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INVARIANCE OF KMS STATES ON GRAPH C^*-ALGEBRAS UNDER CLASSICAL AND QUANTUM SYMMETRY

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Abstract We study the invariance of KMS states on graph C^*-algebras coming from strongly connected and circulant graphs under the classical and quantum symmetry of the graphs. We show that the unique KMS state for strongly connected graphs is invariant under the quantum automorphism group of the graph. For circulant graphs, it is shown that the action of classical and quantum automorphism groups preserves only one of the KMS states occurring at the critical inverse temperature. We also give an example of a graph C^*-algebra having more than one KMS state such that all of them are invariant under the action of classical automorphism group of the graph, but there is a unique KMS state which is invariant under the action of quantum automorphism group of the graph.

Keywords: KMS states; Graph C^*-algebra; quantum automorphism group

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1. Introduction

For a C^*-algebra $\mathcal{A}$, $(\mathcal{A},\sigma)$ is called a C^*-dynamical system if there is a strongly continuous map $\sigma:\mathbb{R}\to \mathrm{Aut}(\mathcal{A})$. When $\mathcal{H}$ is a finite-dimensional Hilbert space, a C^*-dynamical system $(\mathcal{B}(\mathcal{H}),\sigma)$ is given by a self-adjoint operator $H\in\mathcal{B}(\mathcal{H})$ in the sense that $\sigma_t(A)=e^{itH}Ae^{-itH}$. For such a C^*-dynamical system, it is well known that at any inverse temperature $\beta\in\mathbb{R}$, the unique thermal equilibrium state is given by the Gibbs state

$$\omega_\beta(A)=\frac{\mathrm{Tr}(e^{-\beta H}A)}{\mathrm{Tr}(e^{-\beta H})},\quad A\in\mathcal{A}.$$

For a general C^*-dynamical system $(\mathcal{A},\sigma)$, the generalization of the Gibbs states is the KMS (Kubo–Martin–Schwinger) states. A KMS state for a C^*-dynamical system $(\mathcal{A},\sigma)$ at an inverse temperature $\beta\in\mathbb{R}$ is a state $\tau\in\mathcal{A}^*$ that satisfies the KMS condition given by

$$\tau(ab)=\tau(b\sigma_{i\beta}(a)),$$

for a, b in a dense subalgebra of $\mathcal{A}$ called the algebra of analytic elements of $(\mathcal{A}, \sigma)$. For a general C^*-dynamical system, unlike the finite-dimensional case, KMS states do not exist at every temperature. Even if they exist, generally nothing can be said about their uniqueness at a given temperature. It is worth mentioning that in Physics literature, the uniqueness of KMS state often is related to phase transition and symmetry breaking.

One of the mathematically well-studied KMS states are the KMS states for the C^*-dynamical systems on the graph C^*-algebras (see [6, 9]). For a finite-directed graph Γ, the dynamical system is given by the C^*-algebra $C^*(\Gamma)$ and the automorphism σ being the natural lift of the canonical gauge action on $C^*(\Gamma)$. In [6], it is shown that there is a KMS state at the critical inverse temperature $\ln(\rho(D))$ if and only if $\rho(D)$ is an eigenvalue of D with eigenvectors having all entries non- negative, where $\rho(D)$ is the spectral radius of the vertex matrix D of the graph. For a general graph C^*-algebra, we can not say anything about the uniqueness of KMS states. In [6], for strongly connected graphs, such uniqueness result has been obtained. In fact, for strongly connected graphs, there is a unique KMS state occurring only at the critical inverse temperature. However, for another class called the circulant graphs, one can show that KMS states at the critical inverse temperature are not unique. In this context, it is interesting to study invariance of KMS states under some natural added (apart from the gauge symmetry) internal symmetry of the graph C^*-algebra and see if such an invariance could force the KMS state to be unique in certain cases. It is shown in [14] that for a graph Γ, the graph C^*-algebra has a natural generalized symmetry coming from the quantum automorphism group $\mathrm{Aut}^+(\Gamma)$ (see [2]) of the graph itself. This symmetry is generalized in the sense that it contains the classical automorphism group $\mathrm{Aut}(\Gamma)$ of the graph. In this paper, we study the invariance of the KMS states under this generalized symmetry.

For a strongly connected graph Γ, we show that the unique KMS state is preserved by the quantum automorphism group $\mathrm{Aut}^+(\Gamma)$. This result has a rather interesting consequence on the ergodicity of the action of $\mathrm{Aut}^+(\Gamma)$ on the graph. It is shown that for a non-regular strongly connected graph Γ, $\mathrm{Aut}^+(\Gamma)$ can not act ergodically. Then we study another class of graphs called the circulant graphs. Circulant graphs admit KMS states at the inverse critical temperature, but they are not necessarily unique. But due to the transitivity of the action of the automorphism group, it is shown that there exists a unique KMS state which is invariant under the classical or quantum symmetry of the system. In fact, we also show that the only temperature where the KMS state could occur is the inverse critical temperature. Finally, we show by an example that invariance of the KMS state under the action of the quantum symmetry group forces the KMS state to be unique. More precisely, we give an example of a graph with 48 vertices coming from the Linear Binary Constraint system (LBCS, see [11]) where the corresponding graph C^*-algebra has more than one KMS state all of which are preserved by the action of the classical automorphism group of the graph. However, it has a unique $\mathrm{Aut}^+(\Gamma)$ invariant KMS state. In this example also, the only possible inverse temperature at which the KMS state could occur is the inverse critical temperature. This shows that in deed where the classical symmetry fails to fix KMS state, the richer 'genuine' quantum symmetry of the system plays a crucial role to fix a KMS state.

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2. Preliminaries

2.1. KMS states on graph C^*-algebra without sink at the critical inverse temperature

A **finite** directed graph is a collection of finitely many edges and vertices. If we denote the edge set of a graph Γ by $E = (e_1, \ldots, e_n)$ and the vertex set of Γ by $V = (v_1, \ldots, v_m)$ then recall the maps $s, t : E \to V$ from [13] and the vertex (or the adjacency) matrix D which is an $m \times m$ matrix whose ijth entry is k if there are k-number of edges from v_i to v_j. We denote the space of paths by E^* (see [6]). vE^*w will denote the set of paths between two vertices v and w.

Definition 2.1. Γ is said to be without sink if the map $s : E \to V$ is surjective. Furthermore Γ is said to be without any multiple edge if the adjacency matrix D has entries either 1 or 0.

Remark 2.2. Note that the graph C^*-algebra corresponding to a graph without sink is a Cuntz–Krieger algebra. The reader might see [4] for more details on Cuntz–Krieger algebra.

Now we recall some basic facts about graph C^*-algebras. The reader might consult [13] for details on graph C^*-algebras. Let $\Gamma = \{E = (e_1, \ldots, e_n), V = (v_1, \ldots, v_m)\}$ be a finite, directed graph without sink. In this paper, all the graphs are **finite**, without **sink** and without **any multiple edges**. We assign partial isometries S_i's to edges e_i for all $i = 1, \ldots, n$ and projections p_{v_i} to the vertices v_i for all $i = 1, \ldots, m$.

Definition 2.3. The graph C^*-algebra $C^*(\Gamma)$ is defined as the universal C^*-algebra generated by partial isometries $\{S_i\}_{i=1,\ldots,n}$ and mutually orthogonal projections $\{p_{v_k}\}_{k=1,\ldots,m}$ satisfying the following relations:

$$S_i^* S_i = p_{t(e_i)}, \qquad \sum_{s(e_j)=v_l} S_j S_j^* = p_{v_l}.$$

The KMS states at various inverse temperatures on a graph C^*-algebra are well known and we refer the reader to [6] for details. The critical inverse temperature of a graph C^*-algebra is given by $\ln(\rho(D))$ where $\rho(D)$ is the spectral radius of the vertex matrix D of the underlying graph. In this subsection, we mainly collect a few results on the existence of KMS states at the critical inverse temperature on graph C^*-algebras coming from graphs **without** sink. We continue to assume Γ to be a finite, connected graph without sink and with vertex matrix D. We denote the spectral radius of D by $\rho(D)$. With this notation, combining Proposition 4.1 and Corollary 4.2 of [6], we have the following

Proposition 2.4. *The graph C^*-algebra $C^*(\Gamma)$ has a $KMS_{\ln(\rho(D))}$ state if and only if $\rho(D)$ is an eigenvalue of D such that it has eigenvector with all entries being non-negative.*

Lemma 2.5. *Suppose Γ is a finite directed graph without sink with vertex matrix D. If $\rho(D)$ is an eigenvalue of D with an eigenvector $\mathbf{v}$ whose entries are strictly positive such that $\mathbf{v}^T D = \rho(D)\mathbf{v}^T$, then the only possible inverse temperature where the KMS state could occur is $\ln(\rho(D))$.*

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Proof. Suppose $\beta \in \mathbb{R}$ is another possible inverse temperature where a KMS state say ϕ could occur. Then since we have assumed our graph to be without sink, e^β is an eigenvalue of D. Let us denote an eigenvector corresponding to e^β by $\mathbf{w} = (w_1, \ldots, w_m)$ so that $w_i = \phi(p_{v_i})$. Since ϕ is a state, $w_i \geq 0$ for all $i = 1, \ldots, m$ with at least one entry being strictly positive. We have

$$\begin{aligned} D\mathbf{w} &= e^\beta \mathbf{w} \\ &\Rightarrow \mathbf{v}^T D\mathbf{w} = \mathbf{v}^T e^\beta \mathbf{w} \\ &\Rightarrow (\rho(D) - e^\beta)\mathbf{v}^T \mathbf{w} = 0 \end{aligned}$$

By assumption, all the entries of $\mathbf{v}$ are strictly positive and $w_i \geq 0$ for all i with at least one entry being strictly positive which imply that $\mathbf{v}^T\mathbf{w} \neq 0$ and hence $e^\beta = \rho(D)$ i.e. $\beta = \ln(\rho(D))$. □

We discuss examples of two classes of graphs which are **without sink** such that they admit KMS states only at the critical inverse temperature. We shall use them later in this paper.

Strongly connected graphs:

Definition 2.6. A graph is said to be strongly connected if vE^*w is non-empty for all $v, w \in V$.

Definition 2.7. An $m \times m$ matrix D is said to be irreducible if for $i, j \in \{1, \ldots, m\}$, there is some $k > 0$ such that $D^k(i, j) > 0$.

We state the following two well-known results without proof.

Proposition 2.8. *A graph is strongly connected if and only if its vertex matrix is irreducible.*

Proposition 2.9. *An irreducible matrix D has its spectral radius $\rho(D)$ as an eigenvalue with one-dimensional eigenspace spanned by a vector with all its entries being strictly positive (called the Perron–Frobenius eigenvector).*

As a corollary we have

Corollary 2.10. *Let Γ be a strongly connected graph. Then the graph C^*-algebra $C^*(\Gamma)$ has a unique KMS$_{\ln(\rho(D))}$ state. In fact by (b) of Theorem 4.3 of [6], this is the only KMS state.*

Circulant graphs:

Definition 2.11. A graph with m vertices is said to be circulant if its automorphism group contains the cyclic group $\mathbb{Z}_m$.

It is easy to see that if a graph is a circulant, then its vertex matrix is determined by its first-row vector say $(d_0, \ldots, d_{m-1})$. More precisely, the vertex matrix D of a circulant

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graph is given by

$$\begin{bmatrix} d_0 & d_1 \dots & d_{m-1} \\ d_{m-1} & d_0 \dots & d_{m-2} \\ . & & \\ . & & \\ . & & \\ d_1 & d_2 \dots & d_0 \end{bmatrix}.$$

Remark 2.12. Note that a circulant graph is always without sink except the trivial case where it has no edge at all. This is because if ith vertex of a circulant graph is a sink, then the ith row of the vertex matrix will be zero forcing all the rows to be identically zero.

Let ϵ be a primitive mth root of unity. The following is well known (see [10]):

Proposition 2.13. *For a circulant graph with vertex matrix as above, the eigenvalues are given by*

$$\lambda_l = d_0 + \epsilon^l d_1 + \cdots + \epsilon^{(m-1)l} d_{m-1}, \quad l = 0, \dots, (m-1)$$

It is easy to see that $\lambda = \sum_{i=0}^{m-1} d_i$ is an eigenvalue of D and it has a normalized eigenvector given by $(\frac{1}{m}, \dots, \frac{1}{m})$. Since $|\lambda_l| \leq \lambda$, we have

Corollary 2.14. *For a circulant graph Γ with vertex matrix D, D has its spectral radius λ as an eigenvalue with a normalized eigenvector (not necessarily unique) having all its entries non-negative.*

Combining the above corollary with Proposition 2.4, we have

Corollary 2.15. *For a circulant graph Γ, $C^*(\Gamma)$ has a KMS$_{ln(\lambda)}$ state.*

Lemma 2.16. *For a circulant graph Γ without sink, the only possible temperature where a KMS state could occur is the critical inverse temperature.*

Proof. As we have assumed the circulant graphs are without sink, it is enough to show that the vertex matrix D of Γ satisfies the conditions of Lemma 2.5. It is already observed that the eigenvalue λ has an eigenvector with all its entries being positive (column vector with all its entries 1 to be precise). Also since the row sums are equal to column sums which are equal to λ, $(1, \dots, 1)D = \lambda(1, \dots, 1)$. Hence an application of Lemma 2.5 finishes the proof. □

Note that KMS states at the critical inverse temperature are not necessarily unique, since the dimension of the eigenspace of the eigenvalue λ could be strictly larger than 1

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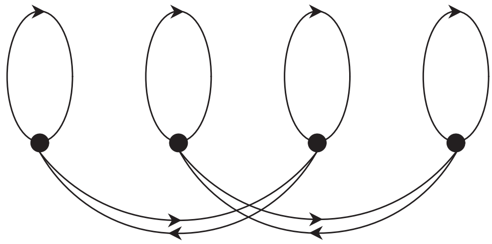

Figure 1. Circulant graph with more than one KMS state.

as the example in Figure 1 illustrates. We take the graph whose vertex matrix is given by

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Hence the graph is circulant with its spectral radius 2 as an eigenvalue with multiplicity 2. So the dimension of the corresponding eigenspace is 2 violating the uniqueness of the KMS state at the critical inverse temperature $\ln(2)$.

2.2. Quantum automorphism group of graphs as symmetry of graph C^*-algebra

2.2.1. Compact quantum groups and quantum automorphism groups

In this subsection, we recall the basics of compact quantum groups and their actions on C^*-algebras. The facts collected in this subsection are well known and we refer the readers to [12, 15, 16] for details. All the tensor products in this paper are minimal.

Definition 2.17. A compact quantum group (CQG) $\mathbb{G}$ is a pair $(C(\mathbb{G}), \Delta_{\mathbb{G}})$ such that $C(\mathbb{G})$ is a unital C^*-algebra and $\Delta_{\mathbb{G}} : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ is a unital C^*-homomorphism satisfying

(i) $(\mathrm{id} \otimes \Delta_{\mathbb{G}}) \circ \Delta_{\mathbb{G}} = (\Delta_{\mathbb{G}} \otimes \mathrm{id}) \circ \Delta_{\mathbb{G}}$.

(ii) $\mathrm{Span}\{\Delta_{\mathbb{G}}(C(\mathbb{G}))(1 \otimes C(\mathbb{G}))\}$ and $\mathrm{Span}\{\Delta_{\mathbb{G}}(C(\mathbb{G}))(C(\mathbb{G}) \otimes 1)\}$ are dense in $C(\mathbb{G}) \otimes C(\mathbb{G})$.

Remark 2.18. Strictly speaking the quantum group $\mathbb{G}$ is the dual object of the pair $(C(\mathbb{G}), \Delta_{\mathbb{G}})$. But following [12], we shall refer the dual object $(C(\mathbb{G}), \Delta_{\mathbb{G}})$ as the quantum group itself without mentioning it explicitly.

Given a CQG $\mathbb{G}$, there is a canonical dense Hopf $*$-algebra $C(\mathbb{G})_0$ in $C(\mathbb{G})$ on which an antipode κ and counit ϵ are defined. Given two CQG's $\mathbb{G}_1$ and $\mathbb{G}_2$, a CQG morphism between them is a unital C^*-homomorphism $\pi : C(\mathbb{G}_1) \to C(\mathbb{G}_2)$ such that $(\pi \otimes \pi) \circ \Delta_{\mathbb{G}_1} = \Delta_{\mathbb{G}_2} \circ \pi$.

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Definition 2.19. Given a (unital) C^*-algebra $\mathcal{C}$, a CQG $\mathbb{G}$ is said to act faithfully on $\mathcal{C}$ if there is a unital C^*-homomorphism $\alpha : \mathcal{C} \to \mathcal{C} \otimes C(\mathbb{G})$ satisfying

(i) $(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta_{\mathbb{G}}) \circ \alpha$.

(ii) $\mathrm{Span}\{\alpha(\mathcal{C})(1 \otimes C(\mathbb{G}))\}$ is dense in $\mathcal{C} \otimes C(\mathbb{G})$.

(iii) The $*$-algebra generated by the set $\{(\omega \otimes \mathrm{id}) \circ \alpha(\mathcal{C}) : \omega \in \mathcal{C}^*\}$ is norm-dense in $C(\mathbb{G})$.

Definition 2.20. An action $\alpha : \mathcal{C} \to \mathcal{C} \otimes C(\mathbb{G})$ is said to be ergodic if $\alpha(c) = c \otimes 1$ implies $c \in \mathbb{C}1$.

Definition 2.21. Given an action α of a CQG $\mathbb{G}$ on a C^*-algebra $\mathcal{C}$, α is said to preserve a state τ on $C(\mathbb{G})$ if $(\tau \otimes \mathrm{id}) \circ \alpha(a) = \tau(a)1$ for all $a \in \mathcal{C}$.

Definition 2.22 (Def 2.1 of [3]). Given a unital C^*-algebra $\mathcal{C}$, the quantum automorphism group of $\mathcal{C}$ is a CQG $\mathbb{G}$ acting faithfully on $\mathcal{C}$ satisfying the following universal property:
If $\mathbb{B}$ is any CQG acting faithfully on $\mathcal{C}$, there is a surjective CQG morphism $\pi : C(\mathbb{G}) \to C(\mathbb{B})$ such that $(\mathrm{id} \otimes \pi) \circ \alpha = \beta$, where $\beta : \mathcal{C} \to \mathcal{C} \otimes C(\mathbb{B})$ is the corresponding action of $\mathbb{B}$ on $\mathcal{C}$ and α is the action of $\mathbb{G}$ on $\mathcal{C}$.

From now on, we shall drop the suffix of Δ whenever the quantum group is clear from the context.

Example 2.23. If we take a space of n points X_n then the quantum automorphism group of the C^*-algebra $C(X_n)$ is denoted by S_n^+. The underlying C^*-algebra $C(S_n^+)$ is the universal C^* algebra generated by $\{u_{ij}\}_{i,j=1,\ldots,n}$ satisfying the following relations (see Theorem 3.1 of [15]):

$$u_{ij}^2 = u_{ij}, \quad u_{ij}^* = u_{ij}, \quad \sum_{k=1}^{n} u_{ik} = \sum_{k=1}^{n} u_{ki} = 1, \ i, j = 1, \ldots, n.$$

The coproduct on the generators is given by $\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$.

2.2.2. Quantum automorphism group of finite graphs and graph C^*-algebras

Recall the definition of finite, directed graph $\Gamma = ((V = v_1, \ldots, v_m), (E = e_1, \ldots, e_n))$ without any multiple edge from Subsection 2.1.

Definition 2.24. The quantum automorphism group of a graph Γ without any multiple edge will be denoted by $\mathrm{Aut}^+(\Gamma)$. The underlying C^*-algebra $C(\mathrm{Aut}^+(\Gamma))$ is defined to be the quotient $C(S_n^+)/(AD - DA)$, where $A = ((u_{ij}))_{i.j=1,\ldots,m}$, and D is the adjacency matrix for Γ. The coproduct on the generators is again given by $\Delta(u_{ij}) = \sum_{k=1}^{m} u_{ik} \otimes u_{kj}$.

For the classical automorphism group $\mathrm{Aut}(\Gamma)$, the commutative C^*-algebra $C(\mathrm{Aut}(\Gamma))$ is generated by u_{ij} where u_{ij} is a function on S_n taking value 1 on the permutation which sends ith vertex to jth vertex and takes the value zero on other elements of the group.

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It is a quantum subgroup of $\mathrm{Aut}^+(\Gamma)$. The surjective CQG morphism $\pi : C(\mathrm{Aut}^+(\Gamma)) \to C(\mathrm{Aut}(\Gamma))$ sends the generators to generators.

Remark 2.25. Since $\mathrm{Aut}^+(\Gamma)$ is a quantum subgroup of S_n^+, it is a Kac algebra and hence $\kappa(u_{ij}) = u_{ji}^* = u_{ji}$. Applying κ to the equation $AD = DA$, we get $A^T D = DA^T$, where $A^T = ((u_{ji}))$.

With analogy of vertex-transitive action of the automorphism group of a graph, we have the following

Definition 2.26. A graph Γ is said to be quantum vertex transitive if the generators u_{ij} of $C(\mathrm{Aut}^+(\Gamma))$ are all non-zero.

Remark 2.27. It is easy to see that if a graph is vertex transitive, it must be quantum vertex transitive.

Proposition 2.28 (Corollary 3.7 of [11]). *The action of $Aut^+(\Gamma)$ on $C(V)$ is ergodic if and only if the action is quantum vertex transitive.*

Remark 2.29. For a graph $\Gamma = (V, E)$, when we talk about ergodic action, we always take the corresponding C^*-algebra to be $C(V)$.

In the next proposition, we shall see that in fact for a finite, connected graph Γ without multiple edge the CQG $\mathrm{Aut}^+(\Gamma)$ has a C^*-action on the infinite-dimensional C^*-algebra $C^*(\Gamma)$.

Proposition 2.30 (see Theorem 4.1 of [14]). *Given a directed graph Γ without multiple edge, $Aut^+(\Gamma)$ has a C^*-action on $C^*(\Gamma)$. The action is given by*

$$\alpha(p_{v_i}) = \sum_{k=1}^{m} p_{v_k} \otimes u_{ki}$$

$$\alpha(S_j) = \sum_{l=1}^{n} S_l \otimes u_{s(e_l)s(e_j)} u_{t(e_l)t(e_j)}.$$

Proposition 2.31. *Suppose $\Gamma = (V = (v_1, \ldots, v_m), E = (e_1, \ldots, e_n))$ is a finite, directed graph without any multiple edge as before. For a KMS_β state τ on the graph C^*-algebra $C^*(\Gamma)$, $Aut^+(\Gamma)$ preserves τ if and only if $(\tau \otimes id) \circ \alpha(p_{v_i}) = \tau(p_{v_i})1$ for all $i = 1, \ldots, m$.*

We start with proving the following lemma which will be used to prove the above proposition. In the following lemma, $\Gamma = (V, E)$ is again a finite, directed graph without any multiple edge.

Lemma 2.32. *Suppose $Aut^+(\Gamma)$ preserves some linear functional τ on $C(V)$. Then $\tau(p_{v_i}) \neq \tau(p_{v_j}) \Rightarrow u_{ij} = 0$.*

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Proof. Let i, j be such that $\tau(p_{v_i}) \neq \tau(p_{v_j})$. By the assumption,

$$(\tau \otimes \mathrm{id}) \circ \alpha(p_{v_i}) = \tau(p_{v_i})1$$
$$\Rightarrow \sum_k \tau(p_{v_k})u_{ki} = \tau(p_{v_i})1.$$

Multiplying both sides of the last equation by u_{ji} and using the orthogonality, we get $\tau(p_{v_j})u_{ji} = \tau(p_{v_i})u_{ji}$ i.e. $(\tau(p_{v_j}) - \tau(p_{v_i}))u_{ji} = 0$ and hence $u_{ji} = 0$ as $\tau(p_{v_i}) \neq \tau(p_{v_j})$. Applying κ, we get $u_{ij} = u_{ji} = 0$. □

Proof. If $\mathrm{Aut}^+(\Gamma)$ preserves τ, then $(\tau \otimes \mathrm{id}) \circ \alpha(p_{v_i}) = \tau(p_{v_i})1$ for all $i = 1, \ldots, m$ trivially. For the converse, given $(\tau \otimes \mathrm{id}) \circ \alpha(p_{v_i}) = \tau(p_{v_i})1$ for all $i = 1, \ldots, m$, we need to show that $(\tau \otimes \mathrm{id}) \circ \alpha(S_\mu S_\nu^*) = \tau(S_\mu S_\nu^*)1$ for all $\mu, \nu \in E^*$. The proof is similar to that of Theorem 3.5 of [8]. It is easy to see that for $|\mu| \neq |\nu|$, $(\tau \otimes \mathrm{id}) \circ \alpha(S_\mu S_\nu^*) = 0 = \tau(S_\mu S_\nu^*)$. So let $|\mu| = |\nu|$. For $\mu = \nu = e_{i_1} e_{i_2} \ldots e_{i_p}$, we have $S_\mu S_\mu^* = S_{i_1} \ldots S_{i_p} S_{i_p}^* \ldots S_{i_1}^*$. So

$$(\tau \otimes \mathrm{id}) \circ \alpha(S_\mu S_\mu^*) = (\tau \otimes \mathrm{id}) \Big(\sum S_{j_1} \cdots S_{j_p} S_{j_p}^* \cdots S_{j_1}^* \otimes u_{s(e_{j_1})s(e_{i_1})} u_{t(e_{j_1})t(e_{i_1})} \cdots u_{s(e_{j_p})s(e_{i_p})} u_{t(e_{j_p})t(e_{i_p})} u_{s(e_{j_p})s(e_{i_p})} \cdots u_{t(e_{j_1})t(e_{i_1})} u_{s(e_{j_1})s(e_{i_1})} \Big)$$

By the same argument as given in the proof of the Theorem 3.5 of [8], for $S_{j_1} \cdots S_{j_p} = 0$, $u_{s(e_{j_1})s(e_{i_1})} u_{t(e_{j_1})t(e_{i_1})} \cdots u_{s(e_{j_p})s(e_{i_p})} u_{t(e_{j_p})s(t_{i_p})} = 0$ and hence the last expression equals to

$$\sum e^{-\beta|\mu|} \tau(p_{t(e_{j_p})}) u_{s(e_{j_1})s(e_{i_1})} u_{t(e_{j_1})t(e_{i_1})} \ldots u_{s(e_{j_p})s(e_{i_p})} u_{t(e_{j_p})t(e_{i_p})} u_{s(e_{j_p})s(e_{i_p})} \ldots u_{t(e_{j_1})t(e_{i_1})} u_{s(e_{j_1})s(e_{i_1})}$$

Observe that any KMS_β state restricts to a state on $C(V)$ so that by Lemma 2.32, for $\tau(p_{t(e_{j_p})}) \neq \tau(p_{t(e_{i_p})})$, $u_{t(e_{j_p})t(e_{i_p})} = 0$. Using this, the last summation reduces to

$$e^{-\beta|\mu|} \sum \tau(p_{t(e_{i_p})}) u_{s(e_{j_1})s(e_{i_1})} u_{t(e_{j_1})t(e_{i_1})} \cdots u_{s(e_{j_p})s(e_{i_p})} u_{t(e_{j_p})t(e_{i_p})} u_{s(e_{j_p})s(e_{i_p})} \ldots u_{t(e_{j_1})t(e_{i_1})} u_{s(e_{j_1})s(e_{i_1})}.$$

Using the same arguments used in the proof of Theorem 3.13 in [7] repeatedly, it can be shown that the last summation actually equals to $e^{-\beta|\mu|}\tau(p_{t(e_{i_p})}) = \tau(S_\mu S_\mu^*)$. Hence

$$(\tau \otimes \mathrm{id}) \circ \alpha(S_\mu S_\mu^*) = \tau(S_\mu S_\mu^*)1.$$

With similar reasoning, it can easily be verified that for $\mu \neq \nu$, $(\tau \otimes \mathrm{id}) \circ \alpha(S_\mu S_\nu^*) = 0 = \tau(S_\mu S_\nu^*)1$. Hence by linearity and continuity of τ, for any $a \in C^*(\Gamma)$, $(\tau \otimes \mathrm{id}) \circ \alpha(a) = \tau(a).1$. □

If we apply Proposition 2.31 to the action of classical automorphism group of a graph on the corresponding graph C^*-algebra, we get the following

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Lemma 2.33. *Given a* KMS_β *state* τ *on* $C^*(\Gamma)$, *we denote the vector* $(\tau(p_{v_1}), \ldots, \tau(p_{v_m}))$ *by* $\mathcal{N}^\tau$. *If we denote the permutation matrix corresponding to an element* $g \in Aut(\Gamma)$ *by* B, *then* $Aut(\Gamma)$ *preserves* τ *if and only if* $B\mathcal{N}^\tau = \mathcal{N}^\tau$.

Proof. Follows from the easy observation that $(\tau \otimes \mathrm{id}) \circ \alpha(p_{v_i}) = \tau(p_{v_i})1$ implies $B\mathcal{N}^\tau = \mathcal{N}^\tau$ for the classical automorphism group of the graph. □

3. Invariance of KMS states under the symmetry of graphs

3.1. Strongly connected graphs

Recall the unique KMS state of $C^*(\Gamma)$ for a strongly connected graph Γ with vertex matrix D. We denote the ijth entry of D by d_{ij}. The unique KMS state at the critical inverse temperature $\ln(\rho(D))$ is determined by the unique normalized Perron–Frobenius eigenvector of D corresponding to the eigenvalue $\rho(D)$. If the state is denoted by τ, the eigenvector is given by $((\tau(p_{v_i})))_{i=1,\ldots,m}$ where m is the number of vertices. Now we prove the main result of this subsection.

Theorem 3.1. *For a strongly connected graph* Γ, $Aut^+(\Gamma)$ *preserves the unique KMS state of* $C^*(\Gamma)$.

Proof. Note that by Proposition 2.31, it suffices to show that $(\tau \otimes \mathrm{id}) \circ \alpha(p_{v_i}) = \tau(p_{v_i})1 \forall\ i = 1, \ldots, m$. Recall the action of $\mathrm{Aut}^+(\Gamma)$ on $C^*(\Gamma)$. We continue to denote the matrix $((u_{ij}))$ by A. For a state ϕ of $C(\mathrm{Aut}^+(\Gamma))$, we denote the vector whose ith entry is $(\tau \otimes \phi) \circ \alpha(p_{v_i})$ by $\mathbf{v}_\phi$. Then

$$\begin{aligned}
(D\mathbf{v}_\phi)_i &= \sum_j d_{ij}(\tau \otimes \phi) \circ \alpha(p_{v_j}) \\
&= \sum_{j,k} \tau(p_{v_k})\phi(d_{ij}u_{kj}) \\
&= \sum_k \tau(p_{v_k})\phi\left(\sum_j d_{ij}u_{kj}\right) \\
&= \sum_k \tau(p_{v_k})\phi\left(\sum_j u_{ji}d_{jk}\right)(DA^T = A^T D \text{by Remark 2.25}) \\
&= \sum_{j,k} d_{jk}\tau(p_{v_k})\phi(u_{ji}) \\
&= \rho(D)\sum_j \tau(p_{v_j})\phi(u_{ji})(D((\tau(p_i)))^T = \rho(D)((\tau(p_{v_i})))^T) \\
&= \rho(D)(\tau \otimes \phi) \circ \alpha(p_{v_i}).
\end{aligned}$$

Hence $\mathbf{v}_\phi$ is an eigenvector of D corresponding to the eigenvalue $\rho(D)$. By the one dimensionality of the eigenspace, we have some constant C_ϕ such that $(\tau \otimes \phi) \circ \alpha(p_{v_i}) = C_\phi\tau(p_{v_i})$ for all $i = 1, \ldots, m$. To determine the constant C_ϕ, we take the summation over

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i on both sides and get

$$\begin{aligned}\sum_i(\tau\otimes\phi)\circ\alpha(p_{v_i}) &= C_\phi\sum_i\tau(p_{v_i})\\ &\Rightarrow \sum_{i,j}\tau(p_{v_j})\phi(u_{ji}) = C_\phi\\ &\Rightarrow \sum_j\tau(p_{v_j})\sum_i\phi(u_{ji}) = C_\phi\\ &\Rightarrow C_\phi = 1.\end{aligned}$$

Hence $\phi((\tau\otimes \mathrm{id})\circ\alpha(p_{v_i})) = \phi(\tau(p_{v_i})1)$ for all i and for all state ϕ which implies that $(\tau\otimes\mathrm{id})\circ\alpha(p_{v_i}) = \tau(p_{v_i})1$ for all $i = 1,\ldots,m$. □

We end this subsection with a proposition about the non-ergodicity of the action of $\mathrm{Aut}^+(\Gamma)$ on $C(V)$ for a strongly connected graph $\Gamma = (V,E)$. Note that the following proposition does not deal with states on the infinite-dimensional C^*-algebra $C^*(\Gamma)$.

Proposition 3.2. *For a strongly connected graph Γ, if the Perron–Frobenius eigenvector is not a multiple of $(1,\ldots,1)$, then the action of $Aut^+(\Gamma)$ is non-ergodic.*

Proof. It follows from Lemma 2.32, Theorem 3.1 and Proposition 2.28. □

Remark 3.3. It is known that non-regular graphs can never be quantum vertex transitive (see Lemma 3.2.3 of [5]). However, Proposition 3.2 gives an alternative proof of the result in case of the strongly connected graphs since the vertex matrix of a non-regular graph can not have $(1,\ldots,1)$ as an eigenvector.

3.2. Circulant graphs

Consider a finite graph $\Gamma = (V,E)$ with m-vertices $(v_1,\ldots,v_m)$. Recall the notation $\mathcal{N}^\tau$ for a KMS_β state τ on $C^*(\Gamma)$.

Lemma 3.4. *Given a graph Γ such that $Aut(\Gamma)$ acts transitively on its vertices, $B\mathcal{N}^\tau = \mathcal{N}^\tau$ for all $B\in Aut(\Gamma)$ if and only if $\mathcal{N}_i^\tau = \mathcal{N}_j^\tau$ for all $i,j = 1,\ldots,m$.*

Proof. If $\mathcal{N}_i^\tau = \mathcal{N}_j^\tau$ for all $i,j = 1,\ldots,m$, then $B\mathcal{N}^\tau = \mathcal{N}^\tau$ for all $B\in\mathrm{Aut}(\Gamma)$ trivially. For the converse, let $\mathcal{N}_i^\tau \neq \mathcal{N}_j^\tau$ for some i,j. Since the action of the automorphism group is transitive, there is some $B\in\mathrm{Aut}(\Gamma)$ so that $B(v_i) = v_j$ and hence $B\mathcal{N}^\tau\neq\mathcal{N}^\tau$. □

Now recall from the discussion following Corollary 2.15 and Lemma 2.16 that for a circulant graph, the KMS states exist only at the critical inverse temperature, but they are not necessarily unique. We shall prove that if we further assume the invariance of such a state under the action of the automorphism group of the graph, then it is unique.

Proposition 3.5. *For a circulant graph Γ there exists a unique $Aut(\Gamma)$ invariant KMS state on $C^*(\Gamma)$.*

Proof. Since for a circulant graph, the automorphism group acts transitively on the set of vertices, by Lemma 3.4 and Lemma 2.33, a KMS_β state τ is $\mathrm{Aut}(\Gamma)$ invariant if and only if $\tau(p_{v_i}) = \frac{1}{m}$ for all i. This coupled with the fact that $(\frac{1}{m}, \ldots, \frac{1}{m})$ is an eigenvector corresponding to the eigenvalue λ (= spectral radius) finishes the proof of the proposition. □

Remark 3.6. We remark that the group invariant KMS state is also invariant under the action of quantum automorphism group of the underlying graph. Since $\tau(p_{v_i}) = \tau(p_{v_j})$, it is easy to see that for the action of $\mathrm{Aut}^+(\Gamma)$, $(\tau \otimes \mathrm{id}) \circ \alpha(p_{v_i}) = \tau(p_{v_i})1$ for all $i = 1, \ldots, m$. Hence an application of Proposition 2.31 finishes the proof of the claim.

3.3. Graph of the Mermin–Peres magic square game

We start this subsection by clarifying a few notation to be used in this subsection. Given an undirected graph Γ, we make it directed by declaring that both (i, j) and (j, i) are in the edge set whenever there is an edge between two vertices v_i and v_j. The vertex matrix of such a directed graph is symmetric by definition. In this subsection, we use the notation $\overrightarrow{\Gamma}$ for the directed graph coming from an undirected graph Γ in this way. Γ will always denote an undirected graph.

Remark 3.7. By definition, $\mathrm{Aut}^+(\overrightarrow{\Gamma}) \cong \mathrm{Aut}^+(\Gamma)$ and hence $\mathrm{Aut}(\overrightarrow{\Gamma}) \cong \mathrm{Aut}(\Gamma)$ (see [2]).

Given two graphs $\Gamma_1 = (V_1, E_1), \Gamma_2 = (V_2, E_2)$, their disjoint union $\Gamma_1 \cup \Gamma_2$ is defined to be the graph $\Gamma = (V, E)$ such that $V = V_1 \cup V_2$. There is an edge between two vertices $v_i, v_j \in V_1 \cup V_2$ if both the vertices belong to either Γ_1 or Γ_2 and they have an edge in the corresponding graph.

Proposition 3.8. *Let Γ_1, Γ_2 be two non-isomorphic connected graphs. Then the automorphism group of $\overrightarrow{\Gamma_1 \cup \Gamma_2}$ is given by $Aut(\overrightarrow{\Gamma_1}) \times Aut(\overrightarrow{\Gamma_2})$.*

Proof. The result follows from Theorem 2.5 of [17] and Remark 3.7. □

Proposition 3.9. *Let Γ_1 and Γ_2 be two non-isomorphic connected graphs such that $\overrightarrow{\Gamma_1}$ and $\overrightarrow{\Gamma_2}$ have symmetric vertex matrices D_1 and D_2 having equal spectral radius say λ. Then for the graph $\Gamma = \Gamma_1 \cup \Gamma_2$, $C^*(\overrightarrow{\Gamma})$ has infinitely many KMS states at the critical inverse temperature $ln(\lambda)$ such that all of them are invariant under the action of $Aut(\overrightarrow{\Gamma}) \cong Aut(\overrightarrow{\Gamma_1}) \times Aut(\overrightarrow{\Gamma_2})$.*

To prove the proposition, we require the following

Lemma 3.10. *Let $A \in M_n(\mathbb{C})$ and $B \in M_m(\mathbb{C})$. Then the spectral radius of the matrix $\begin{bmatrix} A & 0_{n\times m} \\ 0_{m\times n} & B \end{bmatrix}$ is equal to $\max\{sp(A), sp(B)\}$.*

Proof. It follows from the simple observation that any eigenvalue of the matrix $\begin{bmatrix} A & 0_{n\times m} \\ 0_{m\times n} & B \end{bmatrix}$ is either an eigenvalue of A or an eigenvalue of B. □

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Proof of Proposition 3.9. We assume that $\overrightarrow{\Gamma}_1$ has n-vertices and $\overrightarrow{\Gamma}_2$ has m-vertices. Let us denote the vertex matrix of $\overrightarrow{\Gamma}$ by D. D is given by the matrix

$$\begin{bmatrix} D_1 & 0_{n\times m} \\ 0_{m\times n} & D_2 \end{bmatrix}.$$

Then the spectral radius is equal to λ by Lemma 3.10. Also, it is easy to see that the spectral radius is an eigenvalue of the matrix D. Now λ has a one-dimensional eigenspace for D_1 spanned by say w_1 and a one-dimensional eigenspace for D_2 spanned by say w_2 as both the graphs are connected and hence strongly connected as directed graphs. We take both the eigenvectors normalized for convenience. Then for D, the eigenspace corresponding to λ is two dimensional spanned by the vectors $\mathbf{w_1} = (w_1, 0_m)$ and $\mathbf{w_2} = (0_n, w_2)$ where 0_k is the zero k-tuple. So the eigenspace of D corresponding to the eigenvalue λ is given by $\{\xi\mathbf{w_1} + \eta\mathbf{w_2} : (\xi, \eta) \in \mathbb{C}^2 - (0, 0)\}$. For any $(\xi, \eta) \in \mathbb{C}^2$, $\xi\mathbf{w_1} + \eta\mathbf{w_2} = (\xi w_1, \eta w_2)$. It is easy to see that there are infinitely many choices of ξ, η such that corresponding eigenvector is normalized with all its entries being non-negative which in turn give rise to infinitely many KMS states. The set of normalized vectors is given by $\{(\xi w_1, (1-\xi)w_2) : 0 \leq \xi \leq 1\}$. We shall show that any $B \in \mathrm{Aut}(\overrightarrow{\Gamma})$ keeps such a normalized eigenvector invariant. Let $\mathbf{w} = (\xi w_1, (1-\xi)w_2)$ be one such choice. By Proposition 3.8, any $B \in \mathrm{Aut}(\overrightarrow{\Gamma})$ can be written in the matrix form $\begin{bmatrix} B_1 & 0_{n\times m} \\ 0_{m\times n} & B_2. \end{bmatrix}$, for $B_i \in \mathrm{Aut}(\overrightarrow{\Gamma}_i)$ and $i = 1, 2$. Then $B\mathbf{w} = (\xi B_1 w_1, (1-\xi)B_2 w_2)$. Since w_1, w_2 are Perron–Frobenius eigenvectors of D_1, D_2 respectively with both the graphs $\overrightarrow{\Gamma}_1, \overrightarrow{\Gamma}_2$ strongly connected, by Proposition 3.1, $B_i(w_i) = w_i$ for $i = 1, 2$. So $B\mathbf{w} = (\xi w_1, (1-\xi)w_2) = \mathbf{w}$. Hence an application of Lemma 2.33 completes the proof of the proposition. □

Now we turn to the main object of study of this subsection. A linear binary constraint system (LBCS) $\mathcal{F}$ consists of a family of binary variables $x_1, \ldots, x_n$ and constraints $C_1, \ldots, C_m$, where each C_l is a linear equation over $\mathbb{F}_2$ in some subset of the variables i.e. each C_l is of the form $\sum_{x_i \in S_l} x_i = b_l$ for some $S_l \subset \{x_1, \ldots, x_n\}$. Corresponding to every LBCS $\mathcal{F}$, one can associate a graph (see section 6.2 of [1]) to be denoted by $\mathcal{G}(\mathcal{F})$. The following is an example of an LBCS.

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 & x_1 + x_4 + x_7 &= 0 \\ x_4 + x_5 + x_6 &= 0 & x_2 + x_5 + x_8 &= 0 \\ x_7 + x_8 + x_9 &= 0 & x_3 + x_6 + x_9 &= 1, \end{aligned}$$

where the addition is over $\mathbb{F}_2$. From now on $\mathcal{F}$ will always mean the above LBCS.

Definition 3.11. Given an LBCS $\mathcal{F}$, its homogenization $\mathcal{F}_0$ is defined to be the LBCS obtained by assigning zero to the right-hand side of every constraint C_l.

In light of the Theorem 6.2, 6.3 and 6.4 of [1], the corresponding graphs $\mathcal{G}(\mathcal{F})$ and $\mathcal{G}(\mathcal{F}_0)$ are quantum isomorphic, but not isomorphic. The graph $\mathcal{G}(\mathcal{F})$ is called the graph of **Mermin–Peres magic square** game.

Both the graphs $\mathcal{G}(\mathcal{F})$ and $\mathcal{G}(\mathcal{F}_0)$ are vertex transitive (in fact they are Cayley as mentioned in [11]) and hence quantum vertex transitive by Remark 2.27. Combining the

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facts that $\mathcal{G}(\mathcal{F})$ and $\mathcal{G}(\mathcal{F}_0)$ are quantum isomorphic and quantum vertex transitive with Lemma 4.15 of [11], we get

Lemma 3.12. *For the LBCS $\mathcal{F}$, the disjoint union of $\mathcal{G}(\mathcal{F})$ and $\mathcal{G}(\mathcal{F}_0)$ is quantum vertex transitive.*

By Remark 3.7,

Corollary 3.13. $\overrightarrow{\mathcal{G}(\mathcal{F}) \cup \mathcal{G}(\mathcal{F}_0)}$ *is quantum vertex transitive.*

It can be verified that both the graphs $\mathcal{G}(\mathcal{F})$ and $\mathcal{G}(\mathcal{F}_0)$ are connected with 24 vertices each such that the vertex matrices of the graphs $\overrightarrow{\mathcal{G}(\mathcal{F})}$ and $\overrightarrow{\mathcal{G}(\mathcal{F})_0}$ have spectral radius 9 (see Figures 2 and 3). Then since they are non-isomorphic, by Proposition 3.9, $C^*(\overrightarrow{\mathcal{G}(\mathcal{F}) \cup \mathcal{G}(\mathcal{F}_0)})$ has infinitely many KMS states at the critical inverse temperature $\ln(9)$ all of which are invariant under the action of the classical automorphism group of $\overrightarrow{\mathcal{G}(\mathcal{F}) \cup \mathcal{G}(\mathcal{F}_0)}$. But as mentioned earlier, if we further assume that the KMS state at the critical inverse temperature is invariant under the action of $\mathrm{Aut}^+(\overrightarrow{\mathcal{G}(\mathcal{F}) \cup \mathcal{G}(\mathcal{F}_0)})$, then it is necessarily unique. We prove it in the next theorem.

Theorem 3.14. *For the LBCS $\mathcal{F}$, the graph C^*-algebra $C^*(\overrightarrow{\mathcal{G}(\mathcal{F}) \cup \mathcal{G}(\mathcal{F}_0)})$ has a unique $\mathrm{Aut}^+(\overrightarrow{\mathcal{G}(\mathcal{F}) \cup \mathcal{G}(\mathcal{F}_0)})$-invariant KMS state τ given by*

$$\tau(S_\mu S_\nu^*) = \delta_{\mu,\nu} \frac{1}{9^{|\mu|} 48}.$$

Proof. By Corollary 3.13, the graph $\overrightarrow{(\mathcal{G}(\mathcal{F}) \cup \mathcal{G}(\mathcal{F}_0))}$ is quantum vertex transitive. Hence by Lemma 2.32, for any $\mathrm{Aut}^+(\overrightarrow{\mathcal{G}(\mathcal{F}) \cup \mathcal{G}(\mathcal{F}_0)})$ invariant KMS state τ on $C^*(\overrightarrow{\mathcal{G}(\mathcal{F}) \cup \mathcal{G}(\mathcal{F}_0)}))$ at the critical inverse temperature $\ln(9)$, we have $\tau(p_{v_i}) = \tau(p_{v_j})$ for all i, j. That forces $\tau(p_{v_i})$ to be $\frac{1}{48}$ for all $i = 1, \ldots, 48$. Since $(\frac{1}{48}, \ldots, \frac{1}{48})$ is an eigenvector corresponding to the eigenvalue 9, there is a unique KMS state on $C^*(\overrightarrow{\mathcal{G}(\mathcal{F}) \cup \mathcal{G}(\mathcal{F}_0)})$ at the critical inverse temperature $\ln(9)$ satisfying

$$\tau(S_\mu S_\nu^*) = \delta_{\mu,\nu} \frac{1}{9^{|\mu|} 48}.$$

$\mathrm{Aut}^+(\overrightarrow{\mathcal{G}(\mathcal{F}) \cup \mathcal{G}(\mathcal{F}_0)})$ preserves the above KMS state by following the same line of arguments as given in Remark 3.6. To complete the proof, we need to show that the only possible inverse temperature where a KMS state could occur is the critical inverse temperature. For that first notice that the graph $\overrightarrow{\mathcal{G}(\mathcal{F}) \cup \mathcal{G}(\mathcal{F}_0)}$ is without sink. We have already observed that the spectral radius 9 has an eigenvector with all entries being strictly positive (column vector with all its entries $\frac{1}{48}$). Also the vertex matrix of the graph $\overrightarrow{\mathcal{G}(\mathcal{F}) \cup \mathcal{G}(\mathcal{F}_0)}$ is symmetric implying that $(\frac{1}{48}, \ldots, \frac{1}{48})D = 9(\frac{1}{48}, \ldots, \frac{1}{48})$. Hence an application of Lemma 2.5 finishes the proof of the theorem. □

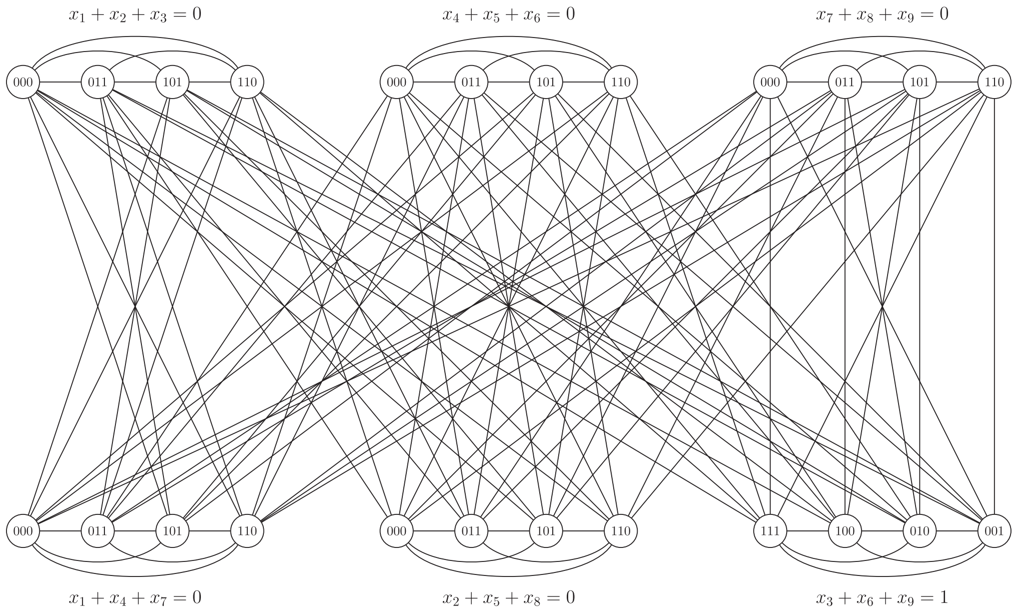

Fig. 2 - B/W online, B/W in print

Figure 2. Graph $\mathcal{G}(\mathcal{F})$.

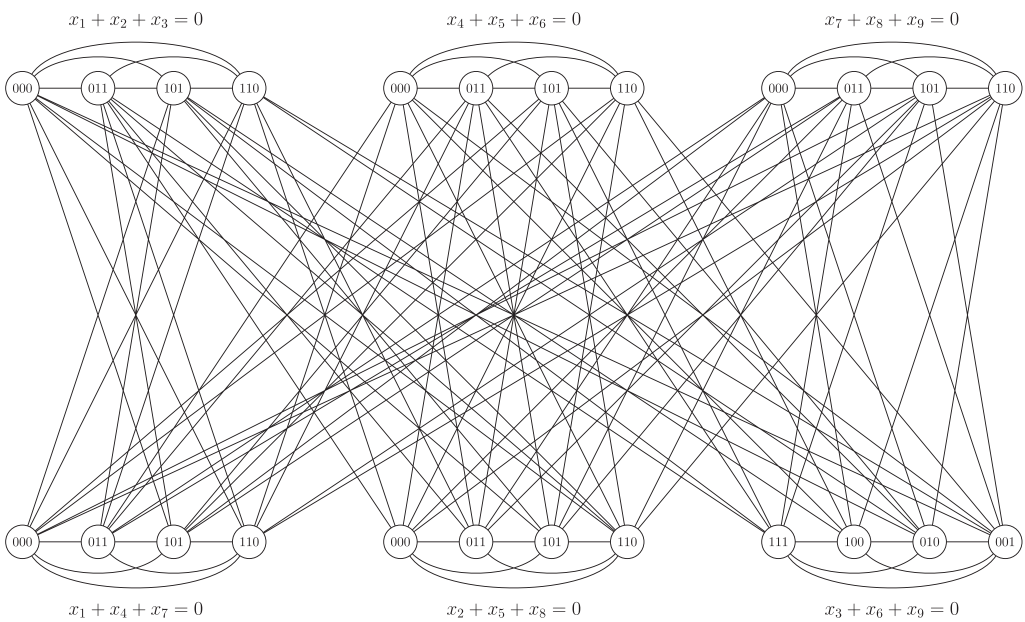

Fig. 3 - B/W online, B/W in print

Figure 3. Graph $\mathcal{G}(\mathcal{F}_0)$.

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4. Concluding remarks

1. In light of the Theorem 3.1, for strongly connected graphs, we can relax the condition on the graph in [8]. In that paper regularity of the underlying graph was assumed to ensure that $\mathrm{Aut}^+(\Gamma)$ belongs to the category $\mathcal{C}_\tau^\Gamma$ (see [8] for notation) for the unique KMS state τ on a strongly connected graph Γ. Now we have for a strongly connected graph Γ (regular or not) with its unique KMS state τ, the category $\mathcal{C}_\tau^\Gamma$ contains $\mathrm{Aut}^+(\Gamma)$.

2. In all the examples considered in this paper, KMS states always occur at the critical inverse temperature. But, in general, it might be interesting to see if some natural symmetry could also fix the inverse temperature. In this context, one can possibly look at the graphs with sink which has a richer supply of KMS states (see [9]).

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