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Expansion of source equation of elastic line

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SUMMARY

The paper is concerned with the relationship between the equation of elastic line motion, the "Euler-Bernoulli approach" (EBA), and equation of motion at the point of elastic line tip, the "Lumped-mass approach" (LMA). The Euler–Bernoulli equations (which have for a long time been used in the literature) should be expanded according to the requirements of the motion complexity of elastic robotic systems. The Euler–Bernoulli equation (based on the known laws of dynamics) should be supplemented with all the forces that are participating in the formation of the bending moment of the considered mode. This yields the difference in the structure of Euler–Bernoulli equations for each mode. The stiffness matrix is a full matrix. Mathematical model of the actuators also comprises coupling between elasticity forces. Particular integral of Daniel Bernoulli should be supplemented with the stationary character of elastic deformation of any point of the considered mode, caused by the present forces. General form of the elastic line is a direct outcome of the system motion dynamics, and can not be described by one scalar equation but by three equations for position and three equations for orientation of every point on that elastic line. Simulation results are shown for a selected robotic example involving the simultaneous presence of elasticity of the joint and of the link (two modes), as well the environment force dynamics.

KEYWORDS: Robot; Modeling; Elastic deformation; Gear; Link; Coupling; Dynamics; Kinematics; Trajectory planning.

1. Introduction

All elements in a robotics system are not as rigid as we would like. Stiffness of elements significantly simplifies our work. Including the elasticity of gears and (or) links makes a robotic system more complex and opens perspectives for other problems, for example has been new control laws. "Harmonic-Drive" (HD) reducers have a specific degree of transmission with significant elastic effects. These gears are in wide usage because of their wonderful characteristics (great transmission ratio with almost null clearance). Besides HD reducers, an elastic effect can originate from tug transmission of motor power on robot links as well as from the consequences of torsion axle for transmission of motor power on robot links.

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Elasticity of withy, long-spread links, constructed of light materials, also requires elasticity analysis. We cannot ignore the fact that flexibility is a natural characteristic of a material, but it is "unpleasant" for modeling. Research from this area can also be applied in making and modeling muscles. They are constructed by combining springs in humanoid robotics, which is a subject of today's top-level research. This could represent a further trend in the development of world-eminent laboratories for robotics. Many constructions (e.g., industrial robots, humanoid robots, etc.) are from time-to-time or continuously exposed to effects of environment force. Because of this reason, this topic presents a challenge for researchers and there are many reasons for its "return" into the focus of their interest.

In ref. [1], the control of robots with elastic joints in contact with dynamic environment is considered. In ref. [2], the feedback control was formed for the robot with flexible links (two-beam, two-joint systems) with distributed flexibility, robots with flexible links being also dealt with in. ref. [3]. In ref. [4] a nonlinear control strategy for tip position trajectory tracking of a class of structurally flexible multi-link manipulators is developed.

Authors of refs. [5, 6] derived dynamic equations of the joint angle, the vibration of the flexible arm, and the contact force.

Reference [7] presents an approach to end-point control of elastic manipulators based on the nonlinear predictive control theory. References [8, 9] present method for the generation of efficient kinematics and dynamic models of flexible robots. In ref. [10] author discusses the force control problem for flexible joint manipulators.

In ref. [11] the authors extend the integral manifold approach for the control of flexible joint robot manipulators from the known parameter case to the adaptive case. The author of ref. [12] designed a control law for local regulation of contact force and position vectors to desired constant vectors. In paper, ref. [13] different from conventional approaches, authors focus on the design of the rigid part motion control and the selection of bandwidth of which is rigid subsystem. Work reference [14] presents the derivation of the equations of motion for application mechanical manipulators with flexible links. In ref. [15] the equations are derived using Hamilton's principle and are nonlinear integro-differential equations. The method of separation of variables and the Gelarkin's approach are suggested in ref. [16] for the boundary-value problem with time-dependent boundary condition. First detailed presentation of the procedure for creating reference trajectory was given in ref. [17] and later in ref. [18].

Mathematical model of a mechanism with one degree of freedom (DOF), with one elastic gear was defined by Spong ref. [19] in 1987. Based on the same principle, elasticity of gears is introduced into the mathematical model in this paper, as also in refs. [20–22]. However, when the introduction of link flexibility into the mathematical model is concerned, it is necessary to point out some essential problems in this domain.

In this paper we do not use “assumed modes technique”, proposed by Meirovitch in ref. [23] (and used by all authors until today, e.g. in ref. [5, 8, 9, 18, 24–29] etc.), as we disagree with him.

Our opinion is that elastic deformation and frequency of present modes are consequences of the overall dynamics motion of the robotic system.

Type of each joint which can appear in any robotic configuration is defined in refs. [21, 22].

In this paper following the idea of Euler and Bernoulli, their equation is expanded with several aspects. Any elastic deformations can be described by superposing of stationary solution, forced character + particular solutions, oscillatory characteristic of Daniel Bernoulli. Stationary forces are characterized by permanent character of action namely gravity, inertial, centrifugal, Coriolis, coupled forces, and environment force. Disturbance forces are characterized by instantaneous character of action, and environment force can also have such a character. Euler–Bernoulli equations are not equal for each mode, as shown due to the presence of coupling. General form of robot elastic line is defined with six equations, three of which define the position of each point on the elastic line and the other three define the orientation of each point on the elastic line. Damping characteristic is included. Stiffness matrix (and also damping matrix) is a full matrix, as a consequence of coupling between flexible forces. The EBA and LMA (used in refs. [30–32]) are two comparable approaches addressing the same problem but from different aspects, also mentioned in ref. [20, 21]. (see Fig. 1). A new formulation of mathematical model of the motor is defined. Most of these phenomena are mentioned in refs. [20, 21], while they are explained in detail in this paper through the form of mathematical model of flexible robotic system in a classical form.

The reference trajectory is defined in such a manner that the flexible characteristics are partially known in the system. It is assumed that all elasticity characteristics in the system

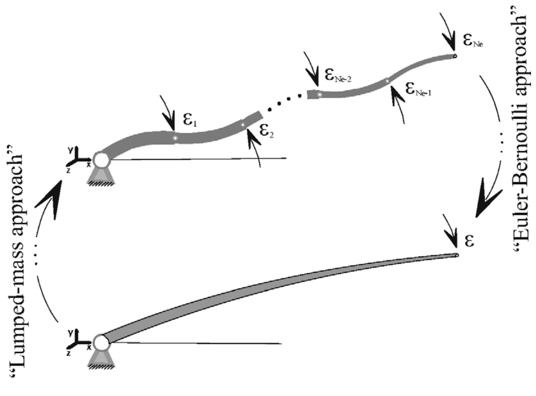

Fig. 1. LMA and EBA as two approaches of comparable worth.

(both of stiffness and damping) are “known”, at least partly and at this level can be included into the process of defining the reference motion and control law, as explained in ref. [21].

Section 2 analyzes the source equations of elastic line. In Section 3 is given a supplement to the source equations of flexible line and a general form of the equation of flexible line of a complex robotic system of arbitrary configuration, using EBA. The flexible line equation extends by a damping component. Also, we demonstrate that the particular integral of D. Bernoulli is just a component of flexural deformation of any point of the mode considered, to which is necessary to add the component of flexural deformation of the stationary regime. Section 4 demonstrates the relationship between the equation of elastic line motion and equation of motion at any point of the elastic line. Section 5 analyzes movement dynamics of a multiple DOF elastic robotic pair with elastic gear and flexible link in the presence of the second mode and environment force. Section 6 gives some concluding remarks.

2. Analysis of the Source Equations of Elastic Line

Equation of the elastic line of beam bending is of the following form:

$$\hat{M}_{1,1} + \beta_{1,1} \cdot \frac{\partial^2 \hat{y}_{1,1}}{\partial \hat{x}_{1,1}^2} = 0, \tag{1}$$

i.e.,

$$\hat{M}_{1,1} + \hat{\varepsilon}_{1,1} = 0, \tag{2}$$

$$\hat{\varepsilon}_{1,1} = \beta_{1,1} \cdot \frac{\partial^2 \hat{y}_{1,1}}{\partial \hat{x}_{1,1}^2}, \tag{3}$$

where $\hat{M}_{1,1}[Nm]$ is the load moment. In these source equations encompassing only inertia, $\hat{\varepsilon}_{1,1}$ bending moment, $\beta_{1,1}[Nm^2]$ is the flexural rigidity.

General solution of motion (see Fig. 2), i.e., the form of transversal oscillations of flexible beams, can be found by the method of particular integrals of D. Bernoulli, i.e.,

$$\hat{y}_{to1,1}(\hat{x}_{1,1}, t) = \hat{X}_{1,1}(\hat{x}_{1,1}) \cdot \hat{T}_{to1,1}(t), \tag{4}$$

$$\begin{aligned} \hat{X}_{1,1}(\hat{x}_{1,1}) = C_{1,(1,1)} \cos\, k_{1,1}\hat{x}_{1,1} + C_{2,(1,1)} \sin\, k_{1,1}\hat{x}_{1,1} \\ + C_{3,(1,1)} Ch\, k_{1,1}\hat{x}_{1,1} + C_{4,(1,1)} Sh\, k_{1,1}\hat{x}_{1,1} \end{aligned} \tag{5}$$

$$\hat{T}_{to1,1}(t) = A_{1,1} \cos\, p_{1,1}t + B_{1,1} \sin\, p_{1,1}t. \tag{6}$$

By superimposing the particular solutions (4), any transversal oscillation can be presented in the following form

$$\hat{y}_{to1}(\hat{x}_{1,j}, t) = \sum_{j=1}^{\infty} \hat{X}_{1,j}(\hat{x}_{1,j}) \cdot \hat{T}_{to1,j}(t). \tag{7}$$

$x_{1,1}$, $y_{1,1}$, $z_{1,1}$ is a local coordinate frame, which is set in the base of considered mode. In this case (See Fig. 2), it is the first mode of the first link, while in a general case this is

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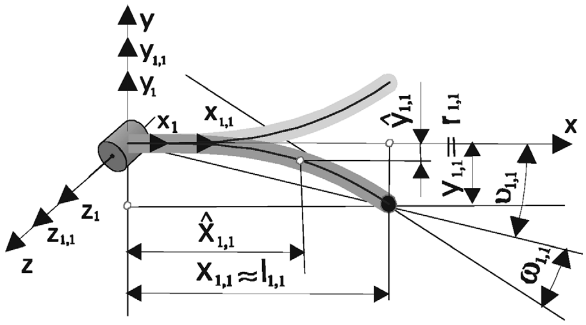

Fig. 2. Idealized motion of elastic body according to D. Bernoulli.

$x_{i,j}, y_{i,j}, z_{i,j}$, where j is the serial number of the mode of considered link $j = 1, 2, 3 \ldots n_i$, and i is the serial number of the link of the considered robotic system $i = 1, 2, 3 \ldots m$. x_1, y_1, z_1 is a local coordinate frame, which is set in the base of the considered link. In a general case it is x_i, y_i, z_i. x, y, z is the basic coordinate frame, which is set in the root of the considered robotic system. $\vartheta_{1,1}$ is the bending angle of the first mode of the first link. $\omega_{1,1}$ is the rotation angle of the top of the same mode (see ref. [33]).

Remark 1. Equations (1–7) need a short explanation, which should be assumed, but is missing from the original literature ref. [34]. Euler and Bernoulli wrote Eq. (7) based on 'vision'. They did not define the mathematical model of a link with an infinite number of modes, which has a general form of Eq. (8), but they did define the motion solution (shape of elastic line) of such a link, which is presented in Eq. (7). They left the task of link modeling with infinite number of modes to their successors. Transversal oscillations defined by Eq. (7) describe the motion of elastic beam to which we assigned an infinite number of DOFs (modes), and which can be described by a mathematical model composed of an infinite number of equations, in the form:

$$\begin{gathered} \hat{M}_{1,j} + \hat{\varepsilon}_{1,j} = 0 \\ j = 1, 2, \ldots, j, \ldots \infty \end{gathered}. \tag{8}$$

Dynamics of each mode is described by one equation. The equations in the model (8) are not of equal structure as our contemporaries, authors of numerous works presently interpret it. We think that the coupling between the modes involved leads to structural diversity among the equations in the model (8). This explanation is of key importance and is necessary for understanding our further discussion.

Remark 2. The symbol "$\wedge$" denotes generally the quantities that are related to an arbitrary point of the elastic line of the mode, e.g., $\hat{y}_{1,1}, \hat{x}_{1,1}, \hat{\varepsilon}_{1,1}$. The same quantities that are not designated by "$\wedge$" are defined for the mode tip, e.g., $y_{1,1}, x_{1,1}, \varepsilon_{1,1}$.

Remark 3. Under a mode we understand the presence of coupling between all the modes present in the system. We analyze the system in which the action of coupling forces (inertial, Coriolis', and elasticity forces) exists between the present modes. To differentiate it from "mode shape" or "assumed mode", we call it a coupled mode or, shorter, in the text to follow, a mode. This yields the difference in the structure of Euler–Bernoulli equations for each mode.

3. Equation of the Elastic Line of a Complex Robotic System

The Bernoulli solution (4–6) describes only partially the nature of motion of real elastic beams. More precisely, it is only one component of motion. Euler–Bernoulli equations (1–7) should be expanded from several aspects in order to be applicable in a broader analysis of elasticity of robot mechanisms. By supplementing these equations with the expressions that come out directly from the motion dynamics of elastic bodies, they become more complex.

As already mentioned, Eqs. (1–7) were defined under the assumption that the elasticity force is opposed only by the inertial force proper. Besides, it is supposed by definition that the motion in Eq. (1) is caused by an external force, suddenly added and then removed. The solution (4–6) of D. Bernoulli satisfies these assumptions.

The motion of the considered robotic system mode is far more complex than the motion of the body presented in Fig. 2. This means that the equations that describe the robotic system (and its elements) must also be more complex than the Eqs. (1–7), formulated by Euler and Bernoulli. This fact is overlooked and the original equations are widely used in the literature to describe the robotic system motion. This is far inadequate because valuable pieces of information about the complexity of the elastic robotic system motion are thus lost. Hence, the necessity of expanding the source equations for the purpose of modeling robotic systems should be especially emphasized, and this should be done in the following way:

- Based on the known laws of dynamics, Eq. (3) *is to be supplemented by all the forces that participate in the formation of the bending moment of the considered mode.* It is assumed that the forces of coupling (inertial, Coriolis, and elastic) between the present modes are also involved, which yields structural difference between equations (3) in the model (8).
- Equation (4) is to be supplemented by *the stationary character of the elastic deformation caused by the forces involved.*

Let us consider the motion of the first mode of the given link. The link has in all n_1 modes. First mode is the bracket (support) of uniformly distributed mass along the mode, loaded by the moment $\hat{M}_{1,1}$. The load moment $\hat{M}_{1,1}$ is composed of all the forces acting on the first mode of the link, and these are inertial forces (own and coupled inertia forces of the other modes), centrifugal, gravitational, Coriolis forces (own and coupled), forces due to relative motion of one mode with respect to the other, coupled elasticity forces of the other modes, as well as the force of the environment dynamics, which is via Jacobian matrix transferred to the motion of the first mode. This means that all these forces participate in generating of bending moment that is in forming elastic deformation as well as of the elasticity line of the first mode. In that case the model of elastic line of the first mode of the

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elastic link is of the form:

$$\hat{H}_{1,j}\frac{d^2\hat{y}_{1,j}}{dt^2}+\hat{h}_{1,1}+j_{1,1}^T F_{uk}+z_{1,j}\cdot\varepsilon_1 + \beta_{1,1}\cdot\frac{\partial^2(\hat{y}_{1,1}+\eta_{1,1}\cdot\dot{\hat{y}}_{1,1})}{\partial\hat{x}_{1,1}^2}=0. \quad (9)$$

j is the ordinal number of the considered mode, $j = 1, 2, 3\ldots n_1$.

Vectors in Eq. (9) are of the following form: $\hat{H}_{1,j} = [\hat{H}_{1,(1,1)}\hat{H}_{1,(1,2)}\ \hat{H}_{1,(1,3)}\cdots\hat{H}_{1,(1,n_1)}]$, the vector characterizing the inertia of the first mode.

$$\frac{\partial^2\hat{y}_{1,j}}{\partial t^2}=\left[\frac{\partial^2\hat{y}_{1,1}}{\partial t^2}\quad\frac{\partial^2\hat{y}_{1,2}}{\partial t^2}\quad\frac{\partial^2\hat{y}_{1,3}}{\partial t^2}\cdots\frac{\partial^2\hat{y}_{1,n_1}}{\partial t^2}\right]^T.$$

$\hat{h}_{1,1}$ is the centrifugal, gravitational and Coriolis forces of the first mode.

$j_{e1,1}^T = [J_{e1,(1,1)}\ \ J_{e1,(1,2)}\ \ J_{e1,(1,3)}\cdots J_{e1,(1,6)}]$, first row of the Jacobian matrix serving to map the impact of the dynamic force of contact F_{uk} on the behavior of the first mode.

$$z_{1,j}=\left[0\quad-\frac{1}{2^1}\quad+\frac{1}{2^2}\quad-\frac{1}{2^3}\cdots(-1)^{(n_1-1)}\frac{1}{2^{(n_1-1)}}\right].$$

The vector $z_{1,j}$ characterizer the effect of elasticity forces of the other modes on the first mode. $z_{1,j}$ is obtained by modeling different link structures (with one, two, three... modes). Moment of bending defined for the tip of any mode of the considered link is:

$$\varepsilon_{1,j}=F_{1,j}\cdot l_{1,j}=C_{s1,j}\cdot r_{1,j}\cdot l_{1,j}+B_{s1,j}\cdot\dot{r}_{1,j}\cdot l_{1,j}. \quad (10)$$

The rigidity and damping characteristic for the tip of any mode is designated as $C_{s1,j}[N/m]$ and $B_{s1,j}[N\cdot s/m]$ respectively, maximal deflection is $r_{1,j}$, the mode length is $l_{1,j}$. The vector of bending moments is ε_1.

$$\varepsilon_1=[\varepsilon_{1,1}\ \ \varepsilon_{1,2}\ \ \varepsilon_{1,3}\ \ \varepsilon_{1,4}\ \ \varepsilon_{1,5}\ldots.\varepsilon_{1,n_1}]^T.$$

Bending moment defined for an arbitrary point of the first mode is:

$$\hat{\varepsilon}_{1,1}=\beta_{1,1}\cdot\frac{\partial^2(\hat{y}_{1,1}+\eta_{1,1}\cdot\dot{\hat{y}}_{1,1})}{\partial\,\hat{x}_{1,1}^2}.$$

The force acting on the formation of elastic line of an arbitrary mode of the considered link is $\hat{F}_{1,j}$. Load moment $\hat{M}_{1,1}$ from Eq. (9) is defined as:

$$\hat{H}_{1,j}\frac{d^2\hat{y}_{1,j}}{dt^2}+\hat{h}_{1,1}+j_{1,1}^T F_{uk}+z_{1,j}\cdot\varepsilon_1 + \beta_{1,1}\cdot\frac{\partial^2(\hat{y}_{1,1}+\eta_{1,1}\cdot\dot{\hat{y}}_{1,1})}{\partial\hat{x}_{1,1}^2}=0. \quad (11)$$

Thus Eq. (9) can be now written in a simpler form

$$\hat{M}_{1,1}+\hat{\varepsilon}_{1,1}=0. \quad (12)$$

Equation (12) was defined under the assumption that the elasticity moment $\hat{\varepsilon}_{1,1}$ is opposed by the load moment $\hat{M}_{1,1}$, which, among the other forces, encompasses also the *coupled elasticity of the other modes*. In a stationary regime of robotic task realization, the mentioned moments that oppose the elasticity moment $\hat{\varepsilon}_{1,1}$ continuously change during the robotic task realization. On this system can also act disturbance forces, which may be of an instant or permanent character.

Therefore, elastic deformations of a given body can be generated by

- *disturbance* forces, causing oscillatory motion.
- *stationary* forces, causing stationary motion.

By superimposing the particular solutions of oscillatory character and stationary solution of forced character, any elastic deformation can be presented in the following general form:

$$\begin{aligned}\hat{y}_{1,1}&=\hat{X}_{1,1}(\hat{x}_{1,1})\cdot(\hat{T}_{st1,1}(t)+\hat{T}_{to1,1}(t))\\&=\hat{a}_{1,1}(\hat{x}_{1,1},\hat{T}_{st1,1},\hat{T}_{to1,1},t),\end{aligned} \quad (13)$$

Where, $\hat{T}_{st1,1}(t)$ is the stationary part of elastic deformation caused by stationary forces that may continuously change in time.

In case the robot is in state of inaction, then stationary forces are gravity forces. In case the robot is in state of motion, then stationary forces are gravity, inertial, centrifugal, Coriolis, and of course coupled forces of all forces and environment force (if it is continuous). This means that stationary forces are all forces which change continuously in time.

$\hat{T}_{to1,j}(t)$ is the oscillatory part of elastic deformation as in (6). This component of elastic deformation is caused by disturbance force (which acts instantaneously) and can appear in state of robot inaction and also in state of robot motion.

Environment force can be of

- Disturbing character (e.g., when the robot is moving without limitation and only in one moment enters into the contact with environment), or
- Stationary character (e.g., when the robot is continuously under the influence of environment force).

Total motion of the considered mode, defined by the sum of stationary and oscillatory motion, is given by Eq. (13).

Orientation of any point of the first mode is defined by

$$\hat{\psi}_{1,1}=\hat{d}_{1,1}(\hat{x}_{1,1},\hat{T}_{st1,1},\hat{T}_{to1,1},t). \quad (14)$$

Just as we defined the elastic line model of the first mode by Eq. (9), similarly we can also define the model of elastic line of the second, third ... n_1th mode of the elastic link. The elastic line model of the first link that has n_1 modes is given

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in a matrix form by the following equation:

$$\hat{H}_1 \cdot \frac{d^2\hat{y}_{1,j}}{dt^2} + \hat{h}_1 + j_{e1}^T \cdot F_{uk} + z_1 \cdot \varepsilon_1 + \hat{\varepsilon}_1 = 0. \tag{15}$$

Matrixes and vectors in Eq. (15) are of the following form:

$$\hat{H}_1 = \begin{bmatrix} \hat{H}_{1,(1,1)} & \hat{H}_{1,(1,2)} & \dots & \hat{H}_{1,(1,n_1)} \\ \hat{H}_{1,(2,1)} & \hat{H}_{1,(2,2)} & \dots & \hat{H}_{1,(2,n_1)} \\ \dots & \dots & \dots & \dots \\ \hat{H}_{1,(n_1,1)} & \hat{H}_{1,(n_1,2)} & \dots & \hat{H}_{1,(n_1,n_1)} \end{bmatrix},$$

the matrix characterizing the inertia of the each mode, $\hat{h}_1 = [\hat{h}_{1,1}\ \hat{h}_{1,2} \cdots \hat{h}_{1,n_1}]^T$, the vector characterizing effect of centrifugal, gravitational and Coriolis forces of the each mode,

$$j_{e1}^T = \begin{bmatrix} J_{e1,(1,1)} & J_{e1,(1,2)} & \dots & J_{e1,(1,6)} \\ J_{e1,(2,1)} & J_{e1,(2,2)} & \dots & J_{e1,(2,6)} \\ \dots & \dots & \dots & \dots \\ J_{e1,(n_1,1)} & J_{e1,(n_1,2)} & \dots & J_{e1,(n_1,6)} \end{bmatrix},$$

the Jacobian matrix serving to map the impact of the dynamic force of contact F_{uk} on the behavior of the each mode.

$$z_1 = \begin{bmatrix} 0 & -\frac{1}{2^1} & \frac{1}{2^2} & -\frac{1}{2^3} & \cdots & (-1)^{(n_1-1)}\frac{1}{2^{(n_1-1)}} \\ 0 & 0 & -\frac{1}{2^1} & \frac{1}{2^2} & \cdots & (-1)^{(n_1-2)}\frac{1}{2^{(n_1-2)}} \\ 0 & 0 & 0 & -\frac{1}{2^1} & \cdots & (-1)^{(n_1-3)}\frac{1}{2^{(n_1-3)}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Matrix z_1 characterizer the mutual effect of elasticity forces of the presented modes on each mode,

$$\hat{\varepsilon}_1 = \left[\beta_{1,1} \cdot \frac{\partial^2(\hat{y}_{1,1} + \eta_{1,1} \cdot \dot{\hat{y}}_{1,1})}{\partial \hat{x}_{1,1}^2} \ \dots\ \beta_{1,n_{1l}} \cdot \frac{\partial^2(\hat{y}_{1,n_{1l}} + \eta_{1,n_{1l}} \cdot \dot{\hat{y}}_{1,n_{1l}})}{\partial \hat{x}_{1,n_{1l}}^2}\right]^T.$$

The load moment $\hat{M}_1$ is defined by

$$\hat{M}_1 = \hat{H}_1 \cdot \frac{d^2\hat{y}_{1,j}}{dt^2} + \hat{h}_1 + j_{e1}^T \cdot F_{uk} + z_1 \cdot \varepsilon_1. \tag{16}$$

Equation (15) can be written in a simpler form as

$$\hat{M}_1 + \hat{\varepsilon}_1 = 0. \tag{17}$$

Equation (17) represents the equation of motion *of elastic line of the first link.*

To describe the behavior of the one-link robotic system having n_1 modes, the vector Eq. (17) should be supplemented by the mathematical model of the motor. The motor's mathematical model can be defined by writing the equation of motion of all the moments that act on the motor shaft. In the case of a rigid robotic system the motor moment is opposed by the mechanism moment. With elastic robotic systems we have a somewhat different situation: the motor moment is opposed by the bending moment of the first elastic mode that comes after the motor and, partly, opposed by the bending moments of the other elastic modes that are connected in series after the motor. All the modes that come after the motor, due to their position, exert certain influence on the motor dynamics. The effect of the first mode bending moment is defined by the factor $+\frac{1}{2^0}$, of the second by $-\frac{1}{2^1}$, of the third by $+\frac{1}{2^2}$, of the fourth by $-\frac{1}{2^3}$, of the fifth by $+\frac{1}{2^4}$ etc.

We add all these elasticity moments to the motor model because they are just to oppose the rotation moment of the motor shaft. The mathematical model of motor is of the following form:

$$\begin{aligned} &u_1 = R_1 \cdot i_1 + C_{E1} \cdot \dot{\bar{\theta}}_1 \\ &C_{M1} \cdot i_1 = I_1 \cdot \ddot{\bar{\theta}}_1 + B_{u1} \cdot \dot{\bar{\theta}}_1 - S_1 \cdot \varepsilon_{m\,1} \end{aligned} \left| \begin{array}{l} \sum M = 0 \\ \text{about the rotation axis} \\ \text{of the first motor} \end{array} \right. . \tag{18}$$

Where, $R_1[\Omega]$ is the rotor circuit resistance; $i_1[A]$ is the rotor current, $C_{E1}[V/(rad/s)]$ and $C_{M1}[Nm/A]$ are the proportionality constants of the electromotive force and moment, respectively, $B_{u1}[Nm/(rad/s)]$ is the coefficient of viscous friction; $I_1[kgm^2]$ is the inertia moments of the rotor and reducer; S_1 is the expression defining the reducer geometry, and $\varepsilon_{m\,1}$ is the equivalent elasticity moment that opposes the rotation moment of the motor shaft.

$$\begin{aligned} &\varepsilon_{m\,1} = z_{m\,1,j} \cdot \varepsilon_1, \\ &z_{m\,1,j} = \left[+\frac{1}{2^0} - \frac{1}{2^1} + \frac{1}{2^2} - \frac{1}{2^3} \ \dots.. (-1)^{(n_1-1)}\frac{1}{2^{(n_1-1)}}\right]. \end{aligned}$$

The vector $z_{m\,1,j}$ characterizes the influence of the elasticity moment of each mode on the motor dynamics.

It has not be explained how we have obtained expression $C_{M1} \cdot i_1$, $I_1 \cdot \ddot{\bar{\theta}}_1$, $B_{u1} \cdot \dot{\bar{\theta}}_1$ in Eq. (18), because this is already known from the literature but the procedure of obtaining equivalent elasticity moment $\varepsilon_{m\,1}$ has been explained. The potential energy in top of jth mode of the first link is $E_{pels\,1,j} = \frac{1}{2}C_{s\,1,j} \cdot \vartheta_{1,j}^2 \cdot l_{1,j}^2$, while dissipative energy is:

$$\Phi_{els\,1,j} = \tfrac{1}{2}B_{s\,1,j} \cdot \dot{\vartheta}_{1,j}^2 \cdot l_{1,j}^2.$$

All quantities should be expressed in dependence of generalized coordinates. One of them is also angle $\bar{\theta}_1$.

By applying Lagrange's equations on the expressions $E_{pels\,1,1}$, $E_{pels\,1,2}, \dots, E_{pels\,1,j}, \dots, E_{pels\,1,n_1}$, and $\Phi_{els\,1,1}$, $\Phi_{els\,1,2}, \dots, \Phi_{els\,1,j}, \dots, \Phi_{els\,1,n\,1}$ with respect to the generalized coordinate $\bar{\theta}_1$, we obtain the equivalent elasticity moment $\varepsilon_{m\,1}$, that opposes to the rotation moment of the first motor shaft.

(In case of presence of elastic gear behind the motor, we have its potential energy $E_{pel\xi} = \frac{1}{2} \cdot C_\xi \cdot \xi^2$, and dissipative: $\Phi_{el\xi} = \frac{1}{2} \cdot B_\xi \cdot \dot{\xi}^2$ energy. These quantities should also be expressed in dependence of generalized coordinates and Lagrange's equation should be applied.)

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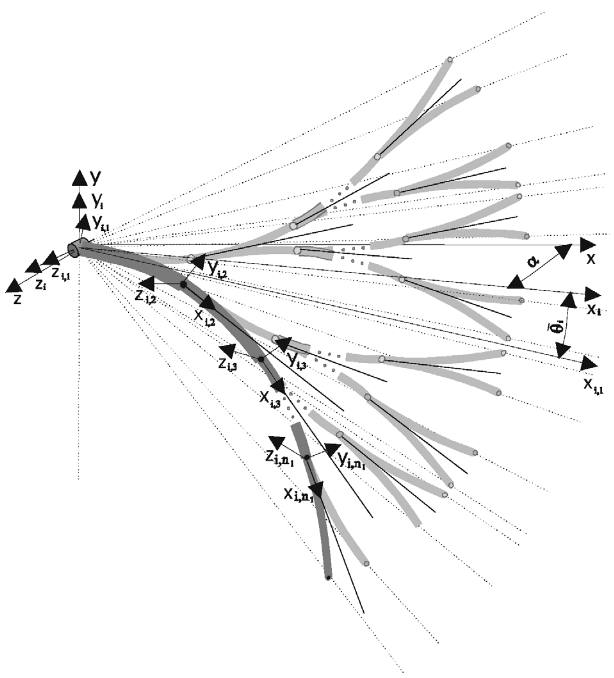

Fig. 3. Possible positions of the tip of elastic line with $n_1$modes.

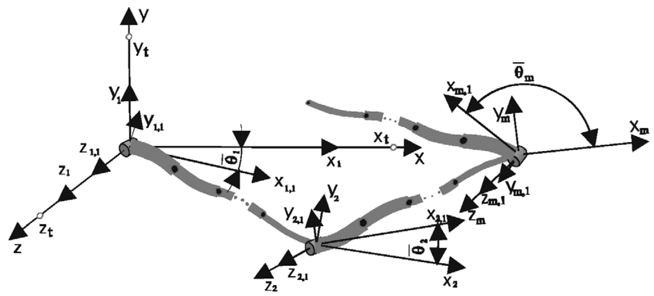

Fig. 4. The elastic line of the complex robotic system with m links.

All this is explained in detail at modeling of considered example in ref. [21], (title 4.A)]. The overall order of the system (17–18) is $n_1 + 1$.

Like we defined the motion of any point on the first mode elastic line by Eqs. (13–14), we can also define the motion of any point on the elastic line of the second, third $\ldots n_1$th mode of the elastic link.

By superimposing the solution (13–14) for all the present modes of the first link and adding to it the dynamics of motor motion that drives it, we obtain total solution of the system (17–18) in the form

$$\hat{y}_1 = R_1(\bar{\theta}_1, t) + \sum_{j=1}^{n_1} \hat{X}_{1,j}(\hat{x}_{1,j})(\hat{T}_{st1,j}(t) + \hat{T}_{to1,j}(t))$$
$$= \hat{a}_1(\hat{x}_{1,j}, \hat{T}_{st1,j}, \hat{T}_{to1,j}, \bar{\theta}_1, t). \tag{19}$$

On considering Fig. 3 we can see that the position $\hat{x}_1$ should also be defined, which is not only $\sum_{j=1}^{n_1} \hat{x}_{1,j}$ (because the directions of the axes $\hat{x}_{1,1}, \hat{x}_{1,2} \ldots \hat{x}_{1,n_1}$ most often do not coincide with the direction of the axis $\hat{x}_1$), but also includes to a significant extent the geometry and characteristics of the mechanism bending, i.e., the mechanism's dynamics,

$$\hat{x}_1 = N_1(\bar{\theta}_1, t) + \sum_{j=1}^{n_1} \hat{K}_{1,j}(\hat{x}_{1,j})(\hat{T}_{st1,j}(t) + \hat{T}_{to1,j}(t))$$
$$= \hat{b}_1(\hat{x}_{1,j}, \hat{T}_{st1,j}, \hat{T}_{to1,j}, \bar{\theta}_1, t). \tag{20}$$

Any form of elastic line and the pertinent transversal oscillations, as well as the motor motion, can be presented by Eqs. (19–20). To this equation one should also add the equation defining the orientation of each point on the elastic line of the link

$$\hat{\psi}_1 = \hat{d}_1(\hat{x}_{1,j}, \hat{T}_{st1,j}, \hat{T}_{to1,j}, \bar{\theta}_1, t). \tag{21}$$

In Fig. 3 is sketched the possible forms of elastic line of the ith link having n_i modes that appear in the plane $x_i - y_i$. The plane $x_i - y_i$ is rotated by the angle α, characterizing in the figure the position of the link base with respect to the main coordinate frame $x - y - z$. In the same figure is presented, only some of the possible forms of elastic line. The link tip can assume very different positions in the plane $x_i - y_i$.

Let us consider a robotic system with m links, where by the first link contains n_1 modes, second link contains n_2 modes, and $\ldots m$th link contains n_m modes (See Fig. 4). Model of the elastic line of this complex elastic robotic system is given in the matrix form by the following equation:

$$\hat{H} \cdot \frac{d^2\hat{y}}{dt^2} + \hat{h} + j_e^T \cdot F_{uk} + z \cdot \Theta \cdot \varepsilon + \hat{\varepsilon} = 0. \tag{22}$$

If we define $k = \sum_{i=1}^{m} n_i$ then we have $\hat{H} \in R^{kxk}$, matrix characterizing the inertia; $\hat{h} \in R^{kx\,1}$, vector of the centrifugal, gravitational, and Coriolis forces; $j_e^T \in R^{kx\,6}$, Jacobian matrix mapping the effect of the dynamic contact force F_{uk}; $\Theta \in R^{kxk}$, matrix characterizing the robot configuration.

$$z = \begin{bmatrix} 0 & -\frac{1}{2^1} & \frac{1}{2^2} & -\frac{1}{2^3} & \ldots & (-1)^{(k-1)}\frac{1}{2^{(k-1)}} \\ 0 & 0 & -\frac{1}{2^1} & \frac{1}{2^2} & \ldots & (-1)^{(k-2)}\frac{1}{2^{(k-2)}} \\ 0 & 0 & 0 & -\frac{1}{2^1} & \ldots & (-1)^{(k-3)}\frac{1}{2^{(k-3)}} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

$z \in R^{kxk}$ is the matrix characterizing the mutual influence of the forces of elastic modes of all the links,

$$\varepsilon = [\varepsilon_{1,1}\ \varepsilon_{1,2} \ldots \varepsilon_{1,n_1}\varepsilon_{2,1}\ \varepsilon_{2,2} \ldots \varepsilon_{2,n_2} \ldots\ldots \varepsilon_{m,n_m}]^T.$$

$$\hat{\varepsilon} = \left[\beta_{1,1} \cdot \frac{\partial^2(\hat{y}_{1,1} + \eta_{1,1} \cdot \dot{\hat{y}}_{1,1})}{\partial\, \hat{x}_{1,1}^2} \ldots \beta_{n_m,n_{ml}} \cdot \frac{\partial^2(\hat{y}_{n_m,n_{ml}} + \eta_{n_m,n_{ml}} \cdot \dot{\hat{y}}_{n_m,n_{ml}})}{\partial\, \hat{x}_{n_m,n_{ml}}^2}\right]^T.$$

If we define from (22) the load moment $\hat{M}$ as

$$\hat{M} = H \cdot \frac{d^2\hat{y}}{dt^2} + \hat{h} + j_e^T \cdot F_{uk} + z \cdot \Theta \cdot \varepsilon, \tag{23}$$

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then Eq. (22) can be written in the form

$$\hat{M} + \hat{\varepsilon} = 0. \tag{24}$$

Equation (24) represents the *equation of motion of the elastic line of the overall robotic system.* In order to describe the behavior of a robotic system having m links (each of them containing n_i modes), we have to add to the vector Eq. (24) the mathematical model of all the motors written in a vector form. Let us define it by setting for each motor the equation of motion of all the moments acting about the rotation axis of the given motor. It has the form of the mathematical model of the motor of a rigid robotic system, but the difference being in that the moment of the ith motor is not opposed by the mechanism moment (as with rigid robotic systems). The motor moment is opposed by the bending moment of the first elastic mode that comes after the motor, and also in part, by the bending moments of the other elastic modes that are connected in series after the given motor. All the modes after the motor, due to their position, influence the dynamics of motor motion. Mathematical model of all m motors can be written in a vector form as:

$$\left.\begin{array}{l} u = R \cdot i + C_E \cdot \dot{\bar{\theta}} \\ C_M \cdot i = I \cdot \ddot{\bar{\theta}} + B_u \cdot \dot{\bar{\theta}} - S \cdot \varepsilon_m \end{array}\right| \begin{array}{l} \sum M = 0 \\ \text{about the rotation axis} \\ \text{of the each motor} \end{array} . \tag{25}$$

In Eq. (25) we have m equations of motors.

$$\varepsilon_m = z_m \cdot \Theta \cdot \varepsilon. \tag{26}$$

$$z_m = \begin{bmatrix} \frac{1}{2^0} & -\frac{1}{2^1} & \ldots & (-1)^{(n_1-1)}\frac{1}{2^{(n_1-1)}} & \ldots & (-1)^{(k-n_m-1)}\frac{1}{2^{(k-n_m-1)}} & (-1)^{(k-n_m)}\frac{1}{2^{(k-n_m)}} & \ldots & (-1)^{(k-1)}\frac{1}{2^{(k-1)}} \\ 0 & 0 & \ldots & 0 & \ldots & (-1)^{(k-n_m-n_1-1)}\frac{1}{2^{(k-n_m-n_1-1)}} & (-1)^{(k-n_m-n_1)}\frac{1}{2^{(k-n_m-n_1)}} & \ldots & (-1)^{(k-n_1-1)}\frac{1}{2^{(k-n_1-1)}} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & \ldots & \frac{1}{2^0} & -\frac{1}{2^1} & \ldots & (-1)^{(n_m-1)}\frac{1}{2^{(n_m-1)}} \end{bmatrix}$$

$z_m \in R^{mxk}$ is the matrix characterizing the effect of elasticity moment of each mode on the motor motion dynamic.

The potential energy in top of jth mode of the ith link is $E_{pels\, i,j} = \frac{1}{2} C_{s\, i,j} \cdot \vartheta_{i,j}^2 \cdot l_{i,j}^2$, while dissipative energy is $\Phi_{els\, i,j} = \frac{1}{2} B_{s\, i,j} \cdot \dot{\vartheta}_{i,j}^2 \cdot l_{i,j}^2$.

All quantities should be expressed in dependence of generalized coordinates. One of them is also angle $\bar{\theta}_i$.

By applying Lagrange's equations on the expressions $E_{pels\,1,1}, E_{pels\,1,2}, \ldots, E_{pels\,i,j}, \ldots, E_{pels\;m,n\,1}$, and $\Phi_{els\,1,1}, \Phi_{els\,1,2}, \ldots, \Phi_{els\,i,j}, \ldots, \Phi_{els\;m,n\,1}$ with respect to the generalized coordinate $\bar{\theta}_i$, we obtain the equivalent elasticity moment $\varepsilon_{m\;i}$ that opposes to the rotation moment of the ith motor shaft. $\varepsilon_m = [\varepsilon_{m\,1}\; \varepsilon_{m\,2}\; \ldots\; \varepsilon_{m\,i}\; \ldots\; \varepsilon_{m\,m}]^T$.

The overall order of the system (24–25) is $k + m$. Full model is planned on classical principles of the mechanics as in ref. [34].

It is known that the robot configuration can substantially influence the mutual position of elastic lines of particular links (see Fig. 4). Solution of the system (24–25), i.e., the form of its elastic line, can be obtained by superimposing the solutions (19–21) for all the links involved in the presence of the dynamics (angle) of rotation of each motor, as well as by taking into account the robotic configuration, i.e., the angle between the axes z_{i-1} and z_i.

$$\begin{aligned} \hat{y} &= \sum_{i=1}^{m} D_i(\alpha_i) \cdot R_i(\theta_i, t) \\ &\quad + \sum_{i=1}^{m} (D_i(\alpha_i) \sum_{j=1}^{n_1} \hat{X}_{i,j}(\hat{x}_{i,j})(\hat{T}_{st\,i,j}(t) + \hat{T}_{to\,i,j}(t))). \\ &= \hat{a}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{toi,j}, \bar{\theta}, \alpha, t) \end{aligned} \tag{27}$$

$$\begin{aligned} \hat{x} &= \sum_{i=1}^{m} A_i(\alpha_i) \cdot N_i(\theta_i, t) \\ &\quad + \sum_{i=1}^{m} (A_i(\alpha_i) \sum_{j=1}^{n_1} \hat{K}_{i,j}(\hat{x}_{i,j})(\hat{T}_{st\,i,j}(t) + \hat{T}_{to\,i,j}(t))). \\ &= \hat{b}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{toi,j}, \bar{\theta}, \alpha, t) \end{aligned} \tag{28}$$

$$\begin{aligned} \hat{z} &= \sum_{i=1}^{m} L_i(\alpha_i) \cdot M_i(\theta_i, t) \\ &\quad + \sum_{i=1}^{m} (L_i(\alpha_i) \sum_{j=1}^{n_1} \hat{P}_{i,j}(\hat{x}_{i,j})(\hat{T}_{st\,i,j}(t) + \hat{T}_{to\,i,j}(t))). \\ &= \hat{c}(\hat{x}_{i,j}, T_{sti,j}, T_{sti,j}, \bar{\theta}, \alpha, t). \end{aligned} \tag{29}$$

D_i, A_i, and L_i is the function that maps the rotation angle between the axes z_{i-1} and z_i . R_i, N_i, M_i is the function mapping the rotation angle of the motor shaft θ_i with respect to the y, x, and z axis, respectively.

Of course, it is also necessary to define orientation at each point of the elastic line. The orientation of any point of the elastic line of the given robot is defined by

$$\hat{\psi} = \hat{d}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{toi,j}, \bar{\theta}, \alpha, t). \tag{30}$$

$$\hat{\xi} = \hat{e}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{toi,j}, \bar{\theta}, \alpha, t). \tag{31}$$

$$\hat{\varphi} = \hat{f}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{toi,j}, \bar{\theta}, \alpha, t). \tag{32}$$

Thus we define the position and orientation of each point of the elastic line in the space of Cartesian coordinates. It should be pointed out that the form of elastic line comes directly out from the dynamics of the system motion.

4. Relationship between the Equation of Elastic Line Motion and Equation of Motion at any Point of the Elastic Line

Robotic man is especially interested in the motion of the first mode tip of each link, of the link tip and finally, of the robot tip motion. At the point mode tip inertial forces (own and the coupled ones of the other modes), centrifugal, gravitational, Coriolis forces (own and coupled), forces due to the relative motion of one mode with respect to the other, coupled elasticity forces of the other modes, as well as the environment force; the effect of the latter on the motion of the considered mode being transferred through the Jacobian matrix. The sum of all these forces denoted by the force $F_{1,1}$ and called it elasticity force.

All the forces forming the force $F_{1,1}$ acting at the distance $l_{1,1}$ from the base of the first mode form the elasticity moment $\varepsilon_{1,1}$ cause the deflection of the first mode $r_{1,1}$ (see Eq. (10)).

The equation of motion of the forces involved at any point of the elastic line of first mode, including the point of the first mode tip, can be defined from the equation of motion of elastic line (9). The equation of motion of all forces at the first mode tip for the given boundary conditions can be defined by the following equation:

$$H_{1,j}\frac{d^2 y_{1,j}}{dt^2} + h_{1,1} + j_{e1,1}^T F_{uk} + z_{1,j}\varepsilon_{1,j} + \varepsilon_{1,1} = 0 \left| \begin{array}{l} \sum F = 0 \\ \text{at the point of} \\ \text{first mode tip} \end{array} \right. . \quad (33)$$

Obviously, the term $z_{1,j}\varepsilon_{1,j} + \varepsilon_{1,1}$ in Eq. (33) could be written in a more compact form, but here we purposely wrote it in a split form, to indicate the presence of elasticity forces of the other modes characterized by $z_{1,j}\varepsilon_{1,j}$, and which influence the deformation of the considered (first) mode. Equation (33) is interesting because it allows one to calculate the position of the first mode tip. If we know position of each mode tip we can always calculate the position of the link tip too and eventually the position of the robot tip.

The motion of the mode tip, its position, and orientation are defined by the sum of the stationary and oscillatory motion (cf. Eqs. (13–14)).

$$y_{1,1} = a_{1,1}(x_{1,1}, T_{st1,1}, T_{to1,1}, t) \left| \begin{array}{l} \text{tip position of the} \\ \text{first mode in the} \\ \text{direction of the axis } y_{1,1} \end{array} \right. . \quad (34)$$

$$\psi_{1,1} = d_{1,1}(x_{1,1}, T_{st1,1}, T_{to1,1}, t) \left| \begin{array}{l} \text{orientation of the} \\ \text{first mode tip} \\ \text{about the axis } z_{1,1} \end{array} \right. . \quad (35)$$

The equation of motion of all the forces at the tip of each mode of the first link can be defined from Eq. (15) for the preset boundary conditions

$$H_1\frac{d^2 y_{1,j}}{dt^2} + h_1 + j_{e1}^T F_{uk} + z_1\varepsilon_{1,j} + \varepsilon_{1,j} = 0 \left| \begin{array}{l} \sum F = 0 \\ \text{at the point of} \\ \text{each mode tip} \\ \text{of the first link} \end{array} \right. . \quad (36)$$

This equation should be supplemented with the mathematical model of motor, defined by Eq. (18). The motion of the first link tip is defined by the sum of the stationary and oscillatory motion of the tip of each mode plus the dynamics of motor motion $\bar{\theta}_1$ (cf. Eqs. (19–21)):

$$y_1 = a_1(x_{1,j}, T_{st1,j}, T_{to1,j}, \bar{\theta}, t) \left| \begin{array}{l} \text{first link} \\ \text{tip position in the} \\ \text{direction of the axis } y_1 \end{array} \right. . \quad (37)$$

$$x_1 = b_1(x_{1,j}, T_{st1,j}, T_{to1,j}, \bar{\theta}, t) \left| \begin{array}{l} \text{first link} \\ \text{tip position in the} \\ \text{direction of the axis } x_1 \end{array} \right. . \quad (38)$$

$$\psi_1 = d_1(x_{1,j}, T_{st1,j}, T_{to1,j}, \bar{\theta}, t) \left| \begin{array}{l} \text{first link} \\ \text{tip orientation} \\ \text{about the axis } z_1 \end{array} \right. . \quad (39)$$

The equation of motion of all the forces at the point of each mode tip of any link can be defined from Eq. (22) by setting the boundary conditions. Vector Equation of all the forces involved for each mode tip of any link is

$$H\frac{d^2 y}{dt^2} + h + j_e^T \cdot F_{uk} + z \cdot \Theta \cdot \varepsilon + \varepsilon = 0 \left| \begin{array}{l} \sum F = 0 \\ \text{at the tip of} \\ \text{any mode of the} \\ \text{link considered} \end{array} \right. . \quad (40)$$

This equation should be supplemented by the vector Eq. (25) of the mathematical model of motor. The robot tip motion is defined by the sum of the stationary and oscillatory motion of each mode tip plus the dynamics of motion of the motor powering each link, as well by the included robot configuration (cf. Eqs. (27–32)).

$$y = a(x_{i,j}, T_{sti,j}, T_{toi,j}, \bar{\theta}, \alpha, t) \left| \begin{array}{l} \text{tip position of} \\ \text{the robotic system} \\ \text{in the direction of the axis } y \end{array} \right. . \quad (41)$$

$$x = b(x_{i,j}, T_{sti,j}, T_{toi,j}, \bar{\theta}, \alpha, t) \left| \begin{array}{l} \text{tip position of} \\ \text{the robotic system} \\ \text{in the direction of the axis } x \end{array} \right. . \quad (42)$$

$$z = c(x_{i,j}, T_{sti,j}, T_{toi,j}, \bar{\theta}, \alpha, t) \left| \begin{array}{l} \text{tip position of} \\ \text{the robotic system} \\ \text{in the direction of the axis } z \end{array} \right. . \quad (43)$$

$$\psi = d(x_{i,j}, T_{sti,j}, T_{toi,j}, \bar{\theta}, \alpha, t) \left| \begin{array}{l} \text{tip orientation of} \\ \text{the robotic system} \\ \text{about the axis } z \end{array} \right. . \quad (44)$$

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$$\xi = e(x_{i,j}, T_{sti,j}, T_{toi,j}, \bar{\theta}, \alpha, t) \left| \begin{array}{l} \text{tip orientation of} \\ \text{the robotic system.} \\ \text{about the axis } y \end{array} \right. \quad (45)$$

$$\varphi = f(x_{i,j}, T_{sti,j}, T_{toi,j}, \bar{\theta}, \alpha, t) \left| \begin{array}{l} \text{tip orientation of} \\ \text{the robotic system.} \\ \text{about the axis } x \end{array} \right. \quad (46)$$

From Eqs. (41–46) we can calculate the position of each mode tip, of each link, and finally of the robot tip motion.

Generally, we can derive the following conclusion:

To define the form of elastic line of the considered robotic system it is necessary to expand the previously known solutions, namely:

- Supplement it by adding stationary solution to the particular solution of D. Bernoulli, which is of oscillatory character. This means that the given solution depends directly on the overall system dynamics.
- As the link elastic line does not usually conform to the direction of the preset axes but extends in the space, we cannot define it by only one equation. General form of the elastic line is a direct outcome of the dynamics of system motion and cannot be represented by one scalar equation as three equations are needed to define position and three equations to define orientation of each point on the elastic line.
- The equation of elastic line of the robotic system should also encompass the angles of motor shaft rotation $\bar{\theta}$ as in ref. [18], and also the robot configuration, i.e., the angles between the axes z_{i-1} and z_i.

5. Simulation Example

Robot starts from point "A" (Fig. 5) and moves toward point "B" in the predicted time $T = 2[s]$. Dynamics of the environment force as in ref. [35] is included into the dynamics of system's motion. The adopted velocity profile is trapezoidal ($\dot{q}^o_{\max} = 0.9817[{}^{rad}/_{s}]$), with the acceleration/deceleration period of $0.2 \cdot T(\ddot{q}^o_{\max} = \pm 2.4544[{}^{rad}/_{s^2}]$).

The same example is analyzed in ref. [21] with only somewhat different parameter flexibility.

Elastic deformation is a quantity which is at least partly encompassed by the reference trajectory as explained in ref. [21] (2.1 under 2).

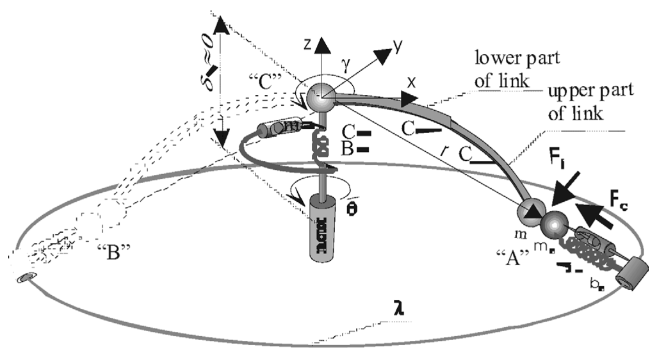

Fig. 5. The robotic mechanism.

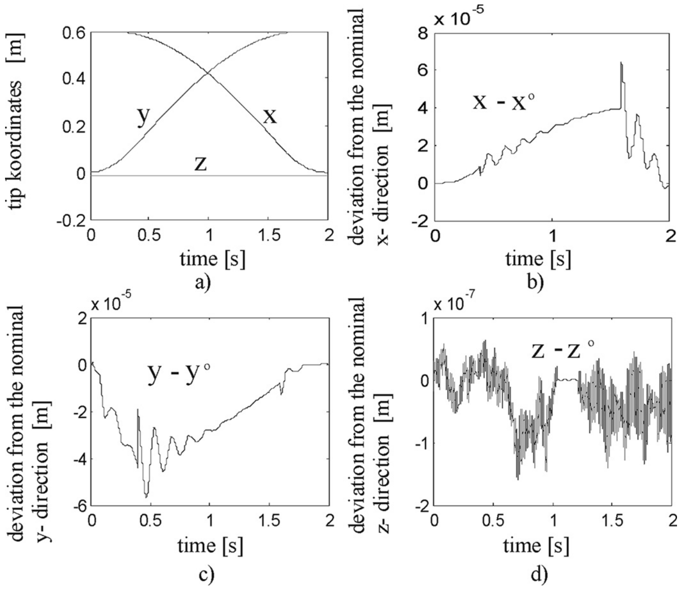

Fig. 6. Tip coordinates and deviation of position from the reference level (Example 1).

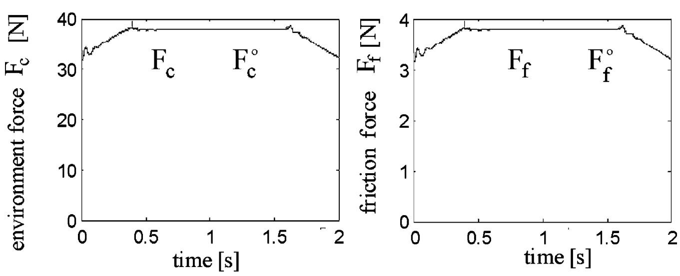

Fig. 7. Dynamic force of the environment (Example 1).

Example 1: The rigidity characteristic for the tip of first mode is $C_{s1,1} = 6.1569 * 10^5\,[N/m]$ and $B_{s1,1} = 0$ $[N \cdot s/m]$, and for the tip of the second mode is $C_{s1,2} = 1.873 * 10^4\,[N/m]$ and $B_{s1,2} = 60\,[N \cdot s/m]$.

All other characteristics of the system and environment are the same as in paper ref. [21].

The characteristics of stiffness and damping of the gear in the real and reference regimes are not the same and neither are the stiffness and damping characteristics of the link.

$$C_\xi = 0.99 \cdot C^o_\xi, \quad B_\xi = 0.99 \cdot B^o_\xi, \quad C_{s1,1} = 0.99 \cdot C^o_{s1,1},$$
$$B_{s1,1} = 0.99 \cdot B^o_{s1,1}, \quad C_{s1,2} = 0.99 \cdot C^o_{s1,2}, \quad B_{s1,2} = 0.99 \cdot B^o_{s1,2}.$$

As can be seen from Fig. 6 in its motion from point "A" to point "B" the robot tip tracks well the reference trajectory in the space of Cartesian coordinates.

As position control law for controlling local feedback was applied, the tracking of the reference force was directly dependent on the deviation of position from the reference level (see Fig. 7).

In Fig. 8 are given the elastic deformations that are taking place in the vertical plane angle of bending of the lower part of the link (first mode) ϑ_m and the angle of bending of the upper part of the link (second mode) ϑ_e, as well as the elastic

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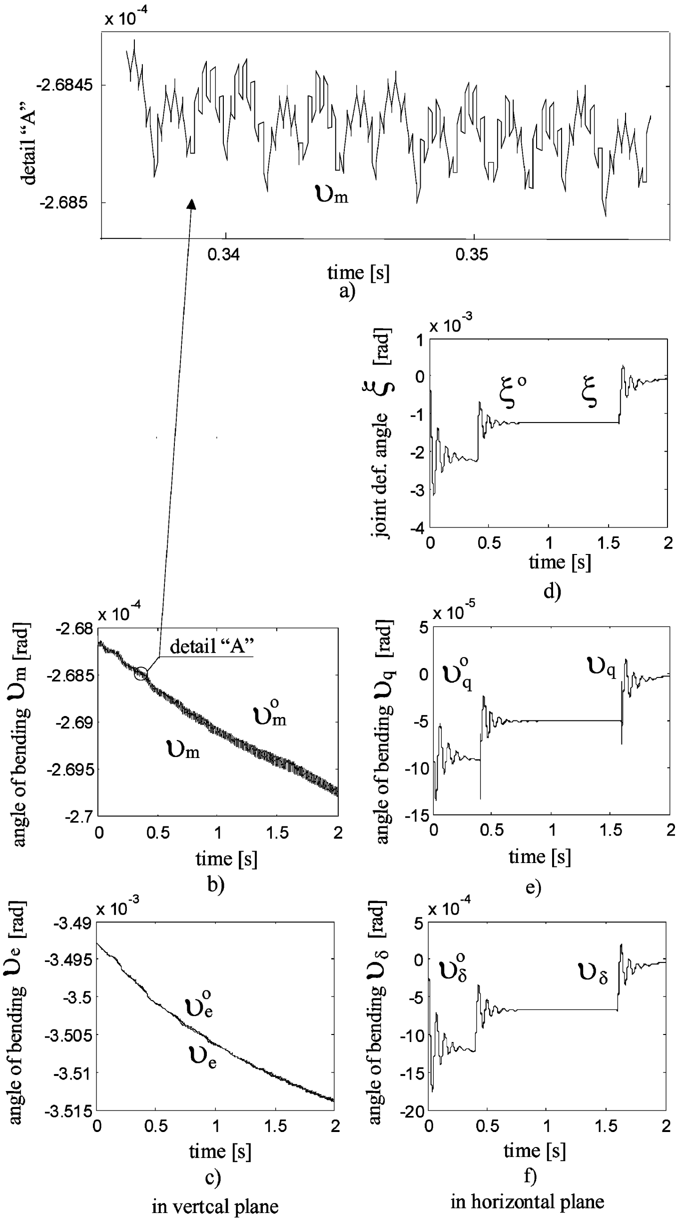

Fig. 8. The elastic deformations (Example 1).

deformations taking place in the horizontal plane, the angle of bending of the lower part of the link (first mode) ϑ_q, the angle of bending of the upper part of the link (second mode) ϑ_δ, and the deflection angle of gear ξ.

The rigidity of the second mode is about ten times lower compared with that of the first mode, it is then logical that the bending angle for the second mode is about ten times larger compared to that of the first mode.

Figure 8(a) exhibits the wealth of different amplitudes and circular frequencies of the present modes of elastic elements.

Example 2: In contrast to Example 1, the characteristics of stiffness and damping of the second mode of the link in the real regime differ significantly from that in the reference regime.

$$C_\xi = 0.99 \cdot C_\xi^o, \quad B_\xi = 0.99 \cdot B_\xi^o, \quad C_{s1,1} = 0.99 \cdot C_{s1,1}^o,$$
$$B_{s1,1} = 0.99 \cdot B_{s1,1}^o, C_{s1,2} = 0.2 \cdot C_{s1,2}^o, B_{s1,2} = 0.2 \cdot B_{s1,2}^o.$$

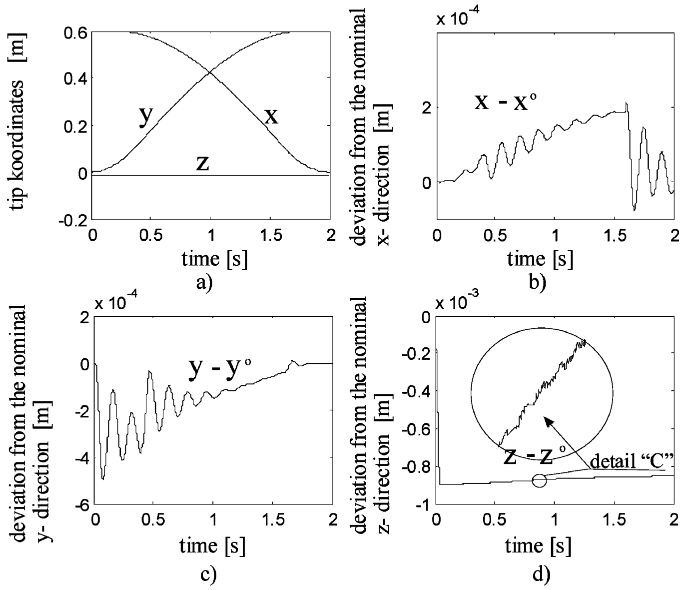

Fig. 9. Tip coordinates and deviation of position from the reference level (Example 2).

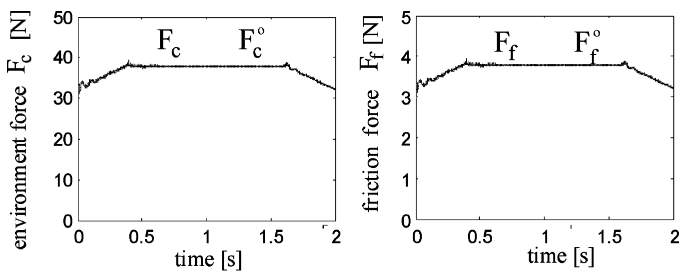

Fig. 10. Dynamic force of the environment (Example 2).

All other characteristics of the system are the same as in Example 1.

As can be seen from Fig. 9, the real robotic tip motion in the x, y, z-directions does not track so well the reference trajectory in the space of Cartesian coordinates as in Example 1. The partial lack of the knowledge of flexibility characteristics only of the second mode in the robotic system may significantly influence the real robotic tip motion, which has now a much larger deviation from the reference trajectory in the space of Cartesian coordinates (cf. Figs. 6 and 9).

In this example, the real force has a more pronounced oscillatory character compared to the reference force (cf. Figs. 7 and 10).

The elastic deformations that take place in the vertical plane, the angle of bending ϑ_m and the angle ϑ_e, as well as elastic deformations taking place in the horizontal plane, the gear deflection angle ξ, the angle ϑ_q and the angle ϑ_δ are shown in Fig. 11.

Flexibility characteristic of the link second mode is an insufficiently known quantity at the reference level, which, because of the significant coupling involved, is also reflected on all other dynamic quantities in the system.

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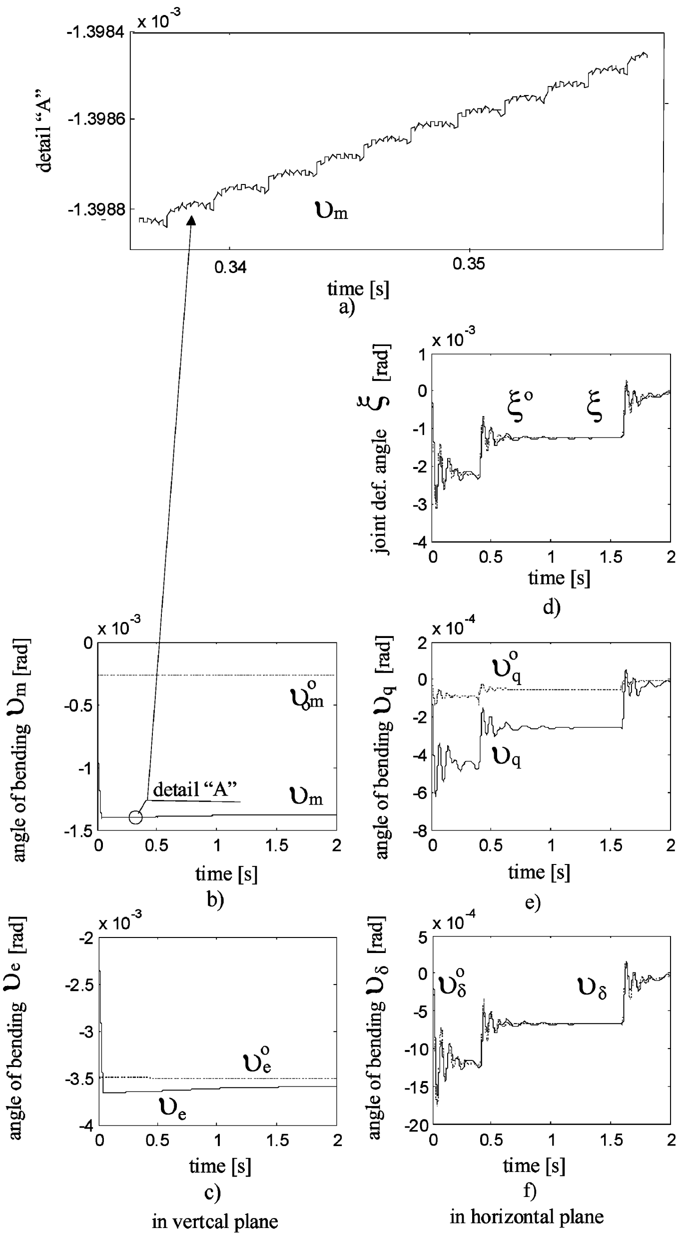

Fig. 11. The elastic deformations (Example 2).

A more significant lack of knowledge of flexibility characteristics of the second mode of the link causes larger deviations of this quantity from the reference in the course of robotic task realization. However, the other elastic deformations in this robotic mechanism have also larger deviations with respect to the reference level of elastic deformations (cf. Figs. 8 and 11).

Let us show the special significance of results from Figs. 8(a) and 11(a). These Figures exhibit the wealth of different amplitudes and circular frequencies of the present modes of elastic elements. We have oscillations within oscillations. This confirms that we have modeled all elastic elements as well as high harmonics (in this case two harmonics of considered link). Comparative analysis of the presented approach and the well-known previous approaches is presented in the following table:

	Presented	Previous
Relationship between EBA and LMA	Defined	Unknown
Procedure obtaining of elastic deformation and frequency of presented modes	Follows from whole robot movement dynamics	Defined in advance and, when defined in this way, imported in robot model
According to forces, which cause it, elastic deformation is	Summary of stationary forced character solution and particular oscillatory character solution of Daniel Bernoulli	Particular solutions of oscillatory character of Daniel Bernoulli
Stiffness matrix, damping matrix	Full	Diagonal
Mathematical model of motor	The motor torque is opposed by the elasticity force of the first elastic element that is coming directly after the motor and also, in part, by the bending moments of the other flexible modes that are connected sequentially after the first mode.	Solution is not unique
Coupling between present modes	Exists. For this reason Euler Bernoulli equations are not equal mutually	Does not exist
Damping	Modeled	Not modeled

6. Conclusions

Based on the EBA is defined the equation of elastic line of each mode of any link of a complex robotic system. Also demonstrated is the equation of motion of all the forces involved at any point that follows directly from the equation of elastic line. If we define boundary conditions for the mode tip as the most interesting point on the elastic line, we obtain the equation of motion at that point, i.e., we obtain *classical form* of the mathematical model of the elastic robotic system considered, which essentially LMA is. Thus the connection of the LMA and EBA is demonstrated. LMA is just a special case of EBA. In addition to the comparative analysis of the EBA and LMA, the paper also analyzes a number of other phenomena that make constitutive parts of the motion dynamics of these systems.

(a) Euler–Bernoulli equations have been expanded from several aspects:

(1) Euler–Bernoulli equation (based on the known laws of dynamic) should be supplemented with all the forces that are participating in the formation of the bending moment of the considered mode causing the difference in the structure of these equations for each mode.
(2) Structure of the stiffness matrix must also have the elements outside the diagonal due to the existence of strong coupling between the elasticity forces involved.
(3) Damping is an omnipresent elasticity characteristic of real systems, so it is naturally included in the Euler–Bernoulli equation.
(4) General form of the transversal elastic deformation is defined by superimposing particular solutions of oscillatory character (solution of Daniel Bernoulli) and stationary solution of the forced character (which is a consequence of the forces involved).
(5) General form of the elastic line is a direct outcome of the dynamics of system motion and cannot be represented by one scalar equation as three equations are needed to define position and three equations to define orientation of each point on the elastic line.

(b) Structure of the mathematical models of actuators:
With elastic robotic systems, the actuator torque is opposed by the bending moment of the first elastic mode, which comes after the motor, and partly by the bending moments of other modes, which are connected in series after the motor considered. All modes coming after the motor, because of their position, exert influence on the dynamics of motor motion. The mathematical model in our work is connected to the rest of the mechanism via the equivalent elasticity moment. *New structures of the stiffness matrix and mathematical model of actuators appear as a consequence of the coupling between the modes of particular links.*

In a word, elastic deformation is a consequence of the overall dynamics of the robotic system.

All this has been presented for a relatively simple robotic system that offered the possibility of analyzing the phenomena involved. Through the analysis and modeling of an elastic mechanism we made an attempt to give a contribution to the development of this area.

References

1. M. Vukobratovic, V. Potkonjak and V. Matijević, "Control of robots with elastic joints interacting with dynamic environment," *J. Intel. Robot Syst.* **23**, 87–100 (1998).
2. Book, Maizza-Neto and Whitney, "Feedback control of two beam, two joint systems with distributed flexibility," *Trans ASME J. Dyn. Syst. Meas. Control* **97**G(4), 424–431 (1975).
3. P. C. Hughes, "Dynamic of a Flexible Manipulator Arm for the Space Shuttle," *AIAA Astrodynamics Conference*, Grand Teton National Park, Wyoming, USA (1977).
4. M. Moallem, K. Khorasani and V. R. Patel, "Tip Position Tracking of Flexible Multi-Link Manipulators: An Integral Manifold Approach," *IEEE International Conference on Robotics and Automation*, Minneapolis, Minnesota (Apr. 1996) pp. 2432–2436.
5. F. Matsuno, K. Wakashiro and M. Ikeda, "Force Control of a Flexible Arm," *International Conference on Robotics and Automation*, San Diego, CA, USA (May 1994) pp. 2107–2112.
6. F. Matsuno and T. Kanzawa, "Robust Control of Coupled Bending and Torsional Vibrations and Contact Force of a Constrained Flexible Arm," *International Conference on Robotics and Automation*, Minneapolis, Minnesota (1996) pp. 2444–2449.
7. W. Yim, "Modified Nonlinear Predictive Control of Elastic Manipulators," *International Conference on Robotics and Automation*, Minneapolis, Minnesota (Apr. 1996) pp. 2097–2102.
8. D. Surdilovic and M. Vukobratovic, "One method for efficient dynamic modeling of flexible manipulators," *Mech. Mach. Theory*. **31**(3), 297–315 (1996).
9. D. Surdilovic and M. Vukobratovic, "Deflection compensation for large flexible manipulators," *Mech. Mach. Theory* **31** (3), 317–329 (1996).
10. M. W. Spong, "On the force control problem for flexible joint manipulators," *IEEE Trans. on Autom. Control* **34**(1) (Jan. 1989) pp. 107–111.
11. F. Ghorbel and M. W. Spong, "Adaptive Integral Manifold Control of Flexible Joint Robot Manipulators," *IEEE International Conference on Robotics and Automation*, Nice, France (May 1992) pp. 707–714.
12. H. Krishnan, "An Approach to Regulation of Contact Force and Position in Flexible-Link Constrained Robots," *IEEE*, Vancouver (1995) pp. 2087–2092.
13. J. Cheong, W. Chung and Y. Youm, "Bandwidth Modulation of Rigid Subsystem for the Class of Flexible Robots," *Proceedings Conference on Robotics and Automation*, San Francisco, USA (Apr. 2000) pp. 1478–1483.
14. H. K. Low, "A systematic formulation of dynamic equations for robot manipulators with elastic links," *J Robot. Syst*. **4**(3), 435–456 (Jun 1987).
15. H. K. Low and M. Vidyasagar, "A Lagrangian formulation of the dynamic model for flexible manipulator systems," *ASME J Dyn. Syst., Meas.t Control* **110**(2), 175–181(Jun 1988).
16. H. K. Low, "Solution schemes for the system equations of flexible robots," *J. Robot. Syst.* **6**(4), 383–405 (Aug. 1989).
17. E. Bayo, "A finite-element approach to control the end-point motion of a single-link flexible robot," *J. Robot. Syst.* **4**(1), 63–75 (1987).
18. A. De Luca, "Feedforward/Feedback Laws for the Control of Flexible Robots," *IEEE Intern Conf. on Robotics and Automation*, San Francisco, CA (Apr. 2000) pp. 233–240.
19. M. W. Spong, "Modeling and control of elastic joint robots," *ASME J. Dyn. Syst Meas. Control* **109**, 310–319 (1987).
20. M. Filipović and M. Vukobratović, "Modeling of Flexible Robotic Systems," *Computer as a Tool, EUROCON 2005, The International Conference*, Belgrade, Serbia and Montenegro, (Nov. 21–24, 2005) Vol. 2, pp. 1196–1199.
21. M. Filipović and M. Vukobratović, "Contribution to modeling of elastic robotic systems," *Eng. Autom. Probl Int. J.* **5**(1), 22–35 (Sep. 23, 2006).
22. M. Filipović, V. Potkonjak and M. Vukobratović: "Humanoid robotic system with and without elasticity elements walking on an immobile/mobile platform," *J. Intel. Robot. Syst. Int. J.* **48**, 157–186 (2007).
23. L. Meirovitch, *Analytical Methods in Vibrations* (*Macmillan: New York*, 1967).
24. F. Matsuno and Y. Sakawa: "Modeling and quasi-static hybrid position/force control of constrained planar two-link flexible manipulators," *IEEE Trans. Robot. Autom.* **10**(3) (Jun 1994).
25. J. S. Kim, K. Siuzuki K. and A. Konno, "Force Control of Constrained Flexible Manipulators," *International Conference*

on Robotics and Automation, Minneapolis, Minnesota (Apr. 1996) pp. 635–640.
26. A. De Luka and B. Siciliano, "Closed-form dynamic model of planar multilink lightweight robots," *IEEE Trans. Syst., Man Cybern*. **21**, 826–839 (Jul./Aug. 1991).
27. H. Jang, H. Krishnan and M. H. Ang Jr., "A Simple Rest-to-Rest Control Command for a Flexible Link Robot," *IEEE International Conference on Robotics and Automation* (1997) pp. 3312–3317.
28. J. Cheong, W. K. Chung and Y. Youm, "PID Composite Controller and Its Tuning for Flexible Link Robots," *Proceedings of the 2002 IEEE/RSJ, International Conference on Intelligent Robots and Systems EPFL*, Lausanne, Switzerland (Oct. 2002) pp. 2122–2128.
29. S. E. Khadem and A. A. Pirmohammadi, "Analytical development of dynamic equations of motion for a three-dimensional flexible link manipulator with revolute and prismatic joints," *IEEE Trans. Syst. Man Cybern., part B: Cybernetics* **33**(2) (Apr. 2003).
30. W. J. Book and M. Majette, "Controller design for flexible, distributed parameter mechanical arms via combined state space and frequency domain techniques," *Trans ASME J. Dyn. Syst. Meas. Control* **105**, 245–254 (1983).
31. W. J. Book, "Recursive Lagrangian dynamics of flexible manipulator arms," *Int. J. Robot. Res.* **3**(3) (1984).
32. W. J. Book, "Analysis of massless elastic chains with servo controlled joints," *Trans. ASME J. Dyn. Syst. Meas. And Control* **101**, 187–192 (1979).
33. J. W. Strutt and Lord Rayligh, "Theory of Sound," *second publish*, Mc. Millan & Co., London and New York (1894–1896) *paragraph 186*.
34. S. Timoshenko and D. H. Young, *Vibration Problems in Engineering* (D. Van Nostrand Company: New York, 1955).
35. V. Potkonjak and M. Vukobratovic, "Dynamics in contact tasks in robotics, Part I: General model of robot interacting with dynamic environment," *Mech. Mach. Theory* **33** (1999).

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