Bull. Aust. Math. Soc. **101** (2020), 426–437
doi:10.1017/S0004972719000819

SYMMETRIC FUNCTIONS AND MULTIPLE ZETA VALUES

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(Received 3 June 2019; accepted 16 June 2019; first published online 24 July 2019)

Abstract

Four classes of multiple series, which can be considered as multifold counterparts of four classical summation formulae related to the Riemann zeta series, are evaluated in closed form.

2010 *Mathematics subject classification*: primary 11M06; secondary 33B10, 65B10.

Keywords and phrases: Riemann zeta function, elementary symmetric function, complete symmetric function, Bernoulli number, Euler number.

1. Introduction and outline

The Riemann zeta function plays an important role in classical analysis. The following values (see Stromberg [10, Ch. 7]) are well known:

$$\sum_{n=1}^{\infty}\frac{1}{n^{2m}}=(-1)^{m-1}\frac{(2\pi)^{2m}}{2(2m)!}B_{2m},$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{2m}}=(-1)^{m-1}\frac{\pi^{2m}}{2(2m)!}B'_{2m},$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{2m}}=(-1)^{m-1}\frac{\pi^{2m}}{2(2m)!}B''_{2m},$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^{2m+1}}=(-1)^{m-1}\frac{\pi^{2m+1}}{4^{m+1}(2m)!}E_{2m},$$

where for brevity, we have introduced the notations

$$B'_{2m}:=(4^m-1)B_{2m}\quad\text{and}\quad B''_{2m}:=(2-4^m)B_{2m},$$

and the Bernoulli and Euler numbers (see Comtet [3, Section 1.14], Graham *et al.* [4, Sections 6.5 and 6.6] and Stromberg [10, Ch. 7]) are defined, respectively, by the generating functions

$$\frac{x}{e^x-1}=\sum_{m=0}^{\infty}\frac{x^m}{m!}B_m\quad\text{and}\quad\frac{2e^x}{1+e^{2x}}=\sum_{m=0}^{\infty}\frac{x^m}{m!}E_m.$$

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Let $\mathbb{N}$ be the set of natural numbers. For a sequence of indeterminates $\{y_n\}_{n\in\mathbb{N}}$, let $\sigma_m(y_n|n \in \mathbb{N})$ and $\hbar_m(y_n|n \in \mathbb{N})$ be the mth elementary and complete symmetric functions, respectively. They can be expressed by the following multiple sums (see MacDonald [6, Ch. 1]):

$$\sigma_m(y_n|n \in \mathbb{N}) = \sum_{1\le n_1<n_2<\cdots<n_m<\infty} \prod_{\imath=1}^{m} y_{n_\imath},$$

$$\hbar_m(y_n|n \in \mathbb{N}) = \sum_{1\le n_1\le n_2\le\cdots\le n_m<\infty} \prod_{\imath=1}^{m} y_{n_\imath}.$$

The aim of this paper is to examine multiple series counterparts of the four Riemann zeta series evaluations displayed at the beginning of the paper. Several summation formulae will be established in the next four sections evaluating the elementary symmetric functions $\sigma_m(y_n|n \in \mathbb{N})$ in closed form when y_n is replaced by one of

$$\left\{\frac{1}{n^{2\lambda}}, \quad \frac{1}{(2n-1)^{2\lambda}}, \quad \frac{(-1)^{n\lambda}}{n^{2\lambda}}, \quad \frac{(-1)^{n\lambda}}{(2n-1)^{\lambda}}\right\}.$$

The paper ends with a brief comment about evaluations of the corresponding complete symmetric functions $\hbar_m(y_n|n \in \mathbb{N})$.

Throughout, for $\lambda \in \mathbb{N}$, we fix a (2λ)th root of unity by

$$\omega := \omega_\lambda = \exp\Big(\frac{\pi i}{\lambda}\Big).$$

It has the property

$$\{\omega^k\}_{k=0}^{2\lambda-1} = \{\omega^k\}_{k=0}^{\lambda-1} \cup \{-\omega^k\}_{k=0}^{\lambda-1}.$$

2. Formulae for $\sigma_m((1/n^{2\lambda})|n \in \mathbb{N})$

Performing the replacement $x \to x\omega^k$ in the equation

$$\frac{\sin \pi x}{\pi x} = \prod_{n=1}^{\infty}\left\{1 - \frac{x^2}{n^2}\right\}$$

and then multiplying the resulting equations for $0 \le k < \lambda$ gives

$$\prod_{k=0}^{\lambda-1} \frac{\sin \pi x\omega^k}{\pi x\omega^k} = \prod_{n=1}^{\infty}\left\{1 - \frac{x^{2\lambda}}{n^{2\lambda}}\right\} = 1 + \sum_{m=1}^{\infty}(-1)^m x^{2m\lambda}\sigma_m\Big(\frac{1}{n^{2\lambda}}\Big|n \in \mathbb{N}\Big),$$

where the mth elementary symmetric function is defined by the multiple sum

$$\sigma_m\Big(\frac{1}{n^{2\lambda}}\Big|n \in \mathbb{N}\Big) = \sum_{1\le n_1<n_2<\cdots<n_m<\infty} \prod_{\imath=1}^{m} \frac{1}{n_\imath^{2\lambda}}.$$

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In order to evaluate σ_m, we have to expand the product

$$\prod_{k=0}^{\lambda-1}\frac{\sin\pi x\omega^k}{\pi x\omega^k}=\prod_{k=0}^{\lambda-1}\frac{\exp(\pi ix\omega^k)-\exp(-\pi ix\omega^k)}{2\pi ix\omega^k}$$
$$=\frac{i(-1)^\lambda}{(2\pi x)^\lambda}\prod_{k=0}^{\lambda-1}\sum_{\varepsilon_k=\pm1}\varepsilon_k\exp(\varepsilon_k\pi ix\omega^k)$$
$$=\frac{i(-1)^\lambda}{(2\pi x)^\lambda}\sum_{\substack{\varepsilon_k=\pm1\\0\le k<\lambda}}\Big\{\prod_{k=0}^{\lambda-1}\varepsilon_k\Big\}\exp\Big(\sum_{k=0}^{\lambda-1}\varepsilon_k\pi ix\omega^k\Big)$$
$$=\frac{i(-1)^\lambda}{(2\pi x)^\lambda}\sum_{m=0}^{\infty}\sum_{\substack{\varepsilon_k=\pm1\\0\le k<\lambda}}\frac{(\pi x)^m}{m!}\Big\{\prod_{k=0}^{\lambda-1}\varepsilon_k\Big\}\Big(\sum_{k=0}^{\lambda-1}\varepsilon_k i\omega^k\Big)^m.$$

Denote by $[x^m]\phi(x)$ the coefficient of x^m in the formal power series $\phi(x)$. Then

$$\sigma_m\Big(\frac{1}{n^{2\lambda}}\Big|n\in\mathbb{N}\Big)=(-1)^m[x^{2m\lambda}]\prod_{k=0}^{\lambda-1}\frac{\sin\pi x\omega^k}{\pi x\omega^k}$$
$$=\frac{i(-1)^{m+\lambda}}{(2\pi)^\lambda}\sum_{\substack{\varepsilon_k=\pm1\\0\le k<\lambda}}\frac{\pi^{2m\lambda+\lambda}}{(2m\lambda+\lambda)!}\Big\{\prod_{k=0}^{\lambda-1}\varepsilon_k\Big\}\Big(\sum_{k=0}^{\lambda-1}\varepsilon_k i\omega^k\Big)^{2m\lambda+\lambda}$$
$$=\frac{\pi^{2m\lambda}(-1)^\lambda}{2^\lambda(2m\lambda+\lambda)!}\sum_{\substack{\varepsilon_k=\pm1\\0\le k<\lambda}}\Big\{i\prod_{k=0}^{\lambda-1}\varepsilon_k\Big(\sum_{k=0}^{\lambda-1}\varepsilon_k i\omega^k\Big)^\lambda\Big\}^{2m+1}.$$

This leads us to the following interesting summation theorem.

THEOREM 2.1. *For $m,\ \lambda\in\mathbb{N}$, we have the evaluation formula*

$$\sigma_m\Big(\frac{1}{n^{2\lambda}}\Big|n\in\mathbb{N}\Big)=\frac{\pi^{2m\lambda}(-1)^\lambda}{2^\lambda(2m\lambda+\lambda)!}\sum_{\beta\in\mathcal{A}(\lambda)}\beta^{2m+1},$$

where the multiset $\mathcal{A}(\lambda)$ of complex numbers is given by

$$\mathcal{A}(\lambda)=\Big\{i\prod_{k=0}^{\lambda-1}\varepsilon_k\Big(\sum_{k=0}^{\lambda-1}\varepsilon_k i\omega^k\Big)^\lambda\Big|\varepsilon_k=\pm1\ \text{ for }\ 0\le k<\lambda\Big\}.$$

This formula was derived explicitly by the author [2] and rediscovered by Nakamura [8]. Its initial case $\lambda=1$ was obtained independently by Merca [7].

Observe that $\mathcal{A}(\lambda)$ is, in general, a multiset. We can represent it by pairs $[m(\beta):\beta]$, where $m(\beta)$ denotes the multiplicity of $\beta\in\mathcal{A}(\lambda)$. By making use of *Mathematica*, we

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are able to determine the first six examples:

$$\mathcal{A}(1) = \{[2 : -1]\},$$
$$\mathcal{A}(2) = \{[4 : 2]\},$$
$$\mathcal{A}(3) = \{[2 : 0], [6 : -8]\},$$
$$\mathcal{A}(4) = \{[8 : 8(3 \pm 2\sqrt{2})]\},$$
$$\mathcal{A}(5) = \{[2 : 0], [10 : -32], [10 : -16(11 \pm 5\sqrt{5})]\},$$
$$\mathcal{A}(6) = \{[4 : 0], [12 : \pm 64i], [12 : 512], [12 : 64(26 \pm 15\sqrt{3})]\}.$$

From Theorem 2.1, we recover the following six elegant summation formulae, derived by Borwein *et al.* [1].

PROPOSITION 2.2 (Multiple series identities: $m \in \mathbb{N}$).

$$\sigma_m\Big(\frac{1}{n^2}\Big|n \in \mathbb{N}\Big) = \frac{\pi^{2m}}{(2m+1)!},$$
$$\sigma_m\Big(\frac{1}{n^4}\Big|n \in \mathbb{N}\Big) = \frac{2^{2m+1}}{(4m+2)!}\pi^{4m}$$
$$\sigma_m\Big(\frac{1}{n^6}\Big|n \in \mathbb{N}\Big) = \frac{3 \cdot 2^{6m+1}}{(6m+3)!}\pi^{6m}$$
$$\sigma_m\Big(\frac{1}{n^8}\Big|n \in \mathbb{N}\Big) = \frac{2^{6m+2}}{(8m+4)!}\pi^{8m}\Big\{(3 \pm 2\sqrt{2})^{2m+1}\Big\},$$
$$\sigma_m\Big(\frac{1}{n^{10}}\Big|n \in \mathbb{N}\Big) = \frac{5 \cdot 2^{8m}}{(10m+5)!}\pi^{10m}\Big\{2^{2m+1} + (11 \pm 5\sqrt{5})^{2m+1}\Big\},$$
$$\sigma_m\Big(\frac{1}{n^{12}}\Big|n \in \mathbb{N}\Big) = \frac{3 \cdot 2^{12m+2}}{(12m+6)!}\pi^{12m}\Big\{8^{2m+1} + (26 \pm 15\sqrt{3})^{2m+1}\Big\}.$$

The first identity above was conjectured by Moen and confirmed by Hoffman [5]. Different proofs can be found in Chu [2], Merca [7] and Zagier [11].

3. Formulae for $\sigma_m((1/(2n-1)^{2\lambda})|n \in \mathbb{N})$

Performing the replacement $x \to x\omega^k$ in the equation

$$\cos \pi x = \prod_{n=1}^{\infty}\Big\{1 - \frac{4x^2}{(2n-1)^2}\Big\}$$

and then multiplying the resulting equations for $0 \le k < \lambda$ gives

$$\prod_{k=0}^{\lambda-1}\cos \pi x\omega^k = \prod_{n=1}^{\infty}\Big\{1 - \frac{(2x)^{2\lambda}}{(2n-1)^{2\lambda}}\Big\}$$
$$= 1 + \sum_{m=1}^{\infty}(-1)^m(2x)^{2m\lambda}\sigma_m\Big(\frac{1}{(2n-1)^{2\lambda}}\Big|n \in \mathbb{N}\Big),$$

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where the mth elementary symmetric function is defined by the multiple sum

$$\sigma_m\left(\frac{1}{(2n-1)^{2\lambda}}\middle|n\in\mathbb{N}\right)=\sum_{1\le n_1<n_2<\cdots<n_m<\infty}\prod_{\iota=1}^{m}\frac{1}{(2n_\iota-1)^{2\lambda}}.$$

In order to evaluate the above σ_m, we need the power series expansion

$$\begin{aligned}\prod_{k=0}^{\lambda-1}\cos\pi x\omega^k&=\prod_{k=0}^{\lambda-1}\frac{\exp(\pi ix\omega^k)+\exp(-\pi ix\omega^k)}{2}\\&=2^{-\lambda}\prod_{k=0}^{\lambda-1}\sum_{\varepsilon_k=\pm1}\exp(\varepsilon_k\pi ix\omega^k)\\&=2^{-\lambda}\sum_{\substack{\varepsilon_k=\pm1\\0\le k<\lambda}}\exp\left(\sum_{k=0}^{\lambda-1}\varepsilon_k\pi ix\omega^k\right)\\&=2^{-\lambda}\sum_{m=0}^{\infty}\sum_{\substack{\varepsilon_k=\pm1\\0\le k<\lambda}}\frac{(\pi x)^m}{m!}\left(\sum_{k=0}^{\lambda-1}\varepsilon_k i\omega^k\right)^m.\end{aligned}$$

Then we can extract the coefficient

$$\begin{aligned}\sigma_m\left(\frac{1}{(2n-1)^{2\lambda}}\middle|n\in\mathbb{N}\right)&=\frac{(-1)^m}{2^{2m\lambda}}[x^{2m\lambda}]\prod_{k=0}^{\lambda-1}\cos\pi x\omega^k\\&=\frac{(-1)^m}{2^{2m\lambda+\lambda}}\sum_{\substack{\varepsilon_k=\pm1\\0\le k<\lambda}}\frac{\pi^{2m\lambda}}{(2m\lambda)!}\left(\sum_{k=0}^{\lambda-1}\varepsilon_k i\omega^k\right)^{2m\lambda}\\&=\frac{\pi^{2m\lambda}(-1)^m}{2^{2m\lambda+\lambda}(2m\lambda)!}\sum_{\substack{\varepsilon_k=\pm1\\0\le k<\lambda}}\left\{\sum_{k=0}^{\lambda-1}\varepsilon_k i\omega^k\right\}^{2m\lambda}.\end{aligned}$$

THEOREM 3.1. *For $m,\ \lambda\in\mathbb{N}$, we have the evaluation formula*

$$\sigma_m\left(\frac{1}{(2n-1)^{2\lambda}}\middle|n\in\mathbb{N}\right)=\frac{\pi^{2m\lambda}(-1)^m}{2^{2m\lambda+\lambda}(2m\lambda)!}\sum_{\beta\in\mathcal{B}(\lambda)}\beta^m,$$

where the multiset $\mathcal{B}(\lambda)$ of complex numbers is given by

$$\mathcal{B}(\lambda)=\left\{\left(\sum_{k=0}^{\lambda-1}\varepsilon_k i\omega^k\right)^{2\lambda}\middle|\varepsilon_k=\pm1\ \textit{for}\ 0\le k<\lambda\right\}.$$

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With the help of *Mathematica*, the first six multisets are determined as follows:

$$\mathcal{B}(1) = \{[2 : -1]\},$$
$$\mathcal{B}(2) = \{[4 : -4]\},$$
$$\mathcal{B}(3) = \{[2 : 0], [6 : -64]\},$$
$$\mathcal{B}(4) = \{[8 : -64(17 \pm 12\sqrt{2})]\},$$
$$\mathcal{B}(5) = \{[2 : 0], [10 : -1024], [10 : -512(123 \pm 55\sqrt{5})]\},$$
$$\mathcal{B}(6) = \{[4 : 0], [12 : -2^{18}], [24 : 2^{12}], [12 : -2^{12}(1351 \pm 780\sqrt{3})]\}.$$

Consequently, we find the following remarkable closed formulae.

Proposition 3.2 (Multiple series identities: $m \in \mathbb{N}$).

$$\sigma_m\Big(\frac{1}{(2n-1)^2}\Big|n \in \mathbb{N}\Big) = \frac{\pi^{2m}}{2^{2m}(2m)!},$$
$$\sigma_m\Big(\frac{1}{(2n-1)^4}\Big|n \in \mathbb{N}\Big) = \frac{\pi^{4m}}{2^{2m}(4m)!},$$
$$\sigma_m\Big(\frac{1}{(2n-1)^6}\Big|n \in \mathbb{N}\Big) = \frac{3\pi^{6m}}{4(6m)!},$$
$$\sigma_m\Big(\frac{1}{(2n-1)^8}\Big|n \in \mathbb{N}\Big) = \frac{\pi^{8m}}{2^{2m+1}(8m)!}\Big\{(17 \pm 12\sqrt{2})^m\Big\},$$
$$\sigma_m\Big(\frac{1}{(2n-1)^{10}}\Big|n \in \mathbb{N}\Big) = \frac{5\pi^{10m}}{2^{m+4}(10m)!}\Big\{2^m + (123 \pm 55\sqrt{5})^m\Big\},$$
$$\sigma_m\Big(\frac{1}{(2n-1)^{12}}\Big|n \in \mathbb{N}\Big) = \frac{3\pi^{12}}{2^4(12m)!}\Big\{2^{6m} - 2 + (1351 \pm 780\sqrt{3})^m\Big\}.$$

4. Formulae for $\sigma_m(((-1)^{n\lambda}/n^{2\lambda})|n \in \mathbb{N})$

Performing the replacement $x \to x\omega^k$ in the equation

$$\frac{\sin \pi x}{\pi x}\cosh \pi x = \prod_{n=1}^{\infty}\Big\{1 - \frac{4x^2(-1)^n}{n^2}\Big\}$$

and then multiplying the resulting equations for $0 \le k < \lambda$ gives

$$\prod_{k=0}^{\lambda-1}\frac{\sin \pi x\omega^k}{\pi x\omega^k}\cosh \pi x\omega^k = \prod_{n=1}^{\infty}\Big\{1 - \frac{(2x)^{2\lambda}(-1)^{n\lambda}}{n^{2\lambda}}\Big\}$$
$$= 1 + \sum_{m=1}^{\infty}(-1)^m(2x)^{2m\lambda}\sigma_m\Big(\frac{(-1)^{n\lambda}}{n^{2\lambda}}\Big|n \in \mathbb{N}\Big),$$

where the mth elementary symmetric function is defined by the multiple sum

$$\sigma_m\Big(\frac{(-1)^{n\lambda}}{n^{2\lambda}}\Big|n \in \mathbb{N}\Big) = \sum_{1 \le n_1 < n_2 < \cdots < n_m < \infty}\prod_{\imath=1}^{m}\frac{(-1)^{\lambda n_\imath}}{n_\imath^{2\lambda}}.$$

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By expanding the following product into power series

$$\begin{aligned}\prod_{k=0}^{\lambda-1}\frac{\sin\pi x\omega^k}{\pi x\omega^k}\cosh\pi x\omega^k &= \prod_{k=0}^{\lambda-1}\frac{\exp(\pi ix\omega^k)-\exp(-\pi ix\omega^k)}{2\pi ix\omega^k}\,\frac{\exp(\pi x\omega^k)+\exp(-\pi x\omega^k)}{2}\\ &= \frac{i(-1)^\lambda}{(4\pi x)^\lambda}\prod_{k=0}^{\lambda-1}\sum_{\varepsilon_k=\pm1}\varepsilon_k\exp(\varepsilon_k i\pi x\omega^k)\sum_{\epsilon_k=\pm1}\exp(\epsilon_k\pi x\omega^k)\\ &= \frac{i(-1)^\lambda}{(4\pi x)^\lambda}\sum_{\substack{\varepsilon_k,\epsilon_k=\pm1\\0\le k<\lambda}}\Big\{\prod_{k=0}^{\lambda-1}\varepsilon_k\Big\}\exp\Big(\sum_{k=0}^{\lambda-1}(\varepsilon_k i+\epsilon_k)\pi x\omega^k\Big)\\ &= \frac{i(-1)^\lambda}{(4\pi x)^\lambda}\sum_{m=0}^{\infty}\sum_{\substack{\varepsilon_k,\epsilon_k=\pm1\\0\le k<\lambda}}\frac{(\pi x)^m}{m!}\Big\{\prod_{k=0}^{\lambda-1}\varepsilon_k\Big\}\Big(\sum_{k=0}^{\lambda-1}(\varepsilon_k i+\epsilon_k)\omega^k\Big)^m,\end{aligned}$$

we can evaluate the multiple alternating series

$$\begin{aligned}\sigma_m\Big(\frac{(-1)^{n\lambda}}{n^{2\lambda}}\Big|n\in\mathbb{N}\Big) &= \frac{(-1)^m}{2^{2m\lambda+\lambda}}[x^{2m\lambda}]\prod_{k=0}^{\lambda-1}\frac{\sin\pi x\omega^k}{\pi x\omega^k}\cosh\pi x\omega^k\\ &= \frac{i(-1)^{m+\lambda}}{2^{2m\lambda+2\lambda}}\sum_{\substack{\varepsilon_k,\epsilon_k=\pm1\\0\le k<\lambda}}\frac{\pi^{2m\lambda}}{(2m\lambda+\lambda)!}\Big\{\prod_{k=0}^{\lambda-1}\varepsilon_k\Big\}\Big(\sum_{k=0}^{\lambda-1}(\varepsilon_k i+\epsilon_k)\omega^k\Big)^{2m\lambda+\lambda}\\ &= \frac{(-1)^\lambda\pi^{2m\lambda}}{2^{2m\lambda+2\lambda}(2m\lambda+\lambda)!}\sum_{\substack{\varepsilon_k,\epsilon_k=\pm1\\0\le k<\lambda}}\Big\{i\prod_{k=0}^{\lambda-1}\varepsilon_k\Big(\sum_{k=0}^{\lambda-1}(\varepsilon_k i+\epsilon_k)\omega^k\Big)^\lambda\Big\}^{2m+1}.\end{aligned}$$

Theorem 4.1. *For $m,\ \lambda\in\mathbb{N}$, we have the evaluation formula*

$$\sigma_m\Big(\frac{(-1)^{n\lambda}}{n^{2\lambda}}\Big|n\in\mathbb{N}\Big)=\frac{(-1)^\lambda\pi^{2m\lambda}}{2^{2m\lambda+2\lambda}(2m\lambda+\lambda)!}\sum_{\beta\in C(\lambda)}\beta^{2m+1},$$

where the multiset $C(\lambda)$ of complex numbers is given by

$$C(\lambda)=\Big\{i\prod_{k=0}^{\lambda-1}\varepsilon_k\Big(\sum_{k=0}^{\lambda-1}(\varepsilon_k i+\epsilon_k)\omega^k\Big)^\lambda\Big|_{\epsilon_k=\pm1}^{\varepsilon_k=\pm1}\ \textit{for}\ \ 0\le k<\lambda\Big\}.$$

We remark that when λ is even, this theorem evaluates the same multiple zeta series as Theorem 2.1, even though they have different expressions.

The first four multisets $\beta\in C(\lambda)$ are determined by *Mathematica* as follows:

$$\begin{aligned}&C(1)=\{[2:-1\pm i]\},\\ &C(2)=\{[4:0],[4:8],[4:\pm4i]\},\\ &C(3)=\{[4:0],[12:-8],[6:\pm8i],[6:16\pm16i],[6:-(4\pm4i)(5\pm3\sqrt{3})]\},\\ &C(4)=\left\{\begin{matrix}[16:0],[32:\pm16i],[8:128(3\pm2\sqrt{2}],[8:16(\pm7i\pm4\sqrt{2})],\\ [16:\pm64i],[16:32(\pm3\pm2\sqrt{2}],[8:16i(\pm17\pm12\sqrt{2})]\end{matrix}\right\}.\end{aligned}$$

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The next proposition gives the two summation formulae for $\lambda = 1$ and $\lambda = 3$.

PROPOSITION 4.2 (Multiple series identities: $m \in \mathbb{N}$).

$$\sigma_m\Big(\frac{(-1)^n}{n^2}\Big|n \in \mathbb{N}\Big) = \frac{\pi^{2m}}{2^{2m+1}(2m+1)!}\Big\{(1 \pm i)^{2m+1}\Big\},$$

$$\sigma_m\Big(\frac{(-1)^n}{n^6}\Big|n \in \mathbb{N}\Big) = \frac{3\pi^{6m}}{2^{2m+3}(6m+3)!}\left\{\begin{matrix} 2^{2m+2} - 2^{4m+2}(1 \pm i)^{2m+1} \\ +(1 \pm i)^{2m+1}(5 \pm 3\sqrt{3})^{2m+1} \end{matrix}\right\}.$$

5. Formulae for $\sigma_m(((-1)^{n\lambda}/(2n-1)^\lambda)|n \in \mathbb{N})$

Performing the replacement $x \to x\omega^{2k}$ in the equation

$$\cos\pi x - \sin\pi x = \prod_{n=1}^{\infty}\Big\{1 + \frac{4x(-1)^n}{2n-1}\Big\}$$

and then multiplying the resulting equations for $0 \le k < \lambda$ gives

$$\begin{aligned}\prod_{k=0}^{\lambda-1}\{\cos\pi x\omega^{2k} - \sin\pi x\omega^{2k}\} &= \prod_{n=1}^{\infty}\Big\{1 - \frac{(4x)^\lambda(-1)^{n\lambda+\lambda}}{(2n-1)^\lambda}\Big\} \\ &= 1 + \sum_{m=1}^{\infty}(-1)^{m+m\lambda}(4x)^{m\lambda}\sigma_m\Big(\frac{(-1)^{n\lambda}}{(2n-1)^\lambda}\Big|n \in \mathbb{N}\Big),\end{aligned}$$

where the mth elementary symmetric function is defined by the multiple sum

$$\sigma_m\Big(\frac{(-1)^{n\lambda}}{(2n-1)^\lambda}\Big|n \in \mathbb{N}\Big) = \sum_{1\le n_1<n_2<\cdots<n_m<\infty}\prod_{\iota=1}^{m}\frac{(-1)^{\lambda n_\iota}}{(2n_\iota-1)^{2\lambda}}.$$

By manipulating the product,

$$\begin{aligned}\prod_{k=0}^{\lambda-1}\{\cos\pi x\omega^{2k} - \sin\pi x\omega^{2k}\} &= \prod_{k=0}^{\lambda-1}\frac{(1+i)\exp(\pi i x\omega^{2k}) + (1-i)\exp(-\pi i x\omega^{2k})}{2} \\ &= 2^{-\lambda}\prod_{k=0}^{\lambda-1}\sum_{\varepsilon_k=\pm1}(1+i\varepsilon_k)\exp(\varepsilon_k\pi i x\omega^{2k}) \\ &= 2^{-\lambda}\sum_{\substack{\varepsilon_k=\pm1 \\ 0\le k<\lambda}}\prod_{k=0}^{\lambda-1}(1+i\varepsilon_k)\exp\Big(\sum_{k=0}^{\lambda-1}\varepsilon_k\pi i x\omega^{2k}\Big) \\ &= 2^{-\lambda}\sum_{m=0}^{\infty}\sum_{\substack{\varepsilon_k=\pm1 \\ 0\le k<\lambda}}\frac{(\pi x)^m}{m!}\prod_{k=0}^{\lambda-1}(1+i\varepsilon_k)\Big(\sum_{k=0}^{\lambda-1}i\varepsilon_k\omega^{2k}\Big)^m,\end{aligned}$$

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we derive

$$\sigma_m\Big(\frac{(-1)^{n\lambda}}{(2n-1)^\lambda}\Big|n\in\mathbb{N}\Big)=\frac{(-1)^{m+m\lambda}}{2^{2m\lambda}}[x^{m\lambda}]\prod_{k=0}^{\lambda-1}\{\cos\pi x\omega^{2k}-\sin\pi x\omega^{2k}\}$$
$$=\frac{(-1)^{m+m\lambda}}{2^{2m\lambda+\lambda}}\sum_{\substack{\varepsilon_k=\pm1\\0\le k<\lambda}}\frac{\pi^{m\lambda}}{(m\lambda)!}\prod_{k=0}^{\lambda-1}(1+i\varepsilon_k)\Big(\sum_{k=0}^{\lambda-1}i\varepsilon_k\omega^{2k}\Big)^{m\lambda}$$
$$=\frac{\pi^{m\lambda}(-1)^{m+m\lambda}}{2^{2m\lambda+\lambda}(m\lambda)!}\sum_{\substack{\varepsilon_k=\pm1\\0\le k<\lambda}}\prod_{k=0}^{\lambda-1}(1+i\varepsilon_k)\Big(\sum_{k=0}^{\lambda-1}i\varepsilon_k\omega^{2k}\Big)^{m\lambda}.$$

THEOREM 5.1. *For $m,\ \lambda\in\mathbb{N}$, we have the evaluation formula*

$$\sigma_m\Big(\frac{(-1)^{n\lambda}}{(2n-1)^\lambda}\Big|n\in\mathbb{N}\Big)=\frac{\pi^{m\lambda}(-1)^{m+m\lambda}}{2^{2m\lambda+\lambda}(m\lambda)!}\sum_{(\alpha,\beta)\in\mathcal{D}(\lambda)}\alpha\beta^m,$$

where the multiset $\mathcal{D}(\lambda)$ of complex pairs is given by

$$\mathcal{D}(\lambda)=\Big\{\Big(\prod_{k=0}^{\lambda-1}(1+i\varepsilon_k),\Big(\sum_{k=0}^{\lambda-1}i\varepsilon_k\omega^{2k}\Big)^\lambda\Big)\Big|\varepsilon_k=\pm1\ \textit{for}\ 0\le k<\lambda\Big\}.$$

When λ is even, this theorem evaluates the same multiple zeta series as Theorem 3.1. However, the formula in this theorem is more complicated.

Here, $\mathcal{D}(\lambda)$ is a multiset of pairs (α,β), which can be represented by the triplets $[m(\alpha,\beta):\alpha,\beta]$, where $m(\alpha,\beta)$ denotes the multiplicity of $(\alpha,\beta)\in\mathcal{D}(\lambda)$. We are able to determine the first six examples by using *Mathematica*:

$$\mathcal{D}(1)=\{[1:1+i,i],[1:1-i,-i]\},$$
$$\mathcal{D}(2)=\{[1:\pm2i,0],[2:2,-4]\},$$
$$\mathcal{D}(3)=\{[1:-2\pm2i,0],[3:2+2i,8i],[3:2-2i,-8i]\},$$
$$\mathcal{D}(4)=\{[2:\pm4,0],[4:4,-64],[4:\pm4i,16]\},$$
$$\mathcal{D}(5)=\left\{\begin{array}{r}[1:-4\pm4i,0],[5:-4+4i,-32i],[5:4+4i,16i(11\pm5\sqrt{5})],\\ [5:-4-4i,32i],[5:4-4i,-16i(11\pm5\sqrt{5})]\end{array}\right\},$$
$$\mathcal{D}(6)=\left\{\begin{array}{l}[2:8,0],[12:\pm8,-64],[6:8,-4096],\\ [4:\pm8i,0],[6:\pm8i,-64],[6:\pm8i,1728]\end{array}\right\}.$$

Three summation formulae corresponding to odd λ are displayed below.

PROPOSITION 5.2 (Multiple series identities: $m\in\mathbb{N}$).

$$\sigma_m\Big(\frac{(-1)^n}{2n-1}\Big|n\in\mathbb{N}\Big)=\frac{\pi^m(-1)^{\lceil m/2\rceil}}{2^{2m}m!},$$
$$\sigma_m\Big(\frac{(-1)^n}{(2n-1)^3}\Big|n\in\mathbb{N}\Big)=\frac{3\pi^{3m}(-1)^{\lceil m/2\rceil}}{2^{3m+1}(3m)!},$$
$$\sigma_m\Big(\frac{(-1)^n}{(2n-1)^5}\Big|n\in\mathbb{N}\Big)=\frac{5\pi^{5m}(-1)^{\lceil m/2\rceil}}{2^{6m+2}(5m)!}\Big\{(11\pm5\sqrt{5})^m-2^m\Big\}.$$

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6. Complete symmetric functions $\hbar_m(y_n|n \in \mathbb{N})$

Recall that elementary symmetric functions $\sigma_m(y_n|n \in \mathbb{N})$ and complete symmetric functions $\hbar_m(y_n|n \in \mathbb{N})$ satisfy the orthogonal relation

$$\sum_{k=0}^{m}(-1)^k \sigma_k(y_n|n \in \mathbb{N})\hbar_{m-k}(y_n|n \in \mathbb{N}) = 0 \quad \text{for } m \in \mathbb{N},$$

which can be reformulated as the recurrence relation

$$\hbar_m(y_n|n \in \mathbb{N}) = \sum_{k=1}^{m}(-1)^{k-1}\sigma_k(y_n|n \in \mathbb{N})\hbar_{m-k}(y_n|n \in \mathbb{N}). \tag{6.1}$$

Therefore for $m, \lambda \in \mathbb{N}$, there are four recurrence relations for the four multiple zeta series corresponding to

$$y_n \in \left\{\frac{1}{n^{2\lambda}}, \quad \frac{1}{(2n-1)^{2\lambda}}, \quad \frac{(-1)^{n\lambda}}{n^{2\lambda}}, \quad \frac{(-1)^{n\lambda}}{(2n-1)^{\lambda}}\right\}. \tag{6.2}$$

By making use of generating functions, the author [2] derived the following general formulae (where a typo in the second one is corrected). The first formula has been reproved recently by Merca [7] by induction.

Proposition 6.1 (Multiple series identities: $m \in \mathbb{N}$).

$$\hbar_m\left(\frac{1}{n^2}\middle|n \in \mathbb{N}\right) = (-1)^m \pi^{2m}\frac{B''_{2m}}{(2m)!},$$

$$\hbar_m\left(\frac{1}{n^4}\middle|n \in \mathbb{N}\right) = \pi^{4m}\sum_{k=0}^{2m}(-1)^k\frac{B''_{2k}}{(2k)!}\frac{B''_{4m-2k}}{(4m-2k)!}.$$

By carrying out the same approach as in [2], we can establish further summation formulae.

Proposition 6.2 (Multiple series identities: $m \in \mathbb{N}$).

$$\hbar_m\left(\frac{1}{(2n-1)^2}\middle|n \in \mathbb{N}\right) = (-1)^m\frac{\pi^{2m}E_{2m}}{2^{2m}\,(2m)!},$$

$$\hbar_m\left(\frac{1}{(2n-1)^4}\middle|n \in \mathbb{N}\right) = \frac{\pi^{4m}}{2^{4m}}\sum_{k=0}^{2m}(-1)^k\frac{E_{2k}}{(2k)!}\frac{E_{4m-2k}}{(4m-2k)!};$$

$$\hbar_m\left(\frac{(-1)^n}{n^2}\middle|n \in \mathbb{N}\right) = \frac{\pi^{2m}}{2^{2m}}\sum_{k=0}^{m}(-1)^k\frac{B''_{2k}}{(2k)!}\frac{E_{2m-2k}}{(2m-2k)!},$$

$$\hbar_m\left(\frac{(-1)^n}{2n-1}\middle|n \in \mathbb{N}\right) = (-1)^m\frac{\pi^m S_m}{4^m\,m!}.$$

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In the final formula, $\mathcal{S}_m$ denotes the Springer number (see [9, A001586]), whose exponential generating function is $1/(\cos x - \sin x)$.

When λ becomes bigger, the closed formulae for complete symmetric functions $\hbar_m(y_n|n \in \mathbb{N})$ corresponding to the four choices in (6.2) are very complicated. However, for specific integer values of m and λ, it is possible to compute them by means of the recurrence relation (6.1). We record here, as examples, two positive series:

$$\hbar_3\Big(\frac{1}{n^6}\Big|n \in \mathbb{N}\Big) = \frac{2^6 \cdot 61 \cdot 3659\pi^{18}}{(21)!},$$

$$\hbar_3\Big(\frac{1}{(2n-1)^6}\Big|n \in \mathbb{N}\Big) = \frac{4810397\pi^{18}}{12 \cdot (17)!};$$

and five alternating series

$$\hbar_3\Big(\frac{(-1)^n}{n^6}\Big|n \in \mathbb{N}\Big) = \frac{-22 \cdot 2576328919\pi^{18}}{(21)!},$$

$$\hbar_3\Big(\frac{(-1)^n}{(2n-1)^3}\Big|n \in \mathbb{N}\Big) = \frac{-3 \cdot 29 \cdot 139\pi^9}{2^{10} \cdot 9!},$$

$$\hbar_4\Big(\frac{(-1)^n}{(2n-1)^3}\Big|n \in \mathbb{N}\Big) = \frac{5 \cdot 13 \cdot 21121\pi^{12}}{2^{15} \cdot (11)!},$$

$$\hbar_5\Big(\frac{(-1)^n}{(2n-1)^3}\Big|n \in \mathbb{N}\Big) = \frac{-3 \cdot 11 \cdot 419 \cdot 209809\pi^{15}}{2^{16} \cdot (15)!},$$

$$\hbar_3\Big(\frac{(-1)^n}{(2n-1)^5}\Big|n \in \mathbb{N}\Big) = \frac{-5 \cdot 23 \cdot 67 \cdot 139 \cdot 5563\pi^{15}}{2^{17} \cdot (15)!}.$$

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https://doi.org/10.1017/S0004972719000819 Published online by Cambridge University Press

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https://doi.org/10.1017/S0004972719000819 Published online by Cambridge University Press