

Maxwell-Huygens, Newton-Cartan, and Saunders-Knox Space-Times

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I address a question recently raised by Simon Saunders concerning the relationship between the space-time structure of Newton-Cartan theory and that of what I will call “Maxwell-Huygens space-time.” This discussion will also clarify a connection between Saunders’s work and a recent paper by Eleanor Knox.

In a recent paper, Simon Saunders (2013) argues that proper reflection on Corollary VI to Newton’s laws of motion reveals that the space-time structure presupposed by Newton’s laws is one on which there is a privileged class of “nonrotating” states of motion but not a privileged class of non-accelerating, or inertial, states of motion. Thus Saunders’s view differs markedly from the received view, according to which Newton’s laws presuppose Galilean space-time, a structure in which one does have a privileged class of inertial trajectories (Stein 1967; Earman 1989; DiSalle 2008). Saunders calls his alternative structure “Newton-Huygens space-time,” though he acknowledges that the same structure was introduced by Earman (1989, sec. 2.3) under the moniker “Maxwellian space-time.”¹ (As a compromise, I will call it “Maxwell-Huygens space-time” here.)

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1. Actually, Saunders uses “Newton-Huygens space-time” to refer to the space-time structure understood as a modified affine space; Maxwellian space-time refers to its characterization in terms of fields on a manifold. For present purposes, the difference is inconsequential, and I will limit attention to the latter.

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At the end of his paper, Saunders poses the following question: “What is the relation between a theory of gravity (and other forces) formulated in [Maxwell-Huygens] space-time and one based on Newton-Cartan space-time?” (2013, 46). It is this question that I hope to answer in what follows. Doing so, I will argue, reveals a certain inadequacy in Saunders’s account of classical space-time structure. It also shows a close, and perhaps not altogether obvious, connection between Saunders’s work and recent work by Knox (2014).

To begin, I will give a precise characterization of Maxwell-Huygens space-time. This structure, which may be denoted by $(M, t_a, h^{ab}, [\nabla])$, consists of a smooth four-dimensional manifold M , which we take to be diffeomorphic to \mathbb{R}^4 ; a “temporal metric” t_a , which is a smooth one form of signature $(1, 0, 0, 0)$;² a spatial metric h^{ab} , which is a smooth symmetric tensor field of signature $(0, 1, 1, 1)$ satisfying $h^{ab}t_a = \mathbf{0}$; and a “standard of rotation,” which I will denote by $[\nabla]$. The temporal and spatial metrics allow one to distinguish between “timelike vectors” (vectors ξ^a at a point $p \in M$ for which $\xi^a t_a \neq 0$) and “spacelike vectors” (any other vector); to assign temporal lengths to arbitrary vectors at a point; and to assign spatial lengths to spacelike vectors at a point. This structure also permits one to foliate space-time into constant-time hypersurfaces; in what follows, I assume that these hypersurfaces are diffeomorphic to \mathbb{R}^3 and complete relative to the metric induced by h^{ab} .

The temporal and spatial metrics are now standard notions. (See Earman [1989, chap. 2] or Malament [2012, sec. 4.1] for further details.) The standard of rotation, however, requires further comment. The idea is that we want enough structure to say whether a smooth timelike vector field ξ^a , representing, say, the motion through space-time of bits of some body, is “rotating.” The class of nonrotating vector fields, then, would represent the privileged states of nonrotation mentioned above.

To capture the idea of a standard of rotation, let me begin by describing Galilean space-time in more detail (Earman 1989, sec. 2.4).³ Galilean space-time consists of a structure that may be written (M, t_a, h^{ab}, ∇) , where M , t_a , and h^{ab} are all as described above. The additional piece of structure is a flat covariant derivative operator ∇ on M that we require to be compatible with the spatial and temporal metrics in the sense that $\nabla_a t_b = \mathbf{0}$ and $\nabla_a h^{bc} = \mathbf{0}$.⁴

2. By defining the temporal metric as a one form, we have implicitly assumed that Maxwell-Huygens space-time is “temporally oriented” (see Malament 2012, 251). Note, too, that here (and throughout) we are working in the abstract index notation, which is explained in Malament’s sec. 1.4.

3. See also Malament (2012, sec. 4.1). For Malament, the structures I call Galilean space-times are “classical space-times” with a flat derivative operator.

4. Here and throughout I restrict attention to torsion-free derivative operators.

Such an object provides a standard of constancy for vector fields; it also provides a notion of inertial motion, corresponding to the constant timelike vector fields. By providing a standard of nonacceleration, a derivative operator also provides a natural standard of rotation. This can be captured as follows. Given a unit timelike vector field ξ^a (i.e., a field such that $|\xi^a t_a| = 1$) in Galilean space-time, we say that ξ^a is *nonrotating* if and only if $\nabla^{[a}\xi^{b]} = \mathbf{0}$, where $\nabla^a \xi^b = h^{am} \nabla_m \xi^b$.⁵

Defining nonrotation in this way certainly allows us to make the determinations we require. But in fact it provides too much structure, since as we have seen, a derivative operator also allows one to define a class of preferred inertial states of motion. What we wanted was a standard of rotation and no more. One way of making precise the sense in which a derivative operator gives us strictly more structure than just a standard of rotation is to observe that, in general, many flat derivative operators are compatible with the classical metric structure;⁶ moreover, many of these will agree on whether a given vector field is nonrotating, even though they disagree on other matters, such as whether a given vector field is constant. This suggests that we can capture the idea of a standard of rotation by considering an equivalence class of flat, metric-compatible derivative operators, all of which agree on whether any given vector field is rotating:⁷

$$[\nabla] = \{ \nabla' : R'^a{}_{bcd} = \mathbf{0}; \nabla'_a t_b = \mathbf{0}; \nabla'_a h^{bc} = \mathbf{0}; \\ \text{and } |\xi^a t_a| = 1 \Rightarrow (\nabla'^{[a}\xi^{b]} = \mathbf{0} \Leftrightarrow \nabla'^a \xi^b = \mathbf{0}) \}.$$

Indeed, this is how Earman characterizes Maxwellian space-time as well.

Let us now turn to Newton-Cartan theory, which is a theory both of space-time structure and of gravitation.⁸ It will once again be helpful to begin with Galilean space-time. In that setting, one may think of Newtonian gravitation as a theory relating a gravitational potential, which is represented by a scalar field φ on M , to the distribution of mass density in space-time, represented by a scalar field ρ . These together satisfy Poisson's equation, which may be

5. The brackets indicate antisymmetrization over the indices. More generally, we may define a *rotation tensor* ω_{ab} associated with ξ^a that captures the magnitude and direction of its rotation; this tensor vanishes iff the nonrotation condition just stated holds. See Malament (2012, 264) for details.

6. Note that this is a strong disanalogy between the metrical structure of classical space-time and that of (pseudo-)Riemannian geometry (see Malament 2012, proposition 4.1.3).

7. Here $R'^a{}_{bcd}$ is the Riemann curvature tensor associated with ∇' ; requiring this tensor to vanish makes precise the requirement that ∇' is flat. For more on curvature tensors, see Malament (2012, sec. 1.8).

8. For background on Newton-Cartan theory, see Trautman (1965) and Malament (2012, chap. 4).

written as $\nabla_a \nabla^a \varphi = 4\pi\rho$. In the presence of (only) a gravitational potential, massive point particles will traverse trajectories with acceleration given by $\xi^n \nabla_n \xi^a = -\nabla^a \varphi$, where ξ^a is the tangent to the particle’s world-line. Models of the theory may be written $(M, t_a, h^{ab}, \nabla, \varphi)$.⁹

Models of Newton-Cartan theory, meanwhile, which I will call “Newton-Cartan space-times,” may be written as $(M, t_a, h^{ab}, \tilde{\nabla})$, where M, t_a , and h^{ab} are as above, and $\tilde{\nabla}$ is a derivative operator compatible with t_a and h^{ab} . In contrast to Galilean space-times, $\tilde{\nabla}$ is no longer required to be flat. Instead, its associated Ricci curvature tensor is required to satisfy $\tilde{R}_{ab} = 4\pi\rho t_a t_b$, where ρ is again some smooth scalar field representing the mass density.¹⁰ In this theory, as in general relativity, gravitation is not conceived as a force; instead, it is a manifestation of space-time curvature in the sense that, in the absence of external forces, bodies will traverse timelike geodesics of the curved derivative operator. It is common to assume that $\tilde{\nabla}$ satisfies two further curvature conditions, which we will take for granted in what follows: $\tilde{R}^a{}_b{}^c{}_d = \tilde{R}^c{}_d{}^a{}_b$ and $\tilde{R}^{ab}{}_{cd} = 0$.¹¹ Note that although a model of Newton-Cartan theory amounts to a space-time structure in much the same way that Maxwell-Huygens space-time and Galilean space-time do, it also does a bit more, since it determines the allowed trajectories of bodies even in the presence of other matter. In Galilean space-time, one needs an additional field—the gravitational potential—to determine those allowed trajectories.

Newtonian gravitation in Galilean space-time and Newton-Cartan theory are systematically related. The relationship is given by the following two results, originally due to Trautman (1965).

Theorem 1 (Geometrization Lemma). Let (M, t_a, h^{ab}, ∇) be a Galilean space-time and let φ and ρ be smooth scalar fields on M satisfying $\nabla_a \nabla^a \varphi = 4\pi\rho$. Then there exists a unique derivative operator $\tilde{\nabla}$ such that (1) $(M, t_a, h^{ab}, \tilde{\nabla})$ is a model of Newton-Cartan theory with mass density ρ , and (2) for all timelike curves with unit tangent field ξ^a , $\xi^n \nabla_n \xi^a = -\nabla^a \varphi$ if and only if $\xi^n \tilde{\nabla}_n \xi^a = 0$.

Theorem 2 (Recovery Theorem). Let $(M, t_a, h^{ab}, \tilde{\nabla})$ be a model of Newton-Cartan theory with mass density ρ . Then there exists a smooth

9. Here and throughout, I am suppressing ρ in specifications of models of both Newton-Cartan theory and Newtonian gravitation, since one can uniquely recover ρ from the other fields and the geometrized Poisson and Poisson equations, respectively.

10. For a given derivative operator ∇ , the associated Ricci tensor is defined by $R_{ab} = R^n{}_{abn}$, where $R^a{}_{bcd}$ is the Riemann curvature tensor.

11. These two conditions are necessary for the Trautman recovery theorem below. For more on the significance of the conditions, see Malament (2012, sec. 4.3); for the consequences of dropping the second condition, see Malament’s sec. 4.5.

scalar field φ and a derivative operator ∇ such that (1) (M, t_a, h^{ab}, ∇) is a Galilean space-time, (2) ∇ and φ satisfy $\nabla_a \nabla^a \varphi = 4\pi\rho$, and (3) for all timelike curves with unit tangent field ξ^a , $\xi^n \nabla_n \xi^a = -\nabla^a \varphi$ if and only if $\xi^n \tilde{\nabla}_n \xi^a = \mathbf{0}$. Moreover, the pair (∇, φ) is not unique. A second pair (∇', φ') will satisfy the same conditions iff (1) $\nabla^a \nabla^b (\varphi' - \varphi) = \mathbf{0}$ and (2) $\nabla' = (\nabla, C^a_{bc})$, where $C^a_{bc} = t_a t_b \nabla^a (\varphi' - \varphi)$.

Given a model of Newton-Cartan theory, I will call the associated models of Newtonian gravitation in Galilean space-time given by theorem 2 “Trautman recoveries.”

I am now in a position to state a first result concerning the relationship between Newton-Cartan theory and Maxwell-Huygens space-times.

Proposition 3. Let $(M, t_a, h^{ab}, \tilde{\nabla})$ be a model of Newton-Cartan theory satisfying the conditions of theorem 2 and let $(M, t_a, h^{ab}, \nabla, \varphi)$ be a Trautman recovery of this space-time. Then given any other flat derivative operator ∇' on M that is compatible with t_a and h^{ab} , there exists a smooth scalar field φ' such that $(M, t_a, h^{ab}, \nabla', \varphi')$ is a Trautman recovery of $(M, t_a, h^{ab}, \tilde{\nabla})$ iff ∇ and ∇' agree on a standard of rotation.

The proof of this proposition appears in the appendix. The proposition provides a precise characterization of one relationship between Maxwell-Huygens space-time and Newton-Cartan theory: namely, given a Newton-Cartan space-time, the collection of flat derivative operators corresponding to Trautman recoveries determines a unique standard of rotation $[\nabla]$.

Of course, as I noted above, any derivative operator generates a standard of rotation, in the sense that relative to any derivative operator, one can determine whether a given smooth timelike vector field ξ^a is rotating. So it is not surprising that the derivative operator associated with Newton-Cartan theory does so. But as I also noted, a derivative operator provides more than a standard of rotation. What is striking here, then, is not that a Newton-Cartan space-time allows one to determine whether a given vector field (or body) is rotating, but rather that the only structure agreed on by the flat derivative operators associated with Trautman recoveries of $(M, t_a, h^{ab}, \tilde{\nabla})$ is the standard of rotation $[\nabla]$.¹² In other words, although a Newton-Cartan space-time gives rise to many Galilean space-times via the recovery theorem, there is a sense in which it naturally gives rise to a unique, privileged Maxwell-Huygens space-time, insofar as Maxwell-Huygens space-time is the shared space-time structure associated with all the Trautman recoveries of a given Newton-Cartan space-time.

12. Of course, they also agree on parallel transport of spacelike vectors and on other structures that all elements of $[\nabla]$ agree on, such as flatness.

It is worth noting an important way in which this result depends on the definition of a standard of rotation above. Following Earman and Saunders (insofar as he cites Earman's definition), I defined a standard of rotation as an equivalence class of flat derivative operators. But one might imagine different definitions, which might not yield results as clean as proposition 3. For instance, if one defined a standard of rotation as an equivalence class of derivative operators with arbitrary curvature (satisfying the other conditions), one would not recover the full equivalence class in the way described by proposition 3, since all Trautman recoveries involve flat derivative operators.

This last observation raises several interesting questions. For instance, are there different, essentially inequivalent, ways of characterizing a standard of rotation?¹³ And, if we do not limit attention to Trautman recoveries, is there still a sense in which the only derivative operators relative to which the geodesics of a given Newton-Cartan space-time may be naturally expressed as "accelerated by some force field" are those that agree with that Newton-Cartan space-time on a standard of rotation? I will not address the first question here, but an answer to the second is essentially given by proposition 1 of Weatherall and Manchak (2014). There it is shown that given a Newton-Cartan space-time $(M, t_a, h^{ab}, \tilde{\nabla})$ and any derivative operator ∇ compatible with the classical metrics, there is a smooth, antisymmetric field, F_{ab} , naturally interpreted as a "force field," such that for any timelike curve with unit tangent field ξ^a , $\xi^n \tilde{\nabla}_n \xi^a = \mathbf{0}$ if and only if $\xi^n \nabla_n \xi^a = h^{am} F_{mn} \xi^n$. In particular, ∇ need not agree with $\tilde{\nabla}$ on a standard of rotation. This helps clarify just what is shown in proposition 3: we get a unique Maxwell-Huygens space-time only after we limit attention to flat derivative operators and require that the force field be expressible as in ordinary Newtonian gravitation, that is, as the gradient of a scalar field, rather than as a more general object, such as F_{ab} .

What about going in the other direction? Given a Maxwell-Huygens space-time, can we always recover a unique Newton-Cartan space-time? Not as we have set things up thus far. But the reason is clear: Newton-Cartan theory is a gravitational theory, whereas Maxwell-Huygens space-time is a space-time structure in which a gravitational theory may be expressed. The question, then, is how to express Newtonian gravitation in Maxwell-

13. Malament (2002) takes up a related question, of whether there is an unambiguous criterion for when a body is rotating that answers to all our classical desiderata, in the context of general relativity, and finds that the answer is "no." Here the issue is somewhat different, since we already have one criterion that apparently does answer to all our desiderata—namely, the one described above. But this criterion is described by considering an equivalence class of derivative operators, and one might wonder if (a) there is an intrinsic characterization of a standard of rotation that does not require us to appeal to structure we then forget, and (b) if so, if it (always) determines a unique equivalence class of derivative operators, as described here.

Huygens space-time. To do so, one will need to provide some prescription for determining how a body will move, given some distribution of masses in the universe. Certainly our ordinary ways of doing this will no longer work, since space-time is not endowed with a fixed derivative operator. In other words, since Maxwell-Huygens space-time does not have any privileged class of inertial trajectories relative to which one can describe acceleration, one cannot make sense of a distinction between forced and unforced motion. In particular, then, Newtonian gravitation may not be conceived as a theory of gravitational force in anything like the standard sense.¹⁴

That said, proposition 3 provides a natural indirect option. That proposition, in conjunction with theorem 2 and standard results concerning the existence of solutions to differential equations, shows that given a Maxwell-Huygens space-time $(M, t_a, h^{ab}, [\nabla])$, for any $\nabla \in [\nabla]$, there exists some scalar field φ such that (1) $\nabla_a \nabla^a \varphi = 4\pi\rho$, where ρ is the mass density distribution of space-time;¹⁵ and (2) the allowed trajectories of bodies are curves γ whose acceleration (relative to ∇) is given by $\xi^a \nabla_n \xi^a = \nabla^a \varphi$. Note that, since Poisson's equation admits homogeneous solutions, there will not be a unique scalar field φ satisfying Poisson's equation for a given mass density field ρ ; moreover, different choices may lead to different trajectories—and thus to different geometrizations, in the sense of theorem 1.¹⁶ But in this regard, we are in precisely the same situation as in ordinary Newtonian gravitation and the same sorts of heuristic methods for determining a gravitational potential apply: in some cases, for instance, we might choose to work with a representative from $[\nabla]$ according to which the center of mass of some system of interest follows a constant trajectory and then impose boundary conditions on φ so that it vanishes “at infinity.” In the present context, however, such heuristics have merely pragmatic value, providing a way of calculating the allowed trajectories given some distribution of matter. Similarly, we do not interpret the acceleration of a curve relative to ∇

14. Of course, Saunders (2013) also develops a way of determining how bodies move, by characterizing the evolution of relative positions of particles in Maxwell-Huygens space-time, including in the presence of forces, such as gravitational force, that depend only on the distances between particles. Importantly, this approach requires a radical reconceptualization of force as proportional to relative acceleration, not absolute acceleration. In any case, for present purposes, as long as everyone agrees on what the trajectories that bodies follow are, how one arrives at those trajectories is inconsequential, and the strategy pursued below, though less elegant than Saunders's, is more perspicuous for the point I aim to make.

15. Note that nothing in particular depends on ρ being a smooth scalar field; one might just as well define the gravitational potential φ as for point particles, using the relation $\varphi = GM/r$, and then work locally.

16. Note that this means that we do not have a unique recovery of a Newton-Cartan space-time from a given Maxwell-Huygens space-time, even after stipulating a mass density ρ . One needs to do a bit more.

as representing some fact of the matter about the curve; nor do we need to interpret the gravitational potential or corresponding gravitational field, $\nabla^a\varphi$, as representing facts about force or a fieldlike entity.

What is the invariant physical structure in this theory? For one, as we have seen, there is the standard of rotation shared between the derivative operators. This gives the sense in which this is a theory in Maxwell-Huygens space-time. The other invariant structure, however, is the collection of allowed trajectories for bodies. These are calculated in different ways depending on which representative one chooses from $[\nabla]$, and the acceleration associated with each such curve varies similarly. So we do not have the structure to say that these curves are accelerating or not. But however they are described, that is, whatever acceleration (if any) is attributed to them, the curves themselves are fixed. Indeed, given some distribution of matter in space-time, it is these curves that form the empirical content of Newtonian gravitational theory.

Now suppose that we are given some such collection of curves, $\{\gamma\}_\rho$, relativized to a matter distribution ρ , in Maxwell-Huygens space-time. Suppose, too, that however these curves are determined—whether by the calculational procedure just mentioned or some other method—they agree with the possible trajectories allowed by ordinary Newtonian gravitation. It turns out that with this information, one can uniquely reconstruct a Newton-Cartan space-time.

Proposition 4. Let $\{\gamma\}_\rho$ be the collection of allowed trajectories for a given mass distribution ρ in Maxwell-Huygen space-time $(M, t_a, h^{ab}, [\nabla])$, as described above. Then there exists a unique derivative operator $\tilde{\nabla}$ such that (1) $\{\gamma\}_\rho$ consists of the timelike geodesics of $\tilde{\nabla}$ and (2) $(M, t_a, h^{ab}, \tilde{\nabla})$ is a model of Newton-Cartan theory for mass density ρ .

Again, the proof appears in the appendix.

This last proposition provides an even stronger relationship between a gravitational theory in Maxwell-Huygens space-time and Newton-Cartan theory. In a sense, such a gravitational theory simply is Newton-Cartan theory, insofar as given the invariant structure of such a theory—namely, the allowed trajectories—one can uniquely construct a derivative operator such that those trajectories are the geodesics of the derivative operator.¹⁷ Note, too, that this result—at least as I interpret it here—reveals a certain

17. For further argument to the effect that Newton-Cartan theory should be construed as equivalent to Newtonian gravitation once one has accepted that different choices of flat derivative operator—and, thus, of inertial frames—may be physically equivalent, see Weatherall (2015). For reasons described here, it is very natural to interpret Maxwell-Huygens space-time as the space-time structure presupposed by the theory described there as NG2.

inadequacy in Saunders's account. Saunders insists that there is no privileged standard of acceleration in Maxwell-Huygens space-time. And there are a few senses in which that is right: (1) before accounting for gravitational influences, Maxwell-Huygens space-time does not have enough structure to make sense of acceleration; and (2) even in the presence of dynamical considerations, there is in general no privileged flat derivative operator, and thus no privileged collection of inertial frames in the standard sense, relative to which acceleration may be defined. Nonetheless, it turns out that once one takes the dynamically allowed trajectories into account, one can define a standard of acceleration, namely, the unique one relative to which the allowed trajectories are geodesics.

In retrospect, this result should not be surprising. After all, as noted above, there are empirically privileged trajectories, namely, those that actual bodies follow, given some mass density distribution. Whether one is on such a trajectory may be determined by using a simple device, such as an inert mass suspended by springs in the center of a cube-shaped frame. The privileged trajectories, then, are the ones relative to which the mass stays centered in the frame. In the context of relativity theory, we call such devices "accelerometers"; more generally, given how we usually think of acceleration and force in both relativity theory and Newtonian gravitation, it is natural to think of trajectories on which the mass is stationary as unaccelerated, since it is on these trajectories that the springs need not exert any force on the mass. And indeed, it is precisely these trajectories that are picked out as unaccelerated by the Newton-Cartan derivative operator. From this point of view, there is always a rough-and-ready way to determine a class of privileged trajectories, even if these are not inertial frames in the traditional sense.

I will conclude by elaborating on the connection with Knox (2014) mentioned above. Like Saunders, Knox begins by meditating on the significance of Corollary VI for how we should think of space-time structure in Newtonian gravitation. Her conclusion, however, seems quite different from Saunders's: she argues that Galilean space-time is merely a halfway point on the journey from Newtonian space-time to Newton-Cartan theory.¹⁸ This presents a *prima facie* puzzle: whence the difference? But I take the arguments here to dissolve the puzzle: Knox and Saunders do not end up in different places after all, except insofar as Knox takes gravitational influences into account all the way through her analysis. Once one fully considers the effects of gravitation in Maxwell-Huygens space-time, Newton-Cartan theory is precisely the result.

18. Newtonian space-time is Galilean space-time plus a privileged standard of rest, given by a constant timelike vector field (see Earman 1989, sec. 2.5).

Appendix

Proofs of Propositions

Proof of Proposition 3. We begin with the “only if” direction. Suppose that $(M, t_a, h^{ab}, \nabla, \varphi)$ and $(M, t_a, h^{ab}, \nabla', \varphi')$ are both Trautman recoveries of $(M, t_a, h^{ab}, \tilde{\nabla})$. Then by theorem 2, $\nabla' = (\nabla, C^a_{bc})$, where $C^a_{bc} = t_b t_c \nabla^a(\varphi' - \varphi)$. Now suppose that ξ^a is a timelike vector field such that $\nabla^{[a} \xi^{b]} = \mathbf{0}$. It follows that

$$\nabla'^{[a} \xi^{b]} = h^{n[a} \nabla'_n \xi^{b]} = -h^{n[a} t_n t_m \nabla^{b]} \xi^m = \mathbf{0}.$$

So if ξ^a is nonrotating relative to ∇ , it is nonrotating relative to ∇' . An identical argument establishes the converse, that if ξ^a is nonrotating relative to ∇' , it is nonrotating relative to ∇ .

Now consider the “if” direction. Suppose that $(M, t_a, h^{ab}, \nabla, \varphi)$ is a Trautman recovery of $(M, t_a, h^{ab}, \tilde{\nabla})$ and suppose that ∇' is a flat derivative operator compatible with t_a and h^{ab} and such that given any timelike vector field ξ^a , $\nabla^{[a} \xi^{b]} = \mathbf{0}$ iff $\nabla'^{[a} \xi^{b]} = \mathbf{0}$. Since ∇' is compatible with t_a and h^{ab} , $\nabla' = (\nabla, C^a_{bc})$, where $C^a_{bc} = 2h^{an} t_{(b} \kappa_{c)n}$ for some smooth antisymmetric tensor field κ_{ab} (Malament 2012, proposition 4.1.3). Now consider some unit timelike vector field ξ^a such that $\nabla^{[a} \xi^{b]} = \mathbf{0}$. (Such a field always exists because ∇ is flat and M is connected and simply connected.) It follows that

$$\begin{aligned} \mathbf{0} &= \nabla'^{[a} \xi^{b]} = \nabla^{[a} \xi^{b]} - 2h^{o[b} h^{a]n} t_{(n} \kappa_{m)o} \xi^m = 2h^{o[b} h^{a]n} t_{(n} \kappa_{m)o} \xi^m \\ &= h^{o[b} h^{a]n} \kappa_{no} \xi^m t_m = \kappa^{ab}. \end{aligned}$$

This means that $\kappa_{ab} = t_{[a} \sigma_{b]}$, for some covector σ_b . (Why? Because t_a is the only covector that annihilates h^{ab} .) It follows that $C^a_{bc} = \eta^a t_b t_c$ for some spacelike η^a (possibly $\mathbf{0}$).

Now recall that $\tilde{\nabla}$ and ∇' are both flat by hypothesis, and thus, using the relation

$$R^a_{bcd} = R^a_{bcd} + 2\nabla_{[c} C^a_{d]b} + 2C^n_{b[c} C^a_{d]n}$$

(Malament 2012, eq. 1.8.2), it follows that $2\nabla_{[c} \eta^a t_d] t_b = \mathbf{0}$. Acting on both sides with h^{cn} , we find $t_b \nabla^n \eta^a = \mathbf{0}$. Since $t_a \neq \mathbf{0}$, it follows that $\nabla^a \eta^b = \mathbf{0}$. Finally, invoking proposition 4.1.6 of Malament (2012), which holds globally because we have limited attention to space-times to which spacelike slices are connected and simply connected, we conclude that there exists a smooth scalar field $\tilde{\varphi}$ such that $\eta^a = \nabla^a \tilde{\varphi}$. Finally, we define $\varphi' = \tilde{\varphi} + \varphi$. It follows that $\nabla^a \nabla^b(\varphi' - \varphi) = \nabla^a \eta^b = \mathbf{0}$ and that $\nabla' = (\nabla, C^a_{bc})$, where $C^a_{bc} = t_b t_c \tilde{\nabla}^a(\varphi' - \varphi)$. Thus, by theorem 2, $(M, t_a, h^{ab}, \nabla', \varphi')$ is a Trautman recovery of $(M, t_a, h^{ab}, \tilde{\nabla})$. QED

Proof of Proposition 4. If the curves $\{\gamma\}_\rho$ agree with the possible trajectories allowed by ordinary Newtonian gravitation, then there must exist some derivative operator $\nabla \in [\nabla]$ (indeed, by proposition 3, any will suffice) and some φ , such that $(M, t_a, h^{ab}, \nabla, \varphi)$ is a model of Newtonian gravitation. Proposition 1 then guarantees that there is a unique derivative operator ∇ satisfying parts 1 and 2 in the proposition. QED

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