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Asymptotic solution of the conserved phase field system in the fast relaxation case

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The conserved phase field system with a small parameter in the n-dimensional case ($n \leqslant 3$) is considered. An asymptotic solution, describing the free interface dynamics, is constructed and justified. As the small parameter tends to zero, the limiting solution satisfies the modified Stefan problem with corrected Gibbs–Thomson law.

1 Introduction

We shall consider the conserved phase field system [1]

$$\left.\begin{aligned} \partial_t(\theta + l\phi) &= k\Delta\theta + f(x,t), \quad (x,t)\in Q, \\ -\tau_0\,\partial_t\phi &= \xi^2\Delta(\xi^2\Delta\phi + (2a)^{-1}(\phi-\phi^3) + \kappa_1\theta). \end{aligned}\right\} \tag{1}$$

Here $Q = \Omega\times(0,T)$, $\Omega\subset R^n$ is a bounded domain with smooth (C^∞) boundary $\partial\Omega$, $n \leqslant 3$, $T<\infty$; $\partial_t = \partial/\partial t$, Δ is the Laplace operator; θ is the normalized temperature; ϕ is the order functions; $l>0$, $k>0$, κ_1 are constants; $f(x,t)$ is a smooth function; $\tau_0>0$, $\xi>0$, and $a>0$ are parameters. The physical meaning of τ_0, ξ, a, and the whole model (1) for the phase transitions have been discussed elsewhere [1–3]. General models for non-isothermal phase transitions with a conserved order function are proposed in Alt & Pawlow [4]. Nevertheless, the simplest model (1) is of independent interest, since it qualitatively describes the actual physical processes.

We shall study the structure of the solution of (1) and analyse how to pass to the limit from the microscopic description to the macroscopic one. It is well-known that the form of the limiting problem depends on the relations between the parameters a, ξ and τ_0. We restrict our consideration to the case $a \ll 1$, $\xi \ll 1$, $\tau_0 \ll 1$, $\xi a^{-1/2} = \text{const}$, $\tau_0 a^{-1} = \text{const}$. Let us introduce a small parameter $\epsilon\to 0$ and set

$$a = \epsilon/2, \quad \xi = \sqrt{\epsilon}, \quad \tau_0 = \kappa\epsilon, \quad \kappa = \text{const} > 0. \tag{2}$$

For simplicity, we also assume that $k = l = 1$. Completing (1) with natural initial and boundary conditions, we obtain our basic mathematical model

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$$\left.\begin{aligned}
&\partial_t(\theta+\phi) = \Delta\theta + f(x,t),\\
&-\kappa\epsilon\partial_t\phi = \Delta(\epsilon^2\Delta\phi+\phi-\phi^3+\epsilon\kappa_1\theta),\\
&\quad\theta|_{t=0} = \theta^0(x,\epsilon),\quad \phi|_{t=0}=\phi^0(x,\epsilon),\quad \partial_N\theta|_\Sigma = 0,\quad \partial_N\phi|_\Sigma=0,\quad \partial_N\Delta\phi|_\Sigma = 0.
\end{aligned}\right\}\tag{3}$$

Here N is the external normal to $\partial\Omega$, $\partial_N = \partial/\partial N$, and $\Sigma = [0,T]\times\partial\Omega$.

Like the solution of the Cahn–Hilliard equation (the second equation in (3) with $\theta = \text{const}$), the solution of (3) is very complicated, since its behaviour varies depending on different stages of the phase separation process in binary alloys. (For example, the solution can be oscillatory [5], or of soliton type [6], or of the 'tanh'-type [6–10]). At present, the solution to (3) with arbitrary initial data can be analysed in detail only by numerical methods. However, by setting some special initial data, one can thoroughly study certain types of solutions. In this paper we restrict our consideration to the dynamics of the free boundary Γ_t between the pure phases, which, from the macroscopic viewpoint (as $\epsilon\to 0$), at each instant of time occupy the domains Ω_t^-, where $\bar\phi = -1$, and Ω_t^+, where $\bar\phi = +1$, $\Omega = \Omega_t^-\cup\Omega_t^+\cup\Gamma_t$; (here and below $\bar f(x,t) = \lim_{\epsilon\to 0} f(x,t,\epsilon)$). So we shall assume that the initial data $\theta^0(x,\epsilon)$, $\phi^0(x,\epsilon)$ are smooth functions (for $\epsilon>0$) such that $\bar\phi^0 = \pm 1$, $\bar\theta^0 = \theta_0^\pm(x)$ for $x\in\Omega_0^\pm$, where $\theta_0^\pm$ are sufficiently smooth on $\Omega_0^\pm$ functions such that $[\theta_0^\pm]|_{\Gamma_0} = 0$, $[\partial_\nu\theta_0^\pm]|_{\Gamma_0} \neq 0$, and that the initial interface Γ_0 is a sufficiently smooth closed surface of codimension 1 such that $\Gamma_0\cap\partial\Omega = \varnothing$. Here and below $[f]|_{\Gamma_t}$ is the jump of the function f on Γ_t, $\partial_\nu = \partial/\partial\nu$ is the derivative along the normal to Γ_t, $t\geqslant 0$. For definiteness, we assume that $\partial\Omega_t^+\cap\partial\Omega = \partial\Omega$.

The Cahn–Hilliard equation was proposed by Cahn and Hilliard [11–13] as a simple model for the process of phase separation of a binary alloy at a fixed temperature. Surveys of physical aspects of this model are also given in Novick-Cohen & Segel [5]. The equilibrium theory for the Cahn–Hilliard equation in the one-dimensional case in which $\xi^2 \sim a \ll 1$, $\tau_0 = \text{const}$ is investigated in Alikakos *et al.* [14] and Carr *et al.* [15]. The initial value problem for the Cahn–Hilliard equation is explicitly investigated in the case in which τ_0, ξ, and a are constants, $n\leqslant 3$ and $\theta = \text{const}$. For a description of the results from the mathematical and physical viewpoint, the reader is referred elsewhere [10, 16–19], where the existence and uniqueness theorems, as well as the existence of an attractor are proved. The multidimensional Cahn–Hilliard equation with a small parameter has been considered by Pego [9] and Stoth [10]. System (1) has also been considered [1, 2, 6, 8].

Let us discuss the results of Caginalp [2], related to case (2). By using the matching method, the two first terms for the inner expansion have been found in a neighbourhood $\Gamma_t^{\delta_1}$ of the free boundary $\Gamma_t = \{x\in\Omega, r(x,t) = 0\}$, $t\geqslant 0$:

$$\left.\begin{aligned}
\theta &= \bar\theta^\pm|_{\Gamma_t} + \epsilon\{(\epsilon^{-1}r + r^1)(\partial_\nu\bar\theta^+ + \partial_\nu\bar\theta^-)|_{\Gamma_t} + \sqrt{2}\,[\partial_\nu\bar\theta^\pm]|_{\Gamma_t}\\
&\quad\times ln\,(2\cosh((r/\epsilon + r^1)/\sqrt{2})) + \omega_1(r/\epsilon + r^1, s, t) + \bar\theta_1^\pm|_{\Gamma_t}\}/2,\\
\phi &= \tanh((r/\epsilon + r^1)/\sqrt{2}) + \epsilon\,\Phi_1(r/\epsilon + r^1, s, t).
\end{aligned}\right\}\tag{4}$$

The outer expansion in the domains $\Omega^\pm_{t,\delta_2}\subset\Omega_t^\pm$, $t\geqslant 0$ is:

$$\theta = \bar\theta^\pm(x,t) + \epsilon\,\bar\theta_1^\pm(x,t),\quad \phi = \pm 1 + \epsilon\,\bar\phi_1^\pm(x,t).$$

Here we have set $\kappa_1 = 2$, r is the signed distance function, $\delta_i = \delta_i(\epsilon)\to 0$ as $\epsilon\to 0$, $\Gamma_t^{\delta_1}\cap\Omega^\pm_{t,\delta_2}\neq\varnothing$, $r^1 = r^1(x,t)$ is a smooth function, $s = s(x,t)$ are (local) coordinates on Γ_t;

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$\Phi_1(\eta, s, t) = \bar{\theta}^{\pm}|_{\Gamma_t} + \omega_2(\eta, s, t)$; $\omega_i(\eta, s, t)$, $i = 1, 2$, are smooth functions exponentially decreasing as $\eta \to \pm\infty$; $\bar{\theta}^{\pm}(x, t)$ is the solution of the modified Stefan problem

$$\partial_t \bar{\theta}^{\pm} = \Delta \bar{\theta}^{\pm}, \quad t \geqslant 0, \quad x \in \Omega_t^{\pm},$$
$$[\partial_\nu \bar{\theta}^{\pm}]|_{\Gamma_t} = 2v_\nu, \quad \kappa_2 \mathscr{K}_t = \bar{\theta}^{\pm}|_{\Gamma_t}, \quad \bar{\theta}^{\pm}|_{t=0} = \theta_0^{\pm}(x), \quad \partial_N \bar{\theta}^{+}|_{\Sigma} = 0, \tag{5}$$

where v_ν is the normal velocity of the motion of the boundary, $\mathscr{K}_t$ is the mean curvature of the surface $\Gamma_t, \kappa_2 > 0$ is a constant.

Unfortunately, in Caginalp [2] the construction of the matching method is such that an additional summand is omitted in the Gibbs–Thomson condition in (5). The point is that the construction of the first correction to the outer expansion necessarily implies the following equality:

$$\Delta(\bar{\phi}_1^{\pm} - \bar{\theta}^{\pm}) = 0, \quad \text{in} \quad \Omega \backslash \Gamma_t,$$

i.e. $\bar{\phi}_t^{\pm}$ may differ from $\bar{\theta}^{\pm}$ by a function harmonic in $\Omega\backslash\Gamma_t$. However, in Caginalp [2], this condition was replaced by the following:

$$\bar{\phi}_1^{\pm} = \bar{\theta}^{\pm}. \tag{6}$$

Let us discuss the consequence of this change of conditions. By using the method of Caginalp [2], we calculate the second-order corrections in the domains Ω_t^* (in which the expansions are matched) common for the outer and inner expansion; for the order-function we obtain the outer expansion:

$$\phi = \pm 1 + \epsilon \bar{\theta}^{\pm}(x, t) + \epsilon^2 \bar{\phi}_2^{\pm}(x, t), \quad t > 0, \quad x \in \Omega_t^* \cap \Omega_t^{\pm}, \tag{7}$$

and the inner expansion in the same domain:

$$\phi = \pm 1 + \epsilon \bar{\theta}^{\pm}|_{\Gamma_t} + \epsilon^2 (r/\epsilon + R_{\pm})(C_1 \pm 2^{-5/2} \kappa [\partial_\nu \bar{\theta}^{\pm}]|_{\Gamma_t}). \tag{8}$$

Here $C_1|_{\Gamma_t}$ is an arbitrary function.

The matching conditions for (7) and (8) necessarily imply the relation

$$\kappa 2^{-3/2} [\partial_\nu \bar{\theta}^{\pm}]|_{\Gamma_t} = [\partial_\nu \bar{\theta}^{\pm}]|_{\Gamma_t},$$

which holds only for $\kappa = 2\sqrt{2}$. Thus, in the general case, it is impossible to match the external and internal expansions. This fact follows from the construction presented in Caginalp [2]. In this construction it is assumed that the solution may grow with respect to the fast (internal) variable, which means that the next terms of the expansion contribute to the preceding terms. In fact, by (4) we have $\theta = \bar{\theta}^{\pm}|_{\Gamma_t} + r \partial_\nu \bar{\theta}^{\pm}|_{\Gamma_t} + \mathscr{O}(\epsilon)$ for $t \geqslant 0$ and $x \in \Omega_t^* \cap \Omega_t^{\pm}$, where r is of order $\mathscr{O}(\epsilon^{\mu})$, $1/2 \leqslant \mu < 1$; thus the second summand cannot be considered as a remainder of order $\mathscr{O}(\epsilon)$. Moreover, for the derivative of θ, we obtain that $\partial_\nu \theta = \partial_\nu \bar{\theta}^{\pm}|_{\Gamma_t} + \mathscr{O}(\epsilon)$ for all $t \geqslant 0$ and $x \in \Omega_t^* \cap \Omega_t^{\pm}$ if and only if in (4) we take into account the 'small correction' in curly brackets. In Caginalp [2] this reasoning resulted in the correct Stefan condition in (5) by taking into account the first correction in (4). However, in Caginalp [2] the second corrections were not considered, though there are similar relations for ϕ:

$$\phi = \pm 1 + \epsilon(\bar{\theta}^{\pm}|_{\Gamma_t} + r C^{\pm}|_{\Gamma_t}) + \mathscr{O}(\epsilon^2), \quad t \geqslant 0, \quad x \in \Omega_t^* \cap \Omega_t^{\pm},$$
$$\partial_\nu \phi = \epsilon C^{\pm}|_{\Gamma_t} + \mathscr{O}(\epsilon^2), \quad C^{\pm} = C_1 \pm 2^{-5/2} \kappa [\partial_\nu \bar{\theta}^{\pm}]|_{\Gamma_t}.$$

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We shall now derive the Gibbs–Thomson condition. It appears when we calculate the first correction Φ_1, i.e. when we solve the equation

$$\partial_\eta^2 \Phi_1 + (1-3\chi^2)\,\Phi_1 = -\mathscr{K}_t\,\chi_\eta - 2\bar{\theta}^{\pm}|_{\Gamma_t} + c,$$

where $\chi = \tanh(\eta/\sqrt{2})$, c is a 'constant' of integration, here and below $f_\eta = \partial_\eta f = \partial f/\partial\eta$. We rewrite Φ_1 as $\Phi_1 = \bar{\phi}_1^{\pm}|_{\Gamma_t} + \omega_2(\eta, t)$, where $\omega_2 \to 0$ as $\eta \to \pm\infty$ (see formula (4.25) in Caginalp [2]). Then $c = 2(\bar{\theta}^{\pm} - \bar{\phi}_1^{\pm})|_{\Gamma_t}$. For this equation to hold, we have the necessary and sufficient condition

$$\int_{-\infty}^{\infty} (3(\chi^2-1)\,\bar{\phi}_t^{\pm}|_{\Gamma_t} - \mathscr{K}_t\,\chi_\eta)\,\chi_\eta)\,\mathrm{d}\eta = 0,$$

which implies $\kappa_2\,\mathscr{K}_t = \bar{\phi}_1^{\pm}|_{\Gamma_t}$. The Gibbs–Thomson condition is derived in Caginalp [2] by using relation (6). However, as shown above, we can match the lower terms of the asymptotic expansion if the function $\bar{\phi}_1^{\pm}$ differs from $\bar{\theta}^{\pm}$ by a certain function $\Phi^{\pm}$ harmonic in $\Omega\backslash\Gamma_t$. Since, in the general case, $\Phi^{\pm}|_{\Gamma_t}$ does not vanish, some additional summands appear on the right-hand side of the Gibbs–Thomson condition. Below we will formulate the Gibbs–Thomson condition and set the problem for obtaining the additional function $\Phi^{\pm}$.

The existence of polynomial summands in the inner expansion necessarily requires a very accurate construction of low-order terms, that is, on the ith step of the asymptotic procedure we must take into account the $(i+1)$th corrections to the solution. This fact is well-known. A description of this accurate use of the matching method can be found, for example, in Il'in [20].

At the same time, there is an alternative method [21–24] for constructing asymptotic solutions with localized 'fast' variations. This method, in which a detailed analysis of lower-order terms is not required, is a modification of the two-scale method for obtaining solutions with localized 'fast' variations. A modification of this method for phase transition problems was proposed elsewhere [6–8, 25, 26]. The main idea of this method is to construct such analogues of the inner and outer expansions that they are defined over the whole range of independent variables. This implies that there are no summands polynomially increasing in the 'fast' variable.

Here we have the following important points. Suppose the 'fast' variation of the solution u is localized in a small neighbourhood of the smooth surface Γ_t, for example,

$$u = A(x,t)\tanh(S(x,t)/\epsilon), \quad \Gamma_t = \{x\in\Omega, S(x,t) = 0\}, \quad S, A \in C^\infty.$$

Then, to calculate the function S at each instant of time, it is sufficient to define only its zero surface $\{x, \psi(x) = t\} \equiv \{x, S(x,t) = 0\}$ and the first normal derivative S' on Γ_t, since in an ϵ-neighbourhood of Γ_t the smooth function S cannot vary more than $\mathscr{O}(\epsilon)$ and outside an $\epsilon^{1-\delta}$-neighbourhood of $\Gamma_t\,(0<\delta<1)$, the solution varies slowly with precision up to $\mathscr{O}(\epsilon^\infty)$. Furthermore, in the traditional two-scale method we first construct a rapidly varying solution $u(S/\epsilon, x, t)$ in the 'extended' space $R^1\times\Omega\times[0,T]$, assuming that $u = u(\eta, x, t)$ and that the variables η and x, t are independent. Then we calculate the trace $u(\eta, x, t)|_{\eta=S/\epsilon}$. This approach is justified for solutions with rapid oscillations everywhere. However, for solutions with localized fast variation we can construct the rapidly varying components only in the subspace $R^1\times\mathscr{T}_T^\delta$, where $\mathscr{T}_T^\delta$ is an $\epsilon^{1-\delta}$-neighbourhood of the surface $\mathscr{T}_T = \{(x,t)\in\bar{Q}, t = \psi(x)\} = \bigcup_{t\in[0,T]}\Gamma_t$. That is, we know beforehand that our final goal is only

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the trace $u(\eta, x, t)|_{\eta=S/\epsilon}$. In its turn, since $u(\eta, x, t)$ smoothly depends on 'slowly' varying variables x and t, we can calculate the solution in two stages: in the first stage we define the trace $\check{u} = u(\eta, x, t)|_{t=\psi(x)}$ of u on the section $R^1 \times \mathscr{T}_T$; in the second stage we construct a sufficiently smooth continuation of $\check{u}$ outside $R^1 \times \mathscr{T}_T$. Needless to say, this continuation must be constructed sufficiently accurately, but there is some freedom in choosing this continuation. Namely, this freedom provides the boundedness (uniformly in $\eta \in R^1, x, t \in Q$) of all terms of the asymptotic expansion.

Let us return to the problem concerning the choice of the initial data to problem (3). The limiting functions $\theta_0^{\pm}(x)$ may be taken arbitrarily: they must satisfy only the natural matching conditions for the limiting initial-boundary value problem. Nevertheless, we cannot choose the regularization $\theta^0(x, \epsilon)$, $\phi^0(x, \epsilon)$ arbitrarily if we want to stay in the chosen class of prelimiting solutions. It is well-known that the problem of the dynamics of the free boundary is described by the Van der Waals-type solution

$$\phi^0(x, \epsilon) = \tanh(r_0(x)/\epsilon\sqrt{2}) + \epsilon\,\phi_1^0(x, \epsilon), \tag{9}$$

where r_0 is the signed distance function, $r_0 > 0$ in Ω_0^+ and $r_0 < 0$ in Ω_0^-, ϕ_1^0 is a function bounded in C. The tanh-type form of the solution (in the principal term) appears in the phase field system (for example, [16, 25, 27–31]), for the Cahn–Hilliard equation [5, 9, 10, 14, 16, 18], and for the conserved phase field system [2, 7, 8] not only when we construct the asymptotic solution, but also when we prove the solvability of the initial-boundary value problem with a small parameter [27, 28, 30, 31]. Such a universal form of the principal term of the solution is related to the fact that the chemical potential (the energy of the phase field system)

$$\mathscr{F}[\phi] = \int_\Omega \left\{\frac{1}{2}\epsilon|\nabla\phi|^2 + \frac{1}{4\epsilon}(\phi^2-1)^2\right\} \mathrm{d}x$$

takes its minimum at the hyperbolic tangent function; $\mathscr{F}[\tanh(r_0/\epsilon\sqrt{2})] \leqslant \text{const}$ as $\epsilon \to 0$. Hitherto, the problem about the degree of freedom in choosing ϕ_1^0 and the problem of how to smooth the temperature has remained unsolved. The asymptotic analysis of the phase field system shows that $\theta^0(x, \epsilon)$ can be arbitrary outside an ϵ-neighbourhood of Γ_0, and that this function must be of a fixed form inside this neighbourhood; thus the choice of $\phi_1^0(x, t)$ must be matched with the choice of $\theta^0(x, \epsilon)$. The latter fact corresponds to the relation between θ and ϕ arising when we consider the free energy

$$\mathscr{F}_\theta[\phi] = \int_\Omega \left\{\frac{1}{2}\epsilon|\nabla\phi|^2 + \frac{1}{4\epsilon}(\phi^2-1)^2 - \kappa_1\,\theta\phi\right\} \mathrm{d}x.$$

However, the analysis of the solution of the phase field system [27, 28] shows that we stay in the sharp-fronted situation, if we add sufficiently small perturbations to the fixed initial data θ_{fix}^0, ϕ_{fix}^0, i.e. if

$$\|\theta^0 - \theta_{\text{fix}}^0; L^2(\Omega)\| + \|\phi^0 - \phi_{\text{fix}}^0; L^2(\Omega)\| \leqslant C\,\epsilon^{n/2}. \tag{10}$$

Therefore, the last bound describes the stability domain for the tanh-type solution of the phase field system. We note that a condition close to (10) was also obtained in Soner [30], although quite different methods were used in Soner [30] and Omel'yanov *et al.* [27, 28]. Nevertheless, the question of whether condition (10) is optimal remains open.

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In the present paper we show how to choose the initial data for the conserved phase field system: by constructing an asymptotic solution, we find some fixed initial data $\theta^0 = \theta^0_{\rm fix}(x,\epsilon), \phi^0 = \phi^0_{\rm fix}(x,\epsilon)$ and, justifying the asymptotics, we obtain the stability domain for our solution.

The main result of the paper is the derivation of the limiting problem in $C(\bar{Q})$ as $\epsilon \to 0$ for a sequence of tanh-type solutions of (3). This problem, namely, the 'superposition' of the modified Stefan problem for the limiting temperature $\bar{\theta}(x,t)\,(\bar{\theta} = \theta^{\pm}$ as $x \in \Omega^{\pm}_t, t \geqslant 0)$ and of the Mullins–Sekerka problem for an auxiliary function $\bar{\Phi}(x,t)\,(\bar{\Phi} = \Phi^{\pm}$ as $x \in \Omega^{\pm}_t$, $t \geqslant 0)$ has the form

$$\partial_t \theta^{\pm} = \Delta\theta^{\pm} + f(x,t), \quad \Delta\Phi^{\pm} = 0, \quad x \in \Omega^{\pm}_t, \quad t > 0, \tag{11}$$

$$\left.\begin{aligned} &\theta^{\pm}{}_{t=0} = \theta^{\pm}_0(x), \quad x \in \Omega^{\pm}_0, \quad \partial_N \theta^{+}|_{\Sigma} = 0, \quad \partial_N \Phi^{+}|_{\Sigma} = 0, \\ &[\theta^{\pm}]|_{\Gamma_t} = 0, \quad [\Phi^{\pm}]|_{\Gamma_t} = 0, \end{aligned}\right\} \tag{12}$$

$$[\partial_\nu \theta^{\pm}]|_{\Gamma_t} = -2v_\nu, \quad [\partial_\nu \Phi^{\pm}]|_{\Gamma_t} = -\kappa v_\nu, \tag{13}$$

$$\operatorname{div}(|\nabla\psi|^{-1}\nabla\psi) = \kappa_2(2^{-1}\kappa_1\check{\theta}^{\pm} + \check{\Phi}^{\pm}), \quad \psi|_{\Gamma_0} = 0. \tag{14}$$

Here $\Gamma_t = \{x \in \Omega, \psi(x) = t\}$ is the interface at the time instant t; $v_\nu = -1/|\nabla\psi|$ is the normal velocity of motion of Γ_t; $\Omega = \Omega^+_t \cup \Omega^-_t \cup \Gamma_t$; $\check{F} = F^{\pm}(x,\psi(x))$; $\partial_\nu = -(\nabla\psi/|\nabla\psi|, \nabla)$, $\kappa_2 = 3\sqrt{2}$. It is easy to see that the left-hand side in (14) is the mean curvature $\mathscr{K}_t$ of Γ_t. The uniqueness of the functions $\Phi^{\pm}$ follows from the normalization condition

$$\int_{\Omega^+_t}\left(\Phi^+ + \frac{\kappa_1}{2}\theta^+\right)\mathrm{d}x + \int_{\Omega^-_t}\left(\Phi^- + \frac{\kappa_1}{2}\theta^-\right)\mathrm{d}x = K_1,$$

where K_1 is the first coefficient of the expression

$$\int_{\Omega}\phi^0(x,\epsilon)\,\mathrm{d}x = \sum_{i \geqslant 0}\epsilon^i K_i.$$

The following theorem (which can be proved in a way similar to Radkevich [26]) implies the classical solvability of problem (11)–(14) (in the Hölder space C^l).

Theorem 1 *Let the initial* Γ_0 *belong to the class* C^{l_0+2} *and let* $\theta^{\pm}_0 \in C^{l_0}(\Omega^{\pm}_0)$, *where* $l_0 > 2m$ *is a non-integer number and* $m > 2$. *Suppose that* $\operatorname{dist}(\Gamma_0, \partial\Omega) \geqslant \text{const} > 0$ *and all matching conditions for problem* (11)–(14) *hold with precision up to order m. Suppose*

$$\Phi^{\pm}_0 = \Phi^{\pm}|_{t=0} \in C^{l_0}(\Omega^{\pm}_0)$$

satisfies the problem: $\Delta\Phi^{\pm}_0 = 0$ *in* $\Omega^{\pm}_0$, $\partial_N \Phi^{+}_0 = 0$ *on* $\partial\Omega$ *and* $[\Phi^{\pm}_0]|_{\Gamma_0} = 0$, $\kappa_1[\partial_\nu \Phi^{\pm}_0]|_{\Gamma_0} = 2[\partial_\nu \theta^{\pm}_0]|_{\Gamma_0}$.

$$\partial_\nu \theta^{\pm}_0|_{\Gamma_0} \geqslant \Lambda_0(\Gamma_0) \quad \textit{and} \quad (\partial_\nu \Phi^+_0 + \partial_\nu \Phi^-_0)|_{\Gamma_0} > 0,$$

where Λ_0 *is a sufficiently large constant depending on the principal curvatures of the surface* Γ_0. *Then there exists a sufficiently small number* $T > 0$ *such that on the time interval* $(0, T)$

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there exists a classical solution of problem (11)–(14), $\Phi^{\pm}$, $\theta^{\pm} \in C^{l,l/2}(Q_T^{\pm})$, *where the non-integer number* l *satisfies the condition* $[l/2] = m-2$, $[l/2]$ *is the integer part of* $l/2$, $Q_T^{\pm} = \{x \in Q_t^{\pm}, t \in (0,T)\}$.

2 Asymptotic solution

Let us consider the general method for constructing the asymptotic solution of problem (3) with an arbitrary precision. We shall need some classes of functions. Let $\mathscr{S} = S(R^1_\eta; C^\infty(\bar{\Omega}))$, where S is the Schwartz space, $\mathscr{H} = \{f(\eta, x, t) \in C^\infty(R^1 \times \bar{Q}), \partial_\eta f \in \mathscr{S}\}$, and let $\mathscr{P} = \{f(\tau, x', t) \in C^\infty(R^1_+ \times \Sigma)\}$ be the class of exponentially vanishing (w.r.t. τ) boundary–layer functions.

Let us note that, for the Van der Waals-type solution, the limit of ϕ as $\epsilon \to 0$ is a Heaviside function discontinuous on Γ_t and the limit of θ as $\epsilon \to 0$ has a weak discontinuity on Γ_t. This implies that the asymptotic solution should be taken to have the following form:

$$\left.\begin{aligned} \theta(x,t,\epsilon) &= \vartheta^M(x,t,\epsilon) + \mathscr{V}^M\left(\frac{S(x,t)}{\epsilon}, \frac{x_N}{\epsilon}, x, t, \epsilon\right), \\ \phi(x,t,\epsilon) &= \epsilon\, \Phi^M(x,t,\epsilon) + \mathscr{W}^M\left(\frac{S(x,t)}{\epsilon}, \frac{x_N}{\epsilon}, x, t, \epsilon\right), \end{aligned}\right\} \tag{15}$$

where the functions ϑ^M, Φ^M (uniformly smooth w.r.t. $\epsilon \in [0,1]$, $x, t \in Q$, the so-called 'regular part') give an analogue of the outer expansion:

$$\vartheta^M(x,t,\epsilon) = \sum_{j=0}^{M} \epsilon^j \theta_j(x,t), \quad \Phi^M(x,t,\epsilon) = \sum_{j=1}^{M} \epsilon^{j-1} \phi_j(x,t);$$

the functions $\mathscr{V}^M$, $\mathscr{W}^M$ (rapidly varying near the free and external boundaries) give an analogue of the inner expansion:

$$\begin{aligned} \mathscr{V}^M(\eta, \tau, x, t, \epsilon) &= \rho_0(x,t)\, V_0(\eta, x, t) \\ &\quad + \sum_{j=1}^{M} \epsilon^j \{\rho_j(x,t)\, V_j(\eta,x,t) + U_j(\eta,x,t) + Y_j(\tau,x',t)\}, \\ \mathscr{W}^M(\eta, \tau, x, t, \epsilon) &= \chi(\eta,x,t) \\ &\quad + \sum_{j=1}^{M} \epsilon^j \{\gamma_j(x,t)\, G_j(\eta,x,t) + W_j(\eta,x,t) + Z_j(\tau,x',t)\}. \end{aligned}$$

Here x_N is the distance from a point $x \in \Omega$ to a point $x' \in \partial\Omega$ along the interior normal; S, ρ_i, γ_j, θ_i, $\phi_j \in C^\infty(\bar{Q})$, $\partial_t S|_{\Gamma_t} \neq 0$; $Y_j(\tau, x', t)$, $Z_j(\tau, x', t) \in \mathscr{P}$; the other functions belong to the space $\mathscr{H}$. We assume that

$$\chi^{\pm} = \pm 1, \quad U_j^+ = 0, \quad W_j^+ = 0, \quad \rho_i|_{\Gamma_t} = 0, \quad \gamma_j|_{\Gamma_t} = 0, \tag{16}$$

where $\Gamma_t = \{x \in \Omega, S(x,t) = 0\}$, $i = 0, 1, \ldots, M$, $j = 1, 2, \ldots, M$. Here and below

$$f^{\pm} = \lim_{\eta \to \pm\infty} f(\eta, x, t) \quad \text{for each} \quad f \in \mathscr{H}.$$

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The definition of the classes $\mathscr{H}$, $\mathscr{P}$ and the above assumptions imply that χ is a regularization of the Heaviside function, the fast variation of which is localized in an ϵ-neighbourhood of the free boundary; Y_j, Z_j, $j = 1, 2, \ldots, M$, are boundary-layer functions localized near the external boundary. For convenience, the functions $\rho_i V_i \in \mathscr{H}$, $i = 0, 1, \ldots, M$ and $\gamma_j G_j \in \mathscr{H}$, $j = 1, 2, \ldots, M$ are grouped as separate summands, since these functions have an additional property: $\rho_i V_i$ and $\gamma_j G_j$ vanish on the free boundary. Thus, in a small neighbourhood of Γ_t the functions $\rho_i V_i$ and $\gamma_j G_j$ present a regularization of the weakly discontinuous functions of the form $f(\eta) = a^+\eta_+ + a^-\eta_-$, where $\eta_+ = \eta$ for $\eta \geqslant 0$ and $\eta_+ = 0$ for $\eta \leqslant 0$, $\eta_- = \eta - \eta_+$, $a^+ \neq a^-$. Obviously, we have $f(\eta = +0) = f(\eta = -0) = 0$, $[\partial f_\eta]|_{\eta=0} = a^+ - a^- \neq 0$. Furthermore, outside small neighbourhoods of the free and external boundaries, i.e. as $\eta \to \pm\infty$, and $\tau \to \infty$ we have $\theta_j + \rho_j V_j + U_j + Y_j \asymp \theta_j + \rho_j V_j^{\pm} + U_j^{\pm}$. Since all the functions in the latter relation are arbitrary for the moment, we can redefine θ_j (for example, by setting $\theta_j := \theta_j + U_j^+$ or $\theta_j := \theta_j + U_j^-$), and thus set one of the limiting values U_j^+ or U_j^- equal to zero. For definiteness, we set $U_j^+ = W_j^+ = 0$. Without loss of generality ([22]), we also assume that $S = t - \psi(x)$, $\chi = \chi(\eta, x)$ and

$$\left.\begin{aligned} &V_j(\eta, x, t) = \alpha_j^+(x, t) + \alpha_j^-(x, t)\,\chi, \quad G_j(\eta, x, t) = \mu_j^+(x, t) + \mu_j^-(x, t)\,\chi, \\ &2\alpha_j^{\pm}(x, t) = V_j^+(x, t) \pm V_j^-(x, t), \quad 2\mu_j^{\pm}(x, t) = G_j^+(x, t) \pm G_j^-(x, t), \end{aligned}\right\} \tag{17}$$

since we can add the soliton-type functions contained in V_j and G_j into U_j and W_j, respectively, $j = 1, 2, \ldots, M$.

First, let us obtain the regular terms of expansion (15). Substituting (15) in (3) and passing to the limit as $\eta \to \pm\infty$, $\tau \to \infty$, $\epsilon \to 0$, we obtain

$$\Delta((1 - (\chi^{\pm})^2)\,\chi^{\pm}) = 0,$$

which does not contradict (16). Let us introduce the notation

$$\left.\begin{aligned} &\theta^{\pm} = \theta_0 + \rho_0 V_0^{\pm}, \quad \theta_j^{\pm} = \theta_j + \rho_j V_j^{\pm} + U_j^{\pm}, \\ &\Phi^{\pm} = \Phi_1^{\pm} - \theta^{\pm}/2, \quad \Phi_j^{\pm} = \phi_j + \gamma_j G_j^{\pm} + W_j^{\pm}, \quad j = 1, 2, \ldots, M. \end{aligned}\right\} \tag{18}$$

Obviously, the functions $\theta^{\pm}$, Φ^{+} satisfy (12) if the assumptions (16) are satisfied and, moreover, $W_1^- = 0$. Further, substituting (15) into (3), passing to the limit as $\eta \to \pm\infty$, $\tau \to \infty$, and setting the terms of the order $\mathcal{O}(\epsilon^j)$ equal to zero, we obtain the relations

$$\left.\begin{aligned} &(\partial_t - \Delta)\,\theta^{\pm} = f(x, t), \quad \Delta(2\Phi_1^{\pm} - \kappa_1\,\theta^{\pm}) = 0, \\ &(\partial_t - \Delta)\,\theta_k^{\pm} = f_{k,\theta}^{\pm}(x, t), \quad \Delta(2\Phi_k^{\pm} - \kappa_1\,\theta_{k-1}^{\pm}) = f_{k,\phi}^{\pm}(x, t). \end{aligned}\right\} \tag{19}$$

Here $f_{k,\theta}^{\pm}(x, t)$, $f_{k,\phi}^{\pm}(x, t)$ are functions of $\theta^{\pm}, \Phi_1^{\pm}, \ldots, \Phi_{k-1}^{\pm}$ and their derivatives. In particular, $f_{1,\theta}^{\pm} = -\partial_t \Phi_1^{\pm}$, $f_{2,\phi}^{\pm} = \kappa\,\partial_t \Phi_1^{\pm} \mp 3\Delta(\Phi_1^{\pm})^2$.

It is clear that the first two equations in (19) lead to equations (11) with respect to functions $\theta^{\pm}$ and $\Phi^{\pm}$.

Now we note that the supports of fast variations of the boundary-layer functions and of functions rapidly varying on a neighbourhood of Γ_t do not intersect (up to terms $\mathcal{O}(\epsilon^{\infty})$). Hence the solution in a neighbourhood of Γ_t and in a neighbourhood of $\partial\Omega$ is constructed differently.

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Let us consider a neighbourhood of Γ_t. Substituting (15) into (3), passing to the limit as $\tau\to\infty$, and setting the terms of the order $\mathcal{O}(\epsilon^{-2+k})$ equal to zero, we have obtained the chain of relations. The first has the form

$$(2\beta^2)^{-1}\partial^2_\eta\check{\chi}+\check{\chi}-\check{\chi}^3=c, \quad \check{\chi}\to 1 \quad \text{as} \quad \eta\to\infty.$$

Here and below we write $\check{F}=F(\eta,x,t)|_{t=\psi(x)}$, $\beta=1/\sqrt{2}|\nabla\psi|$. It can be easily shown that if and only if the constant of integration c is equal to zero, this equation has a solution such that $\check{\chi}\in\mathscr{H}$ and $\check{\chi}^+\neq\check{\chi}^-$. It follows that the solution on $\mathscr{T}_t$ has the Van der Waals-type form $\check{\chi}(\eta,x)=\tanh(\beta(\eta\text{-}\psi_1))$, where another 'constant' of integration $\psi_1=\psi_1(x)$ is a function from $C^\infty(\bar{\Omega})$.

Let us extend $\check{\chi}$ (defined on the section $(x,t)\in\mathscr{T}_T=\bigcup_{t\in[0,T]}\Gamma_t$, $\eta\in R^1$) by the identity to $\chi\equiv\check{\chi}(\eta,x)$ for all $(x,t)\in\bar{Q}$, $\eta\in R^1$. Now we note that $\partial_\eta\mathscr{V}^{\mathscr{M}}=\mathcal{O}(\epsilon)$. Thus, setting the terms $\mathcal{O}(\epsilon^{-2+k})$ equal to zero, we get

$$\partial^2_\eta\check{U}_k=\check{F}^\theta_k, \quad \check{U}_k\to 0 \quad \text{as} \quad \eta\to\infty, \tag{20}$$

$$\partial^2_\eta\hat{L}\check{W}_k=\check{F}^\phi_k, \quad \check{W}_k\to 0 \quad \text{as} \quad \eta\to\infty. \tag{21}$$

Here $\hat{L}=(2\beta^2)^{-1}\partial^2_\eta+1-3\chi^2$, $\check{F}^\phi_k$, $\check{F}^\theta_k$ are functions of $\theta_0, V_0, \ldots, U_{k-1}, W_{k-1}$ and of their derivatives at the point $t=\psi(x)$, $k=1,2,\ldots,M$. In particular

$$\check{F}^\phi_1=\partial^2_\eta\{\hat{\Pi}\,\partial_\eta\chi+3\chi^2\phi_1\}|_{t=\psi}, \tag{22}$$

$$\check{F}^\theta_1=2\beta^2\{2(\nabla\psi,\nabla\rho_0)\,\partial_\eta V_0+\partial_\eta\chi-\eta(2\beta^2)^{-1}\partial_t\rho_0\,\partial^2_\eta V_0\}|_{t=\psi}. \tag{23}$$

It is not too difficult to prove the following statement [6, 22]:

Lemma 1 *The solutions* $\check{U}_k, \check{W}_k\in\mathscr{H}$ *of* (20), (21) *exist if and only if* $\check{F}^\theta_k\in\mathscr{S}$, $\check{F}^\phi_k\in\mathscr{S}$, *and*

$$\left.\begin{aligned} &\int_{-\infty}^{\infty}\check{F}^\theta_k\,d\eta=0, \quad \int_{-\infty}^{\infty}\check{F}^\phi_k\,d\eta=0, \quad \int_{-\infty}^{\infty}\check{f}^\phi_k\,\partial_\eta\chi\,d\eta=0,\\ &\check{f}^\phi_k=\int_{-\infty}^{\eta}\int_{-\infty}^{\eta'}\check{F}^\phi_k(\eta'',x)\,d\eta''\,d\eta'-\int_{-\infty}^{\infty}\int_{-\infty}^{\eta'}\check{F}^\phi_k(\eta'',x)\,d\eta''\,d\eta'.\end{aligned}\right\} \tag{24}$$

where

By (19), $\check{U}_k, \check{W}_k\in\mathscr{H}$ for all $k\geqslant 1$. Let us consider conditions (24) for $k=1$. Using (22), (23), it is easy to see that the second condition in (24) holds automatically. The first condition in (24) leads to the relation

$$(2(\nabla\psi,\nabla\rho_0)+(2\beta^2)^{-1}\partial_t\rho_0)|_{t=\psi}[\check{V}_0]+2=0, \tag{25}$$

where $[V]=V^+-V^-$. Note that $\nabla\rho|_{\Gamma_t}=-\rho_t\nabla\psi|_{\Gamma_t}$ for any smooth function ρ such that $\rho|_{\Gamma_t}=0$. Hence, (25) implies

$$(\nabla\psi,\nabla\rho_0)|_{\Gamma_t}[\check{V}_0]=-2. \tag{26}$$

Since $\theta_0\in C^\infty(\bar{Q})$ and $\rho_0|_{\Gamma_t}=0$, we get the first condition (13).

Furthermore, since $\check{f}^\phi_1=\hat{\Pi}\,\partial_\eta\chi+3\check{\phi}_1(\chi^2-1)$, after simple calculations we obtain that the third condition (24) for $k=1$ is equivalent to (14).

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Now, since (24) are satisfied for $k=1$, we can obtain the functions

$$\check{U}_1 = \check{\zeta}_1^+(x) + \check{\zeta}_1^-(x)\,\chi + u_1(\eta, x), \quad \check{W}_1 = \omega_1(\eta, x).$$

Here

$$\check{\zeta}_1^+ = -\check{\zeta}_1^- = 2\beta^2\psi_1(x), \quad u_1 = \beta^{-1}(\xi\chi - \ln(2\cosh\xi))|_{\xi=\beta(\eta+\psi_1)} \in \mathscr{S},$$
$$\omega_1(\eta, x) = \omega_{1,1}(\eta, x) + \psi_2(x)\,\chi_\eta(\eta, x) \in \mathscr{S},$$

ψ_2 is the 'constant' of integration. Therefore, the functions W_1^+ are actually equal to zero, so we obtain the continuity conditions (11).

Let us define the extensions $\rho_0 V_0$, W_1, $\gamma_1 G_1$

$$\left.\begin{aligned} \rho_0 V_0 &= \rho_0 \alpha_0^+(x,t) + \rho_0 \alpha_0^-(x,t)\,\chi(\eta, x), \\ \gamma_1 G_1 + W_1 &= \gamma_1 \mu_1^+(x,t) + \gamma_1 \mu_1^-(x,t)\,\chi(\eta,x) + \omega_1(\eta,x). \end{aligned}\right\} \tag{27}$$

Here

$$\rho_0\alpha_0^+ = (\theta_c^+ - 2\theta_0 + \theta_c^-)/2, \quad \rho_0\alpha_0^- = (\theta_c^+ - \theta_c^-)/2, \quad \gamma_1\mu_1^+ = (\Phi_{1c}^+ - 2\phi_1 + \Phi_{1c}^-)/2,$$
$$\gamma_1\mu_1 = (\Phi_{1c}^+ - \Phi_{1c}^-)/2, \quad \theta_c^\pm,\ \Phi_{1c}^\pm$$

are sufficiently smooth extensions of $\theta^\pm$, $\Phi_1^\pm = \kappa_1\theta^\pm/2 + \Phi^\pm$ in $\Omega_t^\pm \cup \Gamma_{t,\delta}^\mp$, such that the heat equation and the Laplace equation are satisfied, respectively. Here $0 < \delta \ll 1$ is an arbitrary number. Hence, we have obtained the formulas for the first terms of the asymptotic expansion

$$\left.\begin{aligned} \theta_0^{\mathrm{as}} &= (\theta_c^+(x,t) + \theta_c^-(x,t))/2 + (\theta_c^+(x,t) - \theta_c^-(x,t))\tanh(\beta(\eta+\psi_1(x)))/2, \\ \phi_1^{\mathrm{as}} &= \tanh(\beta(\eta+\psi_1(x))) + \epsilon\{\kappa_1\theta_0^{\mathrm{as}} + (\Phi_c^+(x,t) + \Phi_c^-(x,t)) \\ &\quad + (\Phi_c^+(x,t) - \Phi_c^-(x,t))\tanh(\beta(\eta+\psi_1(x))) + 2\omega_1(\eta,x)\}/2, \end{aligned}\right\} \tag{28}$$

where $\eta = (t - \psi(x))/\epsilon$, $\omega_1 \in \mathscr{S}$. Note that outside Γ_t we have

$$\theta = \theta^\pm + [(\theta_c^+ - \theta_c^-)\,\mathscr{O}(e^{\mp\beta\eta})] + \mathscr{O}(\epsilon), \quad x \in \Omega_t^\pm,$$
$$\phi = \chi + \epsilon\{\Phi_1^\pm + [(\Phi_{1c}^+ - \Phi_{1c}^-)\,\mathscr{O}(e^{\mp\beta\eta})] + \omega_1\} + \mathscr{O}(\epsilon^2).$$

This implies that the expressions in square brackets are of maximal value $\mathscr{O}(\epsilon)$ in an $\epsilon^{1-\mu}$-neighbourhood of Γ_t, $0 < \mu < 1$, and they are exponentially small outside this neighbourhood. Hence the freedom in choosing the extensions $\theta_c^\pm$, $\Phi_c^\pm$ results in corrections of order $\mathscr{O}(\epsilon)$ which automatically are taken into account when we construct the next approximations.

Let us consider equations (20), (21) in the case $k=2$. The right-hand sides of these equations have the form

$$\left.\begin{aligned} \check{F}_2^\theta &= 2\beta^2\{|\nabla\psi|^2\,\partial_t\rho_1\eta\,\partial_\eta^2 V_1 - 2(\nabla\psi, \nabla\rho_1)\,\partial_\eta V_1 - \psi_2\,\partial_\eta^2\chi + f_2^\theta\}|_{t=\psi}, \\ \check{F}_2^\phi &= \{\{2\beta^2\hat{\Pi} - \eta\,\partial_\eta\partial_t\}\,\partial_\eta(\hat{L}(W_1 + \gamma_1 G_1) - 3\phi_1\chi^2 \\ &\quad - \hat{\Pi}\,\partial_\eta\chi + g(\theta_0 + \rho_0 V_0)) - 2\kappa\beta^2\,\partial_\eta\chi + \partial_\eta^2 f_2^\phi\}|_{t=\psi}. \end{aligned}\right\} \tag{29}$$

Here $f_2^\theta \in \mathscr{S}$, $f_2^\phi \in \mathscr{H}$ are functions of ω_{11}, u_1, η, and χ at the point $t=\psi$.

It is easy to see that the first two conditions (24) for $k=2$ have the form

$$\check{\alpha}_1^-(\nabla\psi, \nabla\rho_1)|_{\Gamma_t} = \frac{1}{2}\int_{-\infty}^{\infty} f_2^\theta\,\mathrm{d}\eta, \quad \kappa_1\check{\alpha}_0^-\,\partial_\nu\rho_0 - 2\check{\mu}_1^-\,\partial_\nu\gamma_1|_{\Gamma_t} = \kappa v_\nu. \tag{30}$$

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The first condition (30) can be rewritten as the condition for the jump of the normal derivative $\theta_1^{\pm}$ on Γ_t. By (26), the second condition (30) can be easily transformed to the second condition (13). Finally, after some calculations the third condition (24) for $k = 2$ can be transformed to the linear inhomogeneous equation for the phase correction ψ_1:

$$\mathscr{K}'\psi_1 = f^{\psi_1}(x), \quad \psi_1|_{\Gamma_0} = 0. \tag{31}$$

Here $\mathscr{K}'$ is the variation of the operator $\mathscr{K}_t$ from (14), the right-hand side $f^{\psi_1}(x)$ depends on the functions ψ, $\theta^{\pm}$, $\Phi_1^{\pm}$.

To satisfy the boundary conditions on $\partial\Omega$ we must construct the boundary-layer functions. After some routine calculations (see also [6–8]) we find $Z_j = 0$ for $j = 1, 2, 3$, and $Y_j = 0$ for $j = 1, \ldots, 5$, and

$$Z_4 = -2^{-3/2}\partial_N \Delta\Phi_1^+|_\Sigma \exp(-\sqrt{2}\tau), \quad Y_6 = -2^{-5/2}\partial_N\partial_t\Delta\Phi_1^+|_\Sigma \exp(-\sqrt{2}\tau).$$

It remains to consider the initial conditions. Relation (9) with the fast variable $\eta_0 = r_0(x)/\epsilon$, where r_0 is the distance function, is a natural form of the initial value of ϕ, whereas the asymptotics depends on the variable $\eta = (t-\psi(x))/\epsilon + \psi_1$ with functions ψ and ψ_1 unknown beforehand. Note that $\psi|_{\Gamma_0} = 0$ and $\psi_1|_{\Gamma_0} = 0$, the unit vector $\nabla(\psi/|\nabla\psi|)|_{\Gamma_0}$ is normal to Γ_0 and directed opposite to $\nabla r_0|_{\Gamma_0}$. Therefore, for any functions $f(\eta, x) \in \mathscr{H}$

$$\begin{aligned} f(r_0/\epsilon, x) &= f((-\psi+\epsilon\psi_1)/(|\nabla\psi|\epsilon) + g_1/\epsilon + g_2, x) + f(\eta/|\nabla\psi|, x) \\ &\quad + \epsilon f'_\eta(\eta/|\nabla\psi|, x)\{\eta\,\partial_\nu g_2|_{\Gamma_0} + 0,5\eta^2\partial_\eta^2 g_1|_{\Gamma_0}\} + \mathscr{O}(\epsilon^2), \end{aligned}$$

where $g_1 = r_0 - \psi/|\nabla\psi|$, $g_2 = -\psi_1/|\nabla\psi|$, $\partial_\nu = |\nabla\psi|^{-1}\langle\nabla\psi, \nabla\rangle$, and $\eta = -\psi/\epsilon + \psi_1$ for $t = 0$. We also take into account that $f'_\eta \in \mathscr{S}$, and hence, the function $|\eta^k f'_\eta|$ are bounded in C for all $k \leqslant M$.

Furthermore, in §3 we prove that the initial perturbation $\mathscr{O}(\epsilon^3)$ (in the sense of $L^2(\Omega)$) do not take the Van der Waals-type solution out of the stability domain. Therefore, we fix the initial data only up to the terms $\mathscr{O}(\epsilon^2)$. The above constructions imply that outside a small neighbourhood of Γ_0 the initial value of temperature may be arbitrary; however, the form of $\theta^0(x, \epsilon)$ in an ϵ-neighbourhood of Γ_0 and the form of $\phi^0(x, \epsilon)$ in Ω are fixed.

Finally, analysing our construction, we obtain the statement.

Theorem 2 *Let the assumptions of Theorem* 1 *hold and* $m = m(M)$ *be sufficiently large. Then for any integer* $M \geqslant 0$ *there exist functions*

$$\left.\begin{aligned} \theta_M^{\mathrm{as}} &= \theta_0 + \rho_0 V_0 + \sum_{j=1}^{M}\epsilon^j(\theta_j + \rho_j V_j + U_j + Y_j) + \epsilon^{M+1}(U_{M+1} + Y_{M+1}), \\ \phi_M^{\mathrm{as}} &= \chi + \sum_{j=1}^{M+2}\epsilon^j(\phi_j + \gamma_j G_j + W_j + Z_j) + \epsilon^{M+3} Z_{M+3} \end{aligned}\right\} \tag{32}$$

such that

$$\left.\begin{aligned} \partial_t(\theta_M^{\mathrm{as}} + \phi_M^{\mathrm{as}}) - \Delta\theta_M^{\mathrm{as}} - f(x,t) &= \epsilon^M \mathscr{F}_M^\theta, \\ \kappa\epsilon\partial_t\phi_M^{\mathrm{as}} + \Delta(\epsilon^2\Delta\phi_M^{\mathrm{as}} + \phi_M^{\mathrm{as}} - (\phi_M^{\mathrm{as}})^3 + \epsilon\kappa_1\theta_M^{\mathrm{as}}) &= \epsilon^{M+1}\mathscr{F}_M^\phi, \\ \partial_N\theta_M^{\mathrm{as}}|_\Sigma = \epsilon^{M+1}F_M^\theta, \quad \partial_N\phi_M^{\mathrm{as}}|_\Sigma = 0, \quad \partial_N\Delta\phi_M^{\mathrm{as}}|_\Sigma &= \epsilon^M F_M^\phi, \\ \theta_M^{\mathrm{as}}|_{t=0} &= \theta^0(x,\epsilon) + \tilde{\theta}. \end{aligned}\right\} \tag{33}$$

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Here $\tilde{\theta}$, $\mathscr{F}_M^{\phi,\theta}$, $F_M^{\phi,\theta}$ *are (smooth for* $\epsilon > 0$*) functions such that*

$$\|\tilde{\theta};\, L^2(\Omega)\| \leqslant c_0\sqrt{\epsilon}, \quad \|\mathscr{F}_M^{\theta};\, L^2(\Omega)\| + \|\mathscr{F}_M^{\phi};\, L^2(\Omega)\| \leqslant c_1\sqrt{\epsilon}, \tag{34}$$

$$\|\mathscr{F}_M^{\theta};\, C(\bar{Q})\| + \|\mathscr{F}_M^{\phi};\, C(\bar{Q})\| \leqslant c_2, \quad \|F_M^{\phi};\, C(\Sigma)\| + \|F_M^{\phi};\, C(\Sigma)\| \leqslant c_3, \tag{35}$$

where the constants c_j *are independent of* ϵ.

3 Justification of the asymptotic solution

In this section we shall obtain estimates for the difference between the exact θ, ϕ and asymptotic solutions θ_M^{as}, ϕ_M^{as} of problem (3). Let us introduce the notation $\sigma = \theta - \theta_M^{\mathrm{as}}$, $\omega = \phi - \phi_M^{\mathrm{as}}$ and let us consider only initial data θ^0, ϕ^0 that exhibit a special behaviour to be defined below. Then, from (3) and (33), we get the following problem:

$$\partial_t(\sigma+\omega) - \Delta\sigma = -\epsilon^M \mathscr{F}_M^{\theta}, \tag{36}$$

$$\kappa\,\partial_t\omega + \Delta(\epsilon\,\Delta\omega + \epsilon^{-1}\omega(1 - 3\phi_M^2 - 3\phi_M\,\omega - \omega^2) + \kappa_1\sigma) = -\epsilon^M \mathscr{F}_M^{\phi}, \tag{37}$$

$$\partial_N\sigma|_\Sigma = -\epsilon^{M+1} F_M^{\theta}, \quad \partial_N\omega|_\Sigma = 0, \quad \partial_N\Delta\omega|_\Sigma = -\epsilon^M F_M^{\phi}, \tag{38}$$

$$\sigma|_{t=0} = -\epsilon^{M+1/2} f_M^{\theta}, \quad \omega|_{t=0} = -\epsilon^{M+1/2} f_M^{\phi}. \tag{39}$$

Here $\mathscr{F}_M^{\theta,\phi}$, $F_M^{\theta,\phi}$ are smooth functions satisfying (34), (35), $f_M^{\theta,\phi}$ are functions from $H^2(\Omega)$ such that

$$\|f_M^{\theta};\, L^2(\Omega)\| + \|f_M^{\phi};\, L^2(\Omega)\| \leqslant c\sqrt{\epsilon} \tag{40}$$

with constant c independent of ϵ; $H^k(\Omega)$ denotes the Sobolev space. We also define

$$H_0^2(\Omega) = \{u \in H^2(\Omega), \partial_N u|_{\partial\Omega} = 0\}, \quad H^{2,1}(Q) = \{u \in L^2(0,T;\, H_0^2(\Omega)),\ \partial_t u \in L^2(Q)\},$$
$$H^{4,1}(Q) = \{u \in L^2(0,T;\, H^4(\Omega)), \partial_N \Delta u|_\Sigma = 0, \partial_t u \in L^2(Q)\}.$$

To simplify the notation, we omit the superscript denoting asymptotic solutions.

The main result of this section is

Theorem 3 *Assume that there exists a solution of problem* (3) *such that* $\theta \in H^{2,1}(Q)$, $\phi \in H^{4,1}(Q) \cap C([0,T];\, H_0^2(\Omega))$, *where the quantity* $T > 0$ *is independent of* ϵ. Let also the assumptions of Theorem 1 be satisfied and let $M \geqslant 2$. *Then the estimates*

$$\left.\begin{aligned} \|\omega; L^\infty((0,T); L^2(\Omega))\| + \|\sigma; L^\infty((0,T); L^2(\Omega))\| + \|\nabla\omega; L^2(Q)\| \leqslant c\,\epsilon^{M+1}, \\ \|\nabla\sigma; L^2(Q)\| \leqslant c\,\epsilon^{M+1/2}, \quad \|\Delta\omega; L^2(Q)\| \leqslant c\,\epsilon^M \end{aligned}\right\} \tag{41}$$

hold with constant c independent of ϵ.

The main obstacle to the derivation of *a priori* estimates (41) is the rapidly-varying coefficient ϕ_M in (37). This is typical for nonlinear equations; the summand

$$J = \frac{1}{\epsilon}\int_0^t\int_\Omega (\nabla\omega, \nabla(\phi_M^2\,\omega))\,\mathrm{d}x\,\mathrm{d}t'$$

appears on the right-hand side of the energy inequality, while on the left-hand side we have

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only $\|\omega; L^\infty((0,T; L^2(\Omega))\|^2$ and $\epsilon\|\Delta\omega; L^2(Q)\|^2$. It is clear that trivial estimates (for example, by the maximum modulus) allow us to prove that the discrepancy is bounded only for the time $T_\epsilon \sim \epsilon^3$. The first observation is that, to overcome this difficulty, we can rewrite the 'bad' summand $\omega(1-3\phi_M^2)/\epsilon$ as the sum of $-2\omega/\epsilon$ and $3(1-\phi_M^2)\,\omega/\epsilon$. Thus, we obtain the summand $2\epsilon^{-1}\|\nabla\omega; L^2(Q)\|^2$ on the left-hand side of the energy inequality and the expression

$$J_1 = \frac{3}{\epsilon}\int_0^t\int_\Omega (1-\phi_M^2)|\nabla\omega|^2\,\mathrm{d}x\,\mathrm{d}t' + J_{12}, \quad J_{12} = \frac{3}{\epsilon}\int_0^t\int_\Omega \omega^2\,\Delta(\phi_M^2)\,\mathrm{d}x\,\mathrm{d}t'$$

on the right-hand side. Obviously, if we again estimate the functions $1-\phi_M^2$ and $\Delta(\phi_M^2)$ by the maximum of the modulus, there is no result, but we now can use the fact that with precision up to $\mathcal{O}(\epsilon)$ the functions $1-\phi_M^2$ and $\epsilon^2\,\Delta(\phi_M^2)$ are bounded by a constant (in C) and localized in an ϵ-neighbourhood of the free boundary Γ_t. Here the main point is Lemma 3 about estimating integrals of the form $I = \int_{-\infty}^{\infty} v(x/\epsilon) f(x)\,\mathrm{d}x$, where $v(\eta)\in\mathscr{S}$ is a known function, exponentially decreasing outside the point $\eta = 0$. To estimate f we can use the norms in $L^k(R^1)$ only for $k=1$ and $k=2$. Lemma 3 (which was proved to justify the asymptotic to the KdV equation [23], the complete proof has been presented in [27]) implies that for sufficiently small ϵ the first summand in J_1 is bounded from above by $o(R_\omega)$, where

$$R_\omega = \epsilon^{-1}\|\nabla\omega; L^2(Q)\|^2 + \epsilon\|\Delta\omega; L^2(Q)\|^2.$$

Since on the left-hand side of the energy inequality we have $k\,R_\omega$ with a constant $k>0$ independent of ϵ, we see that the first summand in J_1 is no obstacle to the derivation of *a priori* estimate for all finite T.

The second point is that we can rewrite the second summand in J_1 in the form

$$J_{12} = -\frac{3}{\epsilon^3}\int_0^t\int_\Omega \omega^2\,v\,\mathrm{d}x\,\mathrm{d}t' + \frac{9}{2\epsilon^3}\int_0^t\int_\Omega \omega^2\,v_1\{1-\phi_M^2+\mathcal{O}(\epsilon)\}\,\mathrm{d}x\,\mathrm{d}t',$$

where $v = v(\beta(t-\psi+\epsilon\psi_1)/\epsilon) > 0$ is a non-negative soliton-type function, and hence the first integral in J_{12} can be carried over to the left-hand side of the energy inequality, $|v_1| \leqslant \mathrm{const}\,v$. By Lemma 3 we now can prove that for sufficiently small ϵ the second summand in J_{12} has the upper bound $k(R_\omega + 3\epsilon^{-3}\|\omega\sqrt{v}; L^2(0,t; L^2(\Omega))\|^2)/4$. Therefore, the second summand is no longer an obstacle to the derivation of *a priori* estimate for all finite T.

It should be noted that a statement similar to Lemma 3 has been proved [33] to justify a boundary-layer asymptotic for the semi-linear Dirichlet problem. However, in the boundary-layer situation the rapidly-varying v is localized in a small neighbourhood of the external boundary, and the discrepancy vanishes on the boundary. The condition $f|_{\partial\Omega} = 0$ used in Berges & Fraenkel [33] considerably simplifies the estimation of the integral I. Obviously, in the phase transition problems the remainder does not necessarily vanish on Γ_t.

Proof of Theorem 3 At first, let us derive the auxiliary estimates

$$\left.\begin{aligned} &\|\omega; L^\infty((0,T); L^2(\Omega))\| + \|\sigma; L^\infty((0,T); L^2(\Omega))\| + \|\nabla\sigma; L^2(Q)\| \leqslant c\,\epsilon^{M+1/2}, \\ &\|\nabla\omega; L^2(Q)\| \leqslant c\,\epsilon^{M+1}, \quad \|\Delta\omega; L^2(Q)\| \leqslant c\,\epsilon^{M}. \end{aligned}\right\} \tag{42}$$

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Multiplying equations (36), (37) by σ, ω, respectively, and integrating on Ω, we get the relations

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sigma\|^2+\int_\Omega \omega_t\,\sigma\,\mathrm{d}x+\|\nabla\sigma\|^2=-\epsilon^M\int_\Omega \sigma\,\mathscr{F}^{\,\theta}_M\,\mathrm{d}x-\epsilon^{M+1}\int_{\partial\Omega}\sigma\,F^\theta_M\,\mathrm{d}x', \tag{43}$$

$$\begin{aligned}&\frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|^2+\epsilon\,\|\Delta\omega\|^2+\frac{3}{\epsilon}\|\omega\nabla\omega\|^2+\frac{2}{\epsilon}\|\nabla\omega\|^2\\ &\quad=\frac{3}{\epsilon}\int_\Omega(\nabla\omega,\nabla(\omega(1-\phi_M^2)))\,\mathrm{d}x-\frac{3}{\epsilon}\int_\Omega(\nabla\omega,\nabla(\phi_M\,\omega^2))\,\mathrm{d}x\\ &\qquad+\kappa_2\int_\Omega(\nabla\omega,\nabla\sigma)\,\mathrm{d}x-\epsilon^M\int_\Omega \omega\,\mathscr{F}^\phi_M\,\mathrm{d}x+\epsilon^{M+1}\int_{\partial\Omega}\omega(\kappa_2\,F^\theta_M+F^\phi_M)\,\mathrm{d}x'.\end{aligned} \tag{44}$$

Here and below $\|f\|$ denotes the $L^2(\Omega)$ norm of f. Further, multiplying (36) by ω, integrating on Ω and summing with (43), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\omega\|^2+\|\sigma\|^2+2\int_\Omega \omega\sigma\,\mathrm{d}x\right\}+\|\nabla\sigma\|^2+\int_\Omega(\nabla\omega,\nabla\sigma)\,\mathrm{d}x\\ &\qquad=-\epsilon^M\int_\Omega(\omega+\sigma)\,\mathscr{F}^\theta_M\,\mathrm{d}x-\epsilon^{M+1}\int_{\partial\Omega}(\omega+\sigma)\,F^\theta_M\,\mathrm{d}x'.\end{aligned} \tag{45}$$

Let us fix a constant $K>0$. Now, multiplying (44) by K and summing with (45), we get the equality

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{(1+\kappa K)\,\|\omega\|^2+\|\sigma\|^2+2\int_\Omega \omega\sigma\,\mathrm{d}x\right\}+\frac{2K}{\epsilon}\|\nabla\omega\|^2+\|\nabla\sigma\|^2+\epsilon\,K\|\Delta\omega\|^2+\frac{3K}{\epsilon}\|\omega\nabla\omega\|^2\\ &=\frac{3K}{\epsilon}\int_\Omega(\nabla\omega,\nabla(\omega(1-\phi_M^2)))\,\mathrm{d}x-\frac{3K}{\epsilon}\int_\Omega(\nabla\omega,\nabla(\phi_M\,\omega^2))\,\mathrm{d}x\\ &\quad+(\kappa_2\,K-1)\int_\Omega(\nabla\omega,\nabla\sigma)\,\mathrm{d}x-\epsilon^M\int_\Omega\{(\omega+\sigma)\,\mathscr{F}^\theta_M+\omega\,K\mathscr{F}^\phi_M\}\,\mathrm{d}x\\ &\quad-\epsilon^{M+1}\int_{\partial\Omega}\{(\omega+\sigma)\,F^\theta_M-\omega\,K(\kappa_2\,F^\theta_M+F^\phi_M)\}\,\mathrm{d}x'.\qquad\square\end{aligned} \tag{46}$$

We shall analyse the terms in the right-hand side of (46).

Lemma 2 *Let ϕ_M be the asymptotic expansion* (32). *Then*

$$\frac{1}{\epsilon}\int_\Omega(\nabla\omega,\nabla(\omega(1-\phi_M^2)))\,\mathrm{d}x=-\frac{1}{\epsilon^2}\int_\Omega\frac{\omega^2}{\beta}\chi_t\,\mathrm{d}x+I, \tag{47}$$

$$\begin{aligned}I=&\frac{1}{\epsilon}\int_\Omega|\nabla\omega|^2(1-\chi^2)\,\mathrm{d}x+\frac{3}{2\epsilon^2}\int_\Omega\frac{\omega^2}{\beta}\chi_t(1-\chi^2)\,\mathrm{d}x+\frac{1}{2}\int_\Omega\omega^2\,\chi_t\,\Delta\frac{1}{\beta}\mathrm{d}x\\ &-\int_\Omega|\nabla\omega|^2(2\chi+\epsilon\phi^*_M)\,\phi^*_M\,\mathrm{d}x-\int_\Omega\omega(\nabla\omega,\nabla(2\chi\,\phi^*_M+\epsilon(\phi^*_M)^2))\,\mathrm{d}x\\ &+2\int_\Omega\omega\,\chi_t\left(\nabla\omega,\nabla\frac{1}{\beta}\right)\mathrm{d}x+\frac{1}{2\epsilon}\int_\Omega\frac{\omega^2}{\beta}\partial_t(\{\hat{\Pi}\,\partial_\eta\,\chi(\eta,x)-\epsilon\,\Delta_x\,\chi(\eta,x)\}|_{\eta=(t-\psi/\epsilon)})\,\mathrm{d}x.\end{aligned}$$

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Here $\hat{\Pi} = 2(\nabla\psi, \nabla_x) + \Delta\psi$, $\phi_M^* = (\phi_M - \chi)/\epsilon$, *the variables* η *and* x *in the expression in curly brackets are independent.*

Proof of Lemma 2 Using (38), it is easy to establish that

$$\frac{1}{\epsilon}\int_\Omega (\nabla\omega, \nabla(\omega(1-\phi_M^2)))\,\mathrm{d}x = \frac{1}{\epsilon}\int_\Omega |\nabla\omega|^2(1-\chi^2)\,\mathrm{d}x + \frac{1}{2\epsilon}\int_\Omega \omega^2 \Delta(\chi^2)\,\mathrm{d}x - \int_\Omega |\nabla\omega|^2(2\chi+\epsilon\phi_M^*)\,\phi_M^*\,\mathrm{d}x - \int_\Omega \omega(\nabla\omega, \nabla(2\chi\phi_M^* + \epsilon(\phi_M^*)^2))\,\mathrm{d}x. \quad (48)$$

Since $1-\chi^2 = \epsilon\,\chi_t/\beta$, we have

$$J = \frac{1}{2\epsilon}\int_\Omega \omega^2 \Delta(\chi^2)\,\mathrm{d}x = -\frac{1}{2}\int_\Omega \omega^2 \frac{\partial}{\partial t}\Delta\left(\frac{1}{\beta}\chi\right)\mathrm{d}x.$$

It is clear that

$$J = -\frac{1}{2}\int_\Omega \frac{\omega^2}{\beta}\frac{\partial}{\partial t}\Delta\chi\,\mathrm{d}x + \frac{1}{2}\int_\Omega \omega^2\chi_t\,\Delta\frac{1}{\beta}\mathrm{d}x + 2\int_\Omega \omega\,\chi_t\left(\nabla\omega, \nabla\frac{1}{\beta}\right)\mathrm{d}x. \quad (49)$$

Using the explicit form of the function χ, we obtain

$$\epsilon^2\Delta\chi((t-\psi)/\epsilon, x) = \{\chi^3 - \chi - \epsilon\,\hat{\Pi}\,\partial_\eta\,\chi(\eta,x) + \epsilon^2\Delta_x\,\chi(\eta,x)\}|_{\eta=(t-\psi/\epsilon)}.$$

Thus

$$\int_\Omega \frac{\omega^2}{\beta}\frac{\partial}{\partial t}\Delta\chi\,\mathrm{d}x = \frac{2}{\epsilon^2}\int_\Omega \frac{\omega^2}{\beta}\chi_t\,\mathrm{d}x - \frac{3}{\epsilon^2}\int_\Omega \frac{\omega^2}{\beta}\chi_t(1-\chi^2)\,\mathrm{d}x - \frac{1}{\epsilon}\int_\Omega \frac{\omega^2}{\beta}\frac{\partial}{\partial t}\left(\left\{\hat{\Pi}\frac{\partial\chi(\eta,x)}{\partial\eta} - \epsilon\,\Delta_x\,\chi(\eta,x)\right\}\Big|_{\eta=(t-\psi)/\epsilon}\right)\mathrm{d}x.$$

This equality and (48), (49) complete the proof of Lemma 2. □

Further, by using the embedding theorem and (35), we get

$$4\epsilon^{M+1}\left|\int_{\partial\Omega}[(\omega+\sigma)\,F_M^\theta - \omega\,K(\kappa_2\,F_M^\theta + F_M^\phi)]\,\mathrm{d}x'\right| \leqslant c\,\epsilon^{M+1}(\|\omega; L^2(\partial\Omega)\| + \|\sigma; L^2(\partial\Omega)\|) \leqslant c\,\epsilon^{2M+2} + \|\omega\|_1^2 + \|\sigma\|_1^2.$$

Here and below c denotes a universal constant and $\|f\|_k$ is the $H^k(\Omega)$ norm. It is also easy to see that

$$2\left|\int_\Omega \omega\sigma\,\mathrm{d}x\right| \leqslant \alpha\,\|\omega\|^2 + \frac{1}{\alpha}\|\sigma\|^2, \quad \alpha = \frac{1}{2}(\kappa\,K + \sqrt{(\kappa\,K)^2+4}),$$

$$2\left|\int_\Omega (\nabla\omega, \nabla\sigma)\,\mathrm{d}x\right| \leqslant \epsilon^{-1/2}\,\|\nabla\omega\|^2 + \epsilon^{1/2}\|\nabla\sigma\|^2.$$

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Therefore, integrating (46) w.r.t. t and choosing ϵ small enough, we obtain the inequality

$$\frac{\alpha-1}{2\alpha}\{\|\omega\|^2+\|\sigma\|^2\}(t)+\int_0^t\left\{\frac{K}{\epsilon}\|\nabla\omega\|^2+\frac{1}{2}\|\nabla\sigma\|^2\right.$$

$$\left.+\epsilon K\|\Delta\omega\|^2+\frac{3K}{\epsilon}\|\omega\nabla\omega\|^2+\frac{3K}{\epsilon^2}\int_\Omega\frac{\omega^2}{\beta}\chi_t\,\mathrm{d}x\right\}\mathrm{d}t'$$

$$\leqslant c\,\epsilon^{2M+1}+\int_0^t\left\{3K|I|+c(\|\omega\|^2+\|\sigma\|^2)\right.$$

$$\left.+\frac{3K}{\epsilon}\left|\int_\Omega(\nabla\omega,\nabla(\phi_M\,\omega^2))\,\mathrm{d}x\right|\right\}\mathrm{d}t'. \tag{50}$$

To estimate the integral I we shall need the following:

Lemma 3 (Omel'yanov, 1983). *For any nonnegative functions* $f(x)\in L^2(R^1)\cap L^1(R^1)$, $v(x)\in S(R^1)$, *there exists a constant* $\epsilon_0>0$ *such that, for all* $\epsilon\in(0,\epsilon_0]$,

$$\int_{-\infty}^{\infty}f(x)\,v\left(\frac{x}{\epsilon}\right)\mathrm{d}x\leqslant\delta\|f;L^1(R^1)\|+c_v(\delta)\,\epsilon^{3/2}\rho(\epsilon)\|f;L^2(R^1)\|,$$

where δ *is a constant such that* $\delta\geqslant k\,\epsilon^{1/2-\mu}$, $\mu\in(0,1/2)$, *and* $k>0$ *is a constant. Here* $c_v(\delta)$ *is a constant depending on* δ *and on* $\|v(x);L^2(R^1)\cap L^1(R^1)|$ *such that* $0<c_v(\delta)\leqslant\mathrm{const}/\delta^2$, *and* $\rho(\epsilon)\to0$ *as* $\epsilon\to0$.

Let us estimate the first two terms of I.

Lemma 4 *Let* ϵ *be small enough. Then for arbitrary constants* $\delta_i>0$

$$\frac{1}{\epsilon}\int_\Omega|\nabla\omega|^2(1-\chi^2)\,\mathrm{d}x\leqslant\frac{\delta_1}{\epsilon}\|\nabla\omega\|^2+c_\chi(\delta_1)\,\epsilon^{7/2}\,\|\Delta\omega\|^2, \tag{51}$$

$$\frac{1}{\epsilon^2}\int_\Omega\frac{\omega^2}{\beta}\chi_t(1-\chi^2)\,\mathrm{d}x\leqslant\frac{\delta_2}{\epsilon^2}\int_\Omega\frac{\omega^2}{\beta}\chi_t\,\mathrm{d}x+c_\chi(\delta_2)\,\|\nabla\omega\|^2. \tag{52}$$

Proof Denote by $\mathcal{N}_\mu$ a μ neighbourhood of the interface Γ_t, where $\mu\geqslant0$ is a constant independent of ϵ. Since $1-\chi^2=\mathcal{O}(\epsilon^\infty)$ outside $\mathcal{N}_\mu$, we have

$$\frac{1}{\epsilon}\int_\Omega|\nabla\omega|^2(1-\chi^2)\,\mathrm{d}x=\frac{1}{\epsilon}\int_{\mathcal{N}_\mu}|\nabla\omega|^2(1-\chi^2)\,\mathrm{d}x+\epsilon^3\int_\Omega|\nabla\omega|^2\,\mathrm{d}x.$$

Choosing μ sufficiently small, we pass to the variables $y=(y_1,\ldots,y_n)$ in $\mathcal{N}_\mu$, where y_1 is the coordinate normal to Γ_t. Then in $\mathcal{N}_\mu=\{y,|y_1|\leqslant\mu,\ Y_i^-\leqslant y_i\leqslant Y_i^+,\ i=2,\ldots,n\}$ we have

$$I_1=\int_{\mathcal{N}_\mu}|\nabla\omega|^2(1-\chi^2)\,\mathrm{d}x=\prod_{i=2}^n\int_{Y_i^-}^{Y_i^+}\int_{-\mu}^{\mu}|\widetilde{\nabla\omega}|^2v\left(\frac{y_1}{\epsilon},y,t\right)J\,\mathrm{d}y_1\,\mathrm{d}y_i,$$

where J is the Jacobian of this change of variables, $|\widetilde{\nabla\omega}|=|\nabla_x\omega\|_{x=x(y,t)}$,

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$v(\eta, y, t) = \cosh^{-2}(\beta(\eta+\psi_1))|_{x=x(y,t)}$. By Lemma 3 and the embedding theorem for $n=1$, we get

$$I_1 \leqslant \prod_{i=2}^{n}\int_{Y_i^-}^{Y_i^+}\left\{\delta_1\int_{-\mu}^{\mu}|\widetilde{\nabla\omega}|^2 J\,\mathrm{d}y_1 + c\,\epsilon^{3/2}\rho(\epsilon)\left(\int_{-\mu}^{\mu}|\widetilde{\nabla\omega}|^4 J^2\,\mathrm{d}y_1\right)^{1/2}\right\}\mathrm{d}y_i$$
$$\leqslant \delta_1\,\|\nabla\omega\|^2 + c\,\epsilon^{3/2-k_1-k_2}\rho(\epsilon)\{\epsilon^{4k_1/3}\,\|\nabla\omega\|^2+\epsilon^{4k}{}_4\,(\|\nabla\omega\|^2+\|\Delta\omega\|^2)\}$$
$$\leqslant \delta_1\,\|\nabla\omega\|^2 + c\,\epsilon^{1/2}\rho(\epsilon)\{\|\nabla\omega\|^2+\epsilon^4\,\|\Delta\omega\|^2\},$$

where we choose $k_2 = 1+k_1/3$ and use that $J > 0$ is a bounded smooth function. Here and below we omit the dependence of $c_\chi(\delta)$ on the function χ and on the constants δ. It is clear that, choosing δ, we take into account that $c_\chi(\delta)\to\infty$ as $\delta\to 0$. Similarly,

$$\frac{1}{\epsilon^2}\int_{\mathcal{N}_\mu}\omega^2\,\chi^*(1-\chi^2)\,\mathrm{d}x \leqslant \frac{\delta_2}{\epsilon^2}\int_\Omega \omega^2\,\chi^*\,\mathrm{d}x + c\rho(\epsilon)\,\epsilon^{-1/2}$$
$$\times\prod_{i=2}^{n}\int_{Y_i^-}^{Y_i^+}\|\,\tilde\omega\,\sqrt{J\chi^*}\,;L^2(-\mu,\mu)\|^{3/2}\,\|\tilde\omega\,\sqrt{J\chi^*}\,;H^1(-\mu,\mu)\|^{1/2}\,\mathrm{d}y_i$$
$$\leqslant \delta_2\,\epsilon^{-2}\,\|\omega\,\sqrt{\chi^*}\,\|^2 + c\epsilon\,\rho(\epsilon)\{\epsilon^{-2}\,\|\omega\,\sqrt{\chi^*}\,\|^2+\epsilon^{-1}\,\|\nabla\omega\|^2\},$$

where $\chi^* = \chi_t/\beta$. Lemma 4 is proved. □

Further, we have the trivial estimates

$$\left|\int_\Omega \omega\,\chi_t\!\left(\nabla\omega,\nabla\frac{1}{\beta}\right)\mathrm{d}x\right| \leqslant c\,\|\omega\,\sqrt{\chi_t}\|\,\|\sqrt{\chi_t}\,\nabla\omega\| \leqslant c\epsilon^{-1}\|\omega\,\sqrt{\chi^*}\,\|^2 + c\,\|\nabla\omega\|^2,$$
$$\left|\int_\Omega \omega(\nabla\omega,\nabla\epsilon(\phi_M^*)^2)\,\mathrm{d}x\right| \leqslant c\,\sqrt{\epsilon}\,\|\omega\|^2 + c\|\nabla\omega\|^2/\sqrt{\epsilon},$$

Using notation (15) and (17) for the first correction to the asymptotic expansion ϕ, we now have the equality $\nabla(\chi\phi_M^*) = \nabla(\chi(\phi_1+\gamma_1\,\mu_1^+ + \gamma_1\,\mu_1^-\,\chi + W_1)) + g_1$, where

$$W_1 = \check{W}_1 = g_0\,\eta^2\,\chi_\eta + \psi_2\,\chi_\eta$$

satisfies equation (21) for $k=1$. Here and below by g_i, $i\geqslant 0$, we denote (vector)-functions bounded in $C(\bar{Q})$. By direct calculations we obtain

$$g_2\,\nabla\chi + g_3\,\nabla\chi_\eta = (-g_2+2\chi\,g_3)\,\chi_t\,\nabla\psi + g_4 = \epsilon^{-1/2}\,g_5\,\sqrt{\chi^*} + g_4,$$
$$\nabla(\eta^2\,\chi_\eta) = (\epsilon^{-1}+g_6)\,(g_7\,\eta + g_8\,\eta^2)\cosh^{-2}\eta|_{\eta=\beta((t-\psi)/\epsilon+\psi_1)} = \epsilon^{-1/2}\,g_9\,\sqrt{\chi^*} + g_{10}.$$

Therefore

$$\left|\int_\Omega \omega(\nabla\omega,\nabla(\chi\phi_M^*))\,\mathrm{d}x\right| \leqslant \frac{c}{\sqrt{\epsilon}}\|\omega\,\sqrt{\chi^*}\,\|\,\|\nabla\omega\| + c\,\|\omega\|\,\|\nabla\omega\|$$
$$\leqslant c\,\epsilon^{-1}\,\|\omega\,\sqrt{\chi^*}\,\|^2 + c\,\epsilon^{-1/2}\,\|\nabla\omega\|^2 + c\,\sqrt{\epsilon}\,\|\omega\|^2.$$

In the same way, we have

$$\left|\int_\Omega \frac{\omega^2}{\beta}\frac{\partial}{\partial t}\left\{(\Delta_x\,\chi(\eta,x))\Big|_{\eta=(t-\psi)/\epsilon}\right\}\mathrm{d}x\right| \leqslant c\epsilon^{-1}\int_\Omega \frac{\omega^2}{\beta}\frac{1+\eta^2}{\cosh^2\eta}\Big|_{\eta=\beta((t-\psi)/\epsilon+\psi_1)}\mathrm{d}x$$
$$\leqslant c\,\epsilon^{1/2}\,\|\omega\|^2 + c\,\epsilon^{-3/2}\,\|\omega\,\sqrt{\chi^*}\,\|^2.$$

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Finally,

$$\frac{1}{\epsilon}\left|\int_\Omega \frac{\omega^2}{\beta}\frac{\partial}{\partial t}\left\{\left(\hat{\Pi}\frac{\partial\chi}{\partial\eta}\right)\bigg|_{\eta=(t-\psi)/\epsilon}\right\}\mathrm{d}x\right| = \frac{1}{\epsilon^2}\left|\int_\Omega \omega^2\left(\hat{\Pi}\frac{\partial^2\chi}{\partial\eta^2}\right)\bigg|_{\eta=(t-\psi)/\epsilon}\mathrm{d}x\right|$$
$$\leqslant c\,\epsilon^{-1}\,\|\omega\sqrt{\chi^*}\,\|^2 + \delta_3\,\epsilon^{-1}\,\|\nabla\omega\|^2 + c\,\epsilon^{-2}\|\omega\sqrt{\chi^*(1-\chi^2)}\,\|^2, \quad (53)$$

where $\delta_3 > 0$ is an arbitrary constant. The last term in the right-hand side of (53) was estimated in Lemma 4. By the above constructions we obtain the bound for the integral I:

Lemma 5 *Let ϵ be small enough. Then for arbitrary constants δ', $\delta'' > 0$*

$$|I| \leqslant \delta'\,\epsilon^{-2}\,\|\omega\sqrt{\chi^*}\,\|^2 + \delta''\,\epsilon^{-1}\,\|\nabla\omega\|^2 + c\,\epsilon^{1/2}\,\|\omega\|^2 + c\,\epsilon^{7/2}\,\|\Delta\omega\|^2. \quad (54)$$

Let us estimate the last term in the right-hand side of (50). Using the Galiardo–Nirenberg inequality, we obtain

$$\frac{1}{\epsilon}\left|\int_\Omega (\nabla\omega, \nabla(\phi_M\,\omega^2))\,\mathrm{d}x\right| \leqslant \frac{c}{\epsilon}\int_\Omega |\omega|\,|\nabla\omega|^2\,\mathrm{d}x + \frac{c}{\epsilon^2}\int_\Omega |\nabla\omega|\,\omega^2\,\mathrm{d}x$$
$$\leqslant c\,\epsilon^{-1}\,\|\omega\|^{(8-n)/4}\,\|\omega\|_2^{(4+n)/4} + c\,\epsilon^{-2}\,\|\nabla\omega\|\,\|\omega\|^{(8-n)/4}\,\|\omega\|_2^{n/4}$$
$$\leqslant \delta'''\,\epsilon^{-1}\,\|\nabla\omega\|^2 + \delta'''\,\epsilon(\|\omega\|^2 + \|\Delta\omega\|^2) + e\,\epsilon^{-(12+n)/(4-n)}\,\|\omega\|^{2(8-n)/(4-n)} \quad (55)$$

with an arbitrary constant $\delta''' > 0$. Choosing $\delta' = 1/2$, $\delta'' = \delta''' = 1/12$, and using (54), (55), we can transform (50) as follows:

$$U(t) + c\int_0^t \left\{\frac{1}{\epsilon}\|\nabla\omega\|^2 + \|\nabla\sigma\|^2 + \epsilon\,\|\Delta\omega\|^2 + \frac{1}{\epsilon}\,\|\omega\nabla\omega\|^2 + \frac{1}{\epsilon^2}\left\|\sqrt{\frac{\chi_t}{\beta}}\,\omega\right\|^2\right\}\mathrm{d}t'$$
$$\leqslant c\,\epsilon^{2M+1} + c\int_0^t \{U(t') + \epsilon^{-r}(U(t'))^{1+\lambda}\}\,\mathrm{d}t'. \quad (56)$$

Here $U(t) = \{\|\omega\|^2 + \|\sigma\|^2\}(t)$, $\lambda = 4/(4-n)$, $r = (12+n)/(4-n)$. Let us fix a number $T_1 \in (0, T]$, $T < \infty$, and let $t \in [0, T_1]$. Then, according to the Gronwall lemma, (56) yields

$$z \leqslant c(\epsilon^{2M+1} + \epsilon^{-r}\,T_1\,z^{1+\lambda}), \quad (57)$$

where $z = \max_{t\in[0,T_1]} U(t)$. Now we need the following lemma proved in Maslov & Mosolov [34].

Lemma 6 *Let positive numbers p, q, λ satisfy the estimate*

$$q < \lambda(1+\lambda)^{-1}\,(p(1+\lambda))^{-1/\lambda}. \quad (58)$$

Then the solutions of the inequality

$$0 \leqslant z \leqslant q + p\,z^{1+\lambda}$$

belong to the set $[0, Z_-] \cup [Z_+, \infty)$, *where the numbers* Z_+, Z_- *are such that*

$$0 < Z_- < q(1+\lambda)/\lambda < Z_+.$$

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In our case $p = cT_1 \epsilon^{-r}$. Therefore, since ϵ is small enough, the inequality (58) holds for any $M \geqslant 2$. Since $z = z(T_1)$ depends continuously on T_1 and

$$z(0) \leqslant c\,\epsilon^{2M+1} \leqslant c\,\epsilon^{2M+1}(1+\lambda)/\lambda,$$

we obtain the estimate

$$\max_{t\in[0,T_1]} U(t) \leqslant c\,\epsilon^{2M+1} \tag{59}$$

with a constant c independent of ϵ. It is easy to see that (59) and (56) yield the estimates (42). Finally, repeating the construction [27], we obtain that (42) implies (41). This completes the proof of Theorem 4. □

By using arguments similar to those in Theorem 4 and by (41), we easily estimate the higher-order derivatives.

Theorem 4 *Suppose assumptions of Theorem* 4 *are satisfied and*

$$\|\nabla\sigma^0; L^2(\Omega)\| + \|\nabla\omega^0; L^2(\Omega)\| \leqslant c\,\epsilon^{M-3/2},$$

where $\sigma^0 = \sigma|_{t=0}$, $\omega^0 = \omega|_{t=0}$. *Then*

$$\|\Delta\sigma; L^2(Q)\| + \|\nabla\sigma; L^\infty(0,T; L^2(\Omega))\| + \|\nabla\omega; L^\infty(0,T; L^2(\Omega))\| + \sqrt{\epsilon}\,\|\omega; L^2(0,T; H^3(\Omega))\| \leqslant c\,\epsilon^{M-3/2},$$

where c *is a constant independent of* ϵ.

Theorem 5 *Suppose assumptions of Theorem* 5 *are satisfied*, $M \geqslant 3$, *and also*

$$\|\Delta\omega^0; L^2(\Omega)\| \leqslant c\,\epsilon^{M-5/2}.$$

Then

$$\|\partial_t\omega; L^2(Q)\| + \sqrt{\epsilon}\,\|\omega; L^\infty(0,T; H^2(\Omega))\| \leqslant c\,\epsilon^{M-2},$$

where c *is a constant independent of* ϵ.

4 Conclusion

We have constructed the asymptotic 'tanh'-type solution $u^{as}_{tanh} = (\theta^{as}_{tanh}, \phi^{as}_{tanh})$ of the conserved phase field system (1) related to the case (2). The estimate (41) describes the domain of stability for this solution with respect to the initial data. As in Omel'yanov *et al*. [28], one can prove that u^{as}_{tanh} is an attractor in the space of solutions to (3), and that the radius of the stability set is $\mathcal{O}(\epsilon^{n/2})$. The question whether condition (10) is optimal remains open. Nevertheless, it is clear that the solution becomes 'tanh'-type only after a stage of bifurcations, if the initial data lie sufficiently far from u^{as}_{tanh}. For example, in the case of a large relaxation time ($\xi \sim a^{1/2} \to 0, \tau_0 = \text{const}$) system (1) has the stable 'tanh'-type solution only for large times ($t \sim \xi^{-1}$), while for finite times the solution is of soliton-type

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[6, 8]. As the time increases, a bifurcation necessarily starts, and the soliton-type solution $u^{\mathrm{as}}_{\mathrm{sol}}$ transforms itself to a 'tanh'-type one. It has been recently discovered that this problem also has a rapidly oscillating solution for finite times (this result will be published). As the time grows, a bifurcation occurs and the oscillatory solution $u^{\mathrm{as}}_{\mathrm{osc}}$ transforms itself to the soliton-type one. Such behaviour implies that it is necessary to consider in more detail the stability sets for $u^{\mathrm{as}}_{\mathrm{tanh}}$, for $u^{\mathrm{as}}_{\mathrm{sol}}$, and for $u^{\mathrm{as}}_{\mathrm{osc}}$, the processes of bifurcations, and the process of interaction between stable solutions. In our opinion, the analysis of these problems allows us to describe the solution of basic mathematical models for phase transitions related to arbitrary initial data, and, in particular, to describe the process of the phase decomposition.

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References

[1] Caginalp, G. (1988) A conserved phase system: implication for kinetic undercooling. *Phys. Rev. B* **38**, 789–891.

[2] Caginalp, G. (1990) The dynamics of a conserved phase field system: Stefan like, Hele–Shaw, and Cahn–Hilliard models as asymptotic limits. *IMA J. Appl. Math.* **44**, 77–94.

[3] Novick-Cohen, A. (1993) The Cahn–Hilliard Equation: Mathematical and Modelling Perspectives. In L. A. Peletier (ed.), *Surveys on Nonlinear Diffusion*.

[4] Alt, H. W. & Pawlow, I. (1994) On the Entropy Principle of Phase Transition Models with a Conserved Order Parameter. Rheinische Fridrich-Wilhelms-Universität, Bonn, *Preprint N 365*.

[5] Novick-Cohen, A. & Segel, L. A. (1984) Nonlinear aspects of the Cahn–Hilliard equation. *Physica D* **10**, 277–298.

[6] Omel'yanov, G. A., Radkevich, E. V. & Danilov, V. G. (1997) Soliton type asymptotic solutions of the conserved phase field system. To appear.

[7] Omel'yanov, G. A. (1995) Asymptotic solution of the non-isothermal Cahn–Hilliard system. *Int. Centre for Theor. Phys.*, Trieste, preprint IC/95/73.

[8] Omel'yanov, G. A. (1995) Asymptotic description of two metastable processes of solidification for the case of large relaxation time. *Int. Centre for Theor. Phys.*, Trieste, preprint IC/95/162.

[9] Pego, R. L. (1989) Front migration in the nonlinear Cahn–Hilliard equation. *Proc. Roy. Soc. Lond. A* **422**, 261–278.

[10] Stoth, B. (1907) Convergence of the Cahn–Hilliard equation to the Mullins-Sekerka problem in spherical symmetry. *J. Diff. Eq.* (To appear.)

[11] Cahn, J. W. & Hilliard, J. E. (1958) Free energy of a nonuniform system. I. Interfacial free energy. *J. Chem. Physics* **28**, 258–267.

[12] Cahn, J. W. (1961) On spinodal decomposition. *Acta Metall.* **9**, 795–801.

[13] Cahn, J. W. (1968) Spinodal decomposition. *Trans. Metall. Soc. AIME* **242**, 166–180.

[14] Alikakos, N., Bates, P. W. & Fusco, G. (1991) Slow motion for the Cahn–Hilliard equation in one space dimension. *J. Differential Equations* **90**, 81–135.

https://doi.org/10.1017/S0956792597003227 Published online by Cambridge University Press

[15] Carr, J., Gurtin, M. E. & Slemrod, M. (1984) Structured phase transitions on a finite interval. *Arch. Rat. Mech. Anal.* **86**, 317–354.

[16] Bates, P. W. & Fife, P. C. (1990) Spectral comparison principles for the Cahn–Hillard and phase-field equations, and time scale for coarsening. *Physica D* **43**, 335–348.

[17] Elliott, C. M. & Zheng Songmu (1986) On the Cahn–Hilliard equation. *Arch. Rat. Mech. Anal.* **96**, 339–357.

[18] Elliott, C. M. & French, D. A. (1987). Numerical studies of the Cahn–Hilliard equation for phase separation. *IMA J. Appl. Math.* **38**, 97–128.

[19] Zheng Songmu (1986) Asymptotic behavior of the solution to the Cahn–Hilliard equation. *Appl. Anal.* **23**, 165–184.

[20] Il'in, A. M. (1992) *Matching of Asymptotic Expansions of Solutions of Boundary Value Problems*. AMS, Providence.

[21] Maslov, V. P. & Tsupin, V. A. (1976) Shock wave propagation in isoentropic gas with small viscosity. *Itogi Nauki i Techniki: Sovremennie Problemy Mat.* **8**, VINITI, Moscow, 273–308 (in Russian).

[22] Maslov, V. P. & Omel'yanov, G. A. (1981) Asymptotic soliton-like solution of equations with small dispersion. *Uspekhi Mat. Nauk* **37** (3), 61–126 (in Russian). (English translation in *Rus. Math. Surveys* 36 (3), 73–149.)

[23] Maslov, V. P. & Omel'yanov, G. A. (1983) Hugoniot-type conditions for infinitely narrow solutions of the equation for simple waves. *Sibirskii Mat. Zhurnal* **24** (5), 172–182 (in Russian). (English translation in *Siberian Math. J.* **24** (5), 787–795.)

[24] Maslov, V. P., Danilov, V. G. & Volosov, K. A. (1987) *Mathematical Modelling of Heat and Mass Transfer Processes*, Nauka, Moscow (in Russian). (Kluwer, Dordrecht, 1995 (in English).)

[25] Danilov, V. G., Omel'yanov, G. A. & Radkevich, E. V. (1995) Asymptotic solution of a phase field system and the modified Stefan problem. *Differential'nie Uravneniya* **31** (3), 483–491 (Russian). (English translation in *Differential Equations* **31** (3).)

[26] Radkevich, E. V. (1992) Existence conditions for the classical solution of the modified Stefan problem (Gibbs–Thomson law). *Mat. Sb.* **183** (2), 77–101 (in Russian).

[27] Omel'yanov, G. A., Radkevich, E. V. & Danilov, V. G. (1995) Justification of the phase field system asymptotic solution and a modified Stefan problem. *Mat. Sb.* **186** (12), 62–80 (in Russian).

[28] Omel'yanov, G. A., Radkevich, E. V. & Danilov, V. G. (1995) Stability and 'stabilization' of the solution of the phase field system. *Proceedings of Banach Center, Nonlinear Anal. and Appl.* **6**.

[29] Radkevich, E. V. (1993) About asymptotic solution of a phase field system. *Differential'nie Uravneniya* **29** (3), 487–500 (in Russian). (English translation in *Differential Equations* **39** (3).)

[30] Soner, H. M. (1995) Convergence of the phase field equations to the Mullins–Sekerka problem with kinetic undercooling. *Arch. Rat. Mech. Anal.* **131**, 139–197.

[31] Ward, M. J. (1994) Metastable patterns, layer collapses, and coarsening for a one-dimensional Giunzburg–Landau equation. *Studies in Appl. Math.* **91**, 51–93.

[32] Plotnikov, P. I. & Starovoitov, V. N. (1993) Stefan problem as the limit of the phase field system. *Differential'nie Uravneniya* **39** (3), 461–471 (in Russian). (English translation in *Differential Equations* **29** (3).)

[33] Berger, M. S. & Fraenkel, L. E. (1970) On the asymptotic solution of a nonlinear Dirichlet problem. *J. Math. Mech.* **19** (7), 553–585.

[34] Maslov, V. P. & Mosolov, P. P. (1990) *Equations of One-dimensional Barotropic Gas*. Nauka, Moscow (in Russian).

https://doi.org/10.1017/S0956792597003227 Published online by Cambridge University Press