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A TEMPORAL APPROACH TO THE PARISIAN RISK MODEL

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Abstract

In this paper we propose a new approach to study the Parisian ruin problem for spectrally negative Lévy processes. Since our approach is based on a hybrid observation scheme switching between discrete and continuous observations, we call it a temporal approach as opposed to the spatial approximation approach in the literature. Our approach leads to a unified proof for the underlying processes with bounded or unbounded variation paths, and our result generalizes Loeffen *et al.* (2013).

Keywords: Parisian ruin; Poisson observation; insurance risk model; spectrally negative Lévy process

2010 Mathematics Subject Classification: Primary 60G40; 60G51

1. Introduction

Insurance surplus analysis has long been a central focus in actuarial risk theory. In classical risk theory, the event of ruin is defined as the first time the insurance surplus process drops below level 0, which is essentially a standard first-passage time problem. One major extension in the literature is the so-called Parisian ruin, i.e. the insurer will be granted a grace period if the surplus is observed to be negative, and the Parisian ruin occurs only if the surplus process fails to recover to level 0 within this grace period. The time of a Parisian ruin is referred as the Parisian stopping time.

The Parisian ruin concept was first motivated by Parisian options; see, e.g. [4], [8], [9], [12], [13], and [15]–[17]. Interestingly, researchers also found that the Parisian stopping time provides an elegant mathematical model for Chapter 11 of the United States Bankruptcy Code in corporate finance; see, e.g. [7], [11], [19], [20], and [26].

In the actuarial risk theory literature, researchers mainly focus on the study of Parisian ruin for various insurance surplus processes with (downward) jumps. Dassios and Wu [14] first solved the Parisian ruin probability, i.e. the probability that a Parisian ruin ever occurs, for the classical Cramér–Lundberg model with exponential jumps. This work was later extended by Czarna and Palmowski [10] to more general spectrally negative Lévy processes (SNLPs). Loeffen *et al.* [28] further provided an elegant formula for the Parisian ruin probability of SNLPs via only the so-called scale functions (see, e.g. [21]) and the law of the SNLP at a

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fixed time. Other extensions along this line include [5] and [24] with a random grace period, [30] for renewal risk processes, and [27] for refracted SNLPs.

For the Parisian ruin problem of an SNLP with unbounded variation paths, in the spirit of excursion theory, researchers mainly adopt an approximation approach (see, e.g. [10], [15], [17], [27], and [28]) to address the difficulty that the unbounded variation paths oscillate around the ruin threshold. This approach essentially perturbates the sample paths of the underlying process in the spatial dimension, hence we refer to it as a *spatial approximation approach*. This spatial approximation approach has also shown to be an efficient tool to study the occupation time of Lévy processes and diffusion processes; see, e.g. [23] and [25].

In this paper we propose a new approach which is motivated from the so-called *Poisson-observed ruin*. It is another extension of the classical ruin such that the event of ruin is monitored discretely at independent Poisson arrival times; see, e.g. [1]–[3]. As discussed in the aforementioned works, the Poisson observation scheme not only has many applications in actuarial risk theory and queueing theory, but also provides a bridge between continuous-time and discrete-time observations. Moreover, Poisson observation still leads to explicit exit identities and generalizes continuous-time observation, which can be easily recovered when the Poisson arrival rate goes to infinity.

Our new approach to the Parisian ruin problem is based on a *hybrid observation scheme*; see the mathematical formulation and an illustrative graph in (2.2) and Figure 1, respectively. Heuristically speaking, our hybrid observation scheme can be constructed by following the principle that we *observe discretely in the 'noninterested' zone (when the surplus is positive) and continuously in the 'interested' zone (when the surplus is negative)*. More specifically, the surplus insurance process X is first monitored discretely at Poisson arrival times with rate λ until a negative surplus is observed. Then a fixed grace period $b > 0$ is granted to the insurer and X is subsequently observed continuously during this grace period. The insurer is considered as ruined at the end of the grace period unless the surplus recovers to a pre-specified spatial level $a \geq 0$ within the grace period. In the latter case, the observation scheme will be switched back to the discrete Poisson scheme as soon as the surplus recovers to level a. Since we essentially delay the classical Parisian stopping time using Poisson observations, we call our method a *temporal* approach.

Under the SNLP framework, a compact closed-form expression for the ruin probability is solved in terms of the scale functions and the law of SNLPs at a fixed time. Our result generalizes the classical Parisian ruin probability solved by Loeffen *et al.* [28] which can be recovered by letting $a = 0$ and $\lambda \uparrow \infty$. Our approach can also be used to solve more general Gerber–Shiu-type quantities, such as the Laplace transform of the time of ruin. However, we will derive only the ruin probability because the general results are much longer.

We point out that the recovery barrier a is introduced only for a practical generalization and this parameter plays no mathematical role. In other words, the derivation of the ruin probability under the hybrid observation scheme is the same for $a = 0$ and $a > 0$. This is a major difference between our temporal approach and the spatial approximation approach in which a is restricted to be positive when the underlying process has unbounded variation paths. Essentially, since the distributions of an SNLP at discrete observation times are continuous, we are able to bypass the main difficulty of the oscillation of unbounded variation paths around the ruin threshold. Consequently, our temporal approach provides a unified proof for SNLPs with bounded or unbounded variation paths, whereas these two cases need to be treated separately using the traditional spatial approximation approach.

The contributions of this paper are two-fold. First, we propose a new risk model for actuarial risk theory. A discrete observation scheme is adopted as long as the insurance business is healthy, i.e. the surplus is observed to be positive. This is consistent with insurance practice as a less frequent regulatory check is less onerous for both regulators and insurers when the associated checking expenses are considered. But once the surplus is observed to be negative, the observation scheme is enforced using a more stringent continuous scheme during the grace period, which is also consistent with the potential financial seriousness of the situation. If the surplus is successfully restored to a healthy level a, the financial distress is resolved and the observations are switched back to the discrete scheme. Second, of interest from a theoretical point of view, we introduce a new method to study the Parisian stopping time problem. Compared with the spatial approximation approach, our temporal approach has a few advantages. Underlying processes with bounded or unbounded variation paths can be treated in a unified way. It is also more intuitive to design the approximation scheme by following the principle of observing discretely in the 'noninterested' zone and continuously in the 'interested' zone.

The rest of the paper is organized as follows. Section 2 is devoted to the mathematical formulation of the hybrid observation scheme as well as the associated time of ruin. In Section 3 we review some preliminary results for SNLPs. In Section 4 we present the main results of this paper, i.e. the ruin probability under the hybrid observation scheme and its limiting cases (classical Parisian ruin) by taking $a = 0$ and $\lambda \uparrow \infty$. Some numerical examples are provided in Section 5.

2. A hybrid observation scheme and the associated time of ruin

Consider an insurance surplus process $X = \{X_t\}_{t\geq 0}$, modelled by an SNLP defined on a complete probability space $(\Omega, \mathcal{F}, \boldsymbol{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. The first-passage times of X for level $x \in \mathbb{R}$ are defined as

$$\tau_x^{+(-)} = \inf\{t \geq 0 \colon X_t > (<) x\}.$$

In what follows, we follow the convention that $\inf \varnothing = \infty$. We further define a sequence of *discrete observation times* $\{\xi_n\}_{n\in\mathbb{N}}$ as follows. For ease of notation, we denote $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$. Let $\xi_0 = 0$, and, for $n \in \mathbb{N}_+$,

$$\xi_n - \xi_{n-1} = \begin{cases} e_n^\lambda & \text{if } X_{\xi_{n-1}} \geq 0, \\ \tau_a^+ \circ \theta_{\xi_{n-1}} + e_n^\lambda & \text{if } X_{\xi_{n-1}} < 0, \end{cases} \tag{2.1}$$

where $\{e_n^\lambda\}_{n\in\mathbb{N}_+}$ is a sequence of independent and identically distributed (i.i.d.) exponential random variable with mean $1/\lambda > 0$, the constant $a \geq 0$ is called the recovery barrier, and θ is the Markov shift operator such that $X_t \circ \theta_s = X_{s+t}$. Note that the discrete observation scheme (2.1) can be regarded as delayed Poisson arrival times in the sense that the observations will be paused once X is observed to be negative and it will be restarted once the surplus recovers to level a.

We denote

$$T_0^{\lambda,-} = \inf\{\xi_n \colon X_{\xi_n} < 0,\ n \in \mathbb{N}\}$$

as the first time the surplus is observed below level 0 under the observation scheme $\{\xi_n\}_{n\in\mathbb{N}}$.

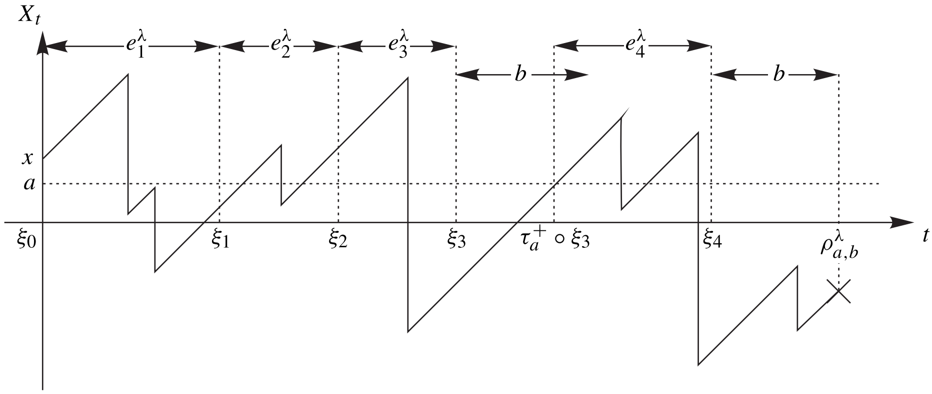

FIGURE 1: Illustration of the ruin time under the hybrid observation scheme.

Clearly, $T_0^{\lambda,-}$ is identical to the ruin time observed at Poisson arrival times, i.e.

$$T_0^{\lambda,-} = \inf\left\{\sum_{i=1}^{n} e_i^\lambda : X_{\sum_{i=1}^n e_i^\lambda} < 0,\ n \in \mathbb{N}\right\}.$$

We then define the time of ruin under a *hybrid observation scheme* with recover barrier $a \geq 0$ and grace period $b > 0$ as

$$\rho_{a,b}^\lambda = \inf\{t \in (\xi_n,\ \tau_a^+ \circ \theta_{\xi_n}) \colon X_{\xi_n} < 0 \text{ and } t - \xi_n \geq b \text{ for } n \in \mathbb{N}\}. \tag{2.2}$$

Under the hybrid observation scheme, the surplus process X is first monitored discretely at Poisson arrival times with rate λ until the surplus is observed to be negative. Then a grace period of length b will be granted to the insurer and the surplus process will be observed continuously during this grace period. The insurer is considered as ruined at the end of the grace period unless the surplus recovers to level a within the grace period. In the latter case, the observation scheme will be switched back to the discrete scheme as soon as the surplus recovers to level a. Note that, if we let $a = 0$ and $\lambda \uparrow \infty$, the time of ruin $\rho_{0,b}^\lambda$ becomes the classical Parisian ruin time studied in, e.g. [9], [10], and [28]. In Figure 1 we illustrate the hybrid observation scheme.

3. Preliminaries on SNLPs

In this section we briefly introduce some preliminary results on SNLPs. Throughout the paper, we denote by $\mathbb{P}_x$ the law of X with $X_0 = x \in \mathbb{R}$. For brevity, we write $\mathbb{P} = \mathbb{P}_0$. To avoid triviality, we assume that $|X|$ is not a subordinator and X satisfies the safe-loading condition, namely,

$$\mathbb{E}[X_1] = \psi'(0+) > 0, \tag{3.1}$$

where $\psi(s) = \log \mathbb{E}[\mathrm{e}^{sX_1}]$, $s \geq 0$, is the Laplace exponent of X. For any given $q \geq 0$, we write

$$\psi_q(s) = \psi(s) - q.$$

The equation $\psi_q(s) = 0$ is known to have at least one positive solution, and we denote the largest root by Φ_q. In particular, due to (3.1), we have $\Phi_0 = 0$. The Laplace transform of the

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first-passage time τ_x^+ is given by

$$\mathbb{E}_u[\mathrm{e}^{-q\tau_x^+}\mathbf{1}_{\{\tau_x^+<\infty\}}] = \mathrm{e}^{-\Phi_q(x-u)}, \qquad u \le x,\ q \ge 0; \tag{3.2}$$

see, e.g. Theorem 3.12 of [22].

Scale functions play a significant role in the fluctuation theory of SNLPs; see, e.g. [21] or [22, Chapter 8]. The scale function $W(x)\colon \mathbb{R} \mapsto [0,\infty)$ is the unique function supported on $[0,\infty)$ with Laplace transform

$$\int_0^\infty \mathrm{e}^{-sx} W(x)\,\mathrm{d}x = \frac{1}{\psi(s)}, \qquad s > 0. \tag{3.3}$$

It is known that W is continuous and strictly increasing on $[0,\infty)$. Further, we have

$$\lim_{x\to\infty} W(x) = \frac{1}{\psi'(0+)}; \tag{3.4}$$

see, e.g. Lemma 3.3 of [21]. We then define the so-called second scale function $Z(x,s)$ as

$$Z(x,s) = \mathrm{e}^{sx}\biggl(1 - \psi(s)\int_0^x \mathrm{e}^{-sy} W(y)\,\mathrm{d}y\biggr), \qquad x, s \ge 0,$$

such that $Z(x,0) = Z(0,s) = 1$. By (3.3), we can also rewrite $Z(x,s)$ as

$$Z(x,s) = \psi(s)\int_0^\infty \mathrm{e}^{-sy} W(x+y)\,\mathrm{d}y, \qquad x \ge 0,\ s > 0. \tag{3.5}$$

Since the hybrid observation scheme introduced in this paper is closely related to Poisson observations, we will use later the following exit identity for SNLPs with Poisson observations, which was proved by Albrecher *et al.* [3, Equation (14)].

Lemma 3.1. *For $u \ge 0$ and $s \ge 0$, we have*

$$\mathbb{E}_u[\mathrm{e}^{sX_{T_0^{\lambda,-}}}\mathbf{1}_{\{T_0^{\lambda,-}<\infty\}}] = \frac{\lambda}{\lambda-\psi(s)}\biggl(Z(u,s) - Z(u,\Phi_\lambda)\frac{\psi(s)\Phi_\lambda}{\lambda s}\biggr). \tag{3.6}$$

In particular, when $s = 0$, it is easy to see that (3.6) reduces to the ruin probability observed at Poisson arrival times given by

$$\mathbb{P}_u(T_0^{\lambda,-} < \infty) = 1 - \psi'(0+)\frac{\Phi_\lambda}{\lambda} Z(u,\Phi_\lambda), \qquad u \ge 0, \tag{3.7}$$

which was first obtained by Landriault *et al.* [23]. Note that $\mathbb{P}_u(T_0^{\lambda,-} < \infty) \equiv 1$ if the safe-loading condition (3.1) fails.

In Lemma 3.2 below, we summarize a few identities involving the scale function and the law of X, which will be used later. They can all be found in [28], and the proof mainly uses Kendall's identity of SNLPs (see, e.g. Corollary VII.3 of [6]), i.e.

$$r\mathbb{P}(\tau_z^+ \in \mathrm{d}r)\,\mathrm{d}z = z\mathbb{P}(X_r \in \mathrm{d}z)\,\mathrm{d}r.$$

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Lemma 3.2. *For $u, a \geq 0$ and $\theta, s > 0$, we have*

$$\int_0^\infty \mathrm{e}^{-\theta s} \int_a^\infty \frac{z}{s} \mathbb{P}(X_s \in \mathrm{d}z)\, \mathrm{d}s = \frac{\mathrm{e}^{-\Phi_\theta a}}{\Phi_\theta}, \tag{3.8}$$

$$\int_0^\infty W(z) \frac{z}{s} \mathbb{P}(X_s \in \mathrm{d}z) = 1, \tag{3.9}$$

$$\int_0^\infty \mathrm{e}^{-\theta s} \int_a^\infty [W(u+z-a) - W(u)] \frac{z}{s} \mathbb{P}(X_s \in \mathrm{d}z)\, \mathrm{d}s = \int_0^\infty \frac{\mathrm{e}^{-\Phi_\theta (a+y)}}{\Phi_\theta} W'(u+y)\, \mathrm{d}y. \tag{3.10}$$

Note that everywhere differentiability of the scale function W is not required as its derivative appears only in integrals.

4. Main results

In this section we aim to first obtain the ruin probability $\mathbb{P}_u(\rho^\lambda_{a,b} < \infty)$ and then show that $\lim_{\lambda\uparrow\infty} \lim_{a\downarrow 0} \mathbb{P}_u(\rho^\lambda_{a,b} < \infty) = \lim_{a\downarrow 0} \lim_{\lambda\uparrow\infty} \mathbb{P}_u(\rho^\lambda_{a,b} < \infty)$, which actually coincides with the compact formula of the Parisian ruin probability obtained by Loeffen *et al.* [28]. Hence, in contrast to the spatial approximation approach used by Czarna and Palmowski [10] and Loeffen *et al.* [28], our paper essentially provides a temporal approach to tackle the Parisian ruin problem.

Theorem 4.1. *For $u, a \geq 0$ and $\lambda, b > 0$, we have*

$$\begin{aligned} &\mathbb{P}_u(\rho^\lambda_{a,b} < \infty) \\ &\quad = 1 - \psi'(0+) \frac{\Phi_\lambda}{\lambda} Z(u, \Phi_\lambda) - \frac{\psi'(0+)(\Phi_\lambda/\lambda) Z(a, \Phi_\lambda)}{1 - \int_0^b \lambda \mathrm{e}^{\lambda(b-s)} g_{a,a,\lambda}(s)\, \mathrm{d}s} \int_0^b \lambda \mathrm{e}^{\lambda(b-s)} g_{u,a,\lambda}(s)\, \mathrm{d}s, \end{aligned}$$

where

$$g_{u,a,\lambda}(s) = \int_a^\infty \left[\frac{\Phi_\lambda}{\lambda} Z(u, \Phi_\lambda) - W(u+z-a)\right] \frac{z}{s} \mathbb{P}(X_s \in \mathrm{d}z). \tag{4.1}$$

Proof. By conditioning on $T_0^{\lambda,-}$, the first time that the surplus is observed to be negative, and using the strong Markov property and spatial homogeneity, we first have

$$\begin{aligned} &\mathbb{P}_u(\rho^\lambda_{a,b} = \infty) \\ &\quad = \mathbb{P}_u(T_0^{\lambda,-} = \infty) + \int_0^\infty \mathbb{P}_u(-X_{T_0^{\lambda,-}} \in \mathrm{d}x, T_0^{\lambda,-} < \infty) \mathbb{P}_{-x}(\tau_a^+ < b) \mathbb{P}_a(\rho^\lambda_{a,b} = \infty) \\ &\quad = \mathbb{P}_u(T_0^{\lambda,-} = \infty) + \mathbb{P}_a(\rho^\lambda_{a,b} = \infty) \int_0^\infty \mathbb{P}_u(-X_{T_0^{\lambda,-}} \in \mathrm{d}x, T_0^{\lambda,-} < \infty) \mathbb{P}(\tau_{x+a}^+ < b). \end{aligned} \tag{4.2}$$

By letting $u = a$ and solving for $\mathbb{P}_a(\rho^\lambda_{a,b} = \infty)$, we obtain

$$\mathbb{P}_a(\rho^\lambda_{a,b} = \infty) = \frac{\mathbb{P}_a(T_0^{\lambda,-} = \infty)}{1 - \int_0^\infty \mathbb{P}_a(-X_{T_0^{\lambda,-}} \in \mathrm{d}x, T_0^{\lambda,-} < \infty) \mathbb{P}(\tau_{x+a}^+ < b)}. \tag{4.3}$$

Substituting the above expression of $\mathbb{P}_a(\rho^\lambda_{a,b}=\infty)$ back into (4.2), and also using (3.7), yields

$$
\begin{aligned}
\mathbb{P}_u&(\rho^\lambda_{a,b}=\infty)\\
&=\mathbb{P}_u(T_0^{\lambda,-}=\infty)+\frac{\mathbb{P}_a(T_0^{\lambda,-}=\infty)\int_0^\infty\mathbb{P}_u(-X_{T_0^{\lambda,-}}\in\mathrm{d}x,\,T_0^{\lambda,-}<\infty)\mathbb{P}(\tau^+_{x+a}<b)}{1-\int_0^\infty\mathbb{P}_u(-X_{T_0^{\lambda,-}}\in\mathrm{d}x,\,T_0^{\lambda,-}<\infty)\mathbb{P}(\tau^+_{x+a}<b)}\\
&=\psi'(0+)\frac{\Phi_\lambda}{\lambda}Z(u,\Phi_\lambda)\\
&\quad+\frac{\psi'(0+)(\Phi_\lambda/\lambda)Z(a,\Phi_\lambda)\int_0^\infty\mathbb{P}_u(-X_{T_0^{\lambda,-}}\in\mathrm{d}x,\,T_0^{\lambda,-}<\infty)\mathbb{P}(\tau^+_{x+a}<b)}{1-\int_0^\infty\mathbb{P}_a(-X_{T_0^{\lambda,-}}\in\mathrm{d}x,\,T_0^{\lambda,-}<\infty)\mathbb{P}(\tau^+_{x+a}<b)}.
\end{aligned}\tag{4.4}
$$

From (4.4), we need to find an explicit expression only for the term

$$\int_0^\infty\mathbb{P}_u(-X_{T_0^{\lambda,-}}\in\mathrm{d}x,\,T_0^{\lambda,-}<\infty)\mathbb{P}(\tau^+_{x+a}<b)$$

in terms of the scale function and the law of X. First, from (3.2), it is easy to see that

$$\int_0^\infty\mathrm{e}^{-\theta b}\mathbb{P}(\tau^+_{x+a}<b)\,\mathrm{d}b=\frac{1}{\theta}\mathbb{E}[\mathrm{e}^{-\theta\tau^+_{x+a}}]=\frac{1}{\theta}\mathrm{e}^{-\Phi_\theta(x+a)}.\tag{4.5}$$

Taking the Laplace transform of $\int_0^\infty\mathbb{P}_u(-X_{T_0^{\lambda,-}}\in\mathrm{d}x,\,T_0^{\lambda,-}<\infty)\mathbb{P}(\tau^+_{x+a}<b)$ with respect to b, and using (3.5), (3.6), (4.5), and Tonelli's theorem, for $\theta>0$, we obtain

$$
\begin{aligned}
\int_0^\infty&\mathrm{e}^{-\theta b}\int_0^\infty\mathbb{P}_u(-X_{T_0^{\lambda,-}}\in\mathrm{d}x,\,T_0^{\lambda,-}<\infty)\mathbb{P}(\tau^+_{x+a}<b)\,\mathrm{d}b\\
&=\int_0^\infty\mathbb{E}_u[\mathbf{1}_{\{-X_{T_0^{\lambda,-}}\in\mathrm{d}x,\,T_0^{\lambda,-}<\infty\}}]\frac{1}{\theta}\mathrm{e}^{-\Phi_\theta(x+a)}\\
&=\frac{1}{\theta}\mathrm{e}^{-\Phi_\theta a}\mathbb{E}_u[\mathrm{e}^{\Phi_\theta X_{T_0^{\lambda,-}}}\mathbf{1}_{\{T_0^{\lambda,-}<\infty\}}]\\
&=\frac{1}{\theta}\mathrm{e}^{-\Phi_\theta a}\frac{\lambda}{\lambda-\theta}\bigg[Z(u,\Phi_\theta)-Z(u,\Phi_\lambda)\frac{\theta\Phi_\lambda}{\lambda\Phi_\theta}\bigg]\\
&=\frac{\lambda\mathrm{e}^{-\Phi_\theta a}}{\lambda-\theta}\bigg[\int_0^\infty\mathrm{e}^{-\Phi_\theta y}W(u+y)\,\mathrm{d}y-\frac{Z(u,\Phi_\lambda)\Phi_\lambda}{\lambda\Phi_\theta}\bigg].
\end{aligned}\tag{4.6}
$$

Since $\Phi_\theta>0$ for $\theta>0$, by (3.4) and integration by parts, we have

$$
\begin{aligned}
\int_0^\infty&\mathrm{e}^{-\Phi_\theta y}W(u+y)\,\mathrm{d}y\\
&=-\frac{1}{\Phi_\theta}\lim_{y\uparrow\infty}[\mathrm{e}^{-\Phi_\theta y}W(u+y)]+\frac{1}{\Phi_\theta}W(u)+\frac{1}{\Phi_\theta}\int_0^\infty\mathrm{e}^{-\Phi_\theta y}W'(u+y)\,\mathrm{d}y\\
&=\frac{1}{\Phi_\theta}W(u)+\frac{1}{\Phi_\theta}\int_0^\infty\mathrm{e}^{-\Phi_\theta y}W'(u+y)\,\mathrm{d}y.
\end{aligned}
$$

Substituting the last equation into (4.6) yields

$$\int_0^\infty \mathrm{e}^{-\theta b} \int_0^\infty \mathbb{P}_u(-X_{T_0^{\lambda,-}} \in \mathrm{d}x,\, T_0^{\lambda,-} < \infty)\mathbb{P}(\tau_{x+a}^+ < b)\,\mathrm{d}b$$
$$= \frac{\lambda}{\theta-\lambda}\bigg[-\frac{\mathrm{e}^{-\Phi_\theta a}}{\Phi_\theta}W(u) - \int_0^\infty \frac{\mathrm{e}^{-\Phi_\theta(a+y)}}{\Phi_\theta}W'(u+y)\,\mathrm{d}y + \frac{Z(u,\Phi_\lambda)\Phi_\lambda \mathrm{e}^{-\Phi_\theta a}}{\lambda\Phi_\theta}\bigg]. \tag{4.7}$$

Next we will apply the inverse Laplace transform to (4.7). Note that

$$\frac{\lambda}{\theta-\lambda} = \int_0^\infty \mathrm{e}^{-\theta b}\lambda \mathrm{e}^{\lambda b}\,\mathrm{d}b, \qquad \theta > \lambda. \tag{4.8}$$

By (3.8), (3.10), and (4.8), we conclude that

$$\int_0^\infty \mathbb{P}_u(-X_{T_0^{\lambda,-}} \in \mathrm{d}x,\, T_0^{\lambda,-} < \infty)\mathbb{P}(\tau_{x+a}^+ < b) = \int_0^b \lambda \mathrm{e}^{\lambda(b-s)} g_{u,a,\lambda}(s)\,\mathrm{d}s, \tag{4.9}$$

where

$$\begin{aligned} g_{u,a,\lambda}(s) &= -\int_a^\infty \frac{z}{s}\mathbb{P}(X_s \in \mathrm{d}z)W(u) - \int_a^\infty [W(u+z-a) - W(u)]\frac{z}{s}\mathbb{P}(X_s \in \mathrm{d}z) \\ &\quad + \frac{Z(u,\Phi_\lambda)\Phi_\lambda}{\lambda}\int_a^\infty \frac{z}{s}\mathbb{P}(X_s \in \mathrm{d}z) \\ &= \int_a^\infty \bigg[\frac{Z(u,\Phi_\lambda)\Phi_\lambda}{\lambda} - W(u+z-a)\bigg]\frac{z}{s}\mathbb{P}(X_s \in \mathrm{d}z). \end{aligned}$$

We complete the proof by substituting (4.9) into (4.4). □

Remark 4.1. Similar to Loeffen *et al.* [28], the closed-form expression of the ruin probability in Theorem 4.1 is in terms of the scale functions and the law of X_s for some fixed time $s > 0$. Unfortunately, the scale functions and the law of X possess explicit expressions for only a few cases, such as Brownian motion, the Cramér–Lundberg model with exponential claims, and the stable process with index $\frac{3}{2}$. In other words, for these examples, the ruin probability in Theorem 4.1 can also be expressed explicitly using the formulas of the scale functions and law of X. In Section 5 numerical examples of the ruin probability are provided for the Brownian motion model and the Cramér–Lundberg model with exponential jumps, respectively.

Remark 4.2. We point out that the proof of Theorem 4.1 can indeed be mimicked to obtain the Laplace transform of time of ruin $\rho_{a,b}^\lambda$. Denote e_s an independent exponential random variable with mean $1/s$. By the memoryless property of an exponential random variable, we have

$$\begin{aligned} \mathbb{P}_u(\rho_{a,b}^\lambda > e_s) &= \mathbb{P}_u(T_0^{\lambda,-} > e_s) + \int_0^\infty \mathbb{P}_u(-X_{T_0^{\lambda,-}} \in \mathrm{d}x,\, T_0^{\lambda,-} < e_s)\mathbb{P}_{-x}(e_s < \tau_a^+ \wedge b) \\ &\quad + \int_0^\infty \mathbb{P}_u(-X_{T_0^{\lambda,-}} \in \mathrm{d}x,\, T_0^{\lambda,-} < e_s)\mathbb{P}_{-x}(\tau_a^+ < b \wedge e_s)\mathbb{P}_a(\rho_{a,b}^\lambda > e_s), \end{aligned}$$

where

$$\mathbb{P}_u(-X_{T_0^{\lambda,-}} \in \mathrm{d}x,\, T_0^{\lambda,-} < e_s) = \mathbb{E}_u[\mathrm{e}^{-sT_0^{\lambda,-}}\mathbf{1}_{\{-X_{T_0^{\lambda,-}} \in \mathrm{d}x,\, T_0^{\lambda,-} < \infty\}}],$$

$$\mathbb{P}_{-x}(e_s < \tau_a^+ \wedge b) = 1 - \mathbb{E}[\mathrm{e}^{-s\tau_{x+a}^+}\mathbf{1}_{\{\tau_{x+a}^+ < b\}}] - \mathrm{e}^{-sb}\mathbb{P}(\tau_{x+a}^+ > b),$$

$$\mathbb{P}_{-x}(\tau_a^+ < b \wedge e_s) = \mathbb{E}[\mathrm{e}^{-s\tau_{x+a}^+}\mathbf{1}_{\{\tau_{x+a}^+ < b\}}].$$

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Back substitution together with simple algebra yields

$$\mathbb{E}_u[\mathrm{e}^{-s\rho^\lambda_{a,b}}\mathbf{1}_{\{\rho^\lambda_{a,b}<\infty\}}] = 1-\mathbb{P}_u(\rho^\lambda_{a,b}>e_s) = I(b)+\mathbb{E}_a[\mathrm{e}^{-s\rho^\lambda_{a,b}}\mathbf{1}_{\{\rho^\lambda_{a,b}<\infty\}}]J(b),$$

where

$$I(b) = \mathrm{e}^{-sb}\int_0^\infty \mathbb{E}_u[\mathrm{e}^{-sT_0^{\lambda,-}}\mathbf{1}_{\{-X_{T_0^{\lambda,-}}\in \mathrm{d}x,\, T_0^{\lambda,-}<\infty\}}]\mathbb{P}(\tau^+_{x+a}>b),$$
$$J(b) = \int_0^\infty \mathbb{E}_u[\mathrm{e}^{-sT_0^{\lambda,-}}\mathbf{1}_{\{-X_{T_0^{\lambda,-}}\in \mathrm{d}x,\, T_0^{\lambda,-}<\infty\}}]\mathbb{E}[\mathrm{e}^{-s\tau^+_{a+x}}\mathbf{1}_{\{\tau^+_{x+a}<b\}}].$$

An explicit expression for $I(b)$ and $J(b)$ can be obtained in a similar fashion as in the proof in Theorem 4.1. Since the Laplace transform result is much longer, to focus on illustrating the temporal approach, we choose to present only the ruin probability.

In order to take limits $a\downarrow 0$ and $\lambda\uparrow\infty$ later, in the following corollary we rewrite the ruin probability in Theorem 4.1 more explicitly.

Corollary 4.1. *For $u, a\geq 0$ and $\lambda, b>0$,*

$$\mathbb{P}_u(\rho^\lambda_{a,b}<\infty) = 1-\psi'(0+)\Phi_\lambda\int_0^\infty \mathrm{e}^{-\Phi_\lambda y}W(u+y)\,\mathrm{d}y - \mathbb{P}_a(\rho^\lambda_{a,b}=\infty) \times\int_0^b \lambda \mathrm{e}^{\lambda(b-s)}g_{u,a,\lambda}(s)\,\mathrm{d}s, \tag{4.10}$$

where

$$\mathbb{P}_a(\rho^\lambda_{a,b}=\infty) = \psi'(0+)\bigg[\frac{\int_0^\infty\int_0^a \lambda \mathrm{e}^{-\lambda s}W(z)(z/(s+b))\mathbb{P}(X_{s+b}\in\mathrm{d}z)\,\mathrm{d}s}{\int_0^\infty W(a+y)\Phi_\lambda \mathrm{e}^{-\Phi_\lambda y}\,\mathrm{d}y} + \int_0^\infty\int_a^\infty \lambda\mathrm{e}^{-\lambda s}\frac{z}{s+b}\mathbb{P}(X_{s+b}\in\mathrm{d}z)\,\mathrm{d}s\bigg]^{-1} \tag{4.11}$$

and

$$\int_0^b \lambda\mathrm{e}^{\lambda(b-s)}g_{u,a,\lambda}(s)\,\mathrm{d}s = \int_0^\infty\int_a^\infty \lambda\mathrm{e}^{-\lambda s}\bigg[W(u+z-a)-\int_0^\infty \Phi_\lambda \mathrm{e}^{-\Phi_\lambda y}W(u+y)\,\mathrm{d}y\bigg] \times\frac{z}{s+b}\mathbb{P}(X_{s+b}\in\mathrm{d}z)\,\mathrm{d}s. \tag{4.12}$$

Proof. First, it is easy to see that (4.10) follows immediately from (4.2), (4.9), and (3.5). Next we prove (4.12). From (4.1), (3.5), integration by parts, and (3.4), it follows that

$$\begin{aligned}\int_0^b &\lambda\mathrm{e}^{\lambda(b-s)}g_{u,a,\lambda}(s)\,\mathrm{d}s\\ &= \lambda\mathrm{e}^{\lambda b}\int_0^b \mathrm{e}^{-\lambda s}\int_a^\infty\bigg[\frac{\Phi_\lambda}{\lambda}Z(u,\Phi_\lambda)-W(u+z-a)\bigg]\frac{z}{s}\mathbb{P}(X_s\in\mathrm{d}z)\,\mathrm{d}s\\ &= \lambda\mathrm{e}^{\lambda b}\int_0^b\int_a^\infty \mathrm{e}^{-\lambda s}\bigg[\Phi_\lambda\int_0^\infty \mathrm{e}^{-\Phi_\lambda y}W(u+y)\,\mathrm{d}y - W(u+z-a)\bigg]\frac{z}{s}\mathbb{P}(X_s\in\mathrm{d}z)\,\mathrm{d}s\end{aligned}$$

$$= \lambda e^{\lambda b} \int_0^b \int_a^\infty e^{-\lambda s} \bigg[W(u) - W(u+z-a) + \int_0^\infty e^{-\Phi_\lambda y} W'(u+y)\,dy \bigg] \times \frac{z}{s} \mathbb{P}(X_s \in dz)\,ds. \tag{4.13}$$

On the other hand, by (3.8) and (3.10), we know that

$$\begin{aligned} &\int_0^\infty \int_a^\infty e^{-\lambda s} \bigg[W(u+z-a) - W(u) - \int_0^\infty e^{-\Phi_\lambda y} W'(u+y)\,dy \bigg] \frac{z}{s} \mathbb{P}(X_s \in dz)\,ds \\ &\quad = \int_0^\infty \frac{e^{-\Phi_\lambda (a+y)}}{\Phi_\lambda} W'(u+y)\,dy - \frac{e^{-\Phi_\lambda a}}{\Phi_\lambda} \int_0^\infty e^{-\Phi_\lambda y} W'(u+y)\,dy \\ &\quad = 0. \end{aligned} \tag{4.14}$$

By (4.13) and (4.14), we deduce

$$\begin{aligned} &\int_0^b \lambda e^{\lambda(b-s)} g_{u,a,\lambda}(s)\,ds \\ &\quad = \lambda e^{\lambda b} \int_b^\infty \int_a^\infty e^{-\lambda s} \bigg[W(u+z-a) - W(u) - \int_0^\infty e^{-\Phi_\lambda y} W'(u+y)\,dy \bigg] \\ &\qquad\qquad \times \frac{z}{s} \mathbb{P}(X_s \in dz)\,ds \\ &\quad = \int_0^\infty \int_a^\infty \lambda e^{-\lambda s} \bigg[W(u+z-a) - W(u) - \int_0^\infty e^{-\Phi_\lambda y} W'(u+y)\,dy \bigg] \\ &\qquad\qquad \times \frac{z}{s+b} \mathbb{P}(X_{s+b} \in dz)\,ds \\ &\quad = \int_0^\infty \int_a^\infty \lambda e^{-\lambda s} \bigg[W(u+z-a) - \int_0^\infty \Phi_\lambda e^{-\Phi_\lambda y} W(u+y)\,dy \bigg] \\ &\qquad\qquad \times \frac{z}{s+b} \mathbb{P}(X_{s+b} \in dz)\,ds, \end{aligned}$$

where the last step is due to integration by parts and (3.4). This proves (4.12).

It is only left to show (4.11). Substituting (3.5), (3.7), and (4.9) into (4.3) yields

$$\mathbb{P}_a(\rho^\lambda_{a,b} = \infty) = \frac{\psi'(0+)(\Phi_\lambda/\lambda) Z(u, \Phi_\lambda)}{1 - \int_0^b \lambda e^{\lambda(b-s)} g_{a,a,\lambda}(s)\,ds} = \frac{\psi'(0+) \int_0^\infty \Phi_\lambda e^{-\Phi_\lambda y} W(a+y)\,dy}{1 - \int_0^b \lambda e^{\lambda(b-s)} g_{a,a,\lambda}(s)\,ds}.$$

Further, by (4.12), it follows that

$$\begin{aligned} \mathbb{P}_a(\rho^\lambda_{a,b} = \infty) = \psi'(0+) &\int_0^\infty \Phi_\lambda e^{-\Phi_\lambda y} W(a+y)\,dy \\ &\times \bigg[1 - \int_0^\infty \int_a^\infty \lambda e^{-\lambda s} \bigg[W(z) - \int_0^\infty \Phi_\lambda e^{-\Phi_\lambda y} W(a+y)\,dy \bigg] \\ &\qquad \times \frac{z}{s+b} \mathbb{P}(X_{s+b} \in dz)\,ds \bigg]^{-1}. \end{aligned} \tag{4.15}$$

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On the other hand, (3.9) implies that

$$\int_0^\infty \lambda \mathrm{e}^{-\lambda s} \int_0^\infty W(z) \frac{z}{s+b} \mathbb{P}(X_{s+b} \in \mathrm{d}z)\, \mathrm{d}s = \int_0^\infty \lambda \mathrm{e}^{-\lambda s}\, \mathrm{d}s = 1. \tag{4.16}$$

Thus, substituting (4.16) into (4.15) yields

$$\begin{aligned}
\frac{1}{\psi'(0+)} \mathbb{P}_a(\rho^\lambda_{a,b} = \infty) &= \biggl[\frac{1 - \int_0^\infty \int_a^\infty \lambda \mathrm{e}^{-\lambda s} W(z)(z/(s+b)) \mathbb{P}(X_{s+b} \in \mathrm{d}z)\, \mathrm{d}s}{\int_0^\infty \Phi_\lambda \mathrm{e}^{-\Phi_\lambda y} W(a+y)\, \mathrm{d}y} \\
&\qquad + \int_0^\infty \int_a^\infty \lambda \mathrm{e}^{-\lambda s} \frac{z}{s+b} \mathbb{P}(X_{s+b} \in \mathrm{d}z)\, \mathrm{d}s \biggr]^{-1} \\
&= \biggl[\frac{\int_0^\infty \int_0^a \lambda \mathrm{e}^{-\lambda s} W(z)(z/(s+b)) \mathbb{P}(X_{s+b} \in \mathrm{d}z)\, \mathrm{d}s}{\int_0^\infty \Phi_\lambda \mathrm{e}^{-\Phi_\lambda y} W(a+y)\, \mathrm{d}y} \\
&\qquad + \int_0^\infty \int_a^\infty \lambda \mathrm{e}^{-\lambda s} \frac{z}{s+b} \mathbb{P}(X_{s+b} \in \mathrm{d}z)\, \mathrm{d}s \biggr]^{-1},
\end{aligned}$$

which proves (4.11). This completes the proof. □

By Corollary 4.1, in the next proposition we show that, when $a \downarrow 0$ and $\lambda \uparrow \infty$, the ruin probability under the hybrid observation scheme $\mathbb{P}_u(\rho^\lambda_{a,b} < \infty)$ reduces to the formula of the Parisian ruin probability obtained by Loeffen *et al.* [28]. The proof of the following proposition mainly utilizes the initial value theorem (IVT) of the Laplace transform; see, e.g. Theorem 3.8.1 of [18], and a more general proof can be found in Theorem 2.2.10 of [29].

Proposition 4.1. *For $u \ge 0$ and $b > 0$,*

$$\lim_{\lambda\uparrow\infty} \lim_{a\downarrow 0} \mathbb{P}_u(\rho^\lambda_{a,b} < \infty) = \lim_{a\downarrow 0} \lim_{\lambda\uparrow\infty} \mathbb{P}_u(\rho^\lambda_{a,b} < \infty) = 1 - \psi'(0+) \frac{\int_0^\infty W(u+z) z \mathbb{P}(X_b \in \mathrm{d}z)}{\int_0^\infty z \mathbb{P}(X_b \in \mathrm{d}z)}.$$

Proof. We first evaluate $\lim_{\lambda\uparrow\infty} \lim_{a\downarrow 0} \mathbb{P}_u(\rho^\lambda_{a,b} < \infty)$. It follows from (4.11), (4.12), and the IVT that

$$\begin{aligned}
\lim_{\lambda\uparrow\infty} \lim_{a\downarrow 0} \mathbb{P}_a(\rho^\lambda_{a,b} = \infty) &= \lim_{\lambda\uparrow\infty} \frac{\psi'(0+)}{\int_0^\infty \lambda \mathrm{e}^{-\lambda s} \int_0^\infty (z/(s+b)) \mathbb{P}(X_{s+b} \in \mathrm{d}z)\, \mathrm{d}s} \\
&= \frac{\psi'(0+)}{\int_0^\infty (z/b) \mathbb{P}(X_b \in \mathrm{d}z)}
\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}
&\lim_{\lambda\uparrow\infty} \lim_{a\downarrow 0} \int_0^b \lambda \mathrm{e}^{\lambda(b-s)} g_{u,a,\lambda}(s)\, \mathrm{d}s \\
&\quad = \lim_{\lambda\uparrow\infty} \int_0^\infty \lambda \mathrm{e}^{-\lambda s} \int_0^\infty \biggl[W(u+z) - \int_0^\infty \Phi_\lambda \mathrm{e}^{-\Phi_\lambda y} W(u+y)\, \mathrm{d}y \biggr] \\
&\qquad\qquad \times \frac{z}{s+b} \mathbb{P}(X_{s+b} \in \mathrm{d}z)\, \mathrm{d}s \\
&\quad = \int_0^\infty \biggl(W(u+z) - \lim_{\Phi_\lambda\uparrow\infty} \int_0^\infty \Phi_\lambda \mathrm{e}^{-\Phi_\lambda y} W(u+y)\, \mathrm{d}y \biggr) \frac{z}{b} \mathbb{P}(X_b \in \mathrm{d}z) \\
&\quad = \int_0^\infty [W(u+z) - W(u)] \frac{z}{b} \mathbb{P}(X_b \in \mathrm{d}z),
\end{aligned} \tag{4.18}$$

where the last step is due to the fact that $\Phi_\lambda \uparrow \infty$ as $\lambda \uparrow \infty$. Substituting (4.17) and (4.18) into (4.10), and using the IVT again, we obtain

$$\begin{aligned}\lim_{\lambda\uparrow\infty}\lim_{a\downarrow 0}\mathbb{P}_u(\rho^\lambda_{a,b}<\infty) &= 1-\psi'(0+)\lim_{\Phi_\lambda\uparrow\infty}\Phi_\lambda\int_0^\infty \mathrm{e}^{-\Phi_\lambda y}W(u+y)\,\mathrm{d}y \\ &\quad - \frac{\psi'(0+)\int_0^\infty[W(u+z)-W(u)](z/b)\mathbb{P}(X_b\in\mathrm{d}z)}{\int_0^\infty(z/b)\mathbb{P}(X_b\in\mathrm{d}z)} \\ &= 1-\psi'(0+)\frac{\int_0^\infty W(u+z)z\mathbb{P}(X_b\in\mathrm{d}z)}{\int_0^\infty z\mathbb{P}(X_b\in\mathrm{d}z)}. \qquad (4.19)\end{aligned}$$

Next we evaluate $\lim_{a\downarrow 0}\lim_{\lambda\uparrow\infty}\mathbb{P}_u(\rho^\lambda_{a,b}<\infty)$. Applying the IVT to (4.11), for $a>0$, we have

$$\begin{aligned}\lim_{\lambda\uparrow\infty}\mathbb{P}_a(\rho^\lambda_{a,b}=\infty) &= \psi'(0+)\bigg[\frac{\int_0^a W(z)(z/b)\mathbb{P}(X_b\in\mathrm{d}z)}{W(a)}+\int_a^\infty\frac{z}{b}\mathbb{P}(X_b\in\mathrm{d}z)\,\mathrm{d}s\bigg]^{-1} \\ &= \psi'(0+)\bigg[\frac{1-\int_a^\infty W(z)(z/b)\mathbb{P}(X_b\in\mathrm{d}z)}{W(a)}+\int_a^\infty\frac{z}{b}\mathbb{P}(X_b\in\mathrm{d}z)\,\mathrm{d}s\bigg]^{-1},\end{aligned}$$

where the last step is due to (3.9). By Lemma 3.1 of [21] and Equation (14) of [28], we know that

$$\lim_{a\downarrow 0}\frac{1-\int_a^\infty W(z)(z/b)\mathbb{P}(X_b\in\mathrm{d}z)}{W(a)}=0,$$

which holds regardless of whether X has bounded variation or unbounded variation sample paths. Thus, it follows that

$$\lim_{a\downarrow 0}\lim_{\lambda\uparrow\infty}\mathbb{P}_a(\rho^\lambda_{a,b}=\infty)=\frac{\psi'(0+)}{\int_0^\infty(z/b)\mathbb{P}(X_b\in\mathrm{d}z)\,\mathrm{d}s}.$$

On the other hand, applying the IVT to (4.12) yields

$$\begin{aligned}&\lim_{a\downarrow 0}\lim_{\lambda\uparrow\infty}\int_0^b\lambda\mathrm{e}^{\lambda(b-s)}g_{u,a,\lambda}(s)\,\mathrm{d}s \\ &\quad=\lim_{a\downarrow 0}\lim_{\lambda\uparrow\infty}\int_0^\infty\lambda\mathrm{e}^{-\lambda s}\int_a^\infty\bigg[W(u+z-a)-\int_0^\infty\Phi_\lambda\mathrm{e}^{-\Phi_\lambda y}W(u+y)\,\mathrm{d}y\bigg] \\ &\qquad\qquad\times\frac{z}{s+b}\mathbb{P}(X_{s+b}\in\mathrm{d}z)\,\mathrm{d}s \\ &\quad=\lim_{a\downarrow 0}\int_a^\infty[W(u+z-a)-W(u)]\frac{z}{b}\mathbb{P}(X_b\in\mathrm{d}z)\,\mathrm{d}s \\ &\quad=\int_0^\infty[W(u+z)-W(u)]\frac{z}{b}\mathbb{P}(X_b\in\mathrm{d}z)\,\mathrm{d}s.\end{aligned}$$

By the same steps as in (4.19), we complete the proof. □

5. Numerical examples

In this section we will provide some numerical examples for the Parisian ruin probability under the hybrid observation scheme. We will study the Brownian motion model and the Cramér–Lundberg model with exponential claims because their scale functions and the law of X possess explicit expressions. For simplicity, we assume that $a=0$ in this section.

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5.1. Brownian motion model

Let $X_t = \mu t + \sigma B_t$, where $\mu, \sigma > 0$ and $\{B_t\}_{t\geq 0}$ is a standard Brownian motion. Then we have $\psi(\theta) = \mu\theta + \frac{1}{2}\sigma^2\theta^2$ and $\Phi_\lambda = (-\mu + \sqrt{\mu^2 + 2\sigma^2\lambda})/\sigma^2$. The scale functions are given by

$$W(x) = \frac{1}{\mu}(1 - \mathrm{e}^{-2\mu\sigma^{-2}x}) \quad \text{and} \quad Z(x,\theta) = \frac{1}{\mu}\psi(\theta)\bigg(\frac{1}{\theta} - \frac{\mathrm{e}^{-2\mu\sigma^{-2}x}}{\theta + 2\mu\sigma^{-2}}\bigg) \quad \text{for } \theta > \Phi.$$

From (4.1), it follows that

$$g_{u,0,\lambda}(s) = \frac{1}{\mu s}\mathrm{e}^{-2\mu\sigma^{-2}u}\bigg[\int_0^\infty \mathrm{e}^{-2\mu\sigma^{-2}z} z\mathbb{P}(X_s \in \mathrm{d}z) - \frac{\Phi_\lambda}{\Phi_\lambda + 2\mu\sigma^{-2}}\int_0^\infty z\mathbb{P}(X_s \in \mathrm{d}z)\bigg].$$

Since $\mathbb{P}(X_s \in \mathrm{d}z) = (1/\sqrt{2\pi\sigma^2 s})\mathrm{e}^{-(z-\mu s)^2/2\sigma^2 s}$, we have

$$\int_0^\infty z\mathbb{P}(X_s \in \mathrm{d}z) = \frac{\sigma\sqrt{s}}{\sqrt{2\pi}}\mathrm{e}^{-\mu^2 s/2\sigma^2} + \mu s\mathcal{N}(\mu\sigma^{-1}\sqrt{s}),$$
$$\int_0^\infty \mathrm{e}^{-2\mu\sigma^{-2}z} z\mathbb{P}(X_s \in \mathrm{d}z) = \int_0^\infty z\mathbb{P}(X_s \in \mathrm{d}z) - \mu s,$$

where $\mathcal{N}(\cdot)$ is the cumulative distribution function of a standard normal random variable. Thus,

$$g_{u,0,\lambda}(s) = \frac{1}{\mu s}\mathrm{e}^{-2\mu\sigma^{-2}u}\bigg[\frac{2\mu\sigma^{-2}}{\Phi_\lambda + 2\mu\sigma^{-2}}\bigg(\frac{\sigma\sqrt{s}}{\sqrt{2\pi}}\mathrm{e}^{-\mu^2 s/2\sigma^2} + \mu s\mathcal{N}(\mu\sigma^{-1}\sqrt{s})\bigg) - \mu s\bigg].$$

By Theorem 4.1,

$$\mathbb{P}_u(\rho_{0,b}^\lambda < \infty) = \frac{\Phi_\lambda \mathrm{e}^{-2\mu\sigma^{-2}u}}{\Phi_\lambda + 2\mu\sigma^{-2}} - \frac{2\mu\sigma^{-2}}{\Phi_\lambda + 2\mu\sigma^{-2}} \frac{\int_0^b \lambda \mathrm{e}^{\lambda(b-s)} g_{u,0,\lambda}(s)\,\mathrm{d}s}{1 - \int_0^b \lambda \mathrm{e}^{\lambda(b-s)} g_{0,0,\lambda}(s)\,\mathrm{d}s}.$$

In Tables 1 and 2 we present the results for the effect of model parameters on the ruin probability of the Brownian motion model. In Table 1, we fix $\mu = 1$, $\sigma = \sqrt{10}$, and $b = 2$. It is seen that the ruin probability increases in λ as the surplus process is observed more frequently, thereby increasing the likelihood of detecting a negative surplus. In Table 2, we fix $\mu = 1$, $\sigma = \sqrt{10}$, and $\lambda = 1$. It is seen that the ruin probability decreases in b because a negative surplus is more likely to be recovered given a longer grace period.

TABLE 1: Effect of λ on the ruin probability.

u	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$	$\lambda = 4$	$\lambda = \infty$
1	1.8639×10^{-1}	2.1541×10^{-1}	2.3596×10^{-1}	2.4905×10^{-1}	2.6546×10^{-1}
5	8.3750×10^{-2}	9.6789×10^{-2}	1.0602×10^{-1}	1.1191×10^{-1}	1.1928×10^{-1}
10	3.0810×10^{-2}	3.5607×10^{-2}	3.9003×10^{-2}	4.1168×10^{-2}	4.3881×10^{-2}
20	4.1697×10^{-3}	4.8188×10^{-3}	5.2785×10^{-3}	5.5715×10^{-3}	5.9386×10^{-3}
30	5.6430×10^{-4}	6.5216×10^{-4}	7.1437×10^{-4}	7.5402×10^{-4}	8.0371×10^{-4}

TABLE 2: Effect of b on the ruin probability.

u	$b=3$	$b=4$	$b=5$	$b=6$	$b=7$
1	1.7086×10^{-1}	1.3977×10^{-1}	1.1668×10^{-1}	9.8831×10^{-2}	8.4657×10^{-2}
5	7.6770×10^{-2}	6.2802×10^{-2}	5.2426×10^{-2}	4.4408×10^{-2}	3.8039×10^{-2}
10	2.8242×10^{-2}	2.3104×10^{-2}	1.9286×10^{-2}	1.6337×10^{-2}	1.3994×10^{-2}
20	3.8222×10^{-3}	3.1267×10^{-3}	2.6101×10^{-3}	2.2109×10^{-3}	1.8939×10^{-3}
30	5.1727×10^{-4}	4.2316×10^{-4}	3.5324×10^{-4}	2.9922×10^{-4}	2.5631×10^{-4}

5.2. Cramér–Lundberg model with exponential claims

Let $X_t = ct - \sum_{i=1}^{N_t} C_i$, where $\{N_t\}_{t\geq 0}$ is a Poisson process with rate η, and C_i are i.i.d. exponential random variables with mean $1/\alpha$, which are independent of the Poisson process. Then

$$\psi(\theta) = c\theta - \eta + \frac{\eta\alpha}{\theta+\alpha} \quad \text{and} \quad \Phi_\lambda = \frac{\sqrt{(\eta+\lambda-c\alpha)^2 + 4c\lambda\alpha} + \eta + \lambda - c\alpha}{2c}.$$

Assume that $c > \eta\alpha^{-1}$ such that $\psi'(0+) = c - \eta\alpha^{-1} > 0$. Further, the scale functions are given by

$$W(x) = \frac{1-(\eta/c\alpha)\mathrm{e}^{(\eta c^{-1}-\alpha)x}}{c-\eta\alpha^{-1}} \quad \text{and} \quad Z(x,\theta) = \frac{\psi(\theta)}{c-\eta\alpha^{-1}}\left[\frac{1}{\theta} - \frac{\eta}{c\alpha}\frac{\mathrm{e}^{(\eta/c-\alpha)x}}{\theta+\alpha-\eta c^{-1}}\right].$$

From (4.1),

$$g_{u,0,\lambda}(s) = \frac{\eta}{c(c\alpha-\eta)s}\mathrm{e}^{(\eta c^{-1}-\alpha)u}\left[\int_0^\infty \mathrm{e}^{(\eta c^{-1}-\alpha)z} z\mathbb{P}(X_s \in \mathrm{d}z) - \frac{\Phi_\lambda}{\Phi_\lambda+\alpha-\eta c^{-1}}\int_0^\infty z\mathbb{P}(X_s\in\mathrm{d}z)\right].$$

Note that

$$\mathbb{P}(X_s \in \mathrm{d}z) = \mathrm{e}^{-\eta r}\left[\delta_0(\mathrm{d}z) + \mathrm{e}^{-\alpha z}\sum_{m=0}^\infty \frac{(\alpha\eta r)^{m+1}}{m!\,(m+1)!} z^m\,\mathrm{d}z\right],$$

where $\delta_0(\mathrm{d}z)$ is the Dirac mass at 0. After some calculation, we arrive at

$$\int_0^\infty z\mathbb{P}(X_s\in\mathrm{d}z) = \mathrm{e}^{-\eta s}\left[cs + \sum_{m=0}^\infty \frac{(\eta s)^{m+1}}{m!\,(m+1)!}\left(cs\Gamma(m+1,\alpha cs) - \frac{1}{\alpha}\Gamma(m+2,\alpha cs)\right)\right],$$

$$\frac{\eta}{c\alpha}\int_0^\infty \mathrm{e}^{(\eta c^{-1}-\alpha)z} z\mathbb{P}(X_s\in\mathrm{d}z) = \int_0^\infty z\mathbb{P}(X_s\in\mathrm{d}z) - \left(c - \frac{\eta}{\alpha}\right)s,$$

where $\Gamma(n,x) := \int_0^x \mathrm{e}^{-t}t^{n-1}\,\mathrm{d}t$ for $n\in\mathbb{N}$, $x\geq 0$ is the incomplete gamma function. By Theorem 4.1,

$$\mathbb{P}_u(\rho_{a,b}^\lambda < \infty) = \frac{\eta}{c\alpha}\mathrm{e}^{(\eta c^{-1}-\alpha)u}\frac{\Phi_\lambda}{\Phi_\lambda+\alpha-\eta c^{-1}} - \frac{\alpha-\eta c^{-1}}{\Phi_\lambda+\alpha-\eta c^{-1}}\frac{\int_0^b \lambda\mathrm{e}^{\lambda(b-s)}g_{u,a,\lambda}(s)\,\mathrm{d}s}{1-\int_0^b\lambda\mathrm{e}^{\lambda(b-s)}g_{a,a,\lambda}(s)\,\mathrm{d}s}.$$

TABLE 3: Effect of λ on the ruin probability.

u	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$	$\lambda = 4$	$\lambda = \infty$
1	2.0507×10^{-1}	2.3546×10^{-1}	2.5681×10^{-1}	2.7035×10^{-1}	2.8723×10^{-1}
5	1.0529×10^{-1}	1.2089×10^{-1}	1.3185×10^{-1}	1.3880×10^{-1}	1.4747×10^{-1}
10	4.5757×10^{-2}	5.2539×10^{-2}	5.7303×10^{-2}	6.0323×10^{-2}	6.4090×10^{-2}
20	8.6424×10^{-3}	9.9232×10^{-3}	1.0823×10^{-2}	1.1396×10^{-2}	1.2105×10^{-2}
30	1.6323×10^{-3}	1.8743×10^{-3}	2.0442×10^{-3}	2.1520×10^{-3}	2.2864×10^{-3}

TABLE 4: Effect of b on the ruin probability.

u	$b = 3$	$b = 4$	$b = 5$	$b = 6$	$b = 7$
1	1.8900×10^{-1}	1.5620×10^{-1}	1.3159×10^{-1}	1.1241×10^{-1}	9.7054×10^{-2}
5	9.7035×10^{-2}	8.0194×10^{-2}	6.7561×10^{-2}	5.7713×10^{-2}	4.9829×10^{-2}
10	4.2171×10^{-2}	3.4852×10^{-2}	2.9362×10^{-2}	2.5082×10^{-2}	2.1656×10^{-2}
20	7.9651×10^{-3}	6.5827×10^{-3}	5.5457×10^{-3}	4.7374×10^{-3}	4.0902×10^{-3}
30	1.5044×10^{-3}	1.2433×10^{-3}	1.0475×10^{-3}	8.9478×10^{-4}	7.7255×10^{-4}

In Tables 3 and 4 we present the results for the effect of model parameters on the ruin probability of the Cramér–Lundberg model with exponential jumps. In Table 3, we fix $c = 6$, $\eta = 5$, $\alpha = 1$, and $b = 2$. In Table 4, we fix $c = 6$, $\eta = 5$, $\alpha = 1$, and $\lambda = 1$. Similar to the Brownian motion model, it is seen that the ruin probability increases in λ and decreases in b.

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