On regular algebraic hypersurfaces with non-zero constant mean curvature in Euclidean spaces

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We prove that there are no regular algebraic hypersurfaces with non-zero constant mean curvature in the Euclidean space $\mathbb{R}^{n+1}$, $n \geqslant 2$, defined by polynomials of odd degree. Also we prove that the hyperspheres and the round cylinders are the only regular algebraic hypersurfaces with non-zero constant mean curvature in $\mathbb{R}^{n+1}, n \geqslant 2$, defined by polynomials of degree less than or equal to three. These results give partial answers to a question raised by Barbosa and do Carmo.

1. Introduction

An algebraic hypersurface in the $(n + 1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$, $n \geqslant 2$, is the zero set $M = P^{-1}(0)$ of a polynomial function $P : \mathbb{R}^{n+1} \to \mathbb{R}$. We say that M is *regular* if the gradient vector field ∇P of P has no zeros on M. The condition of regularity implies that M is a complete properly embedded hypersurface of $\mathbb{R}^{n+1}$.

There are many examples of algebraic hypersurfaces in $\mathbb{R}^{n+1}$ which have constant mean curvature. The basic examples of such hypersurfaces are hyperplanes, hyperspheres and round cylinders. In addition to them, we have, for example, the classical Enneper and Henneberg minimal surfaces in $\mathbb{R}^3$ [5], and the families of algebraic minimal cones in $\mathbb{R}^{n+1}$ constructed in [12]. For other examples, see [4, 6, 8, 10, 11] and the references therein.

Barbosa and do Carmo [3] proved that the only connected regular algebraic surfaces in $\mathbb{R}^3$ with non-zero constant mean curvature are the spheres and the right circular cylinders, a result that was already known for polynomials of degree less

than or equal to three [**7**]. For generalizations of this result for globally subanalytic CMC surfaces in $\mathbb{R}^3$ see [**2**, **9**].

Motivated by the theorem mentioned in the previous paragraph and by the fact that the hyperspheres and the round cylinders are the only examples of regular algebraic hypersurfaces in $\mathbb{R}^{n+1}$ with non-zero constant mean curvature known so far, Barbosa and do Carmo [**3**, p. 177] proposed the following extension of their own result:

The hyperspheres and the round cylinders are the only connected regular algebraic hypersurfaces in $\mathbb{R}^{n+1}$, $n \geqslant 2$, with non-zero constant mean curvature.

From Perdomo and Tkachev [**8**] we know that there are no regular algebraic hypersurfaces with non-zero constant mean curvature in $\mathbb{R}^{n+1}$, $n \geqslant 2$, defined by polynomials of degree 3. Here, we prove that there can be no examples defined by polynomials of any odd degree:

THEOREM 1.1. *Let M^n be a regular algebraic hypersurface in $\mathbb{R}^{n+1}$, $n \geqslant 2$, defined by a polynomial P of degree m. If M^n has non-zero constant mean curvature, then m is even.*

The theorem below shows that the question formulated by Barbosa and do Carmo has an affirmative answer for polynomials of degree less than or equal to three.

THEOREM 1.2. *Let M^n be a regular algebraic hypersurface in $\mathbb{R}^{n+1}$, $n \geqslant 2$, defined by a polynomial P of degree less than or equal to three. If M^n has non-zero constant mean curvature, then M^n is a hypersphere or a round cylinder.*

Using theorem 1.1 one concludes that if $M^n \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, is a regular algebraic hypersurface that has non-zero constant mean curvature and is defined by a polynomial P of degree $m \leqslant 3$, then $m = 2$, i.e., M^n is a quadric hypersurface in $\mathbb{R}^{n+1}$. Thus, in the case $n = 2$, theorem 1.2 follows immediately from theorem 1.1 and the well-known classification of quadric surfaces in $\mathbb{R}^3$. However, the situation is quite different for $n \geqslant 3$, since, to the best of our knowledge, there is not a classification of quadric hypersurfaces in $\mathbb{R}^{n+1}$ for $n \geqslant 3$.

2. Our arguments

A polynomial $P : \mathbb{R}^{n+1} \to \mathbb{R}$ of degree m can be expressed in a unique way as the sum

$$P = \sum_{i=0}^{m} P_i, \tag{2.1}$$

where $P_m \neq 0$ and each P_i is a homogeneous polynomial of degree i. We call P_i, $i = 0, \ldots, m$, the *homogeneous factors* of P, and P_m the *highest order homogeneous factor* of P. Since P_m clearly changes sign when m is odd, theorem 1.1 is a consequence of the stronger theorem below. From now on we consider that a regular algebraic hypersurface $M = P^{-1}(0)$ is oriented by the global unit normal vector field $N = \nabla P / |\nabla P|$.

THEOREM 2.1. *Let M^n be a regular algebraic hypersurface in $\mathbb{R}^{n+1}$, $n \geqslant 2$, given by a polynomial P of degree m. If M^n has non-zero constant mean curvature, then the highest order homogeneous factor P_m of P is semi-definite, i.e., either $P_m(x) \geqslant 0$ for all $x \in \mathbb{R}^{n+1}$ or $P_m(x) \leqslant 0$ for all $x \in \mathbb{R}^{n+1}$.*

Proof. Let (2.1) be the expression of P as the sum of homogeneous polynomials. Writing a point in $\mathbb{R}^{n+1}$ as $x = (x_1, \ldots, x_{n+1})$, let $U^+ = \{x \in \mathbb{R}^{n+1} : P(x) > 0\}$ and $U^- = \{x \in \mathbb{R}^{n+1} : P(x) < 0\}$. Since the mean curvature H of M is a non-zero constant by hypothesis, changing P by $-P$ if necessary, we can assume that $H = c > 0$. This means that the mean curvature vector $\overrightarrow{H}$ of M^n points in the direction of U^+.

By (2.1), for any $t \in \mathbb{R}$ and any vector v in the unit sphere S_1^n we have

$$P(tv) = P_m(v)t^m + P_{m-1}(v)t^{m-1} + \cdots + P_1(v)t + P_0, \qquad (2.2)$$

and so

$$\frac{P(tv)}{t^m} - P_m(v) = \sum_{i=0}^{m-1} \frac{P_i(v)}{t^{m-i}}, \quad v \in S_1^n,\ t \neq 0. \qquad (2.3)$$

Using (2.3) and the compactness of S_1^n, one easily sees that $t^{-m}P(tv) \to P_m(v)$ uniformly on S_1^n when $t \to \infty$.

Suppose, by contradiction, that P_m changes sign. Then, since P_m is homogeneous, there exists a vector w in the unit sphere $S_1^n \subset \mathbb{R}^{n+1}$ such that $P_m(w) > 0$. Hence, by continuity, there is a closed disk W around w in S_1^n such that $P_m(v) > P_m(w)/2$ for all $v \in W$. Since $t^{-m}P(tv) \to P_m(v)$ uniformly on S_1^n, there exists $t_0 > 0$ such that $|t^{-m}P(tv) - P_m(v)| < P_m(w)/4$ for all $t > t_0$ and $v \in S_1^n$. Combining these informations, one obtains

$$t^{-m}P(tv) \geqslant -|t^{-m}P(tv) - P_m(v)| + P_m(v)$$
$$> -\frac{1}{4}P_m(w) + \frac{1}{2}P_m(w) = \frac{1}{4}P_m(w) > 0, \quad t > t_0,\ v \in W, \qquad (2.4)$$

and so

$$\{tv : t > t_0,\ v \in W\} \subset U^+. \qquad (2.5)$$

Let R be an arbitrary positive number. By (2.5), there exists a ball B of radius R in $\mathbb{R}^{n+1}$ such that $B \subset U^+$. Let x_0 be the centre of B and $f : M \to \mathbb{R}$ the function defined by $f(x) = ||x - x_0||^2$. Since M is a closed subset of $\mathbb{R}^{n+1}$, there is a point $p \in M$ such that $f(p) = \inf_M f$. Let v be an arbitrary unit vector of T_pM and $\gamma : I \to M$ a smooth curve such that $\gamma(0) = p$ and $\gamma'(0) = v$. Since the function $f(\gamma(t))$ attains a minimum at $t = 0$, one has

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(\gamma(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \langle \gamma(t) - x_0, \gamma(t) - x_0 \rangle = 2\langle v, p - x_0 \rangle \qquad (2.6)$$

and

$$0 \leqslant \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} f(\gamma(t)) = 2\langle \gamma''(0), p - x_0 \rangle + 2. \qquad (2.7)$$

From (2.6) one obtains that $p - x_0$ is orthogonal to T_pM. Since N points inward U^+ and $x_0 \in U^+$, it follows that $N(p) = (x_0 - p)/||x_0 - p||$. Using this information in (2.7), one obtains

$$\langle Av, v \rangle = -\langle (N \circ \gamma)'(0), \gamma'(0) \rangle = \langle N(p), \gamma''(0) \rangle \leqslant \frac{1}{||x_0 - p||},$$

for any unit vector $v \in T_pM$, where A is the shape operator of M with respect to N. Taking the trace in the above inequality and using the fact that $p \notin B$, one concludes that

$$H = H(p) \leqslant \frac{1}{||x_0 - p||} \leqslant \frac{1}{R},$$

for every $R > 0$, contradicting $H = c > 0$. This contradiction shows that P_m does not change sign, and the theorem is proved. $\qquad\square$

REMARK 2.2. The arguments used in the proof of theorem 2.1 show that the complement of the zero set of any non-zero polynomial, in any number of variables, contains balls of arbitrarily large radius.

In the proof of theorem 1.2, as well as in the proof of corollary 2.5, we will use the following result, which is of interest in its own right. We believe this result is known, but since we were unable to find a reference in the literature, we will provide a proof for it here.

LEMMA 2.3. *Let $P : \mathbb{R}^{n+1} \to \mathbb{R}$ be a polynomial of degree m, $m \geqslant 2$. If for some k, $1 \leqslant k \leqslant n$, $P(x)$ vanishes on $S_r^k(a) \times \mathbb{R}^{n-k}$, where $S_r^k(a)$ is the hypersphere of $\mathbb{R}^{k+1}$ of radius r and centre a, then $P(x)$ is divisible by the polynomial*

$$Q(x) = \sum_{i=1}^{k+1} (x_i - a_i)^2 - r^2.$$

Proof. We will prove the lemma in the case where $m = 2d$, $d \geqslant 1$. The proof in the case that the degree of $P(x)$ is odd is entirely analogous and will be omitted.

Assume first $a = 0$ and $r = 1$. Write P as

$$P = \sum_{i=0}^{d} P_{2i} + \sum_{i=0}^{d-1} P_{2i+1}, \tag{2.8}$$

where P_j, $j = 0, \ldots, 2d$, is a homogeneous polynomial of degree j. By hypothesis, for every $v \in S_1^k(0) \times \mathbb{R}^{n-k}$ one has

$$0 = P(v) = \sum_{i=0}^{d} P_{2i}(v) + \sum_{i=0}^{d-1} P_{2i+1}(v),$$

$$0 = P(-v) = \sum_{i=0}^{d} P_{2i}(v) - \sum_{i=0}^{d-1} P_{2i+1}(v).$$

From the two equalities above one obtains $\sum_{i=0}^{d} P_{2i}(v) = 0 = \sum_{i=0}^{d-1} P_{2i+1}(v)$, which implies

$$P_{2d}(v) = -\sum_{i=0}^{d-1} P_{2i}(v), \quad P_{2d-1}(v) = -\sum_{i=0}^{d-2} P_{2i+1}(v), \ \forall v \in S_1^k(0) \times \mathbb{R}^{n-k}. \quad (2.9)$$

Let $\{e_1, \ldots, e_{n+1}\}$ be the canonical basis of $\mathbb{R}^{n+1}$. For any $x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$ such that $y := \sum_{i=1}^{k+1} x_i e_i \neq 0$ it holds that $x/|y| \in S_1^k(0) \times \mathbb{R}^{n-k}$. By (2.8) and (2.9), for all such x one has

$$
\begin{aligned}
P(x) &= \sum_{i=0}^{d-1} P_{2i}(x) + P_{2d}(x) + \sum_{i=0}^{d-2} P_{2i+1}(x) + P_{2d-1}(x) \\
&= \sum_{i=0}^{d-1} P_{2i}(x) + |y|^{2d} P_{2d}(x/|y|) + \sum_{i=0}^{d-2} P_{2i+1}(x) + |y|^{2d-1} P_{2d-1}(x/|y|) \\
&= \sum_{i=0}^{d-1} P_{2i}(x) - |y|^{2d} \sum_{i=0}^{d-1} P_{2i}(x/|y|) + \sum_{i=0}^{d-2} P_{2i+1}(x) - |y|^{2d-1} \sum_{i=0}^{d-2} P_{2i+1}(x/|y|) \\
&= \sum_{i=0}^{d-1} (1 - |y|^{2(d-i)}) P_{2i}(x) + \sum_{i=0}^{d-2} (1 - |y|^{2(d-i-1)}) P_{2i+1}(x).
\end{aligned}
$$

Notice that the second term on the right-hand side of the above equality vanishes when $d = 1$. Since $(1 - |y|^{2k})$ is divisible by $1 - |y|^2$ for any integer $k \geqslant 1$, it follows from the above equality that

$$P(x) = (|y|^2 - 1)R(x), \quad (2.10)$$

for some polynomial $R(x)$ of degree $2(d-1)$. This proves the lemma in the case $a = 0$ and $r = 1$, since $|y|^2 = \sum_{i=1}^{k+1} x_i^2$. To obtain the lemma in the general case, it suffices to apply (2.10) for the polynomial $\overline{P}(x) := P(rx + a)$. $\qquad \square$

Theorem 2.1 states that if a regular algebraic hypersurface in $\mathbb{R}^{n+1}$ has non-zero constant mean curvature, then the highest order homogeneous factor P_m of its defining polynomial P is semi-definite (in particular, the degree m of P is even). Before proving theorem 1.2, for completeness let us say what happens in the case where P_m is definite, i.e. $P_m(x) \neq 0$ for all $x \in \mathbb{R}^{n+1} - \{0\}$:

THEOREM 2.4. *Let M^n be a regular algebraic hypersurface in $\mathbb{R}^{n+1}$, $n \geqslant 2$, defined by a polynomial P of degree m. If the highest order homogeneous factor P_m of P is definite, then M^n is compact. In particular, if M^n has constant mean curvature, then M^n is a finite union of hyperspheres of $\mathbb{R}^{n+1}$.*

Proof. Let (2.1) be the expression of P as the sum of its homogeneous factors. Since P_m is definite by hypothesis, changing P by $-P$ if necessary we can assume

that $P_m(v) > 0$ for all $v \in S_1^n$. Let $\alpha = \inf\{P_m(v) : v \in S_1^n\} > 0$. By the proof of theorem 2.1,

$$t^{-m}P(tv) \xrightarrow{unif} P_m(v) \text{ on } S_1^n \text{ as } t \to \infty.$$

Then there exists $t_0 > 0$ such that $|t^{-m}P(tv) - P_m(v)| < \alpha/2$, for all $t > t_0$ and $v \in S_1^n$, and so

$$t^{-m}P(tv) \geqslant -|t^{-m}P(tv) - P_m(v)| + P_m(v) > -\frac{\alpha}{2} + \alpha = \frac{\alpha}{2} > 0,$$

for all $t > t_0$ and $v \in S_1^n$. Hence, $P(tv) > 0$ for all $t > t_0$ and $v \in S_1^n$, which implies that the set $U^- = \{x \in \mathbb{R}^{n+1} : P(x) < 0\}$ is bounded. Since $M = \partial(U^-)$, one concludes that M is compact.

Assume now that M^n has constant mean curvature. Since M^n is embedded, by a well known theorem of Alexandrov [1] each connected component of M^n is a hypersphere of $\mathbb{R}^{n+1}$. □

The following result is an immediate consequence of lemma 2.3 and theorem 2.4.

COROLLARY 2.5. *Let M^n be a regular algebraic hypersurface in $\mathbb{R}^{n+1}$, $n \geqslant 2$, defined by an irreducible polynomial P of degree m. If the highest order homogeneous factor P_m of P is definite and M^n has constant mean curvature, then M^n is a hypersphere of $\mathbb{R}^{n+1}$.*

EXAMPLE 2.6. Let $P : \mathbb{R}^3 \to \mathbb{R}$ be the polynomial defined by

$$P(x, y, z) = (x^2 + y^2 + z^2 - 1)((x - 3)^2 + y^2 + z^2 - 1).$$

It is easy to see that the gradient ∇P of P vanishes nowhere in $M^2 := P^{-1}(0)$. Then M^2 is a regular algebraic surface in $\mathbb{R}^3$. The highest order homogeneous factor of P is $P_4 = x^4 + y^4 + z^4 + 2x^2y^2 + 2x^2z^2 + 2y^2z^2$, which is clearly definite. Being the union of two disjoint unit spheres of $\mathbb{R}^3$, M^2 has constant mean curvature. This shows that the hypothesis in corollary 2.5 that P is irreducible cannot be dropped.

Proof of theorem 1.2. Since, by hypothesis, the degree m of P is less than or equal to 3 and M^n has non-zero constant mean curvature, it follows from theorem 1.1 that $m = 2$. Hence, we can express P as

$$P = P_0 + P_1 + P_2, \tag{2.11}$$

where P_i, $i = 0, 1, 2$, is a homogeneous polynomial of degree i in the variables $x_1, \ldots, x_{n+1}$. Changing P by $-P$ if necessary, we can assume that $H = c > 0$. We will establish the theorem by proving that

(†) Up to a rigid motion of $\mathbb{R}^{n+1}$, $M^n = S^k \times \mathbb{R}^{n-k}$ for some k, $1 \leqslant k \leqslant n$, where S^k is a hypersphere of $\mathbb{R}^{k+1}$.

We will prove (†) by induction on n. If $n = 2$, it follows from the well known classification of quadric surfaces in $\mathbb{R}^3$ and from the hypothesis that M has non-zero constant mean curvature that M is a sphere or a right circular cylinder. This shows that (†) holds for $n = 2$.

Assume now $n \geqslant 3$ and that (†) holds for $n - 1$. If P_2 is definite, then M^n is a finite union of hyperspheres of $\mathbb{R}^{n+1}$ by theorem 2.4. Since the degree of P is two, it follows from lemma 2.3 that M^n is a single hypersphere of $\mathbb{R}^{n+1}$. Thus (†) holds for n in the case that P_2 is definite.

If P_2 is indefinite, then there exists $w \in S_1^n$ such that $P_2(w) = 0$. Then, after a change of coordinates given by an orthogonal transformation that sends e_{n+1} to w, the polynomial P can be written as

$$P(x_1, \ldots, x_n, x_{n+1}) = A_1(x_1, \ldots, x_n)x_{n+1} + A_0(x_1, \ldots, x_n), \qquad (2.12)$$

where $A_1(x_1, \ldots, x_n)$ is a polynomial of degree $\leqslant 1$ and $A_0(x_1, \ldots, x_n)$ a polynomial of degree $\leqslant 2$.

Claim. $A_1 = 0$.

Assume that the claim is not true. Since the degree of A_1 is at most 1, given $R > 0$ there exists a closed ball $B \subset \mathbb{R}^n$ of radius R on which $A_1(x_1, \ldots, x_n) \neq 0$. If $A_1 > 0$ on B, using (2.12) one easily verifies that $P(x_1, \ldots, x_n, x_{n+1}) > 0$ for all $(x_1, \ldots, x_n) \in B$ and all $x_{n+1} > \max_B |A_0| / \min_B A_1$. Similarly, if $A_1 < 0$ on B one has $P(x_1, \ldots, x_n, x_{n+1}) > 0$ for all $(x_1, \ldots, x_n) \in B$ and all $x_{n+1} < \max_B |A_0| / \max_B A_1$. In either case, one sees that there is a ball of radius R in $\mathbb{R}^{n+1}$ entirely contained in the set $U^+ = \{x \in \mathbb{R}^{n+1} : P(x) > 0\}$. Then, by an argument used in the proof of theorem 2.1, $H \leqslant 1/R$ for every $R > 0$, contradicting $H = c > 0$. This proves the claim.

By (2.12) and the claim above, $P(x_1, \ldots, x_n, x_{n+1}) = A_0(x_1, \ldots, x_n)$, where $A_0 : \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree two in the variables $x_1, \ldots, x_n$. Since 0 is a regular value for P, so is for A_0. Hence $M^n = M_0^{n-1} \times \mathbb{R}$, where $M_0^{n-1} = A_0^{-1}(0)$ is a regular algebraic hypersurface of $\mathbb{R}^n$. Since M^n has non-zero constant mean curvature, the same is true of M_0^{n-1}. Then, by the induction hypothesis, up to a rigid motion of $\mathbb{R}^n$ one has $M_0^{n-1} = S^k \times \mathbb{R}^{n-1-k}$, for some k, $1 \leqslant k \leqslant n - 1$, where S^k is a hypersphere of $\mathbb{R}^{k+1}$. Hence $M^n = S^k \times \mathbb{R}^{n-k}$, for some k, $1 \leqslant k \leqslant n - 1$. This shows that (†) holds for n also in the case that P_2 is indefinite. Then, by the induction principle, (†) holds for every $n \geqslant 2$, and the theorem is proved. □

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