can be recovered from the f-vector of $P' := \partial P \setminus \{F_1, \ldots, F_n\}$ and n.

2 Preliminaries

For undefined terminology and notation, see [27] for polytopes and complexes or [14, Sec. 2] for framework rigidity.

2.1 Polytopes and Simplicial Complexes

The k-dimensional faces of a polyhedral complex Δ are called k-faces, where the empty face has dimension -1. For a simplicial complex Δ of dimension $d - 1$, the numbers $f_k(\Delta)$ are then related to the h-numbers $h_k(\Delta) := \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}(\Delta)$ by

$$(2.1) \qquad f_{k-1}(\Delta) = \sum_{i=0}^{k} \binom{d-i}{k-i} h_i(\Delta).$$

The h-vector of Δ, $(\ldots, h_k, h_{k+1}, \ldots)$, can be considered as an infinite sequence if we let $h_k(\Delta) = 0$ for $k > d$ and $k < 0$. The g-numbers are defined by $g_k(\Delta) = h_k(\Delta) - h_{k-1}(\Delta)$.

For an ASP pair (P, F), where P is d-dimensional, the following version of the Dehn–Somerville equations applies to the complex $P' = \partial P \setminus \{F\}$.

Proposition 2.1 ([12, Thm. 18.3.6], Dehn–Somerville Equations for P') *The h-vector of the simplicial $(d-1)$-ball P' with boundary ∂F satisfies for $k = 0, \ldots, d$,*

$$(2.2) \qquad\qquad h_k(P') = h_{d-k}(P') + g_k(\partial F).$$

Note that $h_k(P') = 0$ and $h_k(\partial F) = 0$ for $k \geq d$ and $h_{d-1}(\partial F) = 1$.

We proceed with a number of definitions related to simplicial complexes. Let 2^A denote the simplicial complex generated by the set A; it is a simplex. Sometimes we abbreviate this complex by A when the context is clear. Say Δ is *pure* if all its maximal faces, called *facets*, have the same dimension, and a pure simplicial complex Δ is *shellable* if its facets can be ordered $F_1, F_2, \ldots$, such that for each $j > 1$, 2^{F_j} intersects the complex $\bigcup_{i<j} 2^{F_i}$ in a pure codimension 1 subcomplex of 2^{F_j}. Such an order is called a *shelling order* or *shelling process* of Δ. For a shelling order, the set of faces $2^{F_j} \setminus \bigcup_{i<j} 2^{F_i}$ has a unique minimal element, called the *restriction face* of F_j, denoted R_j. For *any* shelling of Δ, $h_i(\Delta)$ equals the number of facets in the shelling whose restriction face has size i; *cf.* [27, Thm. 8.19]. Note that P' is shellable, by a Bruggesser–Mani line shelling [11, Prop. 2].

The *link* of a face F in the simplicial complex Δ is

$$\mathrm{link}_\Delta(F) := \big\{ T \in \Delta : T \cap F = \varnothing, F \cup T \in \Delta \big\},$$

and its *star*, $\mathrm{star}_\Delta(F)$ is the complex $\bigcup_{F \subseteq T} 2^T$. Thus, using the *join* operator on simplicial complexes, we obtain $2^F * \mathrm{link}_\Delta(F) = \mathrm{star}_\Delta(F)$. For a general polyhedral complex, the star of a face F is the polyhedral subcomplex formed by all faces containing F, and their faces. For a vertex v in a polytope Q, its *vertex figure* Q/v is a codimension 1 polytope obtained by intersecting Q with a hyperplane H *below* v; that is, v is on one side of H and the other vertices of Q are on the other side. If $\mathrm{star}_Q(v)$ is simplicial, then the boundary complex of Q/v coincides with $\mathrm{link}_Q(v)$.

A subcomplex K of Δ is *induced* if it contains all the faces in Δ that only involve vertices in K. Note that, for an ASP-pair (P, F), ∂F is an induced subcomplex of P', by convexity.

The underlying set $|\mathcal{C}|$ of a polyhedral complex $\mathcal{C}$ is the point set $\bigcup_{Q \in \mathcal{C}} Q$ of its geometric realization. A *refinement* (or subdivision) of $\mathcal{C}$ is another polyhedral complex $\mathcal{D}$ such that $|\mathcal{D}| = |\mathcal{C}|$, and for any face $F \in \mathcal{D}$, there exists a face $T \in \mathcal{C}$ such that $|F| \subseteq |T|$.

A simplicial complex Δ is a *homology sphere* (over a fixed field $\mathbf{k}$) if for any face $F \in \Delta$, the reduced homology groups

$$\widetilde{H}_i(\mathrm{link}_\Delta(F); \mathbf{k}) \cong \widetilde{H}_i(S^{\dim \Delta - \dim F - 1}; \mathbf{k})$$

for all i, where S^j is the j-dimensional sphere. Say Δ is a *homology ball* if $\widetilde{H}_i(\mathrm{link}_\Delta(F);$ $\mathbf{k})$ vanishes for $i < \dim \Delta - \dim F - 1$ and is isomorphic to either 0 or $\mathbf{k}$ for $i = \dim \Delta - \dim F - 1$. Furthermore, the boundary complex $\partial \Delta$ of Δ, consisting of all faces F for which $\widetilde{H}_{\dim \Delta - \dim F - 1}(\mathrm{link}_\Delta(F); \mathbf{k}) = 0$, is a homology sphere (of codimension 1). In particular, simplicial spheres (resp. balls) are homology spheres (resp. balls). For the ASP-pair (P, F), the complex $P' := \partial P \setminus \{F\}$ is a shellable simplicial $(d-1)$-ball and its boundary complex $\partial P'$, which coincides with ∂F, is a homology sphere.

2.2 Rigidity

We mostly follow the presentation in Kalai's paper [14]. Let $G = (V, E)$ be a graph, and $\text{dist}(a, b)$ denote Euclidean distance between points a and b in a Euclidean space. A d-embedding $\alpha : V \to \mathbb{R}^d$ is called *rigid* if there exists an $\varepsilon > 0$ such that if $\beta : V \to \mathbb{R}^d$ satisfies $\text{dist}(\alpha(v), \beta(v)) < \varepsilon$ for every $v \in V$ and $\text{dist}(\beta(u), \beta(w)) = \text{dist}(\alpha(u), \alpha(w))$ for every $\{u, w\} \in E$, then $\text{dist}(\beta(u), \beta(w)) = \text{dist}(\alpha(u), \alpha(w))$ for every $u, w \in V$. The graph G is said to be *generically d-rigid* if the set of its rigid d-embeddings is open and dense in the metric vector space of all of its d-embeddings. Given a d-embedding $\alpha : V \to \mathbb{R}^d$, a *stress* on α is a function $w : E \to \mathbb{R}$ such that for every vertex $v \in V$,

$$\sum_{u:\{v,u\}\in E} w(\{v, u\})(\alpha(v) - \alpha(u)) = 0.$$

The stresses on α form a vector space, called the *stress space*. Its dimension is the same for all generic d-embeddings (namely, for an open and dense set in the space of all d-embeddings of G). A graph G is called *generically d-stress free* if this dimension is zero.

If a generic $\alpha : V \to \mathbb{R}^d$ is rigid, then $f_1(G) \geq df_0(G) - \binom{d+1}{2}$ [2, Sec. 3]. Thus, if Δ is a simplicial complex of dimension $d - 1$ whose 1-skeleton is generically d-rigid, then $f_1(\Delta) \geq df_0(\Delta) - \binom{d+1}{2}$, and $g_2(\Delta)$ is the dimension of the stress space of any generic embedding. Based on these observations for Δ the boundary of a simplicial d-polytope with $d \geq 3$, and more general complexes, Kalai [14] extended the LBT and characterized the minimizers.

For a d-polytope P with a simplicial 2-skeleton, the so-called toric $g_2(P)$ coincides with $g_2(\partial P) = f_1(P) - df_0(P) + \binom{d+1}{2}$. By a result of Alexandrov (*cf.* Whiteley [26]), the toric $g_2(P)$ equals the dimension of the stress space of the 1-skeleton of P.

For our LBT for ASP, we will need the following very special case of Kalai's monotonicity,[1] which Kalai proved using rigidity arguments.

Theorem 2.2 (Kalai's Monotonicity [15, Thm. 4.1], weak form)   *Let $d \geq 4$, let P be a d-polytope with a simplicial 2-skeleton, and let F be a facet of P. Then*

$$g_2(P) \geq g_2(F).$$

Equivalently, $f_1(P) - f_1(F) \geq (df_0(P) - \binom{d+1}{2}) - ((d-1)f_0(F) - \binom{d}{2})$.

3 A Lower Bound Theorem for Almost Simplicial Polytopes

Before proving the lower bound theorem, we give several definitions that we use in this section.

Let G be a proper face of a polytope Q. A point w is *beyond* G (with respect to Q) if (i) w is not on any hyperplane supporting a facet of Q, (ii) w and the interior of Q lie on different sides of any hyperplane supporting a facet containing G, but (iii) lie on

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[1]Kalai's monotonicity conjecture on the toric g-polynomials, asserting that $g(P) \geq g(F)g(P/F)$ coefficientwise for any face F of P, was first proved for rational polytopes by Braden and MacPherson [9]. Later, using the theory of combinatorial intersection homology, Braden [8] proved Kalai's conjecture in its full generality.

the same side of every other facet-defining hyperplane that does not contain G. For an ASP-pair (P, F), we will consider the simplicial polytope Q obtained as the convex hull of P and a vertex y beyond F.

A polytope is *k-simplicial* if each k-face is a simplex; a $(d-1)$-simplicial d-polytope is simply a simplicial d-polytope. A simplicial d-polytope is called *stacked* if it can be obtained from a d-simplex by repeated *stacking*, namely, adding a vertex beyond a facet and taking the convex hull. While stacked d-polytopes on n vertices, denoted $S(d, n)$, may have different combinatorial structures, they all have the same f-vector, given by

$$f_k\big(S(d, n)\big) = \phi_k(d, n) := \begin{cases} \binom{d}{k}n - \binom{d+1}{k+1}k & \text{for } k = 1, \dots, d-2, \\ (d-1)n - (d+1)(d-2) & \text{for } k = d-1. \end{cases}$$

A homology sphere is *stacked* if it is combinatorially isomorphic to the boundary complex of a stacked polytope.

For any integers $d \geq 3$, $s \geq 0$, and $n \geq d + s + 1$, let F be a stacked $(d-1)$-polytope with $d + s$ vertices. Construct a pyramid over F and then stack $n - d - s - 1$ times over facets of the resulting polytope that are different from F to obtain a polytope $S(d, n, s)$ in $\mathcal{P}(d, n, s)$. One easily computes the f-vector of $S(d, n, s)$, since refining F by its (unique) stacked triangulation refines the boundary complex of $S(d, n, s)$ to a stacked simplicial sphere with f-vector $f(S(d, n))$. We obtain

$$f\big(S(d, n, s)\big) = f\big(S(d, n)\big) - (0, 0, \dots, 0, s, s).$$

We are ready to state the LBT for ASP (Theorem 3.1); its minimizers will be characterized later (see Theorems 3.3 and 3.5). In the proof of Theorem 3.1 we rely on the MPW reduction, which states that if the result is true for the edges of a simplial polytope, then it is true for all faces of all dimensions of the polytope. This reduction is clearly explained in the proof of [5, Thm. 1]. For almost simplicial polytopes, the same reasoning shows that if the result is true for edges, it is true for all faces of dimensions at most $d - 3$; see also [14, Sec. 5].

Theorem 3.1 (LBT for ASP)　*Let $d \geq 3$, $s \geq 0$, $n \geq d + s + 1$. Then for any $P \in \mathcal{P}(d, n, s)$ and $1 \leq i \leq d - 1$, we have*

$$f_i\big(S(d, n, s)\big) \leq f_i(P).$$

Proof　We proceed by induction on d, with the case $d = 3$ as the basis. For $d = 3$ and $n \geq s + 4$, any $P \in \mathcal{P}(3, n, s)$ has f-vector

$$f(P) = (1, n, 3n - 6 - s, 2n - 4 - s) = f(S(3, n, s)).$$

Let $d \geq 4$.

As P is 2-simplicial, by a result of Whiteley [26, Thm. 8.6], the 1-skeleton of P is generically d-rigid, hence $f_1(P) \geq \phi_1(d, n)$, and by the MPW reduction (as usual, counting pairs (v, A) such that v is a vertex in an i-face A), $f_i(P) \geq \phi_i(d, n)$ for all $2 \leq i \leq d - 3$ as well; see [14, Thm. 12.2].[2] The rest of the proof will deal with the cases $i = d - 2, d - 1$.

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[2]Kalai's theorem contains a typo. It includes the case $i = k$, while it holds only for $i < k$, where P is k-simplicial. Our ASP P is $(d - 2)$-simplicial.

Denote by (P, F) the ASP-pair, and by $\deg_P(v)$ the degree of a vertex v in the 1-skeleton of P. We now prove the inequality for the facets, by a variation of the MPW reduction. Note that the vertex figure P/v in P of any vertex $v \in \text{vert } F$ is an ASP (with $\deg_P(v)$ vertices), while for any vertex $v \in \text{vert } P \setminus \text{vert } F$, P/v is a simplicial polytope; *cf.* [10, Thm. 11.5]. Furthermore, for a vertex $v \in \text{vert } F$, letting $s_v := \deg_F(v) - (d-1) \geq 0$ gives $P/v \in \mathcal{P}(d-1, \deg_P(v), s_v)$.

Double counting the number of pairs (v, A) for a vertex v in a facet A of P, we obtain the following inequalities:

$$
\begin{aligned}
d\big(f_{d-1}(P) - 1\big) &+ (d + s) \\
&= \sum_{v \in \text{vert } P} f_{d-2}\big(\text{link}_P(v)\big) \\
&\geq \sum_{v \in \text{vert } P \setminus \text{vert } F} \big((d-2)\deg_P(v) - d(d-3)\big) \\
&\quad + \sum_{v \in \text{vert } F} \big((d-2)\deg_P(v) - d(d-3) - s_v\big) \\
&= 2(d-2)f_1(P) - d(d-3)f_0(P) - 2f_1(F) + (d-1)(d+s) \\
&\geq 2(d-2)\Big[df_0(P) - \binom{d+1}{2} \Big] - d(d-3)f_0(P) \\
&\quad - 2\Big[(d-1)f_0(F) - \binom{d}{2} \Big] + (d-1)(d+s) \\
&= d(d-1)f_0(P) - d(d+1)(d-2) - s(d-1),
\end{aligned}
$$

where the first inequality is by the induction hypothesis, and the second inequality is by Kalai's monotonicity Theorem 2.2 and the LBT inequality for $f_1(P)$. Comparing the LHS with the RHS gives

$$
f_{d-1}(P) \geq \phi_{d-1}(d, n) - s.
$$

The inequality for $f_{d-2}(P)$ follows from the inequality for $f_{d-1}(P)$ by double counting. Since any ridge in P is contained in exactly two facets, counting the number of pairs (R, A) for a ridge R in a facet A of P, we obtain that

$$
2f_{d-2}(P) = d\big(f_{d-1}(P) - 1\big) + f_{d-2}(F).
$$

Applying the classical LBT to the simplicial $(d-1)$-polytope F with $f_0(F) = d + s$, we get

$$
2f_{d-2}(P) \geq d\big(f_{d-1}(P) - 1\big) + (d-2)(d+s) - d(d-3),
$$

and applying the lower bound for $f_{d-1}(P)$ yields, after dividing both sides by 2, the desired lower bound $f_{d-2}(P) \geq \phi_{d-2}(d, n) - s$. ∎

We now turn our attention to characterizing the minimizers of Theorem 3.1. We start with some terminology and background.

A proper subset A of the vertices of a d-polytope P is called a *missing k-face* of P if the cardinality of A is $k + 1$, the simplex on A is not a face of P, but for any proper subset B of A the simplex on B is a face of P. If A is a missing $(d-1)$-face of P (a.k.a. *missing facet*), then adding the simplex A cuts P into two d-polytopes P_1, P_2, glued along the simplex A. We denote this operation by $P = P_1 \# P_2$. Repeating this

procedure on each P_i until no piece P_i contains a missing $(d-1)$-face results in a decomposition $P = P_1 \# P_2 \# \cdots \# P_t$, where intersections along missing $(d-1)$-faces of P define the edges of a tree whose vertices are the P_i's. Call such a decomposition the *prime decomposition* of P, and call each P_i a *prime factor* of P. See [15, Sec. 3.8]. For $d \geq 3$, a prime decomposition of P as above is uniquely defined; this statement follows from the following simple observation.

Lemma 3.2   *Let P be a d-polytope, $d \geq 3$. Then the intersection of any two missing facets of P contains no interior point of P.*

Proof   Let A and B be two missing facets of a P and assume by contradiction that v is an interior point in the intersection of A and B. Denote by H_A (resp. H_B) the unique hyperplane containing A (resp. B). As $d \geq 3$, the intersection of H_A and H_B contains a line ℓ through v. Consider the intersection $\ell \cap A \cap B$ and denote this segment by $[u, w]$. Then v is interior to $[u, w]$. If u is not on the boundary ∂A of A, it must be on ∂B, but the first condition says that u is interior to P, while the second condition says that u is in ∂P, a contradiction. Thus, both u and w are in $\partial A \cap \partial B$.

Let F_u be the minimal face of ∂P containing u, and define F_w similarly. Both F_u and F_w are proper faces of both A and B. As v is interior to A, vert $F_u \cup$ vert $F_w =$ vert A. But then vert $A \subseteq$ vert B, implying that $A = B$, a final contradiction.   ∎

By virtue of Lemma 3.2, for $d \geq 3$, we denote by Δ_P the polyhedral complex defined by *the* prime decomposition of a d-polytope P. Then a simplicial d-polytope P is stacked if and only if all its prime factors are d-simplices. This definition immediately extends to polyhedral spheres where the operation # corresponds to the topological connected sum.

We start with the characterization of the minimizers for the easier case $d > 4$.

Theorem 3.3 (Characterization of minimizers for $d > 4$)   *Let $d > 4$ and $P \in \mathcal{P}(d, n, s)$. Let Δ_F be the polyhedral complex corresponding to the prime decomposition of F, and let Δ be the refinement of the boundary complex ∂P of P obtained by refining F by Δ_F. Assume there is some $1 \leq i \leq d-1$ for which $f_i(P) = f_i(S(d, n, s))$. Then all prime factors in the prime decomposition of Δ are d-simplices. In particular, $f(P) = f(S(d, n, s))$.*

Remark 3.4   Let Q be a polytope, G a facet of Q, and H the hyperplane containing G. An *H-stacking* on Q is the operation of (i) adding a new vertex w in H, beyond a facet of G (with respect to G) such that perturbing w from H to the side of the interior of Q makes w beyond a facet of Q, and (ii) taking the convex hull of w and Q. The minimizers considered in Theorem 3.3 are precisely the polytopes that can be obtained by the following recursive procedure: start with a d-simplex having a facet in a hyperplane H, and repeatedly either H-stack or (usual) stack over a facet not in H.

Clearly this procedure produces a prime decomposition of Δ of Theorem 3.3. Conversely, consider the rooted tree corresponding to the prime decomposition of Δ. Starting with the root, add vertices one by one so that the resulting induced forest is always a tree. Such ordering induces a recursive procedure as above.

Proof of Theorem 3.3 By the classical MPW reduction for $1 \leq i \leq d - 3$ and the variation of it we used in the proof of Theorem 3.1 for $d - 2 \leq i \leq d - 1$, equality for some $1 \leq i \leq d - 1$ implies equality for $i = 1$, so it is enough to consider the case $i = 1$. From Kalai's monotonicity (Theorem 2.2) and our assumption $g_2(P) = 0$, it follows that $g_2(F) = 0$. As F is simplicial of dimension ≥ 4, Kalai's [14, Thm. 1.1(ii)] says that F is stacked, thus Δ is a simplicial $(d-1)$-sphere. Since $g_2(\Delta) = 0$, by [14, Thm. 1.1(ii)] again, Δ is stacked, as desired.

In particular, as $f(P) = f(\Delta) - (0, \ldots, 0, s, s) = f(S(d, n)) - (0, \ldots, 0, s, s)$, we conclude that $f(P) = f(S(d, n, s))$. ∎

For $d = 4$, F need not be stacked. For example, the pyramid over any simplicial 3-polytope is a minimizer. We obtain the following characterization of minimizers.

Theorem 3.5 (Characterization of minimizers for $d = 4$) *Let $P \in \mathcal{P}(4, n, s)$, and keep the notation of Theorem 3.3. Assume there is some $1 \leq i \leq d - 1$ for which $f_i(P) = f_i(S(d, n, s))$. Then the prime factors in the prime decomposition of Δ are either d-simplices with no facet contained in $|F|$ or pyramids over prime factors of F.*

In order to prove this theorem, we first need to show generic d-rigidity for the 1-skeleton of a much larger class of complexes.

Let $\mathcal{C}_k$ be the family of homology k-balls Δ such that:

- the induced subcomplex $\Delta[I]$ on the set I of internal vertices has a connected 1-skeleton, and
- for any edge e in the boundary complex $\partial \Delta$, there exists a 2-simplex T, $e \subset T$, such that T has a vertex in I.

Note that any homology k-ball Δ whose boundary $\partial \Delta$ is an induced subcomplex is in $\mathcal{C}_k$.[3] Indeed, as $\partial \Delta$ is induced, any facet of Δ intersects I nontrivially. Assume by contradiction that the graph $\Delta[I]_{\leq 1}$ is disconnected, and say equals the disjoint union of nontrivial graphs G_1 and G_2. As Δ is facet-connected, it has facets F_1 and F_2 whose intersection S has codimension 1 and F_i has a vertex in G_i for $i = 1, 2$. Then S is disjoint from I, so $S \in \partial \Delta$. This is a contradiction, as S is contained in two facets of Δ, not in one.

In particular, for $P \in \mathcal{P}(d, n, s)$, the simplicial complex $P' = \partial P \setminus \{F\}$ is in $\mathcal{C}_{d-1}$.

Lemma 3.6 *Let $d \geq 4$. The 1-skeleton of any $\Delta \in \mathcal{C}_{d-1}$ is generically d-rigid. Thus, $f_1(\Delta) \geq d f_0(\Delta) - \binom{d+1}{2}$.*

Proof The proof follows from Kalai's proof of the classical LBT. Apply [14, Prop. 6.4] where the tree T is a spanning tree of the 1-skeleton of $\Delta[I]$. ∎

Proof of Theorem 3.5 Consider a prime factor L of Δ. Then L is a 4-polytope that is 2-simplicial, so it has a generically 4-rigid 1-skeleton by [26, Thm. 8.6]. As $g_2(P) = 0$, the 1-skeleton of L, denoted by G, must be generically 4-stress free. Thus, $g_2(L) = 0$.

If L does not contain a facet in Δ_F, then L is simplicial, with $g_2(L) = 0$, and hence is stacked by [14, Thm. 1.1]. Also, being prime, L is a 4-simplex.

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[3]This follows from Alexander duality; however, we preferred to give an easier argument.

Assume then that L contains a facet F'' contained in $|F|$, so (L, F'') is an ASP-pair. If L has a unique vertex outside F'', then L is a pyramid over a prime factor of F, and we are done. Assume the contrary, so there is an edge $vu \in G$ with $v, u \notin F''$ (for concreteness, taking v, u to be the highest two vertices of L above the hyperplane of F works).

First, we show that vu satisfies the link condition $\mathrm{link}_L(v) \cap \mathrm{link}_L(u) = \mathrm{link}_L(vu)$, which guarantees that contracting the edge vu in the simplicial complex $\partial L \setminus \{F''\}$ results in $\widetilde{\Delta} \in \mathcal{C}_3$; see *e.g.,* [23, Prop.2.4].[4] Indeed, if vu fails the link condition, it means that vu is contained in a missing face M, with 3 or 4 vertices. Now, M cannot have 4 vertices, as L is prime. If $M = vuz$, then uz is an edge of L *not* in $\mathrm{link}_L(v)$. Since $\mathrm{link}_L(v)$ is a homology 2-sphere (thus, a simplicial 2-sphere), its 1-skeleton is generically 3-rigid. Consequently, the 1-skeleton of $\mathrm{star}_L(v)$ is generically 4-rigid, and adding uz to it yields a 4-stress in G, a contradiction.

Let m be the number of vertices in the cycle $\mathrm{link}_L(vu)$; then $f_1(\widetilde{\Delta}) = f_1(L) - m - 1$ and $f_0(\widetilde{\Delta}) = f_0(L) - 1$, which implies that $g_2(L) = g_2(\widetilde{\Delta}) + (m - 3)$.

If $m > 3$, then applying Lemma 3.6 to $\widetilde{\Delta}$ yields $g_2(L) > 0$, a contradiction. So assume $m = 3$.

Denote by x, y, z the vertices of $\mathrm{link}_L(vu)$. If the triangle $xyz \in L$, then, as L is prime, both tetrahedra $xyzv$ and $xyzu$ are faces of L, so L is the 4-simplex $xyzuv$, a contradiction (as it has a facet F'' in F).

We are left to consider the case $xyz \notin L$. The argument here is inspired by Barnette [4, Thm. 2]. In this case, the 3-ball formed by the join $vu * \partial(xyz)$ is an induced subcomplex of $\partial L \setminus \{F''\}$. Now replace it by $\partial(vu) * xyz$ (this is a bistellar move) to obtain from $\partial L \setminus \{F''\}$ the complex Δ''. Clearly, Δ'' is a homology 3-ball, and any edge on its boundary is part of a 2-simplex with an internal vertex (just take the same one as in $\partial L \setminus \{F''\}$). To show $\Delta'' \in \mathcal{C}_3$, we are left to show that the graph on the internal vertices I of Δ'' is connected. Assume not, namely removing the edge uv disconnects the induced graph on I in $\partial L \setminus \{F''\}$. In particular, $x, y, z \in F''$. But $xyz \notin L$, so xyz is a missing face of F'', contradicting that F'' is a prime factor of F.

We conclude that $\Delta'' \in C_3$; thus, by Lemma 3.6, $\Delta'' \cup \{vu\}$ has a nonzero 4-stress. However, the 1-skeletons of $\Delta'' \cup \{vu\}$ and of L are equal graphs, so $g_2(L) > 0$, a contradiction. The proof is complete.   ∎

4   An Upper Bound Theorem for Almost Simplicial Polytopes

Throughout this section, we let $P \in \mathcal{P}(d, n, s)$ denote an almost simplicial polytope, (P, F) the ASP-pair, and $P' = \partial P \setminus \{F\}$ the corresponding shellable simplicial $(d-1)$-ball. Recall that $\partial P' = \partial F$ is an induced subcomplex of P'.

4.1   ASP Generalization of Cyclic Polytopes

The *moment curve* in $\mathbb{R}^d$ is defined by $t \mapsto (t, t^2, \ldots, t^d)$ for $t \in \mathbb{R}^d$, and the convex hull of any n points on it gives, combinatorially, the cyclic polytope $C(d, n)$; see, for instance, [27, Example 0.6]. We extend this construction by considering curves $x(t)$

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[4]To apply [23, Prop. 2.4], phrased for homology spheres, simply cone the boundary of the homology ball Δ to obtain a homology sphere.

of the form $(t, t^2, \ldots, t^{d-r}, p_1(t), \ldots, p_r(t))$, where $p_i(t)$ is a continuous function in t for $i = 1, \ldots, r$. Later, a special choice of the curve $x(t)$ and points on it will give, by taking the convex hull, our maximizer polytope $C(d, n, s)$.

We let $V(t_1, \ldots, t_l)$ denote the *Vandermonde determinant* on variables $t_1, \ldots, t_l$:

$$V(t_1, \ldots, t_l) := \begin{vmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_l \\ t_1^2 & t_2^2 & \cdots & t_d^l \\ \vdots & \vdots & \cdots & \vdots \\ t_1^{l-1} & t_2^{l-1} & \cdots & t_l^{l-1} \end{vmatrix} = \prod_{1 \le i < j \le l} (t_j - t_i).$$

Recall that a polytope is *k-neighborly* if each subset of at most k vertices forms the vertex set of a face. A $\lfloor d/2 \rfloor$-neighborly d-polytope is simply called *neighborly*.

Lemma 4.1 *Consider the curve $x(t)$. Then the following hold:*

(i) *Any $d - r + 1$ points on the curve $x(t)$ are affinely independent.*
(ii) *For any n distinct numbers $t_1, \ldots, t_n$, the polytope*

$$Q = \text{conv}\left(\{x(t_1), \ldots, x(t_n)\}\right)$$

   is $(d - r - 1)$-simplicial.
(iii) *The polytope Q is $\lfloor (d - r)/2 \rfloor$-neighborly.*

Proof Consider n real numbers $t_1 < \cdots < t_n$ and the corresponding points $x(t_i)$. From any $d - r + 1$ points $x(t_{i_1}), \ldots, x(t_{i_{d-r+1}})$, we form a matrix by considering them as its columns, in this same order, and adding a row of ones as the first row. The top $(d-r+1) \times (d-r+1)$ square submatrix of this matrix has determinant $V(t_{i_1}, \ldots, t_{i_{d-r+1}})$, which is nonzero, which in turn implies the first assertion. The second assertions follows immediately from the first.

To prove the third assertion proceed as in [13, Sec. 4.7]. Consider a set $S_k = \{x(t_{i_j}) : j = 1, \ldots, k\}, 1 \le i_j \le n$, with $k \le \lfloor (d - r)/2 \rfloor$, and the polynomial

$$\beta(t) = \prod_{i=1}^{k} (t - t_{i_j})^2 = \beta_0 + \beta_1 t + \cdots + \beta_{2k} t^{2k}.$$

Let $b = (\beta_1, \ldots, \beta_{2k}, 0, \ldots, 0)$ be a vector in $\mathbb{R}^d$ and let $H = \{x \in \mathbb{R}^d : x \cdot b = -\beta_0\}$ be a hyperplane in $\mathbb{R}^d$. Here $\cdot$ denotes the dot product of vectors.

All the points in S_k are clearly contained in H, and for any other $x(t_l) \in \{x(t_1), \ldots, x(t_n)\} \setminus S_k$, we have $x(t_l) \cdot b = -\beta_0 + \beta(t_l) > -\beta_0$. Thus, S_k is the vertex set of a simplex face of Q. ∎

Let n and s be fixed integers with $n > d + s$ and $s \ge 0$ and consider the curve $y(t) = (t, t^2, \ldots, t^{d-1}, p(t))$, where

$$p(t) := (n - 1)^{(t-1)(d-1)} t(t + 1) \cdots (t + d + s - 1).$$

Let $t_i = -s - d + i$ for $i = 1, \ldots, n$. The polynomial $p(t)$ has been chosen so that $p(t_i) = 0$ for $i \in \{1, \ldots, d + s\}$ and $p(t_i) > 0$ otherwise. Let $C(d, n, s) := \text{conv}(\{y(t_1), \ldots, y(t_n)\})$. Also, let

$$T = \{t_i : i = 1, \ldots, n\}, \quad I = \{t_i : i = 1, \ldots, d + s\}, \quad y(S) := \{y(t_i) : t_i \in S\}$$

for $S \subset T$.

The following proposition collects a number of properties of the d-polytope $C(d, n, s)$.

Proposition 4.2 *The d-polytope $C(d, n, s)$ $(n > d + s)$ satisfies the following properties.*

(i) $C(d, n, s) \in \mathcal{P}(d, n, s)$.
(ii) *Gale's evenness condition: A d-subset S_d of vert $C(d, n, s)$ such that $S_d \not\subseteq I$ forms a simplex facet if and only if, for any two elements $u, v \in T \setminus S_d$, the number of elements of S_d between u and v on the curve $y(t)$ is even.*

Proof (i) We first show that the first $d+s$ vertices span a facet F. Let $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$ and let

$$D\big((t_1, t_2, \ldots, t_d); z\big) := \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ t_1 & t_2 & \cdots & t_d & z_1 \\ t_1^2 & t_2^2 & \cdots & t_d^2 & z_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ t_1^{d-1} & t_2^{d-1} & \cdots & t_d^{d-1} & z_{d-1} \\ p(t_1) & p(t_2) & \cdots & p(t_d) & z_d \end{vmatrix}.$$

Let $D(z) := D\big((t_1, t_2, \ldots, t_d); z\big)$ and consider the hyperplane

$$H_D := \{z \in \mathbb{R}^d : D(z) = 0\}.$$

The points $y(t_i)$ $(i = 1, \ldots, d + s)$ are all contained in H_D, since the last row of $D(z)$ vanishes at all these points $y(t_i)$; recall that $p(t_i) = 0$ for $i = 1, \ldots, d + s$. Also, by Lemma 4.1, any d of these points $y(t_i)$ $(i = 1, \ldots, d + s)$ are affinely independent. So, in fact, H_D equals the affine span of the points $y(t_i)$ $(i = 1, \ldots, d + s)$. Let $y(t^*) \in$ vert $C(d, n, s) \setminus y(I)$; then $D(y(t^*)) = p(t^*)V(t_1, \ldots, t_d) > 0$, since $p(t^*) > 0$ and $V(t_1, \ldots, t_d) > 0$. Thus, F is a facet of $C(d, n, s)$.

We now show that every other facet is a simplex. Consider any $(d + 1)$-set $\{t_{i_1} < \cdots < t_{i_d} < t_{i_{d+1}} = t^*\} \subset T$ not contained in I. Thus, $t^* \in T \setminus I$. Consider the determinant $E(z) := D\big((t_{i_1}, t_{i_2}, \ldots, t_{i_d}); z\big)$. The hyperplane $H_E := \{z \in \mathbb{R}^d : E(z) = 0\}$ contains all the points $y(t_{i_j})$ $(j = 1, \ldots, d)$. We need to show that $E(y(t^*)) \neq 0$.

Note that $p(t) = 0$ for $t \in I$ and $p(t) > 0$ for $t \in T \setminus I$. Also, note that $|t_a - t_b| \leq n-1$ for $t_a, t_b \in [-s - d + 1, -s - d + n]$. For the sake of clarity, assume that d is odd; the case of even d is analogous. Computing $E(y(t^*))$ by expanding with respect to the last row gives

$$\big(p(t^*)V(t_{i_1}, \ldots, t_{i_d}) - p(t_{i_d})V(t_{i_1}, \ldots, t_{i_{d-1}}, t^*)\big) + \cdots$$
$$+ \big(p(t_{i_2})V(t_{i_1}, t_{i_3} \ldots, t^*) - p(t_{i_1})V(t_{i_2}, \ldots, t^*)\big).$$

The definition of $p(t)$ implies that each pair-summand is nonnegative and the first pair-summand is positive, and so the determinant is positive. Indeed, for $j > 1$, if $p(t_{i_j}) = 0$, then also $p(t_{i_{j-1}}) = 0$, and the corresponding pair-summand vanishes. Otherwise, let $V(j) := V(t_{i_1}, \ldots t_{i_{j-1}}, t_{i_{j+1}}, \ldots, t_{i_{d+1}})$ for short. From the definition of the values of t_i for $i = 1, \ldots, n$, it follows that $t_{i_a} \geq t_{i_b} + 1$ whenever $a > b$, and

consequently, that

$$(4.1) \qquad p(t_{i_j}) = (n-1)^{(d-1)(t_{i_j}-1)} \prod_{\ell=0}^{d+s-1} (t_{i_j} + \ell)$$

$$> (n-1)^{(d-1)(t_{i_{j-1}}-1)}(n-1)^{d-1} \prod_{\ell=0}^{d+s-1} (t_{i_{j-1}} + \ell)$$

$$= (n-1)^{d-1} p(t_{i_{j-1}}).$$

And from the definition of $V(j)$ it follows that

$$(4.2) \qquad \frac{V(j)}{V(j-1)} = \frac{t_{i_{j+1}} - t_{i_{j-1}}}{t_{i_{j+1}} - t_{i_j}} \cdots \frac{t_{i_{d+1}} - t_{i_{j-1}}}{t_{i_{d+1}} - t_{i_j}} \frac{t_{i_{j-1}} - t_{i_{j-2}}}{t_{i_j} - t_{i_{j-2}}} \cdots \frac{t_{i_{j-1}} - t_{i_1}}{t_{i_j} - t_{i_1}}.$$

Since $1 \le t_{i_a} - t_{i_b} \le n-1$ whenever $a > b$, we get

$$(4.3) \qquad \frac{t_{i_\ell} - t_{i_{j-1}}}{t_{i_\ell} - t_{i_j}} \ge \frac{1}{n-1} \quad \text{for } \ell = j+1, \ldots, d+1,$$

$$(4.4) \qquad \frac{t_{i_{j-1}} - t_{i_\ell}}{t_{i_j} - t_{i_\ell}} \ge \frac{1}{n-1} \quad \text{for } \ell = 1, \ldots, j-2,$$

for each of the $d-1$ quotients in Eq. (4.2). In consequence, combining Eqs. (4.1) to (4.3), we finally get that

$$\frac{V(j)}{V(j-1)} p(t_{i_j}) > \frac{1}{(n-1)^{d-1}} (n-1)^{d-1} p(t_{i_{j-1}}),$$

or equivalently that

$$V(j)p(t_{i_j}) > V(j-1)p(t_{i_{j-1}}),$$

as desired. This completes the proof of the first assertion.

(ii) Consider a set $S_d = \{t_{i_1} < \cdots < t_{i_d}\} \notin I$. Let $t^* \in T$, $t_{i_{j-1}} < t^* < t_{i_j}$ (include also the cases $t^* < t_{i_1}$ with $j = 1$ and $t_{i_d} < t^*$ where we put $j = d+1$). From the above reasoning, we see that if the column $y(t^*)$ in the determinant $E(y(t^*))$ is placed between the columns $y(t_{i_{j-1}})$ and $y(t_{i_j})$, then the resulting determinant is positive. To achieve this, we swap $d-j+1$ times the column $y(t^*)$, which gives that the sign of $E(y(t^*))$ is $(-1)^{d-j+1}$. Consequently, on the curve $y(t)$, between $[-s-d+1, -s-d+n]$, the determinant $E(y(t^*))$ changes sign whenever the variable passes through one of the values t_{i_j} ($i = 1, \ldots, d$), and we are done. ∎

A polytope $C(d, n, s)$ will be called *almost cyclic*. Having established in Lemma 4.1 that $C(d, n, s)$ is $\lfloor (d-1)/2 \rfloor$-neighborly, we can compute its h-vector, in steps. Recall that $P' = \partial P \setminus \{F\}$.

Proposition 4.3 *Let $P \in \mathcal{P}(d, n, s)$ be $\lfloor (d-1)/2 \rfloor$-neighborly, and let (P, F) be the ASP-pair. Then*

$$h_k(P') = \binom{n-d-1+k}{k} \qquad \qquad \text{if } 0 \le k \le \lfloor (d-1)/2 \rfloor,$$

$$h_{d-k}(P') = \binom{n-d-1+k}{k} - \binom{s+k-1}{k} \qquad \text{if } 1 \le k \le \lfloor (d-1)/2 \rfloor.$$

Proof   First note that $f_{k-1}(P') = \binom{n}{k}$ for $k \leq \lfloor (d-1)/2 \rfloor$. Thus, it follows that

$$h_k(P') = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i}\binom{n}{i} = \binom{n-d-1+k}{k}.$$

We now consider the remaining values of k. Using that F is a neighborly simplicial $(d-1)$-polytope, we obtain that $g_k(F) = \binom{d+s-(d-1)+k-2}{k}$, for $0 \leq k \leq \lfloor (d-1)/2 \rfloor$. Thus, from Eq. (2.2) of Proposition 2.1 it follows, for $1 \leq k \leq \lfloor (d-1)/2 \rfloor$, that

$$h_{d-k}(P') = \binom{n-d-1+k}{k} - \binom{s+k-1}{k}. \qquad \blacksquare$$

Observe that, for even d, being $\lfloor (d-1)/2 \rfloor$-neighborly does not determine the value of $h_{d/2}(P')$. With the help of Gale's evenness condition we can compute the number of facets of $C(d, n, s)$, and together with Proposition 4.3 and Eq. (2.1), we can compute $h_{d/2}(C(d, n, s))$ for any even d as well.

Proposition 4.4   *For the ASP-pair $(C(d, n, s), F)$ with d even, consider the simplicial ball $C' := C(d, n, s) \setminus \{F\}$. Then*

$$f_{d-1}(C') = \left(\binom{n-d/2-1}{d/2} + \sum_{i=0}^{d/2-1} 2\binom{n-d-1+i}{i} \right) - \binom{s+d/2}{d/2}.$$

Proof   The counting argument for the facets of C', based on Gale evenness, goes as in the proof of the number of facets of cyclic polytopes (*cf.* [27, Cor. 8.28]), with the difference that we discard the Gale d-tuples formed solely by the first $d + s$ vertices; thus, we discard exactly $\binom{s+d/2}{d/2}$ of them. $\qquad \blacksquare$

Corollary 4.5   *The h-numbers of C' are given by*

$$h_k(C') = \binom{n-d-1+k}{k} \qquad\qquad if\ 0 \leq k \leq \lfloor (d-1)/2 \rfloor,$$

$$h_{d-k}(C') = \binom{n-d-1+k}{k} - \binom{s+k-1}{k} \qquad\qquad if\ 1 \leq k \leq \lfloor d/2 \rfloor.$$

Proof   The case of odd d was already established by Proposition 4.3, since $C(d, n, s)$ is $\lfloor (d-1)/2 \rfloor$-neighborly. For the case of even d it remains to compute $h_{d/2}(P)$. Equating the corresponding expression in Proposition 4.4 with the expression of f_{d-1} in Eq. (2.1), after substituting the known values of h_k for $k \neq d/2$, gives

$$h_{d/2}(C') = \binom{n-d/2-1}{d/2} + \sum_{i=0}^{d/2-1} \binom{s+i-1}{i} - \binom{s+d/2}{d/2}$$

$$= \binom{n-d/2-1}{d/2} - \binom{s+d/2-1}{d/2},$$

as desired. $\qquad\qquad \blacksquare$

4.2 An Upper Bound Theorem for Almost Simplicial Polytopes

We are now in a position to state an upper bound theorem for almost simplicial polytopes $P \in \mathcal{P}(d, n, s)$.

Theorem 4.6 (UBT for ASP)   *Any almost simplicial polytope $P \in \mathcal{P}(d, n, s)$ satisfies*

$$(4.5) \qquad h_k(P') \leq \binom{n - d - 1 + k}{k} \qquad\qquad if\, 0 \leq k \leq \lfloor (d-1)/2 \rfloor,$$

$$(4.6) \quad h_{d-k}(P') \leq \binom{n - d - 1 + k}{k} - \binom{s + k - 1}{k} \qquad if\, 1 \leq k \leq \lfloor d/2 \rfloor.$$

Thus,

$$f_{i-1}(P) \leq f_{i-1}(C(d, n, s)) \quad for\, i = 1, 2, \ldots, d,$$

for the almost cyclic d-polytope $C(d, n, s)$.   Equality for some f_{i-1} with $\lfloor (d-1)/2 \rfloor \leq i \leq d$ implies that P is $\lfloor (d-1)/2 \rfloor$-neighborly.

Proof of Theorem 4.6 via [1, Thm. 3.9]   By [1, Thm. 3.9] the inequalities on $h_k(P')$ hold for $0 \leq k \leq d - 1$, as P' is a special case of a homology ball whose boundary is an induced subcomplex. From Corollary 4.5 and Eq. (2.1) the inequality $f_{i-1}(P) \leq f_{i-1}(C(d, n, s))$ follows. Equality for some f_{i-1} with $d \geq i \geq \lfloor (d-1)/2 \rfloor$ implies, by Eq. (2.1), the equality $h_k(P') = \binom{n-d-1+k}{k}$ for all $0 \leq k \leq \lfloor (d-1)/2 \rfloor$, and thus, again by Eq. (2.1), that P is $\lfloor (d-1)/2 \rfloor$-neighborly.   ∎

Remark 4.7 (More maximizers)   As is the case with neighborly polytopes, we expect that there are many combinatorially distinct ASPs achieving the upper bounds in the UBT for ASP. Here we sketch another such construction, based on a certain perturbation of the Cayley polytope constructed by Karavelas and Tzanaki [16, Sec. 5]: there, two neighborly $(d-1)$-polytopes P_1 and P_2 are placed in parallel hyperplanes in $\mathbb{R}^d$ so that the Cayley polytope $P = \text{conv}(P_1 \cup P_2)$ is $\lfloor \frac{d-1}{2} \rfloor$-neighborly and all $\lfloor d/2 \rfloor$-subsets with a vertex in P_1 and a vertex in P_2 form faces of P. Let $n = f_0(P)$ and $d + s = f_0(P_2)$. Thus, for d odd, any small enough perturbation of the vertices of P_1 into general position will, by considering the new convex hull, change P into $Q \in \mathcal{P}(d, n, s)$ $((Q, P_2)$ is the ASP-pair) with $f(Q) = f(C(d, n, s))$; so Q is a maximizer. For d even, in order for Q to be a maximizer, we need the small perturbation be such that all $d/2$-subsets of $\text{vert}(P_1)$ become faces of Q. To achieve this, we recall more from the construction of [16], and make a variation: consider the images of the functions $\gamma_1(t, z_1, z_2) = (t, z_1 t^{d-1}, t^2, t^3, \ldots, t^{d-2}, z_2 t^d) \subset \mathbb{R}^d$ and $\gamma_2(t, z_1) = (z_1 t^{d-1}, t, t^2, \ldots, t^{d-2}, -1) \subset \mathbb{R}^{d-1} \times \{-1\} \subset \mathbb{R}^d$. The vertices of P_1 (resp. P_2) are on appropriate locations on the curve $\gamma_1(t, z_1^*, 0) \subset \mathbb{R}^{d-1} \times \{0\}$ (resp. $\gamma_2(t, z_1^*)$), for small enough fixed $z_1^* > 0$. It is possible to show, by appropriate determinant computation, that for a small enough fixed $z_2^* > 0$, perturbing the vertices $\gamma_1(t_i, z_1^*, 0)$ of P_1 to $\gamma_1(t_i, z_1^*, z_2^*)$ makes all $d/2$-subsets of P_1 faces of Q; so Q is a maximizer.

We proceed by producing an alternative and elementary proof of the UBT for ASP, via shelling. This will take the rest of this section. Our proof follows ideas from the proof of the classical UBT by McMullen (*cf.* [27, Sec. 8.4]) and from a recent work of

Karavelas and Tzanaki [16]. The key new ingredient is Lemma 4.9, for which we need some preparation.

Let $P \in \mathcal{P}(d, n, s)$ and (P, F) an ASP-pair. Let Q be a polytope obtained from P by stacking a new vertex y beyond F. The d-polytope Q is simplicial. The set of proper faces of Q is the disjoint union of the faces of the complex $P' := \partial P \setminus \{F\}$ and the faces of Q that contain y. Consequently, for all $k \geq 0$,

$$(4.7) \qquad\qquad h_k(Q) = h_k(P') + h_{k-1}(F).$$

Recall the star $\mathrm{star}_{\mathcal{C}}(F)$ of a face F in a polytopal complex $\mathcal{C}$ is the polytopal sub-complex generated by all the faces of $\mathcal{C}$ containing F.

We will use a line shelling of Q with some special properties.

Lemma 4.8   *Let (P, F) be an ASP-pair and $v \in F$ a vertex. Then we can choose the aforementioned vertex y beyond F such that, for (P, F, y, Q) as above, there is a line shelling of Q that shells the star of y first and then proceeds to shell the rest of the star of v.*

Proof   For an oriented line ℓ that shells P, with F being first followed by the rest of the facets in the star of v (*cf.* [27, Thm. 8.12, Cor. 8.13][5]), place y on ℓ beyond F to make Q. Now perturb ℓ, intersecting Q near y, to obtain the desired line shelling.   ■

Consider any vertex $v \in \mathrm{vert}\, Q$ and let $S(Q)$ be a shelling of the facets of Q. Then, clearly, the following hold:

- The restriction of $S(Q)$ to $\mathrm{star}_Q(v)$ yields a shelling of $\mathrm{star}_Q(v)$ (*cf.* [27, Lem. 8.7]); denote it by $S_v(Q)$.
- A shelling of $\mathrm{star}_Q(v)$ induces a shelling of $\mathrm{link}_Q(v)$ by deleting v from the facets.
- Since $\partial F = \mathrm{link}_Q(y)$, it follows that $S(Q)$ induces a shelling $S(F)$ of F.
- Recursively, $S(F)$ induces a shelling of $\mathrm{link}_F(v)$ if $v \in \mathrm{vert}\, F$.

Now consider any vertex $v \in F$, a shelling $S(Q)$ as guaranteed in Lemma 4.8, and the induced shellings $S(Q/v)$ of Q/v, $S(F)$ of F and $S(F/v)$ of F/v. Recall that if $\mathrm{star}_K(v)$ is simplicial, then the boundary complex of K/v coincides with $\mathrm{link}_K(v)$.

Following [16, Sec. 4], call the facet F_j of Q *active* if it is the new facet to be added to the shelling process $S(Q)$. Let $F_j|F$ be the active facet of $S(F)$, which is the restriction of F_j to F (if $y \in F_j$). Let F_j/v be the active facet of $S(Q/v)$ induced by F_j (if $v \in F_j$), $F_j|F/v$ be the active facet of $S(F/v)$ induced by $F_j|F$ (if $vy \subset F_j$), and $F_j|v$ be the active facet of $S_v(Q)$ induced by F_j (if $v \in F_j$); so $F_j|v = \{v\} \cup F_j/v$ in this case. Let $R_j \subseteq F_j$, $R_j/v \subseteq F_j/v$, $R_j|F \subseteq F_j|F$, $R_j|F/v \subseteq F_j|F/v$, and $R_j|v$ be the corresponding new minimal faces in the shellings $S(Q)$, $S(Q/v)$, $S(F)$, $S(F/v)$, and $S_v(Q)$, respectively.

Finally, let $h_k^j(Q)$ denote the value of h_k up to step j, namely $h_k(\bigcup_{i \leq j} F_i)$, and similarly for the other complexes.

The following key lemma allows us to relate the difference in h-numbers along a shelling of Q and F to that of Q/v and F/v.

---

[5]Here we use the extension of the notion of shellability to polyhedral complexes.

Lemma 4.9 *For any vertex $v \in \operatorname{vert} F$ and at any step j of the shelling $S(Q)$ that is guaranteed by Lemma 4.8, for all $k \geq 0$ we have*

$$h_k^j(Q) - h_k^j(Q/v) \geq h_k^j(F) - h_k^j(F/v).$$

Proof While shelling $\operatorname{star}_Q(y)$, the minimal face R_j of F_j in $S(Q)$ and the minimal face $R_j|F$ of $F_j|F$ in $S(F)$ coincide at every step, since $F = Q/y$ and $S(Q)$ shells the star of y first. Therefore, while shelling $\operatorname{star}_Q(y)$, it follows that for all $k \geq 0$,

$$h_k^j(Q) = h_k^j(F).$$

For the same reason, if $F_j \in \operatorname{star}_Q(v) \cap \operatorname{star}_Q(y)$, then, regardless of whether $v \in R_j$ or $v \notin R_j$, we have, for all $k \geq 0$, that

$$h_k^j(Q/v) = h_k^j(F/v).$$

Thus, while shelling $\operatorname{star}_Q(y)$, it follows for all $k \geq 0$ that

$$h_k^j(Q) - h_k^j(Q/v) = h_k^j(F) - h_k^j(F/v).$$

After the shelling has left $\operatorname{star}_Q(y)$, we get no new contributions to $h_k(F)$ or $h_k(F/v)$ for all $k \geq 0$, so the RHS does not change.

After shelling $\operatorname{star}_Q(y)$ and while still shelling $\operatorname{star}_Q(v)$, we have that the minimal faces R_j and R_j/v of $S(Q)$ and $S(Q/v)$, respectively, coincide, so the LHS does not change either. To see that $R_j = R_j/v$, first note that $R_j/v \subseteq R_j$ (as the complex at the j-th step of $S(Q)$ contains the complex at the j-th step of $S_v(Q)$). We show the reverse containment. Assume by contradiction that there is a facet F'' of F_j which is in the subcomplex $\bigcup_{i<j} F_i$ of Q but not in the subcomplex $\bigcup_{i<j,\, v\in F_i} F_i$ of $\operatorname{star}_Q(v)$. As we have already left $\operatorname{star}_Q(y)$, $y \notin F_j$ so F'' is a facet of F. However, also $v \in F$, so we must have $v \in F''$, as otherwise, by convexity, $|F_j| \subset |F|$, a contradiction. But then the (unique) facet F_i in $\operatorname{star}_Q(y)$ containing F'' is also in $\operatorname{star}_Q(v)$, a contradiction.

Thus, for all $k \geq 0$,

$$h_k^j(Q) - h_k^j(Q/v) = h_k^j(F) - h_k^j(F/v).$$

After the shelling has left $\operatorname{star}(v, Q)$, we may get new contributions to $h_k(Q)$ but no more to $h_k(Q/v)$. This concludes the proof of the lemma. ∎

Proposition 4.10 *Let $P \in \mathcal{P}(d, n, s)$ and let (P, F) be an ASP-pair. Then, for all $k \geq 0$, we have*

$$h_{d-(k+1)}(P') \leq \frac{n-d+k}{k+1} h_{d-k}(P') + \frac{n-(d+s)}{k+1} g_k(F).$$

Equivalently, for all $k \geq 0$, we have

$$h_{k+1}(P') \leq \frac{n-d+k}{k+1} h_k(P') - \frac{s+k}{k+1} g_k(F) + g_{k+1}(F).$$

Proof The second inequality follows from the first inequality by the Dehn–Sommerville relations ((2.2)). For the first inequality, we have the following

sequence of equalities.

$$
\begin{aligned}
\text{(4.8)} \quad \sum_{v \in \text{vert } Q} & h_k(Q/v) \\
& = (k+1)h_{k+1}(Q) + (d-k)h_k(Q) \\
& = (k+1)h_{d-(k+1)}(Q) + (d-k)h_{d-k}(Q) \\
& = (k+1)\big(h_{d-(k+1)}(P') + h_{d-(k+1)-1}(F)\big) + (d-k)h_{d-k}(P') \\
& \quad + (d-k)h_{d-k-1}(F) \\
& = \big((k+1)h_{d-(k+1)}(P') + (d-k)h_{d-k}(P')\big) + (k+1)h_{k+1}(F) \\
& \quad + (d-1-k)h_k(F) + h_k(F) \\
& = (k+1)h_{d-(k+1)}(P') + (d-k)h_{d-k}(P') + h_k(F) + \sum_{v \in \text{vert } F} h_k(F/v).
\end{aligned}
$$

For the first equality, see, *e.g.*, [27, Eq. 8.27a], while for the second, use Dehn-Sommerville Eq. (2.2) for Q. The third equality follows from Eq. (4.7), the fourth from Eq. (2.2) again, this time for F, and the last equality follows from [27, Eq. 8.27a] again.

As $F \cong Q/y$, Eq. (4.8) then becomes

$$
\begin{aligned}
\text{(4.9)} \quad \sum_{v \in \text{vert } P' \setminus \text{ vert } F} h_k(Q/v) + \sum_{v \in \text{vert } F} \big(h_k(Q/v) - h_k(F/v)\big) = \\
(k+1)h_{d-(k+1)}(P') + (d-k)h_{d-k}(P').
\end{aligned}
$$

From Lemma 4.9 and the fact that any vertex $v \in \text{vert } P' \setminus \text{vert } F$ has the same link in both Q and P', it then follows that

$$
\begin{aligned}
\text{(4.10)} \quad \sum_{v \in \text{vert } P' \setminus \text{ vert } F} h_k(Q/v) + \sum_{v \in \text{vert } F} \big(h_k(Q/v) - h_k(F/v)\big) \leq \\
\sum_{v \in \text{vert } P' \setminus \text{ vert } F} h_k(P'/v) + \sum_{v \in \text{vert } F} \big(h_k(Q) - h_k(F)\big).
\end{aligned}
$$

Let $v \in \text{vert } P' \setminus \text{vert } F$. There is a shelling of P' that shells $\text{star}_{P'}(v)$ first: just perturb a line through an interior point of F and v, so it still intersects P near those two points, to obtain such line shelling. Such shelling shows that $h_k(P'/v) \leq h_k(P')$ for all $k \geq 0$. Consequently, from Eqs. (4.9) and (4.10) it follows that

$$
\begin{aligned}
(k+1) & h_{d-(k+1)}(P') + (d-k)h_{d-k}(P') \\
& \leq \sum_{v \in \text{vert } P' \setminus \text{ vert } F} h_k(P') + \sum_{v \in \text{vert } F} \big(h_k(Q) - h_k(F)\big) \\
& = \big(n - (d+s)\big)h_k(P') + (d+s)\big(h_k(Q) - h_k(F)\big) \\
& = nh_k(P') + (d+s)\big(h_{k-1}(F) - h_k(F)\big) \\
& = n\big(h_{d-k}(P') + g_k(F)\big) - (d+s)g_k(F).
\end{aligned}
$$

The last two equations ensue from Eq. (4.7) and Eq. (2.2), respectively. Hence, the desired inequality follows.   ∎

We are now in a position to prove the inequalities of Theorem 4.6 using the shelling approach.

Proof of Theorem 4.6 via Shellings   The inequalities Eq. (4.5) for P' in Theorem 4.6 hold for any Cohen–Macaulay complex, and can also be proved exactly as in [27, Lem. 8.26]. We prove Eq. (4.6) by induction on k. The case $k = 1$ holds with equality by Eq. (2.2): $h_{d-1}(P') = n - d - s$. Suppose now that the inequality Eq. (4.6) holds for $k - 1 \le \lfloor d/2 \rfloor - 1$. By Proposition 4.10,

$$h_{d-k}(P') \le \frac{n-d+k-1}{k}\left(\binom{n-d-1+k-1}{k-1} - \binom{s+k-2}{k-1}\right)$$
$$+ \frac{n-(d+s)}{k}g_{k-1}(F).$$

From the application of the g-theorem[6] (*cf.* [27, Cor. 8.38]) to F, it follows that $g_{k-1}(F) \le \binom{d+s-(d-1)+k-3}{k-1}$ for $0 \le k - 1 \le \lfloor d/2 \rfloor - 1$. Thus, the previous inequality becomes

$$h_{d-k}(P') \le \frac{n-d+k-1}{k}\left(\binom{n-d-1+k-1}{k-1} - \binom{s+k-2}{k-1}\right)$$
$$+ \frac{n-(d+s)}{k}\binom{s+k-2}{k-1}$$
$$= \binom{n-d-1+k}{k} - \binom{s+k-1}{k},$$

as desired. ∎

Acknowledgments   The authors thank the referees for their detailed comments, as well as Karim Adiprasito, Gil Kalai, and Isabella Novik for helpful feedback on an earlier version of this paper.

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