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Positive periodic solutions of a class of single-species neutral models with state-dependent delay and feedback control

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Sufficient conditions are obtained for the existence of positive periodic solutions of a class of neutral delay differential equations of the form

$$\begin{cases} N'(t) = N(t)F[t, N(t), N(t-\tau(t, N(t))), N'(t-\gamma(t)), P(t), P(t-\mu(t))] \\ P'(t) = -e(t)P(t) + k(t)N(t) + h(t)N(t-\sigma(t)) \end{cases}$$

by using the theory of topological degree. These results extend substantially the existing relevant existence results in the literature. As a demonstration, applying the obtained analysis results to a real complex neutral Lotka-Volterra population model, the existence criterion for positive periodic solutions is easily obtained and an example is used to give an impression of how restrictive these conditions are. Especially, this method is more suitable to state-dependent delay, and which have further applications in many fields.

1 Introduction

Although about 90% of populations in nature do not exhibit sustainable oscillation [30, 27], we tend to pay more attention to those do and often try to model such behaviour. In the literature, there are four typical approaches to modelling such behaviour: (i) introduce more species into the model, and consider higher dimensional systems (like predator-prey interactions); (ii) assume that the per capita growth function is time-dependent and periodic in time; (iii) take into account the time delay effect in the population dynamics [51]; (iv) take into account random effects in nature. Generally speaking, approach (i) is rather artificial, while (ii), (iii) and (iv) emphasize only one aspect of reality. Although all of them are good mechanisms for generating periodic solutions (and therefore offer some explanations to the often observed oscillatory behaviour in population densities), it does not give us much insight as which is the real generating or dominating force behind the oscillatory behaviour if only one of such mechanism is considered. Naturally, more realistic and interesting models of single species growth should take into account both the seasonality of the changing environment and the effects of time delays.

Delay differential equations arise in the modelling of many different physical systems, for examples, control systems [22], cell biology [4], lasers [29] and population growth [31]. As for the species dynamics, the classical non-delayed logistic model is given by

$$N'(t) = rN(t)\left[1 - \frac{N(t)}{K}\right], \tag{1.1}$$

the parameter r denotes the intrinsic growth rate, and K denotes the carrying capacity of the environment. An extensive and captivating review of the history of the logistic equation is given in Kingsland [28]. Hutchinson [26] remarked that for (1.1) to make sense, the biological mechanisms under consideration must operate so rapidly that the time lag between the instant where a given value N is reached and the instant when the effective reproductive rate $1 - N/K$ is updated, is negligible. Arguing that oscillations have been observed in some Daphnia populations, he proposed the following more realistic logistic equation

$$N'(t) = rN(t)\left[1 - \frac{N(t-\tau)}{K}\right], \tag{1.2}$$

which was derived from (1.1) by simply assuming that the net per capita rate of change N'/N might depend on the state of the system τ time units in the past. Indeed, the delayed logistic equation (i.e. Hutchinson's equation) is one of the first examples of a delay differential equation that has been thoroughly examined. Another derivation of (1.1) given by Cunningham [13], he assumed that a population whose per capita rate of change would normally be constant (i.e. $N'/N = A$), is subject to additional effects that decrease the rate of growth A. If these effects are functions of the state of the population at the time $t - \tau$ (the previous generation, for example), then one has the equation

$$N'(t) = N(t)[A - BN(t-\tau)].$$

Note that this equation can be obtained from (1.2) by setting $A = r$ and $B = r/K$. We often refer to (1.2) as the classical logistic delay differential equation. In 1987, Seifert [42] considered the Volterra logistic delay differential equation

$$N'(t) = N(t)[a - bN(t) - N(t-\tau)].$$

For Hutchinson's equation, Zhang & Gopalsamy [50] assumed that the intrinsic growth rate and the carrying capacity are periodic functions of a period ω and that the delay is an integer multiple of the period of the environment. Namely, they considered the periodic delay differential equation of the form

$$N'(t) = r(t)N(t)\left[1 - \frac{N(t-n\omega)}{K(t)}\right], \tag{1.3}$$

where $r(t+\omega) = r(t), K(t+\omega) = K(t)$ for all $t \geqslant 0$. They proved the result on the existence of a unique positive periodic solution of (1.3), which is globally attractive with respect to all other positive solutions. Following the techniques of Zhang & Gopalsamy [50], many scholars have considered the delayed models with the assumption that the coefficients are periodic.

Existing results on the periodic solutions in periodic differential equation population models suggests [31, 50, 20, 34, 36, 47, 45]) that when a strong seasonal force acts,

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regardless of the length of time delay and many other factors, it is often the primary factor causing notable population fluctuation. In addition, in such cases, the fluctuation is often quite robust and synchronizes with the season. When the seasonal effect is weak or absent (this is often the case if the unit of time is long), but the delay length is significant, then delay may be the primary source of destabilization. Indeed, significant delay can often lead to chaotic population behaviour. In such cases, the stability of any fluctuation is often not clear. Indeed, there are often numerous periodic solution coexist.

In this paper, we consider the periodic solutions for general setting of periodic neutral delay differential equations. A neutral delay differential equation is one in which the derivatives of the past history or derivatives of functions of the past history are involved as well as the present state of the system. Although a qualitative theory for general systems has not been developed at the present time, neutral delay differential equations are used in a wide range of applications, particularly the area of population growth modelling [14, 15, 19, 21, 32, 33]. It is pointed out in the book of Hale & Verduyn Lunel [24] that neutral functional differential equations are met when dealing with oscillatory systems with some interconnections between them. As one may already be aware, many real systems are quite sensitive to sudden changes, this fact may suggest that proper mathematical models of the systems should consist of some neutral delay equations. Even though the delay lengths may be short, and the neutral terms relatively small, it is still necessary, for the sake of rigorousness, to justify that the neutral term effects are not important. Indeed, occasionally we may find that neutral term effects can be quite significant. This is largely due to the fact that neutral delay equations are not structurally stable in the sense that the introduction of neutral delay terms may destabilize an asymptotically stable equilibrium. For example, $3x'(t) = -x$ has a globally asymptotically stable trivial solution, while the same solution in $x' + 2x'(t-\tau) = -x$ becomes unstable for any $\tau > 0$. This is because the corresponding characteristic equation of the neutral equation may have roots bifurcating from infinity, a phenomenon that cannot occur in retarded equations. This indicates that it is important to deal with neutral delay equations in real mathematical models.

Now well known, one of the more challenging aspects of mathematical biology is neutral competitive modelling. In 1987, Gopalsamy & Zhang [21] first introduced and investigated the equation

$$N'(t) = rN(t)\left[1 - \frac{N(t-\tau) + cN'(t-\tau)}{K}\right],$$

we may think of N as a species grazing upon vegetation, which takes time τ to recover (for details, see Gopalsamy & Zhang [21]). Since then, some scholars start to consider the neutral equation of population dynamics with delay [32, 19]).

In particular, Kuang [31] proposed the following open problem (Open Problem 9.2) in 1993: How to obtain sufficient conditions for existence of a periodic solution for equation

$$N'(t) = N(t)[a(t) - \beta(t)N(t) - bN(t-\tau(t)) - cN'(t-\tau(t))],$$

where $a(t), \beta(t), b(t), \tau(t), c(t)$ are nonnegative continuous T-periodic functions.

When $a(t), \beta(t), b(t), \tau(t)$ are positive and $c(t) = 0$, for systems such a problem was considered by Freedman & Wu [15] and Tang & Kuang [45]. Li [33] and Fang & Li [14] have investigated the above problem.

Despite the fact that differential delay equations have been used in the modeling of scientific phenomena for many years, often it has been assumed that the delays are either fixed constants or are given as integrals (distributed delays). However, in recent years, the more complicated situation in which the delays depend on the unknown functions has been proposed in models [1, 2, 5, 11, 12, 17, 25, 39, 43], these equations are frequently called equations with state-dependent delay (See Arino *et al.* [3] for a brief review on state-dependent delay models). In population dynamics, it is a well-known fact that many types of structured population models can be reduced to the study of functional differential equations, state-dependent delays result frequently from stage transition thresholds dependent upon the population. Let $N(t)$ denote the size of a population at time t, for example, assume that the number of births is a function of the population size only, the birth rate is thus a density dependent but not age dependent. Assume that the lifespan τ of individuals in the population is variable and is a function of the current population size. If we take into account the crowding effects, then $\tau(\cdot)$ is a decreasing function of the population size. Since the population size $N(t)$ is equal to the total number of living individuals, we have (Béair [5])

$$N(t) = \int_{t-\tau(N(t))}^{t} b(N(s))ds. \tag{1.4}$$

Differentiating with respect to the time t on both sides of (1.4) leads to a state-dependent delay model of the form

$$N'(t) = \frac{b(N(t)) - b(N(t-\tau(N(t))))}{1 - \tau'(N(t))b(N(t-\tau(N(t))))}. \tag{1.5}$$

Note that state-dependent delay equation (1.5) is not equivalent to the integral equation (1.4). It is clear that every solution of (1.4) is a solution of (1.5) but the reverse is not true. In fact, any constant function is a solution of (1.5) but clearly it may not necessarily be a solution of (1.4). The asymptotic behavior of the solutions of (1.5) has been studied by Béair [5]. Recently, great attention has been paid on the study of state-dependent delay equations. Li & Kuang [37] studied the existence of a periodic solution for the following general nonlinear nonautonomous differential equation with state-dependent delay

$$N'(t) = N(t)F[t, N(t-\tau(t, N(t)))].$$

Where they assume that both F and τ are periodic with the same periodicity. In 2005, Yang & Cao [49] studied the existence of periodic solutions in neutral system of the form

$$N'(t) = N(t)F[t, N(t-\tau(t, N(t))), N'(t-\tau(t, N(t)))].$$

Furthermore, frequently, one can find that an ecosystem in the real world is continuously distributed by some forces, which can result in changes in the biological parameters such as survival rates. Of practical interest in ecology is the question of whether or not an ecosystem can withstand those disturbances, which persist for a finite period of time. In the language of control variables, we call the disturbance functions control variables. As a result, some authors try to change the position of the existing periodic solution to keep its stability. This is of significance in the control of ecology balance. One of the methods for us realization of it is to alter the system structurally by introducing feedback control variables so as to get a population stabilizing at another periodic solution. In fact, during the last decade, the qualitative behaviour of the population dynamics with feedback control has been studied extensively. The interested reader is referred to Gopalsamy [18] and Chen *et al.* [10], and the references cited therein.

In this paper, we shall investigate the sufficient conditions for the existence of positive periodic solutions of a class of general neutral delay differential equations of the form

$$\begin{cases} N'(t) = N(t)F[t, N(t), N(t-\tau(t, N(t))), N'(t-\gamma(t)), P(t), P(t-\mu(t))] \\ P'(t) = -e(t)P(t) + k(t)N(t) + h(t)N(t-\sigma(t)) \end{cases}, \tag{1.6}$$

here we assum that $e(t), k(t), h(t), \gamma(t), \mu(t), \sigma(t) \in C(R, [0, +\infty))$ are T-periodic function and T is a positive constant, $\tau \in C(R^2, [0, +\infty))$ is a T-periodic function with respect to its first argument. And

$$F(t, z_1, z_2, z_3, z_4, z_5) \in C(R^6, R),$$

$$F(t+T, z_1, z_2, z_3, z_4, z_5) = F(t, z_1, z_2, z_3, z_4, z_5).$$

It is easy to see that (1.6) includes many mathematical models of delay. Some special cases of (1.6) have been occurred widely in many references [1]–[39], [42]–[51].

The main objective of this paper is to obtain abstract theories for existence of a periodic solution of (1.6), by using the theory of topological degree. We shall see that the conditions are quite weak; from those conditions it is easy to obtain the sufficient conditions for many actual population dynamic models. In addition, the method is more suitable to state-dependent delay, and which have further applications in many fields. In order to illustrates the effectiveness of the theoretical result, we should also study the existence of positive periodic solutions for the following equations

$$\begin{cases} N'(t) = N(t)[a(t) - \beta(t)N(t) - b(t)N(t-\tau(t, N(t))) - c(t)N'(t-\gamma(t)) \\ \qquad\qquad - l(t)P(t) - m(t)P(t-\mu(t))] \\ P'(t) = -e(t)P(t) + k(t)N(t) + h(t)N(t-\sigma(t)) \end{cases}, \tag{1.7}$$

where $a(t), \beta(t), b(t), c(t), l(t), m(t), e(t), k(t), h(t)$ are nonnegative continuous T-periodic functions.

Today, it has been recognized that the theory of existence occupies an important place among exact mathematical methods being used in the design and analysis of control systems. To my knowledge, most research on neutral delay differential equations has been restricted to simple cases of constant delays; few papers consider variable delay and state-dependent delay. And most existing result on the existence of periodic solutions in periodic systems are usually obtained by the technique of bifurcation, by fixed-point theorems, or by monotone semiflow theory, but those conditions for existence are often unnecessary, and difficult to satisfy. Specifically, all of the above methods are ill suited to problems with state-dependent delay equations. Therefore, it is necessary to study neutral model with state-dependent delay and feedback control.

2 Preliminaries

To obtain sufficient conditions for a positive periodic solution of (1.6), we first make the following preparations.

Let X and Z be two Banach spaces, $L : DomL \subset X \longrightarrow Z$ be a linear mapping (a linear mapping is called linear Fredholm mapping with index zero if $\dim KerL = \dim \operatorname{Im} Q < +\infty$ and $\operatorname{Im} L$ is closed in Z), and $N : X \longrightarrow Z$ be a continuous mapping. It follows now from the definition of a Fredholm mapping and from basic results of linear functional analysis that there exist continuous projectors

$$P : X \longrightarrow KerL, \; Q : Z \longrightarrow Z / \operatorname{Im} L,$$

so that $X = KerP \oplus KerL$, $Z = \operatorname{Im} Q \oplus \operatorname{Im} L$, and $\dim KerL = \dim \operatorname{Im} Q$.

If we define $L_P : domL \cap KerP \longrightarrow \operatorname{Im} L$ by $L_P = L|_{domL \cap KerP}$, obviously, L_P is one to one, so that its (algebraic) inverse $K_P : \operatorname{Im} L \longrightarrow domL \cap KerP$ is defined, i.e. $K_P = L_P^{-1}$; we denote $K_{P,Q} : Z \longrightarrow domL \cap KerP$, the generalized inverse of L_P, by $K_{P,Q} = K_P(I - Q)$.

To solve $Lx = Nx$, we embed it in a one-parameter family of equations

$$Lx = N^*(x, \lambda), \; \lambda \in [0, 1], \tag{2.1}$$

where $N^* : X \times [0, 1] \longrightarrow Z$ is continuous and $N^*(x, 1) = Nx$. For the equation $Lx = N^*(x, \lambda)$, we have $L(Px + (I - P)x) = QN^*(x, \lambda) + (I - Q)N^*(x, \lambda)$, that is

$$L(I - P)x = QN^*(x, \lambda) + (I - Q)N^*(x, \lambda).$$

We obtain that

$$(I - P)x = K_P(I - Q)N^*(x, \lambda), \; QN^*(x, \lambda) = 0.$$

Then

$$x = Px + K_P(I - Q)N^*(x, \lambda) + JQN^*(x, \lambda),$$

where $J : \operatorname{Im} Q \longrightarrow KerL = \operatorname{Im} P$ is a linear mapping. Assume $\Omega \subset X$ is a bounded open set and $N^* : \overline{\Omega} \times [0, 1] \longrightarrow Z$ is L-compact (i.e. QN^* and $K_{P,Q}N^*$ are relatively compact on $\overline{\Omega} \times [0, 1]$). Let

$$A(\lambda) = P + K_P(I - Q)N^*(\lambda) + JQN^*(\lambda).$$

Then (2.1) can be replaced $x = A(\lambda)x$.

Define the coincidence degree

$$D\{(L, N^*(x, \lambda)), \Omega\} = \deg\{I - A(\lambda), \Omega, 0\},$$

where deg denotes the Leray–Schauder degree. In the sequel, we shall give the following Lemmas.

Lemma 2.1 Let X and Z be two Banach spaces and L be a Fredholm mapping of index zero. Assume that $\Omega \subset X$ is a bounded open set and $N^* : \overline{\Omega} \times [0, 1] \longrightarrow Z$ is L-compact. For each $x \in domL \cap \partial\Omega$, $Lx \neq N^*(x, \lambda), \lambda \in [0, 1]$, and $D\{(L, N^*(x, \lambda)), \Omega\} \neq 0$. Then $Lx = Nx$ has at least one solution in $domL \cap \Omega$.

Proof For each $x \in domL \cap \partial\Omega$ and $\lambda \in [0,1]$, if $Lx \neq N^*(x,\lambda)$, then

$$x \neq A(\lambda)x, i.e. x - A(\lambda)x \neq 0.$$

By using the property of invariance of the degree under a homology and the condition $D\{(L, N^*(x,\lambda)), \Omega\} \neq 0$, we obtain

$$D\{(L, N^*(x,\lambda)), \Omega\} = \deg\{I - A(\lambda), \Omega, 0\} = \deg\{I - A(1), \Omega, 0\} \neq 0.$$

From the definition and properties of Leray-Schauder degree, it follows that $x = A(1)x$ with $x \in domL \cap \Omega$. Therefore $Lx = Nx$ has at least one solution in $x \in domL \cap \Omega$.

In particular, when $N^*(x,\lambda) = \lambda Nx$, we have

$$A(\lambda) = P + \lambda K_p(I - Q)N + JQN,$$

and hence

$$\begin{aligned} D\{(L, N^*(x,\lambda)), \Omega\} &= \deg\{I - A(\lambda), \Omega, 0\} \\ &= \deg\{I - A(1), \Omega, 0\} \\ &= \deg\{I - A(0), \Omega, 0\} \\ &= \deg\{-JQN|_{KerL\cap\overline{\Omega}}, KerL \cap \Omega, 0\} \\ &= (-1)^n \deg\{JQN|_{KerL\cap\overline{\Omega}}, KerL \cap \Omega, 0\}, \end{aligned} \quad (2.2)$$

where $n = \dim KerL$. □

An important consequence of Lemma 2.1 is the following:

Lemma 2.2 Let X and Z be two Banach spaces and L be a Fredholm mapping of index zero. Assume that $\Omega \subset X$ is a bounded open set and $N : \overline{\Omega} \longrightarrow Z$ is L-compact (i.e. QN and $K_{p,Q}N$ are relatively compact on $\overline{\Omega}$). Then $Lx = Nx$ has at least one solution in $domL \cap \Omega$, if the following conditions are satisfied:

(i) For each $\lambda \in (0,1]$, $x \in domL \cap \partial\Omega$, $Lx \neq \lambda Nx$;
(ii) For each $x \in KerL \cap \partial\Omega$, $QNx \neq 0$ and $\deg\{JQN|_{KerL\cap\overline{\Omega}}, KerL \cap \Omega, 0\} \neq 0$.

Proof From assumption (i), we deduce at once $x \neq A(\lambda)x$, with $x \in domL \cap \partial\Omega$ and $\lambda \in (0,1]$. If $x_0 = A(0)x_0$ for some $x_0 \in domL \cap \partial\Omega$, then $JQNx_0 = 0$ i.e. $QNx_0 = 0$, this contradicts assumption (ii), therefore $x \neq A(\lambda)x$, for each $x \in domL \cap \partial\Omega$, $\lambda \in [0,1]$. That is $Lx \neq \lambda Nx$ with $x \in domL \cap \partial\Omega$, $\lambda \in [0,1]$. And from (2.2), it is easy to verify that $D\{(L, N^*(x,\lambda)), \Omega\} \neq 0$. Thus, the condition of Lemma 2.1 holds. □

Lemma 2.3 $(N(t), P(t))$ is a T-periodic solution of (1.6) if and only if it is a T-periodic solution of the equation

$$\begin{cases} N'(t) = N(t)F[t, N(t), N(t - \tau(t, N(t))), N'(t - \gamma(t)), P(t), P(t - \mu(t))] \\ P(t) = \int_t^{t+T} G(t,s)[k(s)N(s) + h(s)N(s - \sigma(s))]ds := (\Phi N)(t) \end{cases}, \quad (2.3)$$

where $G(t,s) = e^{\int_t^s e(r)dr}(e^{\int_0^T e(r)dr} - 1)^{-1}$

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Proof The proof is similar to that of Lemma 2.2 in Chen *et al.* [10].

If let $(N(t), P(t))$ be a T-periodic solution of (1.6), by using the variation-of-constant formulas, then each T-periodic solution of the equation

$$P'(t) = -e(t)P(t) + k(t)N(t) + h(t)N(t - \sigma(t))$$

is equivalent to that of the equation

$$P(t) = P(0)e^{-\int_0^t e(r)dr} + \int_0^t [k(s)N(s) + h(s)N(s - \sigma(s))]e^{\int_t^s e(r)dr} ds.$$

And then

$$\begin{aligned} &P(t+T) \\ &\quad = P(0)e^{-\int_0^t e(r)dr} \times e^{-\int_t^{t+T} e(r)dr} + \int_0^t [k(s)N(s) + h(s)N(s - \sigma(s))]e^{\int_t^s e(r)dr} \\ &\qquad \times e^{\int_{t+T}^t e(r)dr} ds + \int_t^{t+T} [k(s)N(s) + h(s)N(s - \sigma(s))]e^{\int_t^s e(r)dr} \times e^{\int_t^{t+T} e(r)dr} ds \\ &\quad = P(t)e^{-\int_0^T e(r)dr} + \int_t^{t+T} [k(s)N(s) + h(s)N(s - \sigma(s))]e^{\int_t^s e(r)dr} \times e^{\int_0^T -e(r)dr} ds. \end{aligned}$$

From the periodicity of $P(t)$, it follows that

$$P(t) = \int_t^{t+T} G(t,s)[k(s)N(s) + h(s)N(s - \sigma(s))]ds.$$

On the other hand, assume $(N(t), P(t))$ is a T-periodic solution of (2.3), it is easy to see that

$$\begin{aligned} P'(t) &= \left[\int_t^{t+T} G(t,s)[k(s)N(s) + h(s)N(s - \sigma(s))]ds\right]' \\ &= -e(t)P(t) + k(t)N(t) + h(t)N(t - \sigma(t)). \end{aligned}$$

□

3 Main results

Theorem 3.1 (1.6) has least one T-periodic positive solution, if the following conditions are satisfied:

(a) There exists a number $C > 0$, such that

$$\left| F\left[t, e^{x(t)}, e^{x(t-\tau(t,e^{x(t)}))}, x'(t-\gamma(t))e^{x(t-\gamma(t))}, \Phi(e^x)(t), \Phi(e^x)(t-\mu(t))\right] \right| < C,$$

if $x(t)$ is a C^1 T-periodic function;

(b) There exists a number $R > 0$, such that for every $x_1, x_2, x_3, x_4 \geqslant R$ and uniformly in $t \in [0, T]$,

$$F\left(t, e^{x_1}, e^{x_2}, 0, \Phi(e^{x_3}), \Phi(e^{x_4})\right) < 0 \textit{ and } F\left(t, e^{-x_1}, e^{-x_2}, 0, \Phi(e^{-x_3}), \Phi(e^{-x_4})\right) > 0,$$

or

$$F\left(t, e^{x_1}, e^{x_2}, 0, \Phi(e^{x_3}), \Phi(e^{x_4})\right) > 0 \text{ and } F\left(t, e^{-x_1}, e^{-x_2}, 0, \Phi(e^{-x_3}), \Phi(e^{-x_4})\right) < 0.$$

Proof We shall apply Lemma 2.2 to construct the set Ω by the method of a priori bounds. From Lemma 2.2, let us consider the following equation

$$x'(t) = F[t, e^{x(t)}, e^{x(t-\tau(t,e^{x(t)}))}, x'(t-\gamma(t))e^{x(t-\gamma(t))}, \Phi(e^x)(t), \Phi(e^x)(t-\mu(t))]. \tag{3.1}$$

To use Lemma 2.2 for Eq. (3.1), we take

$$X = \left\{x(t) \in C^1(R, R); x(t+T) = x(t)\right\},$$
$$Z = \left\{z(t) \in C(R, R); z(t+T) = z(t)\right\},$$

and more $||x||_0 = \max_{t\in[0,T]} |x(t)|$, $||x||_1 = \max_{t\in[0,T]} \{||x||_0, ||x'||_0\}$. Then X and Z are Banach spaces when they are endowed with the more $||\cdot||_1$ and $||\cdot||_0$, respectively.

Let $L : domL \subset X \longrightarrow Z$ and $N : X \longrightarrow Z$ be the following maps

$$Lx = x'(t),$$
$$Nx = F\left[t, e^{x(t)}, e^{x(t-\tau(t,e^{x(t)}))}, x'(t-\gamma(t))e^{x(t-\gamma(t))}, \Phi(e^x)(t), \Phi(e^x)(t-\mu(t))\right].$$

Mawhin [38] gave a classical example of a Fredholm mapping, it was shown that every linear mapping $L : X \longrightarrow Z$ is a Fredholm mapping of index 0 (i.e. $\dim X - \dim Z = 0$) when X and Z are finite dimensional. Clearly, L is a Fredholm mapping of index 0.

Define the continuous projective operators P and Q by

$$Px = \frac{1}{T}\int_0^T x(t)dt, \ x \in X; \ Qx = \frac{1}{T}\int_0^T z(t)dt, \ z \in Z.$$

It is easy to check that

$$L_P : z(t) = x'(t) \text{ and } \int_0^T x(t)dt = 0,$$

$$QNx(t) = \frac{1}{T}\int_0^T F[t, e^{x(t)}, e^{x(t-\tau(t,e^{x(t)}))}, x'(t-\gamma(t))e^{x(t-\gamma(t))},$$
$$\Phi(e^x)(t), \Phi(e^x)(t-\mu(t))]dt.$$

Moreover, the restriction L_P of L to $domL \cap KerP$ is one to one and onto $\operatorname{Im} L$. In this case,

$$K_P(z) = \int_0^t z(s)ds - \frac{1}{T}\int_0^T \left[\int_0^t z(s)ds\right] dt$$
$$= \int_T^t z(s)ds + \frac{1}{T}\int_0^T sz(s)ds.$$

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By some computation, since $K_{P,Q}N = K_P(I-Q)N$, we can show that $K_{P,Q}N : \overline{\Omega} \longrightarrow domL \cap KerP$ takes the form

$$
\begin{aligned}
&K_P(I-Q)Nx(t)\\
&= \int_T^t F[s, e^{x(s)}, e^{x(s-\tau(s,e^{x(s)}))}, x'(s-\gamma(s))e^{x(s-\gamma(s))},\\
&\qquad \Phi(e^x)(s), \Phi(e^x)(s-\mu(s))]ds\\
&\quad - \int_T^t ds \frac{1}{T}\int_0^T F[t, e^{x(t)}, e^{x(t-\tau(t,e^{x(t)}))}, x'(t-\gamma(t))e^{x(t-\gamma(t))},\\
&\qquad\qquad \Phi(e^x)(t), \Phi(e^x)(t-\mu(t))]dt\\
&\quad + \frac{1}{T}\int_0^T sF[s, e^{x(s)}, e^{x(s-\tau(s,e^{x(s)}))}, x'(s-\gamma(s))e^{x(s-\gamma(s))},\\
&\qquad\qquad \Phi(e^x)(s), \Phi(e^x)(s-\mu(s))]ds\\
&\quad - \frac{1}{T}\int_0^T ds \frac{s}{T}\int_0^T F[t, e^{x(t)}, e^{x(t-\tau(t,e^{x(t)}))}, x'(t-\gamma(t))e^{x(t-\gamma(t))},\\
&\qquad\qquad \Phi(e^x)(t), \Phi(e^x)(t-\mu(t))]dt\\
&= \int_T^t F[s, e^{x(s)}, e^{x(s-\tau(s,e^{x(s)}))}, x'(s-\gamma(s))e^{x(s-\gamma(s))},\\
&\qquad \Phi(e^x)(s), \Phi(e^x)(s-\mu(s))]ds\\
&\quad + \frac{T-t}{T}\int_0^T F[t, e^{x(t)}, e^{x(t-\tau(t,e^{x(t)}))}, x'(t-\gamma(t))e^{x(t-\gamma(t))},\\
&\qquad\qquad \Phi(e^x)(t), \Phi(e^x)(t-\mu(t))]dt\\
&\quad + \frac{1}{T}\int_0^T sF[s, e^{x(s)}, e^{x(s-\tau(s,e^{x(s)}))}, x'(s-\gamma(s))e^{x(s-\gamma(s))},\\
&\qquad\qquad \Phi(e^x)(s), \Phi(e^x)(s-\mu(s))]ds\\
&\quad - \frac{1}{2}\int_0^T F[t, e^{x(t)}, e^{x(t-\tau(t,e^{x(t)}))}, x'(t-\gamma(t))e^{x(t-\gamma(t))},\\
&\qquad\qquad \Phi(e^x)(t), \Phi(e^x)(t-\mu(t))]dt\\
&= \int_T^t F[s, e^{x(s)}, e^{x(s-\tau(s,e^{x(s)}))}, x'(s-\gamma(s))e^{x(s-\gamma(s))},\\
&\qquad \Phi(e^x)(s), \Phi(e^x)(s-\mu(s))]ds\\
&\quad + \frac{1}{T}\int_0^T sF[s, e^{x(s)}, e^{x(s-\tau(s,e^{x(s)}))}, x'(s-\gamma(s))e^{x(s-\gamma(s))},\\
&\qquad\qquad \Phi(e^x)(s), \Phi(e^x)(s-\mu(s))]ds\\
&\quad + \left(\frac{1}{2}-\frac{t}{T}\right)\int_0^T sF[s, e^{x(s)}, e^{x(s-\tau(s,e^{x(s)}))}, x'(s-\gamma(s))e^{x(s-\gamma(s))},\\
&\qquad\qquad \Phi(e^x)(s), \Phi(e^x)(s-\mu(s))]ds.
\end{aligned}
$$

Under the assumption of (1.6):

$$F[t, N(t), N(t-\tau(t,N(t))), N'(t-\gamma(t)), P(t), P(t-\mu(t))] \in C(R^6, R)$$

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and the condition (a) of Theorem 3.1, we can see that $K_P(I-Q)N$ is a C^1 T-periodic function in the Banach space X for any bounded open set $\Omega \subset X$. Hence

$$\begin{aligned} ||QNx||_0 &\leqslant \max_{t\in[0,T]} \frac{1}{T}\int_0^T |F[t, e^{x(t)}, e^{x(t-\tau(t,e^{x(t)}))}, x'(t-\gamma(t))e^{x(t-\gamma(t))}, \\ &\qquad \Phi(e^x)(t), \Phi(e^x)(t-\mu(t))]|dt \\ &\leqslant C \end{aligned}$$

and

$$\begin{aligned} &||K_P(I-Q)Nx||_0 \\ &\leqslant \max_{t\in[0,T]} \int_T^t |F[s, e^{x(s)}, e^{x(s-\tau(s,e^{x(s)}))}, x'(s-\gamma(s))e^{x(s-\gamma(s))}, \\ &\qquad \Phi(e^x)(s), \Phi(e^x)(s-\mu(s))]|ds \\ &\quad + \max_{t\in[0,T]} \frac{1}{T}\int_0^T |sF[s, e^{x(s)}, e^{x(s-\tau(s,e^{x(s)}))}, x'(s-\gamma(s))e^{x(s-\gamma(s))}, \\ &\qquad \Phi(e^x)(s), \Phi(e^x)(s-\mu(s))]|ds \\ &\quad + \max_{t\in[0,T]} \left(\frac{1}{2}-\frac{t}{T}\right)\int_0^T |F[s, e^{x(s)}, e^{x(s-\tau(s,e^{x(s)}))}, x'(s-\gamma(s))e^{x(s-\gamma(s))}, \\ &\qquad \Phi(e^x)(s), \Phi(e^x)(s-\mu(s))]|ds \\ &\leqslant TC + TC + \frac{TC}{2} < 3TC \end{aligned}$$

for every $x \in \overline{\Omega}$. Moreover, for any $\varepsilon > 0$, there is a $\delta = \varepsilon/(2C)$, such that, for every $x \in \overline{\Omega}$,

$$\begin{aligned} &|K_P(I-Q)Nx(t_1) - K_P(I-Q)Nx(t_2)| \\ &\leqslant \int_{t_2}^{t_1} |F[s, e^{x(s)}, e^{x(s-\tau(s,e^{x(s)}))}, x'(s-\gamma(s))e^{x(s-\gamma(s))}, \\ &\qquad \Phi(e^x)(s), \Phi(e^x)(s-\mu(s))]|ds \\ &\quad + \frac{|t_1-t_2|}{T}\int_0^T |F[s, e^{x(s)}, e^{x(s-\tau(s,e^{x(s)}))}, x'(s-\gamma(s))e^{x(s-\gamma(s))}, \\ &\qquad \Phi(e^x)(s), \Phi(e^x)(s-\mu(s))]|ds \\ &\leqslant 2C|t_1-t_2| < \varepsilon \end{aligned}$$

for $|t_1 - t_2| < \delta$ and $t_1, t_2 \in [0, T]$.

By using the Ascoli–Arzela theorem, for every bounded subset $\Omega \subset X$, then QN and $K_{P,Q}N$ are relatively compact on $\overline{\Omega}$ in C^1 space X, i.e. N is L-compact on $\overline{\Omega}$.

Consider to the operator equation $Lx = \lambda Nx$ with $\lambda \in (0, 1]$, consequently

$$\begin{aligned} x'(t) = \lambda F[t, e^{x(t)}, e^{x(t-\tau(t,e^{x(t)}))}, x'(t-\gamma(t))e^{x(t-\gamma(t))}, \\ \Phi(e^x)(t), \Phi(e^x)(t-\mu(t))]. \end{aligned} \tag{3.2}$$

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Assume that $x(t) \in X$ is a solution of Eq. (3.2), for a certain $\lambda \in (0,1]$, then we have

$$\begin{aligned}\int_0^T & F[t, e^{x(t)}, e^{x(t-\tau(t,e^{x(t)}))}, x'(t-\gamma(t))e^{x(t-\gamma(t))},\\ & \Phi(e^x)(t), \Phi(e^x)(t-\mu(t))]dt\\ =0. \end{aligned} \tag{3.3}$$

From (3.2) and (a), it follows that

$$\begin{aligned}|x'(t)| \leqslant |F[t, e^{x(t)}, e^{x(t-\tau(t,e^{x(t)}))}, x'(t-\gamma(t))e^{x(t-\gamma(t))},\\ \Phi(e^x)(t), \Phi(e^x)(t-\mu(t))]|,\end{aligned}$$

hence

$$\begin{aligned}\int_0^T & |x'|\,dt\\ &= \lambda \int_0^T |F[t, e^{x(t)}, e^{x(t-\tau(t,e^{x(t)}))}, x'(t-\gamma(t))e^{x(t-\gamma(t))},\\ &\quad \Phi(e^x)(t), \Phi(e^x)(t-\mu(t))]|dt\\ &\leqslant \int_0^T |F[t, e^{x(t)}, e^{x(t-\tau(t,e^{x(t)}))}, x'(t-\gamma(t))e^{x(t-\gamma(t))},\\ &\quad \Phi(e^x)(t), \Phi(e^x)(t-\mu(t))]|dt\\ &< TC. \end{aligned} \tag{3.4}$$

From (3.3) and (b), it is easy to check that there exists a point t_1 or t_2^*, t_3^*, t_4^* in $[0,T]$, such that

$$|x(t_1)| < R \text{ or } |x(t_2^* - \gamma(t_2^*))| < R,\ |\Phi(e^x)(t_3^*)| < R,\ |\Phi(e^x)(t_4^* - \mu(t_4^*))| < R.$$

That is

$$|x(t_i)| < R,\ i = 1 \text{ or } 2,3,4, \tag{3.5}$$

where $t_i^* = t_i + mT,\ t_i \in [0,T],\ i = 2,3,4$ and m is a integer. Indeed, if $|x(t)| \geqslant R$ for every $t \in [0,T],\ R > 0$, by (3.3) and (b), this is a contradiction.

From (3.4) and (3.5), and since $x(t) = x(t_i) + \int_{t_i}^t x'(t)dt$, it follows that

$$|x(t)| \leqslant |x(t_i)| + \int_{t_i}^t |x'(t)|\,dt < R + TC.$$

And hence

$$||x||_0 < R + TC.$$

Then, taking $\Omega = \{x(t) \in X;\ ||x||_1 < H\}$ with $H \overset{def}{=} \{R + TC, C\}$, this satisfies condition (i) in Lemma 2.2.

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When $x \in KerL \cap \partial\Omega = R \cap \partial\Omega$, x is a constant with $|x| = H$, we have

$$QNx = \frac{1}{T}\int_0^T F[t, e^{x(t)}, e^{x(t-\tau(t,e^{x(t)}))}, x'(t-\gamma(t))e^{x(t-\gamma(t))}, \Phi(e^x)(t), \Phi(e^x)(t-\mu(t))]dt$$
$$= \frac{1}{T}\int_0^T F[t, e^x, e^x, 0, \Phi(e^x), \Phi(e^x)]\,dt \neq 0$$

We now consider all cases corresponding to assumption (b).

Case 1: If $x = H$ and $F[t, e^x, e^x, 0, \Phi(e^x), \Phi(e^x)] > 0$, let $h(u, x) = ux + (1-u)QNx$ with $u \in [0, 1]$ and $x \in R \cap \partial\Omega$. So that $xh(u, x) > 0$, hence $h(u, x) \neq 0$. According to the invariant of homology, we obtain

$$\deg\{JQN|_{KerL\cap\overline{\Omega}}, KerL \cap \Omega, 0\} = \deg\{I, KerL \cap \Omega, 0\} \neq 0$$

Case 2: If $x = H$ and $F[t, e^x, e^x, 0, \Phi(e^x), \Phi(e^x)] < 0$, let $h(u, x) = -ux + (1-u)QNx$, then

$$\deg\{JQN|_{KerL\cap\overline{\Omega}}, KerL \cap \Omega, 0\} = \deg\{-I, KerL \cap \Omega, 0\} \neq 0$$

Case 3: If $x = -H$ and $F[t, e^x, e^x, 0, \Phi(e^x), \Phi(e^x)] > 0$ or $F[t, e^x, e^x, 0, \Phi(e^x), \Phi(e^x)] < 0$, in the same way of proof as case 1 or case 2, we can get

$$\deg\{JQN|_{KerL\cap\overline{\Omega}}, KerL \cap \Omega, 0\} \neq 0.$$

By now, we have verified all the requirements in Lemma 2.2; hence, (3.1) has at least one T-periodic solution $x^*(t)$. Set $N^*(t) = e^{x^*(t)}$, then $N^*(t)$ is a T-periodic positive solution of (2.3), From Lemma 2.3, we complete the proof of Theorem 3.1. □

Theorem 3.2 Let us assume that conditions of Theorem 3.1 hold, and then the conclusion of Theorem 3.1 holds for the following equation:

$$\begin{cases} N'(t) = -N(t)F[t, N(t), N(t-\tau(t, N(t))), N'(t-\gamma(t)), P(t), P(t-\mu(t))] \\ P'(t) = -e(t)P(t) + k(t)N(t) + h(t)N(t-\sigma(t)) \end{cases}.$$

Proof Its proof is similar to the proof of Theorem 3.1. Here we omit it. □

Remark 1 Even if $\tau(t, N(t)), \gamma(t), \mu(t), \sigma(t)$ is negative, from the proof of Theorem 3.1, it is easy to see that the conclusion of Theorem 3.1 and Theorem 3.2 is still true. In addition, the conditions of Theorem 3.1 are relatively weak. In many special cases, those conditions can be easily checked.

Using Theorem 3.1, we can get many corollaries; the following are examples:

Corollary 3.1 Assume that $x(t)$ is a C^1 T-periodic function and there exist two numbers $C, R > 0$, such that the following conditions hold:

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(1) $\left|F\left[t, e^{x(t)}, e^{x(t-\tau(t,e^{x(t)}))}, x'(t-\gamma(t))e^{x(t-\gamma(t))}, \Phi(e^x)(t), \Phi(e^x)(t-\mu(t))\right]\right| < C$,
if $x(t)$ is a T-periodic function;
(2) $F\left[t, e^{x_1}, e^{x_2}, 0, \Phi(e^{x_3}), \Phi(e^{x_4})\right] < 0,\ F\left[t, e^{-x_1}, e^{-x_2}, 0, \Phi(e^{-x_3}), \Phi(e^{-x_4})\right] > 0$,
with $x_i \geqslant R$, $i = 1, 2, 3, 4$, and uniformly in $t \in [0, T]$.
Then the conclusion of Theorem 3.1 and Theorem 3.2 holds.

Proof Its proof is direct outcome of Theorem 3.1, we omit it. □

Corollary 3.2 Assume that $\tau(t, N(t)) = \gamma(t)$, $x(t)$ is a C^1 T-periodic function and there exist two numbers $C_1, C_2 > 0$, such that the following conditions hold:

(1) $F\left[t, e^{x_1}, e^{x_2}, x_2' e^{x_2}, \Phi(e^{x_3}), \Phi(e^{x_4})\right] \leqslant C_2$ or $F\left[t, e^{x_1}, e^{x_2}, x_2' e^{x_2}, \Phi(e^{x_3}), \Phi(e^{x_4})\right] \geqslant -C_2$, with $\sqrt{x_1^2 + x_2^2 + (x_2')^2 + x_3^2 + x_4^2} > C_1$;
(2) $F\left[t, e^{x_1}, e^{x_2}, 0, \Phi(e^{x_3}), \Phi(e^{x_4})\right] > 0$ and $F\left[t, e^{-x_1}, e^{-x_2}, 0, \Phi(e^{-x_3}), \Phi(e^{-x_4})\right] < 0$,
with $x_1, x_2, x_3, x_4 \geqslant C_1$,or $x_1, x_2, x_3, x_4 \geqslant -C_1$ and uniformly in $t \in [0, T]$.
Then the conclusion of Theorem 3.1 and Theorem 3.2 also holds.

Proof Without loss of generality, we may assume that $F\left[t, e^{x_1}, e^{x_2}, x_2' e^{x_2}, \Phi(e^{x_3}), \Phi(e^{x_4})\right] \leqslant C_2$ when $\sqrt{x_1^2 + x_2^2 + (x_2')^2 + x_3^2 + x_4^2} > C_1$. Since $x(t)$ is a continuous T-periodic function, let

$$\Omega_1 = \left\{ t \in [0, T] \,\middle|\, \sqrt{x_1^2 + x_2^2 + (x_2')^2 + x_3^2 + x_4^2} > C_1 \right\}$$

and

$$\Omega_2 = \left\{ t \in [0, T] \,\middle|\, \sqrt{x_1^2 + x_2^2 + (x_2')^2 + x_3^2 + x_4^2} \leqslant C_1 \right\}.$$

And so, in view of the fact that

$$Nx = F\left[t, e^{x(t)}, e^{x(t-\gamma(t))}, x'(t-\gamma(t))e^{x(t-\gamma(t))}, \Phi(e^x)(t), \Phi(e^x)(t-\mu(t))\right],$$

which implies that F maps a bounded continuous function to a bounded function, there exists a number $C_2' > 0$ so that

$$\left|F\left[t, e^{x(t)}, e^{x(t-\gamma(t))}, x'(t-\gamma(t))e^{x(t-\gamma(t))}, \Phi(e^x)(t), \Phi(e^x)(t-\mu(t))\right]\right| < C_2'$$

in Ω_2. Taking $C = \max\{C_2, C_2'\}$, therefore

$$\left|F\left[t, e^{x(t)}, e^{x(t-\gamma(t))}, x'(t-\gamma(t))e^{x(t-\gamma(t))}, \Phi(e^x)(t), \Phi(e^x)(t-\mu(t))\right]\right| < C.$$

From Theorem 3.1, we derive that the conclusion of Corollary 3.2 holds. □

Theorem 3.3 Assume that
(a) $a(t), \beta(t), e(t) \in C(R, (0, +\infty)),\ c(t) \in C^1(R, [0, +\infty)),\ \gamma(t) \in C^2(R, [0, +\infty))$,
(b) $\underline{u} \overset{def}{=} \min_{t\in[0,T]} \{u(t) \in C(R, R);\ u(t+T) = u(t)\},\ \gamma'(t) < 1$ and $||c||_0 e^{R_2} < 1$.

where

$$d(t) = \frac{c(t)}{1-\gamma'(t)}, \ V = -\frac{d'(t)}{1-\gamma'(t)},$$

$$R_2 = \ln \frac{||a||_0}{\theta_1(\underline{\beta} + \underline{V})} + \frac{||d||_0||a||_0}{\theta_2\underline{\beta}(\underline{1-\gamma'})} + 2T||a||_0.$$

Then (1.7) has at least one positive T-periodic solution.

Proof Consider the equation

$$\begin{aligned} x'(t) = a(t) - \beta(t)e^{x(t)} - b(t)e^{x(t-\tau(t,e^{x(t)}))} - c(t)x'(t-\gamma(t))e^{x(t-\gamma(t))} \\ - l(t)\Phi(e^x)(t) - m(t)\Phi(e^x)(t-\mu(t)). \end{aligned} \tag{3.6}$$

If $x(t) \in X$ is a solution of (3.6), then integrating this identity on $[0, T]$, we have

$$\begin{aligned} &\int_0^T [a(t) - \beta(t)e^{x(t)} - b(t)e^{x(t-\tau(t,e^{x(t)}))} - d(t)(1-\gamma'(t))x'(t-\gamma(t))e^{x(t-\gamma(t))} \\ &\quad - l(t)\Phi(e^x)(t) - m(t)\Phi(e^x)(t-\mu(t))]dt \\ &= \int_0^T [a(t) - \beta(t)e^{x(t)} - b(t)e^{x(t-\tau(t,e^{x(t)}))} + d(t)e^{x(t-\gamma(t))} \\ &\quad - l(t)\Phi(e^x)(t) - m(t)\Phi(e^x)(t-\mu(t))]dt \\ &= 0. \end{aligned}$$

That is

$$\begin{aligned} &\int_0^T [\beta(t)e^{x(t)} + b(t)e^{x(t-\tau(t,e^{x(t)}))} - d'(t)e^{x(t-\gamma(t))} \\ &\quad + l(t)\Phi(e^x)(t) + m(t)\Phi(e^x)(t-\mu(t))]dt \\ &= \int_0^T a(t)dt. \end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we have

$$\begin{aligned} &x'(t) + d(t)\left(e^{x(t-\gamma(t))}\right)' \\ &= a(t) - \beta(t)e^{x(t)} - b(t)e^{x(t-\tau(t,e^{x(t)}))} - l(t)\Phi(e^x)(t) - m(t)\Phi(e^x)(t-\mu(t)), \end{aligned}$$

and hence

$$\begin{aligned}
&\int_0^T \left|\left(x(t)+d(t)e^{x(t-\gamma(t))}\right)'\right| dt \\
&\quad = \int_0^T |a(t)-\beta(t)e^{x(t)} - b(t)e^{x(t-\tau(t,e^{x(t)}))} + d'(t)e^{x(t-\gamma(t))} \\
&\qquad - l(t)\Phi(e^x)(t) - m(t)\Phi(e^x)(t-\mu(t))|dt \\
&\quad \leqslant \int_0^T [\beta(t)e^{x(t)} + b(t)e^{x(t-\tau(t,e^{x(t)}))} - d'(t)e^{x(t-\gamma(t))} \\
&\qquad + l(t)\Phi(e^x)(t) + m(t)\Phi(e^x)(t-\mu(t))]dt + \int_0^T a(t)dt \\
&\quad \leqslant 2\int_0^T a(t)dt \leqslant 2T\|a\|_0. \qquad (3.8)
\end{aligned}$$

It follows from ((3.7) that

$$\begin{aligned}
&\int_0^T a(t)dt \\
&\quad = \theta_1 \int_0^T [\beta(t)e^{x(t)} + b(t)e^{x(t-\tau(t,e^{x(t)}))} - d'(t)e^{x(t-\gamma(t))} \\
&\qquad + l(t)\Phi(e^x)(t) + m(t)\Phi(e^x)(t-\mu(t))]dt \\
&\qquad + \theta_2 \int_0^T [\beta(t)e^{x(t)} + b(t)e^{x(t-\tau(t,e^{x(t)}))} - d'(t)e^{x(t-\gamma(t))} \\
&\qquad + l(t)\Phi(e^x)(t) + m(t)\Phi(e^x)(t-\mu(t))]dt, \qquad (3.9)
\end{aligned}$$

where $\theta_1, \theta_2 > 0$, and $\theta_1 + \theta_2 = 1$.

Let $g = t - \gamma(t),\ t = \varphi(g)$ be the inverse function of $g = t - \gamma(t)$, then

$$\begin{aligned}
\int_0^T d'(t)e^{x(t-\gamma(t))}dt &= \int_{-\gamma(0)}^{T-\gamma(T)} \frac{d'(\varphi(g))}{1-\gamma'(\varphi(g))} e^{x(g)}dt \\
&= \int_0^T \frac{d'(\varphi(g))}{1-\gamma'(\varphi(g))} e^{x(g)}dt.
\end{aligned}$$

By the Mean Value Theorem, this implies that there exists a point for some $\eta_1 \in [0, T]$, such that

$$\int_0^T d'(t)e^{x(t-\gamma(t))}dt = \frac{d'(\eta_1)}{1-\gamma'(\eta_1)} \int_0^T e^{x(t)}dt.$$

Similarly

$$\int_0^T \beta(t)e^{x(t)}dt = \beta(\eta_2)\int_0^T e^{x(t)}dt$$

for some $\eta_2 \in [0, T]$.

Let $V = -d'/\left(1-\gamma'\right)$, it follows that

$$\int_0^T a(t)dt \geqslant (\beta(\eta_2) + V(\eta_1)) \int_0^T e^{x(t)}dt \qquad (3.10)$$

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and hence

$$a(\xi) \geqslant (\beta(\eta_2) + V(\eta_1))\, e^{x(\xi)}$$

for some $\xi \in [0, T]$, that is

$$x(\xi) \leqslant \ln \frac{a(\xi)}{\beta(\eta_2) + V(\eta_1)}.$$

Thus

$$\ln \frac{\underline{a}}{||\beta||_0 + ||V||_0} \leqslant x(\xi) \leqslant \ln \frac{||a||_0}{\underline{\beta} + \underline{V}}$$

Therefore, we obtain

$$|x(\xi)| \leqslant \max\left(\left|\ln \frac{\underline{a}}{||\beta||_0 + ||V||_0}\right|, \ \left|\ln \frac{||a||_0}{\underline{\beta} + \underline{V}}\right| \right) \stackrel{def}{=} R_1. \tag{3.11}$$

Similarly

$$\begin{aligned}\int_0^T \beta(t)e^{x(t)}dt &= \int_{-\gamma(0)}^{T-\gamma(T)} \beta(t-\gamma(t))e^{x(t-\gamma(t))} \left(1-\gamma'(t)\right) dt \\ &= \int_0^T \beta(t-\gamma(t))e^{x(t-\gamma(t))} \left(1-\gamma'(t)\right) dt \\ &= \beta(\delta_1 - \gamma(\delta_1)) \left(1-\gamma'(\delta_1)\right) \int_0^T e^{x(t-\gamma(t))}dt\end{aligned}$$

for some $\delta_1 \in [0, T]$, and

$$\int_0^T d'(t)e^{x(t-\gamma(t))}dt = d'(\delta_2) \int_0^T e^{x(t-\gamma(t))}dt$$

for some $\delta_2 \in [0, T]$. Clearly

$$\int_0^T a(t)dt \geqslant \left(\beta(\delta_1 - \gamma(\delta_1)) \left(1-\gamma'(\delta_1)\right) - d'(\delta_2)\right) \int_0^T e^{x(t-\gamma(t))}dt. \tag{3.12}$$

From (3.9), (3.10) and (3.12), we obtain

$$\begin{aligned}\int_0^T a(t)dt \geqslant{}& \theta_1 \left(\beta(\eta_2) + V(\eta_1)\right) \int_0^T e^{x(t)}dt \\ &+ \theta_2 \left(\beta(\delta_1 - \gamma(\delta_1)) \left(1-\gamma'(\delta_1)\right) - d'(\delta_2)\right) \int_0^T e^{x(t-\gamma(t))}dt.\end{aligned}$$

That is,

$$\begin{aligned}a(\zeta) \geqslant{}& \theta_1 \left(\beta(\eta_2) + V(\eta_1)\right) e^{x(\zeta)} \\ &+ \theta_2 \left(\beta(\delta_1 - \gamma(\delta_1)) \left(1-\gamma'(\delta_1)\right) - d'(\delta_2)\right) e^{x(\zeta-\gamma(\zeta))}\end{aligned}$$

for some $\zeta \in [0, T]$. Therefore, we have

$$x(\zeta) \leqslant \ln \frac{a(\zeta)}{\theta_1 \left(\beta(\eta_2) + V(\eta_1)\right)} \leqslant \ln \frac{||a||_0}{\theta_1(\underline{\beta} + \underline{V})}, \tag{3.13}$$

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$$e^{x(\zeta-\gamma(\zeta))} \leqslant \frac{a(\zeta)}{\theta_2\left(\beta(\delta_1-\gamma(\delta_1))(1-\gamma'(\delta_1))-d'(\delta_2)\right)} \leqslant \frac{||a||_0}{\theta_2\underline{\beta(1-\gamma')}}. \tag{3.14}$$

Form (3.11), (3.13) and (3.14), It follows that

$$\begin{aligned} & x(t)+d(t)e^{x(t-\gamma(t))} \\ & \leqslant x(\zeta)+d(\zeta)e^{x(\zeta-\gamma(\zeta))}+\int_0^T\left|\left(x(t)+d(t)e^{x(t-\gamma(t))}\right)'\right|dt \\ & \leqslant \ln\frac{||a||_0}{\theta_1(\underline{\beta}+\underline{V})}+\frac{||d||_0||a||_0}{\theta_2\underline{\beta(1-\gamma')}}+2T||a||_0 \overset{def}{=} R_2. \end{aligned}$$

And from (2.3) then

$$\begin{aligned} \Phi(e^x)(t) &= \int_t^{t+T} G(t,s)[k(s)e^{x(s)}+h(s)e^{x(s-\sigma(s))}]ds \\ &\leqslant e^{R_2}\left(||k||_0+||h||_0\right)\frac{1}{\underline{e}}\int_t^{t+T} G(t,s)e(s)ds \\ &= e^{R_2}\left(||k||_0+||h||_0\right)\frac{1}{\underline{e}}. \end{aligned}$$

Hence, we have

$$\begin{aligned} |x'(t)| &= |a(t)-\beta(t)e^{x(t)}-b(t)e^{x(t-\tau(t,e^{x(t)}))}-c(t)x'(t-\gamma(t))e^{x(t-\gamma(t))} \\ &\quad -l(t)\Phi(e^x)(t)-m(t)\Phi(e^x)(t-\mu(t))| \\ &\leqslant ||a||_0+||\beta||_0e^{R_2}+||b||_0e^{R_2}+||c||_0||x'(t-\gamma(t))||_0e^{R_2} \\ &\quad +||l||_0\Phi(e^x)(t)+||m||_0\Phi(e^x)(t-\mu(t)) \\ &\leqslant ||a||_0+||\beta||_0e^{R_2}+||b||_0e^{R_2}+||c||_0||x'(t-\gamma(t))||_0e^{R_2} \\ &\quad +\left(||k||_0+||h||_0\right)(||l||_0+||m||_0)\frac{1}{\underline{e}}e^{R_2}. \end{aligned}$$

Since $||c||_0e^{R_2}<1$, hence

$$||x'||_0 < \frac{||a||_0+\left(||\beta||_0+||b||_0+\left(||k||_0+||h||_0\right)(||l||_0+||m||_0)\frac{1}{\underline{e}}\right)e^{R_2}}{1-||c||_0e^{R_2}} \overset{def}{=} M_1. \tag{3.15}$$

From (3.15) then F satisfies conditions (a) of Theorem 3.3. And since

$$\lim_{x_1,\cdots,x_5\longrightarrow+\infty} a(t)-\beta(t)e^{x_1}-b(t)e^{x_2}-c(t)x_3'e^{x_3}-l(t)\Phi(e^{x_4})-m(t)\Phi(e^{x_5})=-\infty,$$

$$\lim_{x_1,\cdots,x_5\longrightarrow-\infty} a(t)-\beta(t)e^{x_1}-b(t)e^{x_2}-c(t)x_3'e^{x_3}-l(t)\Phi(e^{x_4})-m(t)\Phi(e^{x_5})=a(t),$$

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uniformly for $t \in [0, T]$, we can see that there exist a number $R > 0$, such that F satisfies conditions (b) of Theorem 3.3. Therefore, we obtain the conclusion of Theorem 3.3. The proof is complete.

Next, we give an example of (1.7) to demonstrate the conditions of Theorem 3.3 are not contradicting and to give an impression how restrictive these conditions are. □

Example The state-dependent delay differential equation

$$\begin{cases} N'(t) = N(t)[(2+\sin t)e^{-5} - 10e^{\sin^2 t}N(t) - (1+\sin t)N(t - \frac{1}{2}N(t)\cos^2 t) \\ \qquad - (1 - \frac{1}{2}\sin 2t)\sin^2 t N'(t - \frac{1}{2}\sin^2 t) \\ \qquad - (1+\sin t)P(t) - (1+\cos t)P(t - \sin 2t)] \\ P'(t) = -e^{\sin t}P(t) + \sin^2 t N(t) + \cos^2 t N(t - \sin t) \end{cases} \tag{3.16}$$

has a positive 2π-periodic solution.

Indeed, without loss of generality, assume that $\theta_1 = \theta_2 = \frac{1}{2}$. Evidently,

$$a(t) = (2+\sin t)e^{-5} > 0,\ \beta(t) = 10e^{\sin^2 t} > 0,\ b(t) = (1+\sin t) \geqslant 0,$$

$$\tau(t) = \frac{1}{2}N(t)\cos^2 t,\ c(t) = (1 - \frac{1}{2}\sin 2t)\sin^2 t \geqslant 0,\ \gamma(t) = \frac{1}{2}\sin^2 t \geqslant 0,$$

$$1 - \gamma'(t) = 1 - \frac{1}{2}\sin 2t > 0,\ d(t) = \frac{c(t)}{1-\gamma'(t)} = \sin^2 t \geqslant 0,$$

$$V(t) = -\frac{d'(t)}{1-\gamma'(t)},\ \beta(t) + V(t) = 10e^{\sin^2 t} - \frac{\sin 2t}{1 - \frac{1}{2}\sin 2t} > 0.$$

Further

$$\begin{aligned} R_2 &= \ln \frac{||a||_0}{\theta_1(\underline{\beta} + \underline{V})} + \frac{||d||_0||a||_0}{\theta_2\underline{\beta}(\underline{1-\gamma'})} + 2T||a||_0 \\ &= \ln \frac{3e^{-5}}{\frac{1}{2}(10-4)} + \frac{3e^{-5}}{\frac{1}{2}(5-2)} + 12\pi e^{-5} \\ &= -5 + 4e^{-5} + 12\pi e^{-5} < -4, \end{aligned}$$

$$||c||_0 e^{R_2} < 1.$$

By applying Theorem 3.3 to this example with 2π-periodic coefficients, we shall find that all the conditions of Theorem 3.3 are satisfied; and then (3.16) has at least one positive 2π-periodic solution.

Remark 2 Applying the method above, we can also study the following more general periodic systems with several delays:

$$\begin{cases} N'(t) = N(t)F[t, N(t), N(t - \tau_i(t, N(t))), N'(t - \gamma_i(t)), P(t), P(t - \mu_i(t))] \\ P'(t) = -e(t)P(t) + k(t)N(t) + h(t)N(t - \sigma_i(t)) \end{cases}.$$

Moreover, for neutral differential equation with several delays of the form

$$\begin{cases} N'(t) = N(t)[a(t) - \beta(t)N(t) - \sum_{i=1}^{n} b_i(t)N(t - \tau_i(t, N(t))) \\ \qquad\qquad - \sum_{i=1}^{n} c_i(t)N'(t - \gamma_i(t)) - l(t)P(t) - \sum_{i=1}^{n} m_i(t)P(t - \mu_i(t))], \\ P'(t) = -e(t)P(t) + k(t)N(t) + \sum_{i=1}^{n} h_i(t)N(t - \sigma_i(t)) \end{cases}$$

where $a(t), \beta(t), b_i(t), c_i(t), l(t), m_i(t), e(t), k(t), h_i(t)$ are nonnegative continuous T-periodic functions, we can derive some similar results like to Theorems 3.1 and 3.3.

4 Conclusion

If the environment is not temporally constant (e.g., seasonal effects of weather, food supplies, mating habits, etc.), then the parameters become time-dependent. It has been suggested by Nicholson [40] that any periodic change of climate tends to impose its period upon oscillations of internal origin or to cause such oscillations to have a harmonic relation to periodic climatic changes. A very basic and important problem in the study of a population growth model with a periodic environment is the global existence and stability of positive periodic solution, which plays a similar role as a globally stable equilibrium does in an autonomous model. Thus, it is reasonable to seek conditions under which the resulting periodic nonautonomous system would have a positive periodic solution that is globally asymptotically stable.

We can see that the continuation theorem of coincidence degree theory is an effective tool for establishing the existence of periodic solutions in periodic equations with certain dissipativity (such solutions are eventually uniformly bounded). The method presented in this paper is more suitable to general periodic delays; it is conceivable that it can be applied to certain systems of delay differential equations as well. However uniqueness and stability are not guaranteed when the delays are not constant. In fact, it is still an open problem to study the dynamics, such as uniqueness and stability of periodic solutions and bifurcations, for the following simply periodic delay logistic equation

$$N'(t) = r(t)N(t)\left[1 - \frac{N(t - \tau(t))}{K(t)}\right],$$

where $\tau(t)$ is a positive periodic function. As Schley & Gourley [41] showed, the periodic delays can have either a stabilizing effect or a destabilizing one, depending on the frequency of the periodic perturbation.

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References

[1] Aiello, W. G., Freedman, H. I. & Wu, J. (1992) Analysis of a model representing stage-structured population growth with state-dependent time delay. *SIAM J. Appl. Math.* **52**, 855–869.

[2] Alt, W. (1979) *Periodic solutions of some autonomous differential equations with variable time delay*. Lecture Notes in Mathematics, 730, Springer-Verlag.

[3] Arino, O., Sanchez, E. & Fathallah, A. (2001) State-dependent delay differential equations in population dynamics: modeling and analysis. In: T. Faria and P. Freitas (editors), *Topics in Functional Differential and Difference Equations*, Fields Institute Commun. 29, Amer. Math. Soc. Providence, pp. 19–36.

[4] Baker, C., Bocharov, G., Paul, C. & Rihan, F. (1998) Modelling and analysis of time-lags in some basic patterns of cell proliferation. *J. Math. Biol.* **37**, 341–371.

[5] Belair, J. (1991) Population models with state-dependent delays. In: O. Arino, D. E. Axelrod and M. Kimmel (editors), *Mathematical Population Dynamics*, Marcel Dekker, pp. 165–176.

[6] Cao, J. D. (1999) Periodic solutions of a class of higher order neutral equation. *Appl. Mat. mech.* **20**, 647–652 (in Chinese).

[7] Cao, J. D. & He, G. M. (2004) Periodic solutions for higher order neutral differential equations with several delays. *Comput. Math. Appl.* **48**, 1491–1503.

[8] Cao, J. D., Li, Q. & Wan, S. D. (2002) Periodic solutions of the higher dimensional non-autonomous system. *Appl. Math. Comput.* **130**, 369–383.

[9] Chen, F. D. (2005) Positive periodic solutions of neutral Lotka-Volterra system with feedback control. *Appl. Math. Comput.* **162**, 1279–1302.

[10] Chen, F. D., Lin, F. X. & Chen, X. X. (2004) Sufficient conditions for the existence of positive periodic solutions of a class of neutral delay models with feedback control. *Appl. Math. Comput.* **158**, 45–68.

[11] Cooke, K. L. (1967) Functional differential equations: some models and perturbation problems. In: J. K. Hale and J. P. LaSalle (editors), *Differential Equations and Dynamical Systems*, Academic Press.

[12] Cooke, K. L. & Huang, W. (1992) A theorem of George Seifert and an equation with state-dependent delay. In: A. M. Fink, R. K. Miller and W. Kliemann (editors), *Delay and Differential Equations*, World Scientific, Singapore, pp. 65–77.

[13] Cunningham, W. J. (1954) A nonlinear differential-difference equation of growth. *PNAS,* **40**, 708–713.

[14] Fang, H. & Li, J. B. (2001) On the existence of periodic solutions of a neutral delay model of single-species population growth. *J. Math. Anal. Appl.* **259**, 8–17.

[15] Freedman, H. I. & Wu, J. (1992) Periodic solutions of single-species models with periodic delay. *SIAM J. Math. Anal.* **23**, 689–701.

[16] Gaines, R. E. & Mawhin, J. L. (1977) *Coincidence Degree and Nonlinear Differential Equations*. Springer-Verlag.

[17] Gatica, J. A. & Waltman, P. (1984) Existence and uniqueness of solutions of a functional differential equation modeling thresholds. *Nonlinear Anal. TMA,* **8**, 1215–1222.

[18] Gopalsamy, K. (1992) *Stability and Oscillation in Delay Differential Equations of Population Dynamics*. Mathematics and its Applications, Vol. 74, Kluwer Academic.

[19] Gopalsamy, K., He, X. & Wen, L. (1991) On a periodic neutral logistic equation. *Glasgow Math. J.* **33**, 281–286.

[20] Gopalsamy, K., Kulenovic, M. R. S. & Ladas, G. (1990) Environmental periodicity and time delays in a food-limited population model. *J. Math. Anal. Appl.* **147**, 545–555.

[21] Gopalsamy, K. & Zhang, B. G. (1987) On a neutral delay-logistic equation. *Dynamics Stability Systems,* **2**, 183–186.

[22] Gu, K. & Niculescu, S. I. (2003) Survey on recent results in the stability and control of time delay systems. *J. Dynamic Syst., Measure. & Control,* **125**, 158–165.

[23] Hale, J. K. & Mawhin, J. L. (1974) Coincidence degree and periodic solutions of neutral equations. *J. Diff. Eq.* **15**, 295–307.

[24] Hale, J. K. & Verduyn Lunel, S. M. (1993) *Introduction to Functional Differential Equations*. Springer.

[25] Hoppensteadt, F. C. & Waltman, P. (1979) A flow mediated control model of respiration. *Some Mathematical Questions in Biology, 12*, Lectures on Mathematics in the Life Sciences, pp. 211–218.

[26] Hutchinson, G. E. (1948) Circular causal systems in ecology. *Ann. N. Y. Acad. Sci.* **50**, 221–246.

[27] Kendall, B. E., Prendergast, J. & Bjornstad, O. N. (1998) The macroecology of population dynamics: taxonomic and biogeographic patterns in population cycles. *Ecol. Letters,* **1**, 160–164.

[28] Kingsland, S. (1982) The refractory model: the logistic curve and the history of population ecology. *Quart. Rev. Biol.* **57**(1), 29–52.

[29] Krauskopf, B. (2005) Bifurcation analysis of lasers with delay. In: D. M. Kane and K. A. Shore (editors), *Unlocking Dynamical Diversity*, Optical Feedback Erects on Semiconductor Lasers, pp. 147–183, Wiley.

[30] Krukonis, G. & Schaffer, W. M. (1991) Population cycles in mammals and birds: does periodicity scale with body size? *J. Theor. Biol.* **148**, 469–493.

[31] Kuang, Y. (1993) *Delay Differential Equations with Applications in Population Dynamics*. Academic Press.

[32] Kuang, Y & Feldstein, A. (1991) Boundedness of solution of a nonlinear nonautonomous neutral delay equation. *J. Math. Anal. Appl.* **156**, 293–304.

[33] Li, Y. K. (1996) Positive periodic solution for a neutral delay model. *Acta Math. Sinica,* **39**, 789–795 (in Chinese).

[34] Li, Y. (1999) Periodic solutions of a periodic delay predator-prey system. *Proc. Amer. Math. Soc.* **127**, 1331–1335.

[35] Li, Q., Cao, J. D. & Wan, S. D. (1998) Positive periodic solution for a neutral delay model in population. *J. Biomath.* **13**, 435–438 (in Chinese).

[36] Li, Y. & Kuang, Y. (2001) Periodic solutions in periodic delay Lotka-Volterra equations and systems. *J. Math. Anal. Appl.* **255**, 260–280.

[37] Li, Y. & Kuang, Y. (2001) Periodic solutions in periodic state-dependent delay equations and population models. *Proc. Amer. Math. Soc.* **130**, 1345–1353.

[38] Mawhin, J. L. (1979) *Topological Degree Methods in Nonlinear Boundary Value Problems*. AMS, Providence.

[39] Metz, J. A. J. & Diekmann, O. (1986) *The Dynamics of Physiologically Structured Populations*. Lecture Notes in Biomathematics, 68, Springer-Verlag.

[40] Murray, J. D. (1989) *Mathematical Biology*. Biomathematics 19, Springer-Verlag, Berlin.

[41] Schley, D. & Gourley, S. A. (2000) Linear stability criteria for population models with periodic perturbed delays. *J. Math. Biol.* **40**, 500–524.

[42] Seifert, G. (1987) On a delay-differential equation for single specie population variations. *Nonlinear Anal. TMA,* **11**, 1051–1059.

[43] *Smith, H. L.* (1993) Reduction of structured population models to threshold-type delay equations and functional differential equations: a case study. *Math. Biosci.* **113**, 1–23.

[44] Smith, H. L. & Kuang, Y. (1992) Periodic solutions of delay differential equations of threshold-type delays, In: Graef & Hale (editors), *Oscillation and Dynamics in Delay Equations*, pp. 153-176. AMS, Providence.

[45] Tang, B. R. & Kuang, Y. (1997) Existence, uniqueness and asymptotic stability of periodic functional differential systems. *Tohoku Math. J.* **49**, 217–239.

[46] Xia, Y. H., Cao, J. D., Zhang, H. Y. & Chen, F. D. (2004) Almost periodic solutions of n-species competitive system with feedback controls. *J. Math. Anal. Appl.* **294**, 503–522.

[47] Yang, Z. H. (2004) Positive periodic solutions for a class of nonlinear delay equations. *Nonlinear Anal. TMA*, **59**, 1013–1031.

[48] Yang, Z. H. & Cao, J. D. (2003) Sufficient conditions for the existence of positive periodic solutions of a class of neutral delay models. *Appl. Math. Comput.* **142,** 123–142.
[49] Yang, Z. H. & Cao, J. D. (2005) Existence of periodic solutions in neutral state-dependent delays equations and models. *J. Computational & Appl. Math.* **174**, 179–199.
[50] Zhang, B. G. & Gopalsamy, K. (1990) Global attractivity and oscillations in a periodic delay logistic equation. *J. Math. Anal. Appl.* **150**, 274–283.
[51] Zhao, T., Kuang, Y. & Smith, H. L. (1997) Global existence of periodic solutions in a class of delayed Gause-type predator-prey systems. *Nonlinear Anal.* **28**, 1373–1394.