PEF, **5** (1): 1–25, March, 2006.
doi:10.1017/S1474747205002283 Printed in the United Kingdom

Planning for the optimal mix of paygo tax and funded savings

GEORGES DE MENIL*

PSE, Paris, and Stern School, NYU
(*e-mail:* nydemenil@aol.com)

FABRICE MURTIN

CREST (INSEE), and LSE, London
(*e-mail:* fabrice.murtin@ensae.fr)

EYTAN SHESHINSKI

Hebrew University of Jerusalem and Princeton University
(*e-mail:* mseytan@pluto.mscc.huji.ac.il)

Abstract

We analyze the optimal balance between social security taxation and private saving in the provision of retirement income in dynamically efficient economies, a question at the center of policy debates in Europe and the United States. We consider the relative importance for this question of the return to capital, the internal return of the pay-as-you-go system, and the variabilities and correlation (or independence) of labor earnings and the capital return. We analyse these influences theoretically in the context of a two-period, overlapping generations model with uncertainty. We use a new method to calibrate the model using annual data on GDP per worker and the total real return on equities, from 1950 to 2002, from which we infer the stochastic characteristics of lifetime labor income and the return to lifetime savings in the US, UK, France and Japan. We obtain a range of optimal, steady-state values of the social security tax and the rate of lifetime savings. When the relative rate of risk aversion is assumed to be 2.5, the computed optimal tax varies from 5% in the United States to 22% in Japan. France is similar to Japan, and the UK is in between.

1 Introduction

Assessment of the optimal balance between social security taxation and private saving in the provision of retirement income is a recurrent theme in contemporary policy debates in Europe and the United States. In Germany, a pension reform initiative in May 2001 featured the introduction of publicly subsidized and supervised

* Corresponding author.
We are grateful for the comments of three anonymous referees and of Gabrielle Demange, Peter Diamond, Richard Zeckhauser, Stephen Zeldes and the participants of the macroeconomics seminars of PSE, the Stern School and Columbia University. Any errors are our own.

retirement savings instruments, 'Riester Accounts', the initial purpose of which was to substitute for possible shortfalls in public pensions. In France, a new retirement savings vehicle, the 'Plan d'Epargne Retraite Populaire', was similarly introduced as part of a broader reform of the public social security system, in August 2003. The United Kingdom has experimented with successive variants of voluntary opt-outs from social security plans into private savings plans ever since 1979. In its first report in October 2004, the UK Pensions Commision called for a public debate on the advisability of instituting fully mandatory, private savings accounts. In the United States, similar issues are also involved in the national debate about the pros and cons of new mandatory or voluntary 'Private Savings Accounts'.[1]

Our objective is to contribute to this debate by examining the determinants of the optimal balance between intergenerational transfers and private saving. Though the initial, steady-state analysis of Diamond (1965) implied that pay-as-you-go financed social security was only welfare enhancing in dynamically inefficient economies,[2] it is now widely acknowledged that that judgement must be relaxed when wage and capital income are subject to uncertainty. Merton (1983), Gordon and Varian (1988), Gale (1990), Demange and Laroque (1999, 2000), and Demange (2002) have shown that a pay-as-you-go system can enhance welfare, even in dynamically efficient economies, because of the insurance it provides against otherwise uninsurable, macroeconomic risk. In short, it may frequently be desirable for societies to provide for retirement through both channels. However, little attention has been paid to exploring the relationship between the choices of different societies for one system or the other, and differences in the stochastic characteristics of their economies.

Our approach is normative. We calculate the pay-as-you-go (henceforth paygo) tax rate and overall savings rate which a benevolent planner would choose for a society on a steady-state path. Part of this calculation reflects subjective considerations, such as the degree of risk aversion in the underling utility function. However, objective considerations – the variability of wages around their trend, and the variability of the return to capital – also figure centrally in the planner's calculations. We examine those calculations both theoretically, and empirically in the case of four large, developed countries. We use annual data on the growth of earnings and the return to capital in the United States, the United Kingdom, France, and Japan to estimate the stochastic characteristics of the corresponding lifetime variables for these countries. The differences of those means and variabilities from one country to the next imply different optimal steady-state tax and savings rates. We compute these implied rates for different assumed values of relative risk aversion. Dutta, Kappur, and Orszag (2000) – henceforth DKO – were the first to point to the significance of empirical differences in the patterns of wage and capital income in different countries.

[1] See Bonin (2001), Conseil d'orientation des retraites (2004) and UK Pension Commission (2004). In the United States, though the debate is sometimes presented as one between supporters and opponents of privatization, many proposals advocate providing retirement income through both pay-as-you-go and private saving, and differ in the weight and the form envisioned for each. See Feldstein (2005) and Liebman (2005).

[2] See also Aaron (1966).

The portfolio approach which they take to the problem allows them to differentiate between the relative weights different countries will want to give to the two systems, but does not allow them to calculate, as we do, actual preferred levels of the tax rate and the rate of lifetime savings. Matsen and Thogersen (2004) – henceforth MT – use an approach which is similar to ours to calculate preferred tax rates and analyse lifetime portfolio decisions, but their model of the dynamics of lifetime earnings differs from ours in ways which lead them to significantly different conclusions.[3]

The challenge in an exercise such as this, is to estimate the steady-state growth, volatility, and comovements, of *lifetime* earnings and the return to *lifetime* savings in the countries in question. Both DKO and MT approximate the desired lifetime dynamics with measures derived from one or two decade-long observations from 1900 through 2000.[4] In our view, the major structural changes that differentiate post-World War II economies from pre-War economies argue in favor of concentrating on post-War dynamics. But in the period since 1950, all one has observed is one lifecycle. How, therefore, can one estimate the volatility of lifetime earnings and the return to lifetime savings in these post-War economies? Our approach is to use annual data from 1950 to 2002, to estimate the dynamic characteristics of earnings and the return to savings, and to use the estimated, annual, joint densities to compute the variances and covariances of the lifetime earnings and the return to lifetime saving of a representative individual in each country. This allows us to calculate the representative individual's expected lifetime utility on a steady-state path, and to search for the tax and saving rate which maximize that expected utility.

Section 2 describes the model. Section 3 analyses the theoretical determinants, under certain conditions, of optimal paygo tax rates and savings rates. Section 4 discusses the relationship between the dynamics of annual earnings and the annual return to capital, on the one hand, and the variability of lifetime earnings and the lifetime return to saving, on the other. This discussion paves the way for estimates of the stochastic characteristics of these lifetime income concepts in the United States, the United Kingdom, France, and Japan. Section 5 computes – assuming constant absolute risk aversion – the optimal steady-state tax and savings rates implied by these different means and variabilities. Section 6 concludes.

2 The model

Let ω_1 be the lifetime labor income of a representative individual in the current generation of workers and ω_2 be the lifetime income of a representative individual in the next generation of workers (the one which will be making social security contributions to the current generation when it retires).[5] Assume initiallly, for purposes of simplicity, that both the population and the expected, lifetime earnings of

3 Our paper was written without knowledge of the Matsen and Thogerson study, which was published after ours was submitted to the Journal.

4 DKO work with observations of non-overlapping decades. MT estimate the dynamics from overlapping two-decade periods.

5 The model abstracts from intra-generational heterogeneity and related distributional issues.

successive generations are constant. (These simplifying assumptions will be dropped when we use the model to make empirical assessments.)

$$c_1 = (1-\theta)\omega_1 - S \tag{1}$$

$$c_2 = \theta\omega_2 + RS \tag{2}$$

where c_1 and c_2 are consumption during working and retirement years. S is the saving of the representative individual ($S \geqslant 0$), and θ is the contribution rate in a paygo system where the receipts and expenditures are assumed to be balanced each period ($1 \geqslant \theta \geqslant 0$).[6] R is the gross return to the saving the representative individual accumulates during his working years. R, ω_1, and ω_2 are random variables. Let $u(\cdot)$ be the representative individual's thrice differentiable von Neumann–Morgenstern utility function for consumption in each phase of life.[7] Expected lifetime utility is

$$\mathscr{V} = E[u(c_1)] + \beta E[u(c_2)] \tag{3}$$

where β, $0 \leqslant \beta \leqslant 1$ is a discount factor reflecting personal time preference.

Our purpose is to assess the optimality of different values of θ and S. In order to do this, we shall take the point of view of a benevolent planner, who knows the probability distributions of ω_1, ω_2, and R, and knows the preferences of the representative individual. We conduct the following thought experiment. If the planner could somehow rewrite history, and place the economy in stochastic equilibrium, on a steady-state growth path consistent with these distributions, what values of S and θ would he choose? We refer to these steady-state optimal values as S^* and θ^*. This thought experiment abstracts from the actual conditions in which the planner finds the economy, and takes no account of the transitional costs of getting to the optimal steady-state. It nonetheless permits the planner to compare the steady-state path implied by current values of S and θ with what could be a better steady-state path. Because he reasons *ex ante*, the planner effectively maximizes the expected lifetime utility of the representative individual in any generation in the same stochastic steady state.[8]

We leave the possible feedback from S to R and ω (through a closed economy aggregate production function, for instance) out of the analysis. One way to interpret this simplification is to argue that free trade determines factor prices up to structural and institutional country-specific effects.[9] Factors, however, are not mobile, and workers in each country earn the wages paid in their country and the return to savings available in their country.[10]

[6] Theoretically, in an economy with no credit constraints beyond a no-ponzi-game limit on borrowing, either S or θ could be negative. We assume that liquidity and institutional constraints do not permit negative saving or reverse generational transfers.

[7] We abstract from the possibility that the shape of the utility function may not be the same in the active and retirement years.

[8] This *ex ante* calculation corresponds to what Matsen and Thogersen (2004) term a 'Rawlsian approach to risk'.

[9] We will represent these effects with random variables whose distributions vary from country to country. The important simplifying assumption is that these country-specific effects are not influenced by domestic capital accumulation.

[10] In a given country, as domestic savings are invested domestically, and, therefore, national capital accumulates, comparative advantage will lead to increasing production and exports of capital-intensive

In the following section, we will analyse the determinants of θ^* and S^* in the abstract, without reference to any specific country, and without specifying the form of the utility function. In Sections 4 and 5, we will use historical data for four OECD countries, and assume a specific utility function, in order to estimate a range of values of θ^* and S^* for those countries.

3 Comparative statics

Existence of a maximum

Before exploring the comparative statics of the effects of parameter changes on θ^* and S^*, we shall demonstrate that a maximum of (3), subject to (1) and (2) does indeed exist, under reasonable circumstances. Let $u_i = u(c_1)$, $i = 1,2$. The first-order conditions for this maximum are

$$\frac{\partial \mathscr{V}}{\partial S} = \varphi = -E[u_1'] + \beta E[Ru_2'] = 0 \tag{4}$$

$$\frac{\partial \mathscr{V}}{\partial \theta} = \psi = -E[u_1'\omega_1] + \beta E[u_2'\omega_2] = 0 \tag{5}$$

Diminishing marginal utility implies that

$$\frac{\partial^2 \mathscr{V}}{\partial S^2} = \varphi_S = E[u_1''] + \beta E[u_2''R^2] < 0$$

$$\frac{\partial^2 \mathscr{V}}{\partial \theta \partial S} = \varphi_\theta = E[u_1''\omega_1] + \beta E[u_2''R\omega_2] < 0$$

$$\frac{\partial^2 \mathscr{V}}{\partial \theta^2} = \psi_\theta = E[u_1''\omega_1^2] + \beta E[u_2''\omega_2^2] < 0$$

$$\frac{\partial^2 \mathscr{V}}{\partial S \partial \theta} = \psi_S = E[u_1''\omega_1] + \beta E[u_2''R\omega_2] < 0$$

Second-order conditions for a maximum require that $\varphi_S < 0$, $\psi_\theta < 0$, and $\Delta = \varphi_S\psi_\theta - \varphi_\theta\psi_S > 0$. The fact that these conditions depend on the probability distribution of ω and R appears clearly when we develop the expression for the determinant, Δ

$$\begin{aligned}\Delta &= (E[u_1''] + \beta E[u_2''R^2])(E[u_1''\omega_1^2] + \beta E[u_2''\omega_2^2]) - (E[u_1''\omega_1] + \beta E[u_2''R\omega_2])^2 \\ &= E[u_1'']E[u_1''\omega_1^2] + \beta E[u_1'']E[u_2''\omega_2^2] + \beta E[u_2''R^2]E[u_1''\omega_1^2] \\ &\quad + \beta^2 E[u_2''R^2]E[u_2''\omega_2^2] - E[u_1''\omega_1]^2 - \beta^2 E[u_2''R\omega_2]^2 - 2\beta E[u_1''\omega_1]E[u_2''R\omega_2]\end{aligned} \tag{6}$$

It can be shown that if the marginal utility function is convex, but not too sharply so, $\Delta > 0$, and there is a unique solution to the planner's problem, regardless of the shape of the probability distribution. Specifically, the following two conditions

products. We assume that this trade effect eliminates what would otherwise be a tendancy for R to decline.

are sufficient for (3) to have a unique maximum (see Appendix A.)

$$u'''(c) \geqslant 0, \quad \forall c \tag{7}$$

$$-\frac{u'''(c_i)\omega_i}{u''(c_i)} < 1, \quad \forall c_i, i = 1, 2 \tag{7'}$$

How do the optimal θ^* and S^* change when the underlying parameters change? If a maximum with $\theta^* > 0$ and $S^* > 0$ exists, the comparative statics of that maximum can be examined by totally differentiating the first-order conditions, (4) and (5). We consider, the effect on θ^* and S^* of, on the one hand, an increase in level of R, and, on the other, an increase in the variability of ω.

Increase of R

When R is non-random it can be shown, by total differentiation, that a sufficient condition for $\frac{d\theta^*}{dR} < 0$ and $\frac{dS^*}{dR} > 0$ is $\frac{-E[u_2'']E[c_2]}{E[u_2']} < 1$ (see Appendix B).

Mean-preserving spread of ω

One of the rationales for a paygo system in a dynamically efficient economy is that it provides insurance against the risk of being born and working in a period of depressed labor earnings. Each cohort can draw twice from the distribution generating lifetime earnings, once during its active years, and once during retirement, when it receives transfers from the next generation. This insurance should be more valuable the greater is the uncertainty surrounding lifetime earnings, and therefore the greater the variability of ω. In this section we examine the relationship between θ^* and the variability of ω analytically, initially without restricting either the utility function or the probability distribution of ω to any specific form. In Section 5, we estimate numerically the sensitivity of θ^* to the variability of ω under the assumption that the utility function is of the constant absolute risk aversion type.

Consider a mean-preserving spread of the probability distribution of ω

$$\omega_i = \bar{\omega} + \delta(\hat{\omega}_i - \bar{\omega}), \quad i = 1, 2, \quad 0 < \delta < 1 \tag{8}$$

If we substitute (8) into the first-order conditions (4) and (5), we can then differentiate totally, and obtain expressions for $\frac{d\theta^*}{d\delta}$ and $\frac{dS^*}{d\delta}$ (see Appendix C). Inspection of these derivatives suggests that, in the most general case, their signs are ambiguous. They depend on the probability distributions of ω and R before the variability of ω is increased, and on the initial values of θ^* and S^*. The initial value of θ^* is important, because the incomes, the variability of which the representative agent is seeking to balance, are not just ω_1 and ω_2, but $(1-\theta)\omega_1 - S$ and $\theta\omega_2 + RS$. For instance, we show in the appendix that, if the initial value of $\theta^* = 0$ and therefore all the wage variability falls on the first-period, any increase in that variability will increase the desired valuel of θ^*, i.e. at $\theta^* = 0$, $\frac{d\theta^*}{d\delta} > 0$. Things also simplify when the utility function is quadratic. We show in the appendix that, in that case, a necessary and sufficient condition for $\frac{d\theta^*}{d\delta} > 0$ is $\frac{(1-\theta^*)}{\theta^*} > \beta$, which is likely to be the case.

Nonetheless, the remaining ambiguity argues strongly for turning to an empirical assessment of the relationship. In order to do so, we must calibrate our simple model to real-world economies, and perform sensitivity analyses to explore their response to changes in the parameters.

Increased curvature of u

Before turning to calibration, we consider a precautionary tale. We have no evidence on the shape of the utility function, and will, in the end, assume one specific functional form as being reasonable. But θ^* and S^* depend on the specific form that is chosen. We demonstrate here that, given the random process for ω and R, θ^* and S^* will change if the degree of risk aversion changes. We shall proceed by replacing u with its Taylor's series expansion. Expanding in this manner $u'(c_1)$ and $u'(c_2)$ in the neighbourhood of $\bar{c}_1 = E(c_1)$ and $\bar{c}_2 = E(c_2)$, we obtain first-order conditions, which are expressions with higher-order derivates of $u(\bar{c}_1)$ and $u(\bar{c}_2)$, and expectations of first-, second- and higher-order moments of c_1 and c_2. We then disturb these first-order conditions by introducing, ceteris paribus, a small change in u''', $du'''(\bar{c}_1) = du'''(\bar{c}_2) = dz$. We analyse the effects of this change in curvature by solving for $d\theta^*$ and dS^* as functions of dz.

We analyse the effects of $du'''(c)$ in the neighbourhood of $u'''(c) = 0$. This is equivalent to supposing that u starts out as a quadratic, and that we modify it by introducing a small amount of additional curvature. With these assumptions, it can be shown (see Appendix D) that

$$\frac{d\theta^*}{dz} \gtrless 0 \Leftrightarrow R \gtrless 1$$

Moreover, the magnitude of $\frac{d\theta^*}{dz}$ is proportional to $\sigma_\omega^2\, u''$, the product of the steepness of u', and the variance of ω. This result goes in the direction of Kimball (1990), who found, in the context of a two-period investment, that increased curvature leads rational individuals to displace more consumption from the present to the future. He dubs this effect 'precaution'.

4 Calibration of country dynamics

Annual wages and the annual return to capital

Having explored some of the determinants of θ^* and S^* analytically, we now turn to an empirical assessment of tax and savings rates in four large OECD countries, which takes account of the differences in the dynamic characteristics of lifetime earnings from one country to the next. The countries for which we calibrate the simple model of Section 2 are the United States, the United Kingdom, France, and Japan. These countries are, on the one hand, large enough for it to be reasonable to assume that domestic savings are invested in domestic capital and, on the other hand, open enough for factor prices to be determined by trade.

We observe in each country the annual progression of real GDP per person in the labor force, and the real total return of publicly listed stocks. We take real GDP as a proxy for total compensation.[11] Dividing by the labor force produces a measure of real earnings per person, w_t, which directly reflects changes in the employment rate. If the unemployment rate rises, our measure of the average labor earnings of the representative individual will decline, even if the real wage of employed persons remains constant.[12] We take the real total return of publicly listed stocks as a measure of the real annual return to savings, r_t.[13]

We are interested in estimating the dynamic characteristics of w_t and r_t, and inferring from them the dynamic characteristics of the analogous lifetime variables.

We assume that w_t, which we refer to as the real wage rate, varies stochastically around a deterministic trend, which itself converges to a path growing exponentially at the rate g.[14]

$$\begin{aligned} \ln w_t &= a + gt + f(t) + x_t, \\ \lim f(t) &= 0, \quad t \to \infty \end{aligned} \tag{9}$$

where $w_0 = 1$, and x_t is the innovation on annual wages.[15] Different specifications of the function f were tested, and two functional forms have been applied.[16] We also represent the log returns to equity as the sum of a constant mean return $\bar{r}$ and an innovation ε_t

$$\ln(1 + r_t) = \bar{r} + \varepsilon_t \tag{10}$$

We estimate equations (9) and (10), and compute serial correlations and cross correlations of x_t and ε_t. Our principal challenge, then, is to infer from these annual data the stochastic characteristics of the corresponding lifetime concepts.

Calibrating lifetime dynamics

Let us assume that the representative individual works $2T$ years, and lives in retirement for another T years. Using annual data to calibrate the simple, over-lapping

[11] Like DKO and MT, we use real GDP as a proxy for total labor compensation, because of the absence of lengthy time series which are comparable across countries for the latter.

[12] This is equivalent to making the simplifying assumption that the loss of labor earnings caused by unemployment is shared across the labor force.

[13] Annual series for real GDP are taken from IMF, International Financial Statistics. In cases where the IMF data begin later than 1950, we have interpolated it backwards from real GDP series in Mitchell (1998). The labor force is taken from OECD (2005a) from 1970 to 2002. For the US the UK, and Japan, we interpolate backwards from 1970 to 1950 using Mitchell's series on the rate of unemployment and total unemployment. In the case of France, we extrapolate backwards using Carre, Dubois, and Malinvaud (1972), p. 82, Table 12 (series b).

We compute real annual returns on publicly listed stocks from monthly total return series obtained from Global Financial Data (2005). We take annual averages of the monthly rates of return, and deflate them by the December to December annual rate of increase of the CPI.

Our constructed series can be obtained upon request.

[14] Though our conceptual framework is one in which real wages per efficiency unit are determined by trade (see Section 2), this is not inconsistent with different trends in real wages per man hour across countries, if the rate of (labor-augmenting) technical change differs across countries.

[15] Setting $w = 1$ at the beginning of the active life of each cohort ensures that every cohort in every country, no matter when born, views and analyzes lifetime prospects in the same way as every other.

[16] For the US and the UK we used a simple exponential form $\mu e^{-\rho t}$ and for France and Japan, which displayed a larger slowdown in growth from 1950 to 1980, more degrees of freedom were needed. As a consequence, we used the functional form $\mu \log[1 - \nu e^{-\rho t}]$, which is the solution to non-linearized convergence equations.

generations model of Section 2 entails condensing $3T$ years into two periods.[17] A natural way to condense the data is to construct income and savings variables for the two periods by computing weighted averages of the corresponding annual variables. But what weights should one use?

On an optimal, steady-state path, the expected marginal utility of consumption is equal at every point in time. This suggests using the expected marginal utilities of consumption relative to a base year, along an optimal steady-state path, as aggregation weights. In principle, these weights change from one steady-state path to another. In order to avoid the cumbersom procedure of recalculating the aggregate variables with a new set of weights for each steady-state path, we assume that the expected marginal utilities of consumption on any steady-state path are proportional to an annual discount factor[18] expressing the personal rate of time preference, δ, of the representative individual.[19] Thus we attribute, within each aggregate period, the following weight to consumption, income, and saving for the year t

$$\frac{1}{(1+\delta)^{t-t_0}}$$

where t_0 is the base period. Our aggregate measure of the labor earnings of the representative individual in the first-period is[20]

$$\omega_1 = \sum_{t=0}^{2T-1} \frac{w_t}{(1+\delta)^{t-T}} \tag{11}$$

Similarly, our aggregate measure of the amount of second-period labor income-potentially available for paying pensions to each representative individual is[21]

$$\omega_2 = d \sum_{t=2T}^{3T-1} \frac{w_t}{(1+\delta)^{t-2.5T}} \tag{12}$$

The factor d is the ratio of the active labor force to the number of retired persons, the inverse of the dependency ratio. It differs from country to country because of differences in the long-term growth rate of the population and differences (due to custom and mortality) in the ratio of retirement years to active years. On any given

[17] This is, by its nature, an index number problem. As for all index number problems, it has no exact solution. Any index number we construct can only approximate the multi-dimensional reality.

[18] If the expected marginal utility of consumption is independent of changes in the level of consumption over the lifecycle, this condition, which we assume for computational simplicity, holds exactly.

[19] The discount factor, δ, is, like the specific form of the instantaneous utility function, one of the aspects of personal preferences that we are obliged to specify arbitrarily.

[20] In the following expression, w_t is supposed to stand for the real wage of the representative individual in the tth year of his active life. We assume (for lack of better information) that this wage does not depend on seniority, and, therefore, that the representative individual earns the economy-wide average wage rate every year.

[21] In this expression, the income being measured is not that of the representative individual himself. dw_t represents the average labor earnings of the sucessor cohort per retiree. w_t measures precisely, without simplifying assumptions, the labor income in which retirees have, collectively, a pooled interest.

historical trajectory, d varies over time, as demographic patterns change. But, on a steady-state path, d is necessarily constant.[22]

As expressions (11) and (12) make clear, ω_1 and ω_2 are each discounted to the mid point of the period to which they apply. Interperiod temporal preference is captured by the factor β in (3).[23]

We turn now to our aggregate measures of first-period saving, S, and its return. Since saving is to be deducted from first-period income, it must be in the same units. Our aggregate measure is, therefore, constructed by weighting annual saving with the same weights that we apply to annual labor income to obtain our measure of first-period income. For computational simplicity, we assume that annual saving is a constant portion s of the trend value of the annual real wage, $\widetilde{w}_t$.[24] Thus our aggregate measure of first-period saving is

$$S = s \sum_{t=0}^{2T-1} \frac{\widetilde{w}_t}{(1+\delta)^{t-T}} \tag{13}$$

Our aggregate measure of the fund the representative individual will have accumulated at the beginning of his retirement period, valued at the mid point of that period, is

$$F = s(1+\delta)^{T/2} \sum_{t=0}^{2T-1} \frac{\widetilde{w}_t}{(1+\delta)^{t-T}} \prod_{\tau=t}^{2T-1} (1+r_\tau) \tag{14}$$

where r_τ is the annual rate of return on capital.[25] It is then natural to take as our aggregate measure of the return to saving, the ratio

$$R = \frac{F}{S} \tag{15}$$

If R is the factor by which first-period saving is multiplied to obtain the corresponding retirement fund, the factor by which first-period labor income is magnified to obtain the second-period base for paygo pensions is

$$G = \frac{E(\omega_2)}{E(\omega_1)}$$

We have assumed that annual saving is proportional to the trend value of the real wage. Since this trend value is known with certainty, S remains a non-random variable. The rate of time preference is also taken to be non-random. It is clear from their definitions, however, that ω_t, ω_2, and R are stochastic. We shall assume that

[22] In our calibrations, we set d in each country equal to the value at which it is expected to stabilize after the current demographic transition is completed.

[23] The rate of time preference that β measures between aggregate periods is the same as that which δ measures annually. Since the base years of the two periods are $1.5T$ years apart

$$\beta = \frac{1}{(1+\delta)^{1.5T}}$$

[24] In a steady state, the rate of growth of the real wage is the value of g estimated in the previous section.

[25] We value this fund at the mid point of the retirement period, in order that it be in the same units as the individual's paygo pension income, $\theta\omega_2$, to which it is added to obtain total retirement income.

Table 1. *Dynamic characteristics of lifetime earnings and return to capital, four countries*

	d	$G=\frac{E(\omega_2)}{E(\omega_1)}$	$E(R)$	$s_{\omega_{1/2}}=\frac{\sigma_{\omega_{1/2}}}{E(\omega_{1/2})}$	$s_R=\frac{\sigma_R}{E(R)}$	ρ_{ω_2}, R	ρ_{ω_1,ω_2}
US	1.94	1.44	6.93	0.02	0.55	−0.14	0.14
UK	1.48	1.69	6.39	0.02	0.67	−0.26	0.18
France	1.42	1.60	6.81	0.03	0.79	−0.50	0.40
Japan	1.07	1.72	9.51	0.04	0.88	0.10	0.50

Source: authors' calculations.
d from US Bureau Census (2005).

they are jointly normally distributed. Murtin (2003) derives approximations for computing the means, variances, and covariances of these aggregate measures directly from the observed moments of the annual data. Table 1 uses these approximations to calculate the means and expected coefficients of variation and correlation of lifetime earnings and the return to lifetime saving which are implied by the annual movement of real wages and the return to capital in the United States, the United Kingdom, France, and Japan.[26] The table also reports the average value of d, the inverse of the dependency ratio, projected for each country between 2045 and 2065 by the US Bureau of Census. This is the ratio we use for the steady-state calculations reported in the next section.

Table 1 portrays substantial differences in the dynamic characteristics of earnings and the return to saving in the United States, the United Kingdom, France, and Japan. Trend paths differ from country to country, and variabilities around these trend paths differ. In all countries, the return on saving is many times greater than the intergenerational return of the paygo system, captured by the factor G. In the United States, for instance, during an average lifetime, savings are almost multiplied by 7, whereas, if contribution rates remain constant, an average retiree can only expect to receive $1.44 for every $1.00 he contributes to the paygo system. In part, these intergenerational paygo returns are lower than might be expected because they reflect our estimates of the long-term growth rates of GDP per employee to which these countries are tending, which are lower, particularly in France and Japan, than the growth rates observed over the last 50 years. In part, they are also low because of the high demographic dependancy ratios projected, notably again in France and Japan.

Because we construct our aggregate measures of lifetime incomes explicitly from their annual components, we have a framework for decomposing their growth into the effects of real wage rate growth, on the one hand, and demographic structure, on the other. DKO and MT, who take decades of real GDP growth as composite proxies for the evolution of lifetime earnings, agglomerate economic and demographic factors, and cannot directly distinguish between their effects.

[26] The coefficients of variation of ω_1 and ω_2 were estimated to be equal to a first approximation, and are thus reported as one in the table.

Though the expected intergenerational return of the paygo system differs from country to country, the estimated variability of first- and second-period labor earnings around their growth path is modest in every case. We find coefficients of variation of these lifetime earnings between 2% and 4%. These low numbers reflect the well-known moderation of the post-War business cycle (compared with the amplitude of cyclical movement in the inter-war period). The lifetime return to saving, R, on the other hand, displays both much more variability, and more cross-country differences in that variability. The high coefficient of variation of R, more than 20 times that of ω in each country, is clearly the dominant source of uncertainty in every case. The next section will show that, though intergenerational insurance remains a potentially significant feature of paygo systems, differences in the variability of returns to saving are empirically more important to the understanding of intercountry differences in preferred paygo rates.

Table 1 also reports serial correlations between the wage earnings of successive generations, and negative correlations between R and ω_2. Serial correlations are high in France and Japan, which both experienced protracted periods of slow growth. The negative correlation between R and ω_2 in France is an inference based on the poor performance of that country's stock market during years of strong GDP growth.

5 Assessing national paygo tax and savings rates

The CARA utility function and the relative rate of risk aversion

With the lifetime moments described in Table 1, it is possible to compute θ^* and S^* for any given utility function. The next step is, therefore, to specify the form of the utility function, and to fix the parameter values which affect the relative rate of risk aversion. We experimented with both the quadratic and the CARA utility functions, and obtained reasonable numerical results in both cases. However, as an empirical tool for examining policies which imply large changes in lifetime consumption, the quadratic specification presents two disadvantages: marginal utility is positive only within a limited range; and, within that range, the relative rate of risk aversion varies between 0 and ∞. The resulting implication, for instance, that a shift from paygo to funding could change relative risk aversion by a factor of ten or more, is not plausible. The CARA presents neither disadvantage: marginal utility is everywhere positive, and the relative rate of risk aversion is simply proportional to c.[27] Therefore, in what follows, we choose to focus on numerical calculations with the CARA specification.

The assumption that earnings and the return to savings are jointly normally distributed allows us to use the Laplace transform to express the expectation of lifetime utility, (3), directly in terms of the moments presented in Table 1. Specifically, we substitute (1) and (2) into (3), and write

$$V = -\{e^{-\alpha(1-\theta)\bar{w}+\alpha S+\alpha^2(1-\theta)^2\sigma_\omega^2/2} + \beta e^{-\alpha\theta G\bar{w}-\alpha S\,\bar{R}+\alpha^2/2(\theta^2G^2\sigma_w^2+S^2\sigma_R^2+2\theta SG\sigma_w\sigma_R\rho_{wR})}\} \quad (16)$$

[27] Given a, the elasticity of relative risk aversion with respect to consumption is $1+\frac{1}{a-c}$ in the quadratic case. This elasticity is equal to 1 in the CARA case.

Table 2. *Estimates of optimal tax and saving rates (in percents) with CARA utility function*

	RRA*	2	2.5	3	3.5
United States	θ^*	1.2	5.4	13.1	16.0
	$\frac{S^*}{E(\omega_1)}$	15.6	13.2	10.4	9.3
United Kingdom	θ^*	7.7	14.3	18.1	21.9
	$\frac{S^*}{E(\omega_1)}$	12.0	9.5	8.3	7.1
France	θ^*	10.6	19.3	24.9	28.1
	$\frac{S^*}{E(\omega_1)}$	8.9	6.8	5.4	4.6
Japan	θ^*	11.4	22.2	26.2	30.1
	$\frac{S^*}{E(\omega_1)}$	5.5	4.0	3.5	3.0

Notes: $\delta = 0.03$ and $\beta = 0.41$. RRA* = rate of relative risk aversion at $c_1 = (1 - \theta^*)E(\omega_1) - S^*$.

We cannot solve explicitly for the values of θ^* and S^* which maximize this expression. What we do instead is to use numerical methods to search over a grid of values of θ and S for the maximum.

The results are presented in Table 2. In every country, θ^* rises and S^* declines as the assumed relative rate of risk aversion is increased.[28] This could, in principle, reflect either or both of two effects. One is the growing importance, as risk aversion increases, of the intergenerational insurance of labor income which paygo provides. The other is increasing wariness with respect to funded saving, whose higher expected rate of return is accompanied by higher variability. The sensitivity analysis in the next section suggests that the reaction to the variability of R is empirically the more important phenomenon.

The principle message of Table 2 is that there are large differences from one country to the next in the paygo tax rate and lifetime savings rates that a fully knowledgeable benevolent planner would deem optimal in the steady state. The differences are based exclusively on estimated differences in objective characteristics. They are traceable to differences in the rate of return of the paygo system G, the rate of return of lifetime savings R, and the variances and covariances of both. These factors interact in different ways in each country, but higher paygo tax rates and lower savings rates are deemed optimal for France and Japan, and the opposite is deemed optimal for the United States. The United Kingdom lies in between. Table 3 compares the calculated optimal steady-state rates of paygo taxation which correspond to a relative rate of risk aversion of 2.5 with the rates actually in effect in 2003.

The computed numbers cannot be interpreted as targets for policy. They are based on a purely hypothetical choice of the rate of relative risk aversion, and they abstract from the transition costs of moving from one level of paygo taxation to another.

[28] The semi-elasticity parameter in the CARA utility function measures absolute risk aversion. Given the rate of absolute risk aversion, relative risk aversion increases with c. For comparability across experiments, we measure relative risk aversion at the expected value of first-period consumption, when θ and S are set equal to their optimal values $(1 - \theta^*)\bar{\omega}_1 - S^*$.

Table 3. *Comparison of computed and actual paygo tax rates*

	Computed	Actual (2003)
US	5	12
UK	14	15
France	19	37
Japan	22	20

Source: Table 2, computations for RRA* = 2.5; and OECD (2005b), Table 1.3, sum of employer and employee contributions. Figures adjusted to remove contributions for non-pension programs.

But it is interesting to compare the ranking of the two measures. With the exception of France, actual 2003 rates are ranked in the same order as computed rates. France, whose computed paygo tax rate is the third highest, actually had much the highest tax rate in 2003.

More comparative statics

Tables 4 and 5 provide a numerical sensitivity analysis of our computations. The tables were constructed by taking observed stochastic characteristics for France, and varying selected pairs of parameters.[29] Table 4 illustrates the potential importance of insurance against lifetime earnings risk as a motivation for paygo social security. When the coefficient of variation of R is 0.7 (close to its estimated value for France), the optimal paygo tax increases from 17% (when ω is non-random) to 30% (when the coefficient of variation of lifetime earnings is 1.0). However, θ^* is insensitive to variations of $\frac{\sigma_\omega}{\omega}$ in the range of the low values we estimate for France, the US, the UK, and Japan. This suggests that, empirically, in these four countries in the post-War period, the argument that paygo provides insurance against intergenerational variations in labor income is less important than other factors.

Table 4 also illustrates the importance of the notion that when the variability of R increases, the optimal saving rate declines, and the optimal paygo tax rises. When the coefficient of variation of lifetime earnings is 0.1, θ^* rises from 0% to 22% as the coefficient of variation of R goes from 0.3 to 1.0. This effect operates powerfully in the range of the parameters we have estimated.

DKO and MT emphasize what they call the 'portfolio choice' aspect of the balance between paygo and saving. That consideration suggests an important role for $\rho_{\omega_2, R}$, the coefficient of correlation between second-period labor earnings and the return to capital. DKO derive a positive relationship between this correlation and the share of paygo in the provision of retirement income.[30] MT suggest that the relationship may be positive or negative, but report a negative effect in their empirical calculations.[31] Our Table 5 suggests that $\rho_{\omega_2, R}$, is only of modest importance. When the

[29] For Tables 4 and 5, we selected a central value of 2.5 for the relative rate of risk aversion at the expected optimal consumption level in the first period.
[30] See DKO, equation (6).
[31] See MT equations (16) and (22) and Figure 2.

Table 4. *Effects on θ^* and S^* of changes in the variability of ω and R*

		$\frac{\sigma_\omega}{\bar{\omega}}$										
θ^*		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\frac{\sigma R}{\bar{R}}$	0.3	0.0	0.0	0.0	0.0	0.0	6.1	16.6	24.6	29.5	33.1	34.7
	0.4	0.0	0.0	0.0	0.0	3.2	10.1	17.8	24.6	29.5	31.9	33.1
	0.5	6.9	6.9	7.3	7.7	9.7	13.7	19.4	24.6	29.1	30.7	31.5
	0.6	13.3	13.3	13.3	13.7	14.1	16.2	20.2	25.1	28.3	30.3	30.3
	0.7	17.0	17.0	17.0	17.0	17.0	18.6	21.4	25.1	27.9	29.1	29.5
	0.8	19.4	19.4	19.4	19.4	19.4	20.2	22.2	25.1	27.5	28.3	28.7
	0.9	21.0	21.0	21.0	21.0	21.0	21.8	23.0	25.5	27.1	27.9	27.9
	1.0	22.2	22.2	22.2	22.2	22.2	22.6	23.8	25.5	26.7	27.5	27.5
S^*												
$\frac{\sigma R}{\bar{R}}$	0.3	17.0	16.6	16.2	15.4	14.1	10.5	6.5	4.8	5.3	7.3	9.3
	0.4	16.6	16.6	16.2	15.4	12.5	8.9	6.1	4.8	5.3	6.5	8.1
	0.5	12.1	12.1	11.7	10.9	9.3	7.3	5.3	4.4	4.8	5.7	6.9
	0.6	8.5	8.5	8.1	7.7	7.3	6.1	4.8	4.0	4.4	5.3	5.7
	0.7	6.1	6.1	6.1	6.1	5.7	4.8	4.0	3.6	4.0	4.4	4.8
	0.8	4.8	4.8	4.8	4.4	4.4	4.0	3.6	3.2	3.2	3.6	4.0
	0.9	3.6	3.6	3.6	3.6	3.6	3.2	3.2	2.8	2.8	3.2	3.2
	1.0	3.2	3.2	2.8	2.8	2.8	2.8	2.4	2.4	2.4	2.8	2.8

Notes: Computations use a CARA utility function with $\alpha = 0.062$, which corresponds to an estimate of 2.5 for RRA*, the relative risk aversion at the expected optimal consumption in the first period, given central values of stochastic parameters calibrated on French data. $\beta = 0.41$, $s_\omega = \frac{\sigma_\omega}{\bar{\omega}}$, $s_R = \frac{\sigma R}{\bar{R}}$, $\bar{R}$ is the expected return on lifetime savings.

annual rate of return on capital is 7 %, a shift from a positive correlation of 0.8 to a negative correlation of −0.8 only raises the optimal saving rate from 5 % to 6 %. The corresponding decline in the optimal paygo tax is from 19 % to 18 %. This may partly be a reflection of the modest variability of estimated lifetime labor earnings in the countries we examine.

Table 5 also illustrates that both substitution and income effects are operating on the effect of R on S^*. For low values of r (lower than the rate of growth of earnings), $S^* = 0$. As r increases, S^* rises to a maximum of 6 % and then decreases slowly (see progression of S^* when $\rho_{\omega_2 R} = 0$).[32]

6 Conclusion

The optimal balance between pay-as-you-go taxation and funded saving is a central theme in policy debates about the provision of retirement income in Europe and the United States. We take a normative point of view, and consider the influence on

[32] Our value of relative risk aversion does not meet the sufficient condition of Appendix B. Therefore, $\frac{dS}{dR} > 0$ is not guaranteed.

Table 5. *Effects on θ^* and S^* of changes in $\bar{R}$ and the correlation between ω_2 and R*

		$\bar{R}(r)$								
θ^*		1.8 (0.02)	2.4 (0.03)	3.2 (0.04)	4.3 (0.05)	5.7 (0.06)	7.6 (0.07)	10.1 (0.08)	13.3 (0.09)	17.4 (0.10)
	−0.8	27.5	27.5	23.0	20.2	18.6	17.8	17.4	17.4	17.4
	−0.6	27.5	27.5	23.4	20.2	18.6	17.8	17.4	17.4	17.4
	−0.4	27.5	27.5	23.8	20.6	19.0	18.2	17.8	17.8	17.8
	−0.2	27.5	27.5	24.6	20.6	19.0	18.2	18.2	17.8	17.8
$\rho_{\omega_2 R}$	0.0	27.5	27.5	25.1	21.0	19.0	18.2	18.2	17.8	18.2
	0.2	27.5	27.5	25.9	21.8	19.8	19.0	18.2	18.2	18.2
	0.4	27.5	27.5	26.3	22.2	19.8	19.0	18.6	18.2	18.2
	0.6	27.5	27.5	27.1	22.6	20.2	19.0	18.6	18.6	18.2
	0.8	27.5	27.5	27.5	22.6	20.2	19.4	18.6	18.6	19.0
S^*										
	−0.8	0.0	0.0	4.0	6.1	6.5	6.1	5.3	4.4	3.6
	−0.6	0.0	0.0	3.6	6.1	6.5	6.1	5.3	4.4	3.6
	−0.4	0.0	0.0	3.2	5.7	6.1	5.7	4.8	4.0	3.2
	−0.2	0.0	0.0	2.8	5.7	6.1	5.7	4.8	4.0	3.2
$\rho_{\omega_2 R}$	0.0	0.0	0.0	2.4	5.3	6.1	5.7	4.8	4.0	3.2
	0.2	0.0	0.0	1.6	4.8	5.7	5.3	4.8	4.0	3.2
	0.4	0.0	0.0	1.2	4.4	5.7	5.3	4.4	4.0	3.2
	0.6	0.0	0.0	0.4	4.0	5.3	5.3	4.4	3.6	3.2
	0.8	0.0	0.0	0.0	4.0	5.3	4.8	4.4	3.6	2.8

Notes: Computations use a CARA utility function with $\alpha = 0.062$, which corresponds to an estimate of 2.5 for RRA*, the relative risk aversion at the expected optimal consumption in the first period, given central values of stochastic parameters calibrated on French data. $\bar{R}$ is the expected return on lifetime savings, and r the implied annual rate of return on capital.

desirable, steady-state tax rates and savings rates of the long-term rate of return of capital, the long-term rate of return of the pay-as-you-go system, and their variances and covariances. A few general, qualitative results are derived from a simple, overlapping generations model, with general functional forms for utility and the probability distributions of earnings and the return to capital. We apply this model to empirical estimates of the trends and variabilities of labor earnings and the total return to lifetime savings in the United States, the United Kingdom, France, and Japan. We infer the stochastic characteristics of lifetime incomes from observed annual data from 1950 to 2002 by modelling the annual data and estimating the stochastic characteristics of post-War life histories. We find large differences across these countries in the return of the pay-as-you-go system, the return on lifetime savings and their variances and covariances. These differences lead to different computations of optimal, steady-state tax and savings rates across countries. In France and Japan, the variability of long-term incomes makes high tax rates and lower savings rates preferable. The situation is the opposite in the United States.

The United Kingdom lies in between. The principal source of uncertainty in the four countries is the variability of the return to lifetime savings. The expectation of that return is much higher than the expectation of the paygo return. Consequently, if both returns were certain, the return to savings would dominate, and the optimal tax rate would be close to zero. But the high expected variability of the return to saving contributes to raising optimal, steady-state tax rates in all countries. No direct connection can be made between these results and any policy recommendations, because of the arbitrariness of our choice of the relative rate of risk aversion, and because we pay no attention to transitional costs. One can, nonetheless, observe that the high rate of tax prevalent in France in 2003 shifts it from third position in the ranking of estimated, preferred tax rates to a high fourth position in the ranking of actual rates.

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Appendix A: Sufficient conditions for a unique maximum, when R is non-random

Let us write $u(c_i)=u_i$, $u'(c_i)=u_i'$, $u''(c_i)=u_i''$, etc. and $u(c)=u$, $u''(c)=u''$, etc. Society desires to maximize

$$\mathscr{V}=E[u_1]+\beta E[u_2]$$

with respect to θ and S, subject to (1) and (2) in the text.

First-order conditions for a maximum are

$$\frac{\partial \mathscr{V}}{\partial S}=\varphi=0,$$

$$\frac{\partial \mathscr{V}}{\partial \theta}=\psi=0.$$

The second-order conditions are that $\varphi_S<0$, $\psi_\theta<0$ and $\Delta=\varphi_S\psi_\theta-\varphi_\theta\psi_S>0$.

The first two conditions are given by diminishing marginal utility. The problem addressed in this appendix is finding sufficient conditions for the third, in the case of non-random R. From equation (6), collecting terms, and separating those which are independent of R from those in which R is an argument

$$\begin{aligned}\Delta=&E[u_1''\omega_1(\omega_1 E[u_1'']-E[u_1''\omega_1])]\\&+\beta^2 E[u_2''R\omega_2(R\omega_2 E[u_2'']-E[u_2''R\omega_2])]\\&+\beta[E[u_1'']E[u_2''\omega_2^2]+E[u_2'']E[u_1''(R\omega_1)^2]-2E[u_2'']E[u_1''R\omega_1]]\end{aligned} \qquad (A1)$$

Assume that

$$u''' \geqslant 0 \tag{A2}$$

and

$$-\frac{u_i'''\omega_i}{u_i''} < 1 \tag{A3}$$

Under these sufficient assumptions, we shall prove that $\Delta > 0$.

We start by examining the first term of (A1).

Note first that, since $u''' \geqslant 0$

$$cov(u_1'', \omega_1) = E[u_1''\omega_1] - E[u_1'']\bar{\omega} \geqslant 0$$

Consider now the expression $E[u_1''\omega_1] - \omega_1 E[u_1'']$. Once the utility function and probability distribution have been chosen, both expectations are numbers, independent of ω_1. Since $E[u_1''] < 0$, $-\omega_1 E[u_1''] > 0$, and it increases with ω_1. $E[u_1''\omega_1] < 0$. Therefore, the expression $E[u_1''\omega_1] - \omega_1 E[u_1'']$ starts negative, for very low values of ω_1 and rises with ω_1. It must cross the horizontal axis, and changes sign once, at $\tilde{\omega}$.

Assumption (A3) ensures that $-u_1''\omega_1$ is also strictly increasing in ω_1. (This can be seen, by writing the derivative of $-u_1''\omega_1$, and examining its sign, making use of (A3).)

Now we are prepared to examine the product

$$-u_1''\omega_1(E[u_1''\omega_1] - \omega_1 E[u_1'']).$$

The first term is always positive and increasing. The second term is at first negative, then positive. At $\tilde{\omega}$, the second term is zero. Therefore

$$-u_1''\omega_1(E[u_1''\omega_1] - \omega_1 E[u_1'']) > -\tilde{u}_1'''\tilde{\omega}_1(E[u_1''\omega_1] - \omega_1 E[u_1'']) \tag{A4}$$

This follows from the fact that, in the second expression, the negative values of $(E[u_1''\omega_1] - \omega_1 E[u_1''])$ are multiplied by a larger number, and the positive values by a smaller number.[33]

Taking expectations of both sides of (A4), one sees that the first term of equation (A1) becomes

$$E[-u_1''\omega_1(E[u_1''] - \omega_1 E[u_1'']) > -\tilde{u}_1''\tilde{\omega}_1 cov(u_1'', \omega_1) > 0 \tag{A5}$$

We turn now to the second term in (A1). Similarly to before, we can note that $u''' > 0$ implies

$$cov(u_2'', R\omega_2) = E[u_2''R\omega_2] - E[u_2'']R\bar{\omega} > 0.$$

As before, $E[u_2''R\omega_2] - R\omega_2 E[u_2'']$ is strictly increasing in ω_2, and changes sign once, say at $\tilde{\omega}_2$.The expression $-u_2''R\omega_2 > 0$ is also strictly increasing in ω_2. Hence

$$-u_2''R\omega_2(E[u_2''R\omega_2] - R\omega_2 E[u_2'']) > -\tilde{u}_2''R\tilde{\omega}_2(E[u_2''R\omega_2] - R\omega_2 E[u_2'']) \tag{A6}$$

[33] $\omega_1 < \tilde{\omega}_1 \rightarrow -\tilde{u}_1''\tilde{\omega}_1 > -u_1''\omega_1$, with $\tilde{u}_1'' = u''((1-\theta)\tilde{\omega}_1 - S)$. Similarly, $\omega_1 > \tilde{\omega}_1 \rightarrow -\tilde{u}_1''\tilde{\omega}_1 < -u_1''\omega_1$.

Taking expectations on both sides, we find that the second term in (A1) becomes

$$E[-u_2''R\omega_2(E[u_2''R\omega_2]-R\omega_2E[u_2''])] > -\tilde{u}_2''R\tilde{\omega}_2cov(u_2'', R\omega_2) > 0.$$

Finally, we examine the third line of (A1), an expresson in three terms, which we shall call βz

$$z = E[u_1'']E[u_2''\omega_2^2] + E[u_2'']E[u_1''(R\omega_1)^2] - 2E[u_1''R\omega_1]E[u_2''\omega_2]$$

$$= E[u_1'']E[u_2'']\left\{\frac{E[u_2''\omega_2^2]}{E[u_2'']} + \frac{E[u_1''(R\omega_1)^2]}{E[u_1'']} - 2\frac{E[u_1''R\omega_1]}{E[u_1'']}\frac{E[u_2''\omega_2]}{E[u_2'']}\right\} \qquad (A7)$$

We proceed in three steps to demonstrate that, under our assumptions, $(z) > 0$. At each step, we replace one of the terms inside the brackets on the right side of (A7) by a smaller term. We then demonstrate that the sum of these three smaller terms is still > 0.

Our objective in the first step is to show that

$$\frac{E[u_2''\omega_2^2]}{E[u_2'']} > \frac{E[u_2''^2\omega_2^2]}{E[u_2'']^2} \qquad (A8)$$

We demonstrate this by showing that (A8) is equivalent to

$$E\left[\frac{u_2''\omega_2^2}{E[u_2'']}\left(\frac{u_2''}{E[u_2'']} - 1\right)\right] < 0 \qquad (A9)$$

Inequality (A9) follows from two observations. The first is that $\frac{u_2''\omega_2^2}{E[u_2'']}$ is increasing in ω_2. Let us demonstrate this by taking the derivative

$$\frac{1}{E[u_2'']}\frac{\partial(u_2''\omega_2^2)}{\partial\omega_2} = \frac{u_2''}{E[u_2'']}\left(2 + \frac{\omega_2 u_2''}{u_2'''}\right) > 0$$

But, by assumption, $-\frac{\omega_2 u_2'''}{u_2''} < 1 < 2$. Therefore, $\frac{1}{E[u_2'']}\frac{\partial(u_2''\omega_2^2)}{\partial\omega_2} > 0$.

The second observation is that $\left(\frac{u_2''}{E[u_2'']} - 1\right)$ is positive for low values of ω_2, and changes sign once, say at $\tilde{\omega}_2$. That it diminishes follows from the assumption that $u_2''' > 0$. That it changes sign once, follows from the further observation that $E[\frac{u_2''}{E[u_2'']} - 1] = 0$.

Hence

$$\frac{u_2''\omega_2^2}{E[u_2'']}\left(\frac{u_2''}{E[u_2'']} - 1\right) < \frac{\tilde{u}_2''\tilde{\omega}^2}{E[\omega_2'']}\left(\frac{u_2''}{E[u_2'']} - 1\right) \qquad (A10)$$

To the left of $\tilde{\omega}_2$, a positive value of $(\frac{u_2''}{E[u_2'']} - 1)$ is multiplied by a larger positive. To the right of $\tilde{\omega}_2$, a negative value is multiplied by a smaller positive. Taking expectations on both sides of (A10)

$$\frac{E[u_2''^2\omega_2^2]}{E[u_2'']^2} - \frac{E[u_2'']\omega_2^2}{E[u_2'']} < \frac{\tilde{u}_2''\tilde{\omega}_2^2}{E[u_2'']}E\left[\frac{u_2''}{E[u_2'']} - 1\right] = 0 \qquad (A11)$$

Similar proof shows that

$$\frac{E[u_1''^2(R\omega_1)^2]}{E[u_1']^2} - \frac{E[u_1''(R\omega_1)^2]}{E[u_1']} < 0 \qquad \text{(A12)}$$

Finally, since ω_1 and ω_2 are i.i.d

$$\frac{E[u_1''R\omega_1 u_2''\omega_1]}{E[u_1']E[u_2']} = \frac{E[u_1''R\omega_1]E[u_2''\omega_1]}{E[u_1']E[u_2']} \qquad \text{(A13)}$$

(A11), (A12), and (A13) together imply that

$$0 < E\left[\left(\frac{u_1''R\omega_2}{E[u_1']} - \frac{u_2''\omega_2}{E[u_2']}\right)^2\right] = \frac{E[u_1''^2(R\omega_1)^2]}{E[u_1']^2} + \frac{E[u_2''\omega_2]^2}{E[u_2']^2}$$

$$-2\frac{E[u_1''R\omega_1 u_2''\omega_2]}{E[u_1']E[u_2']} < \frac{E[u_1''(R\omega_1)^2]}{E[u_1']} + \frac{E[u_2''\omega_2^2]}{E[u_2']}$$

$$-2\frac{E[u_1''R\omega_1]E[u_2''\omega_2]}{E[u_1']E[u_2']}$$

which is the third line in the expresion for Δ, (A1) above. We have now proved that each of the three lines in that expression is greater than zero, therefore $\Delta > 0$.

Appendix B: Comparative statics of an increase of *R* when it is deterministic

If R is non-random, (4) and (5) in the text, the first-order conditions for a maximum, become

$$\frac{\partial \mathscr{V}}{\partial S} = \varphi = -E[u_1'] + \beta RE[u_2'] = 0 \qquad \text{(B1)}$$

$$\frac{\partial \mathscr{V}}{\partial \theta} = \psi = -E[u_1'\omega_1] + \beta RE[u_2'\omega_2] = 0 \qquad \text{(B2)}$$

Differentiating totally with respect to R, one finds

$$\begin{bmatrix} \varphi_S & \varphi_\theta \\ \psi_S & \psi_\theta \end{bmatrix} \begin{bmatrix} dS^* \\ d\theta^* \end{bmatrix} = \begin{bmatrix} \beta E[u_2'] - \beta RE[u_2'']S \\ -\beta E[u_2''\omega_2]S \end{bmatrix} dR$$

and

$$-\frac{1}{\beta}\frac{dS^*}{dR} = \frac{1}{\Delta}[\psi_\theta(E[u_2'] + RSE[u_2'']) - \varphi_\theta SE[u_2''\omega_2]].$$

Assume that the second-order conditions for a maximum are satisfied. The second term in the brackets on the right is negative. Therefore, if the first term is negative also, then $\frac{dS^*}{dR} > 0$. But the first term can be recounted as

$$\psi_\theta E[u_2']\left(1 + \frac{RS}{RS + \theta\bar{\omega}}\frac{E[u_2'']E[c_2]}{E[u_2']}\right)$$

Thus, if

$$-\frac{E[u_2''](RS+\theta\bar{\omega})}{E[u_2']}<1 \tag{B3}$$

then

$$\frac{dS^*}{dR}>0$$

Similarly

$$-\frac{1}{\beta}\frac{\partial\theta^*}{\partial R}=\frac{1}{\Delta}[\varphi_S E[u_2''\omega_2]S-\psi_S(E[u_2'+RSE[u_2'']])]$$

Under condition (B3) all terms on the right-hand side are positive.
Therefore

$$\frac{d\theta^*}{dR}<0.$$

Appendix C: Comparative statics of a mean-preserving spread of the distributon of ω

We use again the first-order conditions for a maximum when R is non-random, (B1) and (B2). Substituting (8) from the text, and differentiating totally, one obtains

$$\begin{bmatrix} \varphi_S & \varphi_\theta \\ \psi_S\psi_\theta & \end{bmatrix}\begin{bmatrix} dS^* \\ d\theta^* \end{bmatrix} = \begin{bmatrix} E[u_1''(1-\theta)(\hat{\omega}_1-\bar{\omega})]-\beta RE[u_2''\theta(\hat{\omega}_2-\bar{\omega})] \\ E[u_1'(\hat{\omega}_1-\bar{\omega})+\omega_1 u_1''(1-\theta)(\hat{\omega}_1-\bar{\omega})]-\beta E[u_2'(\hat{\omega}_2-\bar{\omega})+\omega_2 u_2''\theta(\hat{\omega}_2-\bar{\omega})] \end{bmatrix} d\delta \tag{C1}$$

and

$$\frac{d\theta^*}{d\delta}=\frac{1}{\Delta}[\varphi_S(E[u_1'(\hat{\omega}_1-\bar{\omega})+\omega_1u_1''(1-\theta)(\hat{\omega}_1-\bar{\omega})]-\beta E[u_2'(\hat{\omega}_2-\bar{\omega})+\omega_2u_2''\theta(\hat{\omega}_2-\bar{\omega})]) \\ -\psi_S(E[u_1''(1-\theta)(\hat{\omega}_1-\bar{\omega})]-\beta RE[u_2''\theta(\hat{\omega}_2-\bar{\omega})])] \tag{C2}$$

Close inspection of (C2) suggests that its sign is likely to depend on the initial value of θ.* Consider what happens when the initial value of $\theta^*=0$. In that case, equation (C1) simplifies, both because terms with θ are set equal to zero, and because both u_2' and u_2'' become constants, since $u_2=u(RS)$. The simplified expression is

$$\begin{bmatrix} \varphi_S & \varphi_\theta \\ \psi_S & \psi_\theta \end{bmatrix}\begin{bmatrix} dS^* \\ d\theta^* \end{bmatrix} = \begin{bmatrix} E[u_1''(\hat{\omega}_1-\bar{\omega})] \\ E[u_1'(\hat{\omega}_1-\bar{\omega})+\omega_1u_1''(\hat{\omega}_1-\bar{\omega})] \end{bmatrix} d\delta \tag{C1$'$}$$

Whence

$$\frac{d\theta^*}{d\delta}=\frac{1}{\Delta}[\varphi_S E[u_1'(\hat{\omega}_1-\bar{\omega})+\omega_1u_1''(\hat{\omega}_1-\bar{\omega})]-\psi_S(E[u_1''(\hat{\omega}_1-\bar{\omega})])] \tag{C2$'$}$$

We will now show that a sufficient condition for this expression to be greater than zero is that $u(c)''' > 0$, and $E[(\hat{\omega}_1 - \bar{\omega})^n] = 0$, for all $n > 2$, i.e. utility function is convex and the probability distribution is normal. First we show that, in the last term in the bracket, $E[u_1''(\hat{\omega}_1 - \bar{\omega})] > 0$. Compare the expression $u''(\bar{\omega})(\hat{\omega}_1 - \bar{\omega})$ with $u''(\hat{\omega}_1 - \bar{\omega})(\hat{\omega}_1 - \bar{\omega})$. Since $u''(\hat{\omega}_1 - \bar{\omega})$ is monotonically increasing

$$u''(\hat{\omega}_1 - \bar{\omega})(\hat{\omega}_1 - \bar{\omega}) > u''(\bar{\omega})(\hat{\omega}_1 - \bar{\omega}) \text{ for all } \hat{\omega}_1$$

If one takes expectations of both sides, one indeed finds that $E[u_1''(\hat{\omega}_1 - \bar{\omega})] > 0$. It remains to be shown that the first expression in brackets on the right side of (C2′) is also positive. First we show that $E[u_1'(\hat{\omega}_1 - \bar{\omega})] < 0$. This follows from the fact that u_1' is monotonically declining, which implies that

$$u'(\omega_1)(\hat{\omega}_1 - \bar{\omega}) < u'(\bar{\omega})(\hat{\omega}_1 - \bar{\omega}) \text{ for all } \hat{\omega}_1$$

If one takes expectations of both sides, one indeed finds that $E[u_1'(\hat{\omega}_1 - \bar{\omega})] < 0$ and, therefore, $\varphi_S E[u_1'(\hat{\omega}_1 - \bar{\omega})] > 0$. The only term remaining to be signed in (C2′) is

$$\varphi_S E[\omega_1 u_1''(\hat{\omega}_1 - \bar{\omega})]$$

Taylor's series expansion of this last expression around $\hat{\omega}_1 = \bar{\omega}$ shows that, under our normality assumption, the expression is $= 0$. The conclusion is that, if $u(c)''' > 0$, and $E[(\hat{\omega}_1 - \bar{\omega})^n] = 0$, for all $n > 2$, then, when $\theta^* = 0$ initially, $\frac{d\theta^*}{d\delta} > 0$.

Finally, it can be shown that if $u(c)$ is quadratic, $\frac{d\theta^*}{d\delta} > 0$ if and only if $\frac{(1-\theta)}{\theta} > \beta$. The proof follows directly from simplification of (C2), using the fact that in the quadratic case

$$u'(c) = a - c, \text{ and } u''(c) = -1$$

Appendix D: Increased curvature

We continue to take R as fixed, and to assume that $\mathrm{E}(\omega_1) = \mathrm{E}(\omega_2) = \overline{\omega}$, and Var $(\omega_1) = \text{Var } (\omega_2) = \sigma^2_\omega$. Two additional simplifying assumptions help us to obtain a tractable result. These are

$$u^n(c) = 0 \quad \forall n > 3 \tag{D1}$$

$$\mathrm{E}(w - \bar{w})^k = 0 \quad \forall k = 2n+1, \; n \geqslant 1 \tag{D2}$$

In principle, all higher-order derivatives in the Taylor expansion can have an effect on θ^* and S^*. (D1) allows us to concentrate on $u'''(c)$. Assumption (D2) states that the distribution of ω is symetrical.

Given these assumptions

$$\bar{c}_1 = (1-\theta)\bar{w} - S$$

$$\bar{c}_2 = \theta\bar{w} + RS$$

$$\sigma^2_{c_1} = (1-\theta)^2 \sigma^2_w$$

$$\sigma^2_{c_2} = \theta^2 \sigma^2_w$$

and

$$u'(c_1)=u'(\bar{c}_1)+u''(\bar{c}_1)(1-\theta)(w-\bar{w})+\frac{u'''(\bar{c}_1)}{2}(1-\theta)^2(w-\bar{w})^2 \tag{D3}$$

$$u'(c_2)=u'(\bar{c}_2)+u''(\bar{c}_2)\theta(w-\bar{w})+\frac{u'''(\bar{c}_2)}{2}\theta^2(w-\bar{w})^2 \tag{D4}$$

Using these approximations, one can write the first-order conditions, (4) and (5) in the text, as

$$\frac{\partial \mathscr{V}}{\partial S}=\varphi=-u'(\bar{c}_1)-u'''(\bar{c}_1)(1-\theta)^2\frac{\sigma_w^2}{2}+\beta R u'(\bar{c}_2)+\beta R u'''(\bar{c}_2)\frac{\theta^2\sigma_w^2}{2}=0 \tag{D5}$$

$$\begin{aligned}\frac{\partial \mathscr{V}}{\partial \theta}=\psi=&-u'(\bar{c}_1)\bar{w}-u''(\bar{c}_1)(1-\theta)\sigma_w^2-u'''(\bar{c}_1)(1-\theta)^2\bar{w}\frac{\sigma_w^2}{2}\\&+\beta u'(\bar{c}_2)\bar{w}+\beta u''(\bar{c}_2)\theta\sigma_w^2+\beta u'''(\bar{c}_2)\frac{\theta^2\bar{w}\sigma_w^2}{2}=0\end{aligned} \tag{D6}$$

We disturb these conditions, by adding $dz>0$ everywhere to $u'''(c)$. The following derivatives are evaluated at $u'''(c)=0$. We use the fact that, at that point, $\forall c\ u''(c)=u''$. We have

$$\varphi_z=[-(1-\theta)^2+\beta R\theta^2]\frac{\sigma_w^2}{2} \tag{D7}$$

$$\varphi_S=u''(1+\beta R^2) \tag{D8}$$

$$\varphi_\theta=u''\bar{w}(1+\beta R) \tag{D9}$$

In evaluating the derivatives of ψ, we use the symmetry assumption, which implies that $\mathrm{E}[(w-\bar{w})^2w]=\mathrm{E}[(w-\bar{w})^2]\bar{w}$

$$\psi_z=[-(1-\theta)^2+\beta\theta^2]\bar{w}\frac{\sigma_w^2}{2} \tag{D10}$$

$$\psi_S=u''\bar{w}(1+\beta R) \tag{D11}$$

$$\psi_\theta=u''(1+\beta)(\bar{w}^2+\sigma_w^2) \tag{D12}$$

We want to examine how θ^* and S^* change when we increase u''', $du'''=dz$. With $\varphi(\theta^*, S^*, z)=0$, $\psi(\theta^*, S^*, z)=0$, by Cramer's rule

$$\frac{d\theta^*}{dz}=\frac{1}{\Delta}\begin{vmatrix}-\varphi_z & \varphi_S\\ -\psi_z & \psi_S\end{vmatrix}$$

$$\frac{dS^*}{dz}=\frac{1}{\Delta}\begin{vmatrix}\varphi_\theta & -\varphi_z\\ \psi_\theta & -\psi_z\end{vmatrix} \tag{D13}$$

$$\Delta=\varphi_\theta\psi_S-\varphi_S\psi_\theta>0$$

Therefore

$$\frac{d\theta^*}{dz} = \frac{1}{\Delta}(-\varphi_z\psi_S + \varphi_S\psi_z) \tag{D14}$$

Using (D7), (D8), (D10), and (D11) one can show that this simplifies to

$$\frac{d\theta^*}{dz} = -\frac{\bar{w}\sigma_w^2 u''}{2\Delta}(R-1)(\beta\theta^2 + \beta R(1-\theta)^2) \gtrless 0 \Leftrightarrow R \gtrless 1 \tag{D15}$$

Note also that $\frac{d\theta^*}{dz}$ is proportional to $\sigma_w^2 u''$. Similarly

$$\frac{dS^*}{dz} = \frac{1}{\Delta}(-\psi_z\varphi_\theta + \varphi_z\psi_\theta) \tag{D16}$$

Using (D7), (D9), (D10), and (D12) one can show that this simplifies to

$$\frac{dS^*}{dz} = \frac{\bar{w}^2\sigma_w^2 u''}{2\Delta}(R-1)(\beta\theta^2 + \beta(1-\theta)^2) + \frac{\sigma_w^4 u''}{2\Delta}(1+\beta)(\beta R\theta^2 - (1-\theta)^2) \tag{D17}$$

This expression can be either positive or negative.