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The initial-value problem for a fourth-order dispersive closed curve flow on the 2-sphere

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A closed curve flow on the 2-sphere evolved by a fourth-order nonlinear dispersive partial differential equation on the one-dimensional flat torus is studied. The governing equation arises in the field of physics in relation to the continuum limit of the Heisenberg spin chain systems or three-dimensional motion of the isolated vortex filament. The main result of the paper gives the local existence and uniqueness of a solution to the initial-value problem by overcoming loss of derivatives in the classical energy method and the absence of the local smoothing effect. The proof is based on the delicate analysis of the lower-order terms to find out the loss of derivatives and on the gauged energy method to eliminate the obstruction.

Keywords: fourth-order dispersive partial differential equation; vortex filament; Heisenberg spin chain systems; local existence and uniqueness; loss of derivatives; gauge transformation

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Secondary 35G61; 35Q40; 35Q55

1. Introduction

This paper is concerned with the initial-value problem (IVP) for a fourth-order nonlinear dispersive partial differential equation (PDE) of the form

$$u_t = \alpha u \wedge \partial_x^3 u_x + \beta(\partial_x u_x, u_x) u \wedge u_x + \gamma |u_x|^2 u \wedge \partial_x u_x + u \wedge \partial_x u_x, \tag{1.1}$$

$$u(0, x) = u_0(x), \tag{1.2}$$

where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the one-dimensional flat torus, $\mathbb{S}^2$ is the standard unit sphere centred at the origin in $\mathbb{R}^3$, $u = u(t, x)\colon \mathbb{R} \times \mathbb{T} \to \mathbb{S}^2$ is an unknown function, a time-dependent closed curve flow on $\mathbb{S}^2$, and $u_0 = u_0(x)\colon \mathbb{T} \to \mathbb{S}^2$ is the initial function. Equation (1.1) describes a three-component PDE system regarding u as an $\mathbb{R}^3$-valued function with a constraint $|u|^2 = 1$, where $\alpha \neq 0$, β, γ are real constants, $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, $\partial_x^k u_x = \partial^{k+1} u/\partial x^{k+1}$ for $k = 1, 2, \dots$ are partial derivatives of u, the exterior product and the inner product in $\mathbb{R}^3$ are denoted by $\wedge$ and $(\cdot, \cdot)$, respectively, and the absolute value in $\mathbb{R}^3$ is denoted by $|\cdot|$.

Equation (1.1) with $\alpha \neq 0$ and $\beta = 2\gamma$ arises in two fields of physics. On the one hand, it was derived by Lakshmanan *et al.* in [14] to model the continuum limit of the Heisenberg spin chain systems with biquadratic exchange interactions. On the other hand, Fukumoto and Moffatt [8, 10] derived a model equation for the three-dimensional motion of a thin isolated vortex filament embedded in an incompressible perfect fluid by taking into account the elliptical deformation effect of the core due to the self-induced strain. The equation reads

$$\boldsymbol{\gamma}_t = -C_1\boldsymbol{\gamma}_x\wedge\partial_x^3\boldsymbol{\gamma}_x + C_1\partial_x\boldsymbol{\gamma}_x\wedge\partial_x^2\boldsymbol{\gamma}_x + (C_b - C_1)|\partial_x\boldsymbol{\gamma}_x|^2\boldsymbol{\gamma}_x\wedge\partial_x\boldsymbol{\gamma}_x + \boldsymbol{\gamma}_x\wedge\partial_x\boldsymbol{\gamma}_x. \quad (1.3)$$

Here $\boldsymbol{\gamma} = \boldsymbol{\gamma}(t,x)$ is the $\mathbb{R}^3$-valued function of t and x, x is required to be an arc-length parameter for the space curve $\boldsymbol{\gamma}(t,\cdot)$ modelling the centreline of the vortex filament at time t with nowhere vanishing curvature, and C_1, C_b are real physical constants. If $\boldsymbol{\gamma}$ is governed by (1.3), then $u := \boldsymbol{\gamma}_x$ satisfies $|u|^2 = 1$ since x is the arc-length parameter, and the PDE satisfied by u coincides with (1.1) for the setting in which $\alpha = -C_1$ and $\beta = 2\gamma = 2(C_2 - 2C_1)$.

The purpose of the paper is to study the existence and uniqueness of a solution to (1.1), (1.2). To state our results precisely, the following notation is introduced. Let m be a non-negative integer. The space $H^m(\mathbb{T};\mathbb{R}^3)$ denotes the standard mth Sobolev space of $\mathbb{R}^3$-valued functions on $\mathbb{T}$ equipped with the norm defined by

$$\|U\|_{H^m} = \left\{\sum_{k=0}^{m}\int_{\mathbb{T}}(\partial_x^k U(x), \partial_x^k U(x))\,\mathrm{d}x\right\}^{1/2}$$

for $U \in H^m(\mathbb{T};\mathbb{R}^3)$. We set $L^2(\mathbb{T};\mathbb{R}^3) = H^0(\mathbb{T};\mathbb{R}^3)$ and $\|\cdot\|_{L^2} = \|\cdot\|_{H^0}$, and denote by $\langle U, V\rangle$ the inner product in L^2 defined by $\langle U,V\rangle = \int_{\mathbb{T}}(U(x),V(x))\,\mathrm{d}x$ for $U, V \in L^2(\mathbb{T};\mathbb{R}^3)$. For a time interval $I \subset \mathbb{R}$ and for a Banach space X, the set of all X-valued continuous (respectively, essentially bounded) functions on I is denoted by $C(I;X)$ (respectively, $L^\infty(I;X)$).

We are now in a position to state our main results.

Theorem 1.1. *Let m be a positive integer satisfying $m \geqslant 6$. Then, for any $u_0 \in C(\mathbb{T};\mathbb{S}^2)$ satisfying $u_{0x} \in H^m(\mathbb{T};\mathbb{R}^3)$, there exists a positive constant*

$$T = T(\|u_{0x}\|_{H^4}) > 0$$

depending on α, β, γ, m and $\|u_{0x}\|_{H^4}$ such that (1.1), (1.2) admits a unique solution $u \in C([-T,T]\times\mathbb{T};\mathbb{S}^2)$ satisfying $u_x \in C([-T,T];H^m(\mathbb{T};\mathbb{R}^3))$.

The solution is extended time-globally under an additional condition on (α,β,γ).

Theorem 1.2. *Let $\beta = 2\gamma = 5\alpha$ and let m be a positive integer satisfying $m \geqslant 6$. Then, for any $u_0 \in C(\mathbb{T};\mathbb{S}^2)$ satisfying $u_{0x} \in H^m(\mathbb{T};\mathbb{R}^3)$, (1.1), (1.2) admits a unique solution $u \in C(\mathbb{R}\times\mathbb{T};\mathbb{S}^2)$ satisfying $u_x \in C(\mathbb{R};H^m(\mathbb{T};\mathbb{R}^3))$.*

To clarify our contribution, we recall the related studies from three directions.

First, we state what is new in the paper. Guo *et al.* [11] showed the existence of a local weak solution to (1.1), (1.2) when $\alpha \neq 0$ and $\beta = 2\gamma = 5\alpha$. The assumption on the constants is the same as that imposed in theorem 1.2 and is the necessary and sufficient condition for (1.1) to be completely integrable in some sense (see, for example, [1,2,7,8,14,18]). The proof in [11] is essentially based on two conservation laws and is not valid without the assumption. Indeed, there have been no results on the existence of a solution to (1.1), (1.2) except for [11], and the problem of the uniqueness remained unsolved even in [11]. In contrast, theorem 1.1 and 1.2 present a positive answer as well as the uniqueness. Specifically, theorem 1.1 is valid without the assumption on the constants except for $\alpha \neq 0$.

Second, one may think of theorem 1.1 and 1.2 as a higher-order analogue of the results for the completely integrable equation for $u\colon \mathbb{R}\times\mathbb{T}\to\mathbb{S}^2$ of the form

$$u_t = u\wedge\partial_x u_x + a\{\partial_x^2 u_x + \tfrac{3}{2}\partial_x u_x\wedge(u\wedge u_x) + \tfrac{3}{2}u_x\wedge(u\wedge\partial_x u_x)\}. \tag{1.4}$$

If $a = 0$, (1.4) coincides with (1.1) with $\alpha=\beta=\gamma=0$ and is the well-known Heisenberg spin model (see, for example, [7, 13]). If $0\neq a\in\mathbb{R}$, (1.4) arises in relation to the vortex filament equation with the axial flow effect (see [9]). Local and global existence of a unique solution to the IVP for (1.4) has been established. For details, see, for example, [12, 19] when $a = 0$, and [16, 20] when $a \neq 0$. In their proof, the classical energy method evaluating $\|u_x\|^2_{H^m}$ by integration by parts works essentially to show the local existence results. Moreover, there exists an infinite number of conservation laws for (1.4), which ensures the global existence results. In contrast, in the case of (1.1) that we consider, so-called loss of derivatives occurs. Indeed, some lower-order terms cannot be handled only by the classical energy method. This is the main difficulty, and is found to be overcome in the present paper.

Third, we state the case in which the spacial domain $\mathbb{T}$ is replaced by the real line $\mathbb{R}$. In the case of $\mathbb{R}$, (1.1) possesses a kind of local dispersive smoothing effect coming from the leading fourth-order term, which is enough to compensate for the loss of derivatives. Indeed, Chihara and Onodera [6] showed the local existence and uniqueness of a solution to (1.1), (1.2), and developed the results from the point of view of geometric analysis, by making full use of the local smoothing effect. Unfortunately, the case of $\mathbb{T}$ does not fall into the scope of the case of $\mathbb{R}$, since the local smoothing effect is absent when the spacial domain is compact. Despite the absence of the local smoothing effect, we succeed in proving theorem 1.1 by finding out the more essential solvable structure of (1.1).

Our proof of theorem 1.1 is based on the delicate analysis of the loss of derivatives occurring in the PDE for higher-order derivatives of the solution, and on the energy estimate for a gauged function of the highest-order derivative to eliminate the obstruction. The method was recently applied to a second- or third-order dispersive equation for maps into some class of compact Riemannian manifolds (see [3, 5, 6, 17]). However, if we turn our eyes to fourth-order dispersive PDEs on $\mathbb{T}$ under different settings, the method has already been established by using pseudo-differential operators. Indeed, Mizuhara [15] established the necessary and sufficient conditions for the L^2-well-posedness of the IVP for a linear fourth-order dispersive equation for complex-valued functions on $\mathbb{T}$. Chihara [4] developed the results to a linear fourth-order dispersive system for $\mathbb{C}^2$-valued functions. We mention that the choice of our gauged function to prove theorem 1.1 is implicitly motivated by [4]. Once theorem 1.1 is established, the proof of theorem 1.2 is straightforward by using conservation laws for (1.1).

For the sake of better understanding, we state a little bit more detail about the proof of theorem 1.1. The gauged function (defined by (3.1)) is chosen by making the following formal observation. Let u be a solution to (1.1), (1.2). We calculate PDEs for $\partial_x^k u_x$ with $0\leqslant k\leqslant m$ to study the energy estimate for u_x in H^m. Specifically, letting $k = m$, we set $U_m = \partial_x^m u_x$. Then, after lengthy calculations, we can write

$$\partial_t U_m = P(\partial_x^4)U_m + \tilde{R}_{(m)}, \tag{1.5}$$

where we denote by $P(\partial_x^4)U_m$ the sum of all terms that include any one of $\partial_x^4 U_m, \dots,$ $\partial_x U_m$, all terms of which can be divided into the following five types.

I. Leading fourth-order term, that is, $\alpha\partial_x^2(u \wedge \partial_x^2 U_m)$.

II. Second-order terms of divergence form with a skew-symmetric operator: a linear combination of $\partial_x(\partial_x u_x \wedge \partial_x U_m)$ and $\partial_x\{(1 + \gamma|u_x|^2)u \wedge \partial_x U_m\}$.

III. First-order terms with symmetry: a linear combination of $T_3(u)\partial_x U_m$ and $T_4(u)\partial_x U_m$.

IV. First-order terms without symmetry but that are harmless in the classical energy estimate thanks to the constraint $|u|^2 = 1$.

V. Lower-order terms that cause loss of derivatives: a linear combination of $(\partial_x^3 U_m, u \wedge u_x)u$, $T_1(u)\partial_x^2 U_m$ and $(\partial_x T_2)(u)\partial_x U_m$.

One can find the precise expression of (1.5) in (2.31)–(2.34) with $k = m$, where $P_4(u)U_m + P_1(u)U_m$ is denoted by $P(\partial_x^4)U_m$ and the sum of all terms of type IV is denoted by $P_1(u)U_m$. The form $\tilde{R}_{(m)}$ includes at most the mth derivative of u_x and is indeed harmless in the classical energy estimate. Among these five types, only the terms of type V cause loss of derivatives. To avoid the difficulty, we consider a gauged function V_m defined (in (3.1)) by the form

$$\begin{aligned} V_m &:= U_m + \Lambda(u)U_m \\ &= U_m + \sum_{i=1}^{2} \Lambda_i(u)U_m, \\ \Lambda_i(u)U_m &:= a_i B_i(u)\partial_x^{m-2} u_x \quad (i = 1, 2), \end{aligned}$$

where for each $i = 1, 2$, a_i is a real constant and $B_i(u)$ is a linear operator acting on $\mathbb{R}^3$-valued functions on $\mathbb{T}$ for each time t. Then the PDE for V_m becomes

$$\partial_t V_m = P(\partial_x^4)V_m - \sum_{i=1}^{2}\{P(\partial_x^4)(\Lambda_i(u)U_m) - \partial_t(\Lambda_i(u)U_m)\} + \text{harmless terms}. \tag{1.6}$$

The detail of the calculations to show (1.6) is described in (3.5)–(3.8). From the second term on the right-hand side of (1.6), we can pick up the commutator of $\alpha\partial_x^2(u \wedge \partial_x^2)$ and $\Lambda_i(u)$ acting on U_m for each i of the form

$$[\alpha\partial_x^2(u \wedge \partial_x^2), \Lambda_i(u)]U_m = \alpha a_i[\partial_x^2(u \wedge \partial_x^2), B_i(u)\partial_x^{-2}]U_m \quad (i = 1, 2).$$

Fortunately, we can choose a_i and $B_i(u)$ so that a linear combination of two commutators eliminates all the terms of type V. This is divided into three parts.

For the first part, by using integration by parts and the constraint $|u|^2 = 1$, we show that the loss of derivatives due to the third-order terms $(\partial_x^3 V_m, u \wedge u_x)u$

included in $P(\partial_x^4)V_m$ are reduced to those due to a linear combination of $T_1(u)\partial_x^2 V_m$ and $(\partial_x T_2)(u)\partial_x V_m$ (see (2.53) and (3.24) for details).

For the second part, we eliminate the second-order terms $T_1(u)\partial_x^2 V_m$ included in $P(\partial_x^4)V_m$ and generated from the first part. By observing that

$$\begin{aligned}
&[\partial_x^2(u \wedge \partial_x^2), B_1(u)\partial_x^{-2}]U_m \\
&\qquad = -2B_1(u)(u \wedge \partial_x^2 U_m) + (B_1(u)u \wedge \cdot + u \wedge B_1(u))\partial_x^2 U_m \\
&\qquad\quad + \text{at most first-order terms},
\end{aligned} \tag{1.7}$$

we set $B_1(u) = (\cdot, u \wedge u_x)u \wedge u_x$. Then we can see that

$$B_1(u)(u \wedge \partial_x^2 U_m) = T_1(u)\partial_x^2 U_m = T_1(u)\partial_x^2 V_m + \text{harmless terms},$$

which combined with the appropriate choice of a_1 eliminates all the second-order terms $T_1(u)\partial_x^2 U_m$. One can see that the first-order term $(\partial_x T_2)(u)\partial_x U_m$ of type V is generated again from the second part. More concretely, the second and the third terms of the right-hand side of (1.7) include second- or first-order derivatives of U_m in addition to $(\partial_x T_2)(u)\partial_x U_m$. Fortunately, however, the form of such terms (except for the terms $(\partial_x T_2)(u)\partial_x U_m$) is essentially limited to types II–IV (see (3.9)–(3.13) for details).

For the third part, we eliminate the first-order terms $(\partial_x T_2)(u)\partial_x V_m$ included in $P(\partial_x^4)V_m$ and generated from the above two parts. By taking $B_2(u) = |u_x|^2\,\mathrm{Id}$, where Id is the identity, we see that

$$[\partial_x^2(u \wedge \partial_x^2), B_2(u)\partial_x^{-2}U_m] = 8(\partial_x T_2)(u)\partial_x U_m + \text{harmless terms},$$

which combined with the appropriate choice of a_2 achieves our aim (see (3.14)–(3.18) for details).

Once the form of $\Lambda(u)$ is decided, we consider the energy $N_m(u)^2 := \|u_x\|_{H^{m-1}}^2 + \|V_m\|_{L^2}^2$, which is equivalent to $\|u_x\|_{H^m}^2$ if we restrict the time interval. As the energy estimate for $N_m(u)^2$ works, we can bound the energy estimate for u_x within H_m, which shows the local existence of a solution. We can make the above argument rigorous by utilizing a fourth-order parabolic regularization. Uniqueness of the solution is proved in the same way, i.e. based on the gauged energy estimate for the difference of two solutions in H^2. The assumptions $m \geqslant 4$ to show the local existence of a solution and $m \geqslant 6$ to show the uniqueness come from the requirement for our gauged energy method to work. Throughout the proof of theorem 1.1 and 1.2, the reader is referred to [16, 20] for basic tools to handle $\mathbb{S}^2$-valued functions.

The paper is organized as follows. In § 2 the classical energy estimate for derivatives of a fourth-order parabolic regularized solution is considered. In § 3 local existence of a solution to (1.1), (1.2) is proved. In § 4 the proofs of theorems 1.1 and 1.2 are completed. The detail of some of the arguments is described in the appendixes.

2. The classical energy estimate

In this section, we consider the IVP for a fourth-order parabolic PDE as an approximation of (1.1) and study the classical energy estimate for derivatives of the regularized solution.

2.1. A fourth-order parabolic regularization

For fixed $\varepsilon \in (0,1]$, we consider the following IVP of the form

$$u_t = -\varepsilon F_4(u, u_x, \dots, \partial_x^3 u_x) + \alpha u \wedge \partial_x^3 u_x u_x$$
$$+ \beta(\partial_x u_x, u_x) u \wedge + \gamma |u_x|^2 u \wedge \partial_x u_x + u \wedge \partial_x u_x \quad \text{in } (0,\infty)\times\mathbb{T}, \tag{2.1}$$
$$u(0,x) = u_0(x) \quad \text{in } \mathbb{T}, \tag{2.2}$$

where $u = u(t,x) \colon [0,\infty)\times\mathbb{T} \to \mathbb{S}^2$ is an unknown function, $u_0 \colon \mathbb{T} \to \mathbb{S}^2$ is the same initial function as that in (1.1), (1.2), and $-\varepsilon F_4(u, u_x, \dots, \partial_x^3 u_x)$ is defined by

$$-\varepsilon F_4(u, u_x, \dots, \partial_x^3 u_x) = -\varepsilon\{\partial_x^3 u_x + 4(\partial_x^2 u_x, u_x)u + 3|\partial_x u_x|^2 u\}. \tag{2.3}$$

Local existence of a unique solution to (2.1), (2.2) follows from the next lemma.

Lemma 2.1. *Let $\varepsilon \in (0,1]$, let m be an integer satisfying $m \geqslant 4$, and let $u_0 \in C(\mathbb{T};\mathbb{S}^2)$ satisfy $u_{0x} \in H^m(\mathbb{T};\mathbb{R}^3)$. Then there exists a positive constant*

$$T_\varepsilon = T(\varepsilon, \|u_{0x}\|_{H^4}) > 0$$

depending on ε, α, β, γ, and on $\|u_{0x}\|_{H^4}$ such that (2.1), (2.2) admits a unique solution $u = u^\varepsilon \in C([0,T_\varepsilon]\times\mathbb{T};\mathbb{S}^2)$ satisfying $u_x^\varepsilon \in C([0,T_\varepsilon]; H^m(\mathbb{T};\mathbb{R}^3))$.

Lemma 2.1 almost falls into the scope of [6, lemma 1] by replacing $\mathbb{R}$ with $\mathbb{T}$ and by restricting a Kähler manifold N to $\mathbb{S}^2$. However, we present the outline of a proof without explicit use of Riemannian geometry in appendix A for interested readers. More precisely, the expression of the added fourth-order parabolic term $F_4(u, u_x, \dots, \partial_x^3 u_x)$ is different from that used in [6]. Our choice comes from the observation that

$$F_4(u, u_x, \dots, \partial_x^3 u_x) = \partial_x^3 u_x - (\partial_x^3 u_x, u)u,$$

the tangent component of $\partial_x^3 u_x$, if $|u|^2 = 1$ is satisfied. Though the difference is not essential, the argument to show that (2.1) is compatible with the constraint $|u|^2 = 1$ becomes a little bit simpler.

2.2. The classical energy estimate

From lemma 2.1, we get a family of solutions to (2.1), (2.2) denoted by $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$. To study the energy estimate for $\|u_x^\varepsilon\|_{H^m}$, we consider PDEs for $\partial_x^k u_x^\varepsilon$ with $k \leqslant m$.

Let $3 \leqslant k \leqslant m$. Set $u = u^\varepsilon$ and $U_k = \partial_x^k u_x$ for simplicity. Then it follows that

$$\partial_t U_k = \partial_x^{k+1} u_t = -\varepsilon \partial_x^{k+1}\{F_4(u, u_x, \dots, \partial_x^3 u_x)\} + P(u, u_x, \dots, \partial_x^{k+4} u_x), \tag{2.4}$$

$$P(u, u_x, \dots, \partial_x^{k+4} u_x) = \alpha \partial_x^{k+1}(u \wedge \partial_x^3 u_x) + \beta \partial_x^{k+1}\{(\partial_x u_x, u_x)u \wedge u_x\}$$
$$+ \gamma \partial_x^{k+1}\{|u_x|^2 u \wedge \partial_x u_x\} + \partial_x^{k+1}(u \wedge \partial_x u_x). \tag{2.5}$$

Each term of the right-hand side of (2.5) can be calculated by the product formula. After lengthy calculations, we obtain that

$$\begin{aligned}
&P(u, u_x, \dots, \partial_x^{k+4} u_x) \\
&\quad = \alpha \partial_x^2 (u \wedge \partial_x^2 U_k) + \alpha (k-1) u_x \wedge \partial_x^3 U_k \\
&\qquad + \tfrac{1}{2}\alpha (k^2 + k - 2) \partial_x (\partial_x u_x \wedge \partial_x U_k) + \partial_x \{(1 + \gamma |u_x|^2) u \wedge \partial_x U_k\} \\
&\qquad + \beta (\partial_x^2 U_k, u_x) u \wedge u_x + \tfrac{1}{6}\alpha (k^3 - 3k^2 - 4k + 6) \partial_x^2 u_x \wedge \partial_x U_k \\
&\qquad + \beta (k+2)(\partial_x U_k, \partial_x u_x) u \wedge u_x + \{\beta (k+1) + 2\gamma\}(\partial_x U_k, u_x) u \wedge \partial_x u_x \\
&\qquad + (\beta + 2k\gamma)(\partial_x u_x, u_x) u \wedge \partial_x U_k + \{k + k\gamma |u_x|^2\} u_x \wedge \partial_x U_k + R_{(k)} \qquad (2.6)
\end{aligned}$$

and

$$\begin{aligned}
R_{(k)} &= ({}_{k+1}\mathrm{C}_{k-1} - 1) \partial_x u_x \wedge U_k \\
&\quad + (\beta + \gamma) \mathcal{O}((|\partial_x^2 u_x|\,|u_x| + |\partial_x u_x|^2 + |\partial_x u_x|\,|u_x|^2)|U_k|) \\
&\quad + \alpha \sum_{j=0}^{k-3} {}_{k+1}\mathrm{C}_j \partial_x^{k-j} u_x \wedge \partial_x^{j+3} u_x + \sum_{j=1}^{k-2} {}_{k+1}\mathrm{C}_j \partial_x^{k-j} u_x \wedge \partial_x^{j+1} u_x \\
&\quad + (\beta + \gamma) \sum_{p_1, p_2, p_3, p_4} \mathcal{O}(|\partial_x^{p_1} u_x||\partial_x^{p_2} u_x||\partial_x^{p_3} u||\partial_x^{p_4} u_x|), \qquad (2.7)
\end{aligned}$$

where the summation in the final line of (2.7) is over all (p_1, p_2, p_3, p_4) satisfying $0 \leqslant p_1, p_2 \leqslant k-1$, $-1 \leqslant p_3 - 1 \leqslant k-1$, $1 \leqslant p_4 \leqslant k-1$, and $p_1 + p_2 + p_3 + p_4 = k+2$. The detail of the calculations to obtain (2.6) and (2.7) is described in appendix B.

We can observe that $R_{(k)}$ includes at most the kth derivative of u_x. Then, from the Sobolev embedding and the Gagliardo–Nirenberg inequality, it follows that

$$\|R_{(k)}\|_{L^2} \leqslant C(\|u_x\|_{H^4}) \|u_x\|_{H^k}. \qquad (2.8)$$

In this section, any non-negative monotonically increasing function in A is denoted by the same $C(A)$, which may depend also on α, β, γ, k but not on ε.

Next we introduce operators $T_i(u)$ and the 'derivative' $(\partial_x T_i)(u)$, $i = 1, 2, \dots, 4$, acting on $\mathbb{R}^3$-valued functions on $\mathbb{T}$ for each t. They are defined by

$$T_1(u)Y = (Y, u_x) u \wedge u_x, \qquad (2.9)$$

$$T_2(u)Y = \tfrac{1}{2}|u_x|^2 u \wedge Y, \qquad (2.10)$$

$$\begin{aligned}
T_3(u)Y = \tfrac{1}{2}\{&(Y, \partial_x u_x) u \wedge u_x + (Y, u_x) u \wedge \partial_x u_x \\
&+ (Y, u \wedge \partial_x u_x) u_x + (Y, u \wedge u_x) \partial_x u_x\}, \qquad (2.11)
\end{aligned}$$

$$T_4(u)Y = (Y, \partial_x u_x + |u_x|^2 u) u \wedge u_x - (Y, u_x) u \wedge \partial_x u_x, \qquad (2.12)$$

$$(\partial_x T_i)(u)Y = \partial_x \{T_i(u)Y\} - T_i(u) \partial_x Y, \qquad (2.13)$$

for any $Y = Y(t, \cdot) \colon \mathbb{T} \to \mathbb{R}^3$. The following propositions will be used frequently.

PROPOSITION 2.2. *For any $Y, Y_1, Y_2 \colon \mathbb{T} \to \mathbb{R}^3$, it follows that*

$$(T_3(u)Y_1, Y_2) = (Y_1, T_3(u)Y_2), \qquad (2.14)$$

$$(T_4(u)Y_1, Y_2) = (Y_1, T_4(u)Y_2), \qquad (2.15)$$

$$(\partial_x T_2)(u)Y = (\partial_x u_x, u_x) u \wedge Y + \tfrac{1}{2}|u_x|^2 u_x \wedge Y. \qquad (2.16)$$

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Proposition 2.3. *For any* $Y\colon \mathbb{T}\to\mathbb{R}^3$, *it follows that*

$$2(Y,\partial_x u_x)u\wedge u_x = (\partial_x T_2)(u)Y + T_3(u)Y + T_4(u)Y - |u_x|^2(Y,u)u\wedge u_x, \tag{2.17}$$

$$2(Y,u_x)u\wedge \partial_x u_x = (\partial_x T_2)(u)Y + T_3(u)Y - T_4(u)Y + |u_x|^2(Y,u)u\wedge u_x, \tag{2.18}$$

$$2(Y,u\wedge u_x)\partial_x u_x = -(\partial_x T_2)(u)Y + T_3(u)Y + T_4(u)Y - |u_x|^2(Y,u\wedge u_x)u, \tag{2.19}$$

$$2(Y,u\wedge \partial_x u_x)u_x = -(\partial_x T_2)(u)Y + T_3(u)Y - T_4(u)Y + |u_x|^2(Y,u\wedge u_x)u. \tag{2.20}$$

Proposition 2.4. *For any* $Y\colon \mathbb{T}\to\mathbb{R}^3$, *it follows that*

$$u_x\wedge Y = (Y,u\wedge u_x)u - (Y,u)u\wedge u_x, \tag{2.21}$$

$$\begin{aligned}\partial_x u_x\wedge Y &= (Y,u\wedge\partial_x u_x)u + (Y,u\wedge u_x)u_x - (Y,u_x)u\wedge u_x\\ &\quad - (Y,u)u\wedge\partial_x u_x,\end{aligned} \tag{2.22}$$

$$\begin{aligned}\partial_x^2 u_x\wedge Y &= -3(\partial_x T_2)(u)Y + \tfrac{3}{2}|u_x|^2 u_x\wedge Y - (Y,u)u\wedge\partial_x^2 u_x\\ &\quad + (Y,u\wedge\partial_x^2 u_x)u.\end{aligned} \tag{2.23}$$

The detail of the proof of propositions 2.2–2.4 is described in appendix C. In their proof and in the calculations below, the fact that

$$|u_x|^2 Y = |u_x|^2(Y,u)u + (Y,u_x)u_x + (Y,u\wedge u_x)u\wedge u_x \tag{2.24}$$

for any $Y\colon \mathbb{T}\to\mathbb{R}^3$ will be used frequently. The proof of (2.24) is short. Indeed, for any $x\in\mathbb{T}$ such that $u_x(x)\neq 0$, $\{u(x), u_x(x)/|u_x(x)|, (u(x)\wedge u_x(x))/|u_x(x)|\}$ forms a basis in $\mathbb{R}^3$, and hence (2.24) holds. For any $x\in\mathbb{T}$ such that $u_x(x)=0$, (2.24) holds since both sides of (2.24) vanish.

We now rewrite the right-hand side of (2.6) by using $T_i(u)$, $i=1,2,\dots,5$. We begin with the second term of the right-hand side of (2.6). By using (2.21), we have

$$u_x\wedge\partial_x^3 U_k = (\partial_x^3 U_k, u\wedge u_x)u - (\partial_x^3 U_k, u)u\wedge u_x.$$

The second term of the right-hand side of the above can be expressed by a form including no third-order derivatives of U_k. The key to observing this is the fact that

$$(\partial_x U_k, u) = -(k+2)(U_k,u_x) - \frac{1}{2}\sum_{j=2}^{k} {}_{k+2}\mathrm{C}_j(\partial_x^{j-1}u_x, \partial_x^{k+1-j}u_x), \tag{2.25}$$

$$(\partial_x^2 U_k, u) = -(k+3)(\partial_x U_k,u_x) - \frac{1}{2}\sum_{j=2}^{k+1} {}_{k+3}\mathrm{C}_j(\partial_x^{j-1}u_x, \partial_x^{k+2-j}u_x), \tag{2.26}$$

$$(\partial_x^3 U_k, u) = -(k+4)(\partial_x^2 U_k,u_x) - \frac{1}{2}\sum_{j=2}^{k+2} {}_{k+4}\mathrm{C}_j(\partial_x^{j-1}u_x, \partial_x^{k+3-j}u_x), \tag{2.27}$$

which can be obtained by taking derivatives in x of both sides of $|u|^2=1$ repeatedly. From (2.9), (2.17) and (2.27) it follows that

$$\begin{aligned}u_x\wedge\partial_x^3 U_k &= (\partial_x^3 U_k, u\wedge u_x)u + (k+4)(\partial_x^2 U_k, u_x)u\wedge u_x\\ &\quad + {}_{k+4}\mathrm{C}_2(\partial_x U_k, \partial_x u_x)u\wedge u_x\\ &\quad + \frac{1}{2}\sum_{j=3}^{k+1} {}_{k+4}\mathrm{C}_j(\partial_x^{j-1}u_x, \partial_x^{k+3-j}u_x)u\wedge u_x\end{aligned}$$

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$$
\begin{aligned}
&= (\partial_x^3 U_k, u \wedge u_x)u + (k+4)T_1(u)\partial_x^2 U_k \\
&\quad + \tfrac{1}{2}\,{}_{k+4}\mathrm{C}_2\{(\partial_x T_2)(u)\partial_x U_k + T_3(u)\partial_x U_k + T_4(u)\partial_x U_k \\
&\qquad\qquad - |u_x|^2(\partial_x U_k, u)u \wedge u_x\} \\
&\quad + \frac{1}{2}\sum_{j=3}^{k+1} {}_{k+4}\mathrm{C}_j(\partial_x^{j-1}u_x, \partial_x^{k+3-j}u_x)u \wedge u_x.
\end{aligned}
\tag{2.28}
$$

Combining (2.9)–(2.13), propositions 2.2–2.4 and (2.28), we obtain

$$
\begin{aligned}
&P(u, u_x, \dots, \partial_x^{k+4}u_x) \\
&= \alpha\partial_x^2(u \wedge \partial_x^2 U_k) + \alpha(k-1)(\partial_x^3 U_k, u \wedge u_x)u + \alpha(k-1)(k+4)T_1(u)\partial_x^2 U_k \\
&\quad + \tfrac{1}{4}\alpha(k-1)(k+3)(k+4)\{(\partial_x T_2)(u)\partial_x U_k + T_3(u)\partial_x U_k + T_4(u)\partial_x U_k \\
&\qquad\qquad - |u_x|^2(\partial_x U_k, u)u \wedge u_x\} \\
&\quad + \tfrac{1}{2}\alpha(k^2+k-2)\partial_x(\partial_x u_x \wedge \partial_x U_k) + \partial_x\{(1+\gamma|u_x|^2)u \wedge \partial_x U_k\} + \beta T_1(u)\partial_x^2 U_k \\
&\quad + \tfrac{1}{6}\alpha(k^3-3k^2-4k+6)\{-3(\partial_x T_2)(u)\partial_x U_k + \tfrac{3}{2}|u_x|^2 u_x \wedge \partial_x U_k \\
&\qquad\qquad - (\partial_x U_k, u)u \wedge \partial_x^2 u_x + (\partial_x U_k, u \wedge \partial_x^2 u_x)u\} \\
&\quad + \tfrac{1}{2}\beta(k+2)\{(\partial_x T_2)(u)\partial_x U_k + T_3(u)\partial_x U_k + T_4(u)\partial_x U_k \\
&\qquad\qquad - |u_x|^2(\partial_x U_k, u)u \wedge u_x\} \\
&\quad + \tfrac{1}{2}(\beta(k+1)+2\gamma)\{(\partial_x T_2)(u)\partial_x U_k + T_3(u)\partial_x U_k \\
&\qquad\qquad - T_4(u)\partial_x U_k + |u_x|^2(\partial_x U_k, u)u \wedge u_x\} \\
&\quad + (\beta + 2k\gamma)\{(\partial_x T_2)(u)\partial_x U_k - \tfrac{1}{2}|u_x|^2 u_x \wedge \partial_x U_k\} \\
&\quad + \{k + k\gamma|u_x|^2\}u_x \wedge \partial_x U_k + \tilde{R}_{(k)} \\
&= \alpha\partial_x^2(u \wedge \partial_x^2 U_k) + \alpha(k-1)(\partial_x^3 U_k, u \wedge u_x)u + \{\alpha(k^2+3k-4)+\beta\}T_1(u)\partial_x^2 U_k \\
&\quad + \tfrac{1}{2}\alpha(k^2+k-2)\partial_x(\partial_x u_x \wedge \partial_x U_k) + \partial_x\{(1+\gamma|u_x|^2)u \wedge \partial_x U_k\} \\
&\quad + \{-\tfrac{1}{4}\alpha(k^3-12k^2-13k+24) + \tfrac{1}{2}\beta(2k+5) + \gamma(1+2k)\}(\partial_x T_2)(u)\partial_x U_k \\
&\quad + \{\tfrac{1}{4}\alpha(k^3+6k^2+5k-12) + \tfrac{1}{2}\beta(2k+3) + \gamma\}T_3(u)\partial_x U_k \\
&\quad + \{\tfrac{1}{4}\alpha(k^3+6k^2+5k-12) + \tfrac{1}{2}\beta - \gamma\}T_4(u)\partial_x U_k \\
&\quad + \{k + \tfrac{1}{4}(\alpha(k^3-3k^2-4k+6) - 2\beta)|u_x|^2\}u_x \wedge \partial_x U_k \\
&\quad - \tfrac{1}{6}\alpha(k^3-3k^2-4k+6)(\partial_x U_k, u)u \wedge \partial_x^2 u_x \\
&\quad + \{-\tfrac{1}{4}\alpha(k^3+6k^2+5k-12) - \tfrac{1}{2}\beta + \gamma\}|u_x|^2(\partial_x U_k, u)u \wedge u_x \\
&\quad + \tfrac{1}{6}\alpha(k^3-3k^2-4k+6)(\partial_x U_k, u \wedge \partial_x^2 u_x)u + \tilde{R}_{(k)},
\end{aligned}
\tag{2.29}
$$

where

$$
\tilde{R}_{(k)} = R_{(k)} + \frac{\alpha(k-1)}{2}\sum_{j=3}^{k+1} {}_{k+4}\mathrm{C}_j(\partial_x^{j-1}u_x, \partial_x^{k+3-j}u_x)u \wedge u_x. \tag{2.30}
$$

Therefore, we conclude that the PDE satisfied by U_k becomes

$$
\partial_t U_k = -\varepsilon\partial_x^{k+1}\{F_4(u, u_x, \dots, \partial_x^3 u_x)\} + P_4(u)U_k + P_1(u)U_k + \tilde{R}_{(k)}, \tag{2.31}
$$

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$$
\begin{aligned}
P_4(u) &= \alpha\partial_x^2(u\wedge\partial_x^2) + \alpha(k-1)(\partial_x^3\cdot, u\wedge u_x)u + A_{1,k}T_1(u)\partial_x^2 \\
&\quad + c_{2,k}\partial_x(\partial_x u_x\wedge\partial_x) + \partial_x\{(1+\gamma|u_x|^2)u\wedge\partial_x\} \\
&\quad + A_{2,k}(\partial_x T_2)(u)\partial_x + c_{3,k}T_3(u)\partial_x + c_{4,k}T_4(u)\partial_x, \qquad (2.32)\\
P_1(u) &= b_{1,k}u_x\wedge\partial_x + b_{2,k}|u_x|^2u_x\wedge\partial_x + b_{3,k}(\partial_x\cdot, u)u\wedge\partial_x^2u_x \\
&\quad + b_{4,k}|u_x|^2(\partial_x\cdot, u)u\wedge u_x + b_{5,k}(\partial_x\cdot, u\wedge\partial_x^2u_x)u, \qquad (2.33)
\end{aligned}
$$

where

$$
\left.
\begin{aligned}
A_{1,k} &= \alpha(k^2+3k-4)+\beta, \\
A_{2,k} &= -\tfrac{1}{4}\alpha(k^3-12k^2-13k+24) + \tfrac{1}{2}\beta(2k+5) + \gamma(1+2k),
\end{aligned}
\right\} \qquad (2.34)
$$

and $c_{j,k}$ $(j=2,3,4)$ and $b_{j,k}$ $(1\leqslant j\leqslant 5)$ are also constants depending only on α, β, γ, k, but the explicit forms are not required.

We now evaluate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U_k\|_{L^2}^2 = \langle\partial_t U_k, U_k\rangle$$

by using (2.31)–(2.33). A simple computation yields

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U_k\|_{L^2}^2 &= -\varepsilon\langle\partial_x^{k+1}\{F_4(u,u_x,\dots,\partial_x^3u_x)\}, U_k\rangle + \alpha\langle\partial_x^2(u\wedge\partial_x^2U_k), U_k\rangle \\
&\quad + \alpha(k-1)\langle(\partial_x^3U_k, u\wedge u_x)u, U_k\rangle + A_{1,k}\langle T_1(u)\partial_x^2U_k, U_k\rangle \\
&\quad + c_{2,k}\langle\partial_x(\partial_x u_x\wedge\partial_x U_k), U_k\rangle + \langle\partial_x\{(1+\gamma|u_x|^2)u\wedge\partial_x U_k\}, U_k\rangle \\
&\quad + A_{2,k}\langle(\partial_x T_2)(u)\partial_x U_k, U_k\rangle + c_{3,k}\langle T_3(u)\partial_x U_k, U_k\rangle \\
&\quad + c_{4,k}\langle T_4(u)\partial_x U_k, U_k\rangle + \langle P_1(u)U_k, U_k\rangle + \langle\tilde{R}_{(k)}, U_k\rangle.
\end{aligned}
$$

We evaluate each term separately. To begin with, integration by parts yields

$$\langle\partial_x^2(u\wedge\partial_x^2U_k), U_k\rangle = \langle u\wedge\partial_x^2U_k, \partial_x^2U_k\rangle = 0, \qquad (2.35)$$

$$\langle\partial_x(\partial_x u_x\wedge\partial_x U_k), U_k\rangle = -\langle\partial_x u_x\wedge\partial_x U_k, \partial_x U_k\rangle = 0, \qquad (2.36)$$

$$\langle\partial_x\{(1+\gamma|u_x|^2)u\wedge\partial_x U_k\}, U_k\rangle = \langle(1+\gamma|u_x|^2)u\wedge\partial_x U_k, \partial_x U_k\rangle = 0. \qquad (2.37)$$

We next recall that $T_3(u)$ and $T_4(u)$ are symmetric matrix-valued functions, as observed in (2.14) and (2.15). By using this, the Sobolev embedding of H^1 into L^∞ and integration by parts, we have

$$\langle T_3(u)\partial_x U_k, U_k\rangle = -\tfrac{1}{2}\langle(\partial_x T_3)(u)U_k, U_k\rangle \leqslant C(\|u_x\|_{H^3})\|U_k\|_{L^2}^2, \qquad (2.38)$$

$$\langle T_4(u)\partial_x U_k, U_k\rangle = -\tfrac{1}{2}\langle(\partial_x T_4)(u)U_k, U_k\rangle \leqslant C(\|u_x\|_{H^3})\|U_k\|_{L^2}^2. \qquad (2.39)$$

We next look at $\langle P_1(u)U_k, U_k\rangle$. We can bound the estimate for $\langle P_1(u)U_k, U_k\rangle$ within H^k with the help of the constraint $|u|^2 = 1$. First, by using (2.25) and the Gagliardo–Nirenberg inequality, we have

$$
\begin{aligned}
\|(\partial_x U_k, u)u\wedge\partial_x^2u_x\|_{L^2} &\leqslant C(\|u_x\|_{H^3})\|u_x\|_{H^k}, \\
\|(\partial_x U_k, u)u\wedge u_x\|_{L^2} &\leqslant C(\|u_x\|_{H^3})\|u_x\|_{H^k},
\end{aligned}
$$

which imply that

$$\langle(\partial_x U_k, u)u\wedge\partial_x^2u_x, U_k\rangle \leqslant C(\|u_x\|_{H^3})\|u_x\|_{H^k}^2, \qquad (2.40)$$

$$\langle(\partial_x U_k, u)u\wedge u_x, U_k\rangle \leqslant C(\|u_x\|_{H^3})\|u_x\|_{H^k}^2. \qquad (2.41)$$

Secondly, by integration by parts, the Sobolev embedding, (2.40) and $(U_k, u) = -(k+1)(\partial_x^{k-1}u_x, u_x) + \cdots$, we have

$$\begin{aligned}\langle(\partial_x U_k, u\wedge\partial_x^2 u_x)u, U_k\rangle &= -\langle(U_k, u\wedge\partial_x^2 u_x)u, \partial_x U_k\rangle - \langle(U_k, u\wedge\partial_x^2 u_x)u_x, U_k\rangle \\ &\quad - \langle(U_k, u\wedge\partial_x^3 u_x)u, U_k\rangle - \langle(U_k, u_x\wedge\partial_x^2 u_x)u, U_k\rangle \\ &\leqslant C(\|u_x\|_{H^3})\|u_x\|_{H^k}^2. \end{aligned} \tag{2.42}$$

Thirdly, (2.21) with $Y = \partial_x U_k$ yields

$$\langle u_x\wedge\partial_x U_k, U_k\rangle = \langle(\partial_x U_k, u\wedge u_x)u, U_k\rangle - \langle(\partial_x U_k, u)u\wedge u_x, U_k\rangle.$$

The right-hand side of the above can be evaluated in the same way as that used to obtain (2.41) and (2.42), which yields

$$\langle u_x\wedge\partial_x U_k, U_k\rangle \leqslant C(\|u_x\|_{H^3})\|u_x\|_{H^k}^2, \tag{2.43}$$

$$\langle|u_x|^2 u_x\wedge\partial_x U_k, U_k\rangle \leqslant C(\|u_x\|_{H^3})\|u_x\|_{H^k}^2. \tag{2.44}$$

Collecting these estimates, we obtain

$$\langle P_1(u)U_k, U_k\rangle \leqslant C(\|u_x\|_{H^3})\|u_x\|_{H^k}^2. \tag{2.45}$$

We next look at $\langle\tilde{R}_{(k)}, U_k\rangle$. In the same way as the estimate (2.8) for $R_{(k)}$, we have $\|\tilde{R}_{(k)}\|_{L^2} \leqslant C(\|u_x\|_{H^4})\|u_x\|_{H^k}$. This implies that

$$\langle\tilde{R}_{(k)}, U_k\rangle \leqslant C(\|u_x\|_{H^4})\|u_x\|_{H^k}^2. \tag{2.46}$$

We next look at $-\varepsilon\partial_x^{k+1}\{F_4(u, u_x, \dots, \partial_x^3 u_x)\}$. In view of (2.3), we write

$$-\varepsilon\partial_x^{k+1}\{F_4(u, u_x, \dots, \partial_x^3 u_x)\} = -\varepsilon\partial_x^4 U_k + \varepsilon N(u, u_x, \dots, \partial_x^{k+3}u_x), \tag{2.47}$$

where, for the term in $N(u, u_x, \dots, \partial_x^{k+3}u_x)$, it can be shown by integration by parts and the Gagliardo–Nirenberg inequality that

$$\langle N(u, u_x, \dots, \partial_x^{k+3}u_x), U_k\rangle \leqslant C(\|u_x\|_{H^3})\|u_x\|_{H^k}\|u_x\|_{H^{k+2}}.$$

The loss of derivative of order 2 can be absorbed by the parabolic term $-\varepsilon\partial_x^4 U_k$. Indeed, by the Young inequality of the form $ab \leqslant a^2/2 + b^2/2$ for any $a, b > 0$ and integration by parts, we can show that

$$\begin{aligned}\langle-\varepsilon\partial_x^{k+1}\{F_4(u, \dots, \partial_x^3 u_x)\}, U_k\rangle &= -\varepsilon\langle\partial_x^4 U_k, U_k\rangle + \varepsilon\langle N(u, \dots, \partial_x^{k+3}u_x), U_k\rangle \\ &\leqslant -\tfrac{1}{2}\varepsilon\|\partial_x^2 U_k\|_{L^2}^2 + C(\|u_x\|_{H^3})\|u_x\|_{H^k}^2. \end{aligned} \tag{2.48}$$

We next evaluate $\langle(\partial_x^3 U_k, u\wedge u_x)u, U_k\rangle$, $\langle T_1(u)\partial_x^2 U_k, U_k\rangle$ and $\langle(\partial_x T_2)(u)\partial_x U_k, U_k\rangle$ with loss of derivatives of order 1. It is easy to see that

$$\langle T_1(u)\partial_x^2 U_k, U_k\rangle \leqslant C(\|u_x\|_{H^2})\|u_x\|_{H^{k+1}}^2, \tag{2.49}$$

$$\langle(\partial_x T_2)(u)\partial_x U_k, U_k\rangle \leqslant C(\|u_x\|_{H^2})\|u_x\|_{H^{k+1}}^2. \tag{2.50}$$

We turn our eyes to $\langle(\partial_x^3 U_k, u\wedge u_x)u, U_k\rangle$. By integration by parts, we have

$$\langle(\partial_x^3 U_k, u\wedge u_x)u, U_k\rangle = E_1 + E_2 + E_3,$$

where

$$E_1 = -\langle(\partial_x^2 U_k, u\wedge u_x)u, \partial_x U_k\rangle,$$
$$E_2 = -\langle(\partial_x^2 U_k, u\wedge u_x)u_x, U_k\rangle,$$
$$E_3 = -\langle(\partial_x^2 U_k, u\wedge \partial_x u_x)u, U_k\rangle.$$

We first look at E_1. By applying (2.25) first to express $(\partial_x U_k, u)$ without $\partial_x U_k$, and next by using the integration by parts and the Gagliardo–Nirenberg inequality, we deduce that

$$\begin{aligned} E_1 &\leqslant (k+2)\langle(\partial_x^2 U_k, u\wedge u_x)u_x, U_k\rangle + {}_{k+2}\mathrm{C}_2\langle(\partial_x^2 U_k, u\wedge u_x)\partial_x u_x, \partial_x^{k-1}u_x\rangle \\ &\quad + C(\|u_x\|_{H^3})\|u_x\|_{H^k}^2 \\ &\leqslant -(k+2)E_2 - {}_{k+2}\mathrm{C}_2\langle(\partial_x U_k, u\wedge u_x)\partial_x u_x, U_k\rangle + C(\|u_x\|_{H^3})\|u_x\|_{H^k}^2. \end{aligned}$$

In the same way, we use $(U_k, u) = -(k+1)(\partial_x^{k-1}u_x, u_x) + \cdots$ to obtain

$$\begin{aligned} E_3 &\leqslant (k+1)\langle(\partial_x^2 U_k, u\wedge \partial_x u_x)u_x, \partial_x^{k-1}u_x\rangle + C(\|u_x\|_{H^3})\|u_x\|_{H^k}^2 \\ &\leqslant -(k+1)\langle(\partial_x U_k, u\wedge \partial_x u_x)u_x, U_k\rangle + C(\|u_x\|_{H^3})\|u_x\|_{H^k}^2. \end{aligned}$$

Hence, it follows that

$$\begin{aligned} E_1 + E_2 + E_3 &\leqslant -(k+1)E_2 - \tfrac{1}{2}(k+1)(k+2)\langle(\partial_x U_k, u\wedge u_x)\partial_x u_x, U_k\rangle \\ &\quad - (k+1)\langle(\partial_x U_k, u\wedge \partial_x u_x)u_x, U_k\rangle + C(\|u_x\|_{H^3})\|u_x\|_{H^k}^2. \end{aligned} \tag{2.51}$$

We look at E_2 on the right-hand side of (2.51). By integrating by parts,

$$\begin{aligned} E_2 &= \langle(\partial_x U_k, u\wedge u_x)u_x, \partial_x U_k\rangle + \langle(\partial_x U_k, u\wedge u_x)\partial_x u_x, U_k\rangle \\ &\quad + \langle(\partial_x U_k, u\wedge \partial_x u_x)u_x, U_k\rangle \\ &= -\langle(U_k, u\wedge u_x)u_x, \partial_x^2 U_k\rangle - \langle(U_k, u\wedge u_x)\partial_x u_x, \partial_x U_k\rangle \\ &\quad - \langle(U_k, u\wedge \partial_x u_x)u_x, \partial_x U_k\rangle + \langle(\partial_x U_k, u\wedge u_x)\partial_x u_x, U_k\rangle \\ &\quad + \langle(\partial_x U_k, u\wedge \partial_x u_x)u_x, U_k\rangle \\ &= -\langle T_1(u)\partial_x^2 U_k, U_k\rangle - \langle(\partial_x U_k, \partial_x u_x)u\wedge u_x, U_k\rangle - \langle(\partial_x U_k, u_x)u\wedge \partial_x u_x, U_k\rangle \\ &\quad + \langle(\partial_x U_k, u\wedge u_x)\partial_x u_x, U_k\rangle + \langle(\partial_x U_k, u\wedge \partial_x u_x)u_x, U_k\rangle. \end{aligned} \tag{2.52}$$

By substituting (2.52) into (2.51), we have

$$\begin{aligned} \langle(\partial_x^3 U_k, u\wedge u_x)u, U_k\rangle &\leqslant (k+1)\langle T_1(u)\partial_x^2 U_k, U_k\rangle + (k+1)\langle(\partial_x U_k, \partial_x u_x)u\wedge u_x, U_k\rangle \\ &\quad + (k+1)\langle(\partial_x U_k, u_x)u\wedge \partial_x u_x, U_k\rangle \\ &\quad - \tfrac{1}{2}(k+1)(k+4)\langle(\partial_x U_k, u\wedge u_x)\partial_x u_x, U_k\rangle \\ &\quad - 2(k+1)\langle(\partial_x U_k, u\wedge \partial_x u_x)u_x, U_k\rangle \\ &\quad + C(\|u_x\|_{H^3})\|u_x\|_{H^k}^2. \end{aligned}$$

Furthermore, by applying (2.17)–(2.20) with $Y = \partial_x U_k$, we deduce that

$$\begin{aligned} &\langle(\partial_x^3 U_k, u\wedge u_x)u, U_k\rangle \\ &\qquad \leqslant (k+1)\langle T_1(u)\partial_x^2 U_k, U_k\rangle + \tfrac{1}{4}(k+1)(k+12)\langle(\partial_x T_2)(u)\partial_x U_k, U_k\rangle \end{aligned}$$

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$$
\begin{aligned}
&+ c_{5,k}\langle T_3(u)\partial_x U_k, U_k\rangle + c_{6,k}\langle T_4(u)\partial_x U_k, U_k\rangle \\
&+ c_{7,k}\langle |u_x|^2(\partial_x U_k, u)u \wedge u_x, U_k\rangle + c_{8,k}\langle |u_x|^2(\partial_x U_k, u\wedge u_x)u, U_k\rangle \\
&+ C(\|u_x\|_{H^3})\|u_x\|_{H^k}^2,
\end{aligned}
$$

where $c_{5,k}, \dots, c_{8,k}$ are real constants depending only on k and the explicit forms are not required. Recalling (2.38), (2.39) and the argument to show (2.40)–(2.42), we immediately obtain

$$
\begin{aligned}
&\langle(\partial_x^3 U_k, u\wedge u_x)u, U_k\rangle \\
&\qquad \leqslant (k+1)\langle T_1(u)\partial_x^2 U_k, U_k\rangle + \tfrac{1}{4}(k+1)(k+12)\langle(\partial_x T_2)(u)\partial_x U_k, U_k\rangle \\
&\qquad\quad + C(\|u_x\|_{H^3})\|u_x\|_{H^k}^2.
\end{aligned}
\tag{2.53}
$$

By combining (2.35)–(2.39), (2.45), (2.46), (2.48), and (2.53), we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U_k\|_{L^2}^2 \leqslant -\frac{\varepsilon}{2}\|\partial_x^2 U_k\|_{L^2}^2 &+ \{\alpha(k^2-1) + A_{1,k}\}\langle T_1(u)\partial_x^2 U_k, U_k\rangle \\
&+ \{\tfrac{1}{4}\alpha(k^2-1)(k+12) + A_{2,k}\}\langle(\partial_x T_2)(u)\partial_x U_k, U_k\rangle \\
&+ C(\|u_x\|_{H^4})\|u_x\|_{H^k}^2.
\end{aligned}
\tag{2.54}
$$

Specifically, by using (2.49) and (2.50), we conclude that

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U_k\|_{L^2}^2 \leqslant -\frac{\varepsilon}{2}\|\partial_x^2 U_k\|_{L^2}^2 + C(\|u_x\|_{H^4})\|u_x\|_{H^{k+1}}^2 \quad \text{for } t\in[0,T_\varepsilon] \tag{2.55}
$$

with $C = C(\|u_x\|_{H^4})$, which depends on α, β, γ, k but not on $\varepsilon\in(0,1]$.

3. Proof of the existence of a solution locally in time

This section is devoted to the proof of local existence of a solution to (1.1), (1.2). More precisely, the goal of the section is to prove the following theorem.

THEOREM 3.1. *Let m be a positive integer satisfying $m \geqslant 4$. Then, for any $u_0 \in C(\mathbb{T};\mathbb{S}^2)$ satisfying $u_{0x}\in H^m(\mathbb{T};\mathbb{R}^3)$, there exists a constant $T = T(\|u_{0x}\|_{H^4}) > 0$ depending on α, β, γ, m, and on $\|u_{0x}\|_{H^4}$ such that (1.1), (1.2) admits a solution $u\in C([0,T]\times\mathbb{T};\mathbb{S}^2)$ that satisfies*

$$
u_x \in L^\infty(0,T;H^m(\mathbb{T};\mathbb{R}^3))\cap C([0,T];H^{m-1}(\mathbb{T};\mathbb{R}^3)).
$$

Proof of theorem 3.1. Let $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ be the family of solutions to (2.1), (2.2) constructed in lemma 2.1. We set

$$
\begin{aligned}
V_m^\varepsilon &:= U_m^\varepsilon + \Lambda(u^\varepsilon)U_m^\varepsilon \\
&= U_m^\varepsilon + \sum_{i=1}^{2}\Lambda_i(u^\varepsilon)U_m^\varepsilon,
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
\Lambda_1(u^\varepsilon)U_m^\varepsilon &= -\frac{d_1}{2\alpha}(\partial_x^{m-2}u_x^\varepsilon, u^\varepsilon\wedge u_x^\varepsilon)u^\varepsilon\wedge u_x^\varepsilon, \\
\Lambda_2(u^\varepsilon)U_m^\varepsilon &= \frac{d_2}{8\alpha}|u_x^\varepsilon|^2\partial_x^{m-2}u_x^\varepsilon,
\end{aligned}
$$

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where $U_m^\varepsilon = \partial_x^m u_x^\varepsilon$, and $d_1, d_2 \in \mathbb{R}$ are real constants, depending only on α, β, γ, m, that will be decided later. We introduce $N_m(u^\varepsilon(t))$, the square of which is defined by

$$N_m^2(u^\varepsilon(t)) = \|u_x^\varepsilon(t)\|_{H^{m-1}}^2 + \|V_m^\varepsilon(t)\|_{L^2}^2. \tag{3.2}$$

We restrict the time interval to $[0, T_\varepsilon^\star]$ with T_ε^* defined by

$$T_\varepsilon^* = \sup\{T > 0 \mid N_4(u^\varepsilon(t)) \leqslant 2N_4(u_0) \text{ for all } t \in [0, T]\}.$$

The restriction ensures the equivalence between $N_m^2(u^\varepsilon)$ and $\|u_x^\varepsilon\|_{H^m}^2$. Indeed, the Sobolev embedding shows that there exists a constant $C = C(\|u_{0x}\|_{H^4}) > 1$ that is independent of $\varepsilon \in (0, 1]$ such that

$$\frac{1}{C} N_m(u^\varepsilon(t)) \leqslant \|u_x^\varepsilon(t)\|_{H^m} \leqslant C N_m(u^\varepsilon(t)) \quad \text{for } t \in [0, T_\varepsilon^\star]. \tag{3.3}$$

We shall show that there exists $T = T(\|u_{0x}\|_{H^4}) > 0$, which is independent of $\varepsilon \in (0, 1]$ and m, such that $T_\varepsilon^\star \geqslant T$ and $\{N_m(u^\varepsilon)\}_{\varepsilon \in (0,1]}$ is bounded in $L^\infty(0, T)$. If this is true, (3.3) yields that $\{u_x^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0, T; H^m(\mathbb{T}; \mathbb{R}^3))$. This implies that $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0, T; H^{m+1}(\mathbb{T}; \mathbb{R}^3))$ since $\mathbb{T}$ is compact. Then the standard compactness argument shows the existence of a $u \in C([0, T] \times \mathbb{T}; \mathbb{S}^2)$ that satisfies $u_x \in L^\infty(0, T; H^m(\mathbb{T}; \mathbb{R}^3)) \cap C([0, T]; H^{m-1}(\mathbb{T}; \mathbb{R}^3))$ and solves (1.1), (1.2).

Bearing this in mind, we evaluate $\{N_m(u^\varepsilon)\}_{\varepsilon\in(0,1]}$. Set $u = u^\varepsilon$, $U_m = U_m^\varepsilon$ and $V_m = V_m^\varepsilon$ for ease of notation. In what follows in this section, any positive constant that depends on α, β, γ, m, $\|u_{0x}\|_{H^4}$ and not on $\varepsilon \in (0, 1]$ will be denoted by the same C. Then the Sobolev embedding yields $\|\partial_x^k u_x\|_{L^\infty((0,T_\varepsilon^\star)\times\mathbb{T})} \leqslant C$ for $k = 0, 1, \dots, 3$.

We begin with the energy estimate for $\|u_x\|_{H^{m-1}}^2$. In view of (2.55) with $k = 3, 4, \dots, m-1$ and a similar computation for $\partial_x^k u_x$ with $k = 0, 1, 2$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_x\|_{H^{m-1}}^2 \leqslant -\frac{\varepsilon}{2}\sum_{k=0}^{m-1}\|\partial_x^2 U_m\|_{L^2}^2 + CN_m^2. \tag{3.4}$$

We next consider the energy estimate for $\|V_m\|_{L^2}^2$. For this purpose, we shall investigate the PDE for V_m and evaluate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|V_m\|_{L^2}^2 = \langle \partial_t V_m, V_m\rangle.$$

From (2.31) and (2.47) with $k = m$ and from the definition of V_m it follows that

$$\begin{aligned}\partial_t V_m &= \partial_t U_m + \partial_t(\Lambda(u)U_m)\\ &= -\varepsilon\partial_x^4 V_m + \varepsilon\{\partial_x^4(\Lambda(u)U_m) + N(u, u_x, \dots, \partial_x^{m+3}u_x)\} + P_4(u)V_m\\ &\quad - \{P_4(u)(\Lambda(u)U_m) - \partial_t(\Lambda(u)U_m)\} + P_1(u)U_m + \tilde{R}_{(m)}.\end{aligned} \tag{3.5}$$

From (2.46) and (3.3) it follows that

$$\langle \tilde{R}_{(m)}, V_m\rangle \leqslant \|\tilde{R}_{(m)}\|_{L^2}\|V_m\|_{L^2} \leqslant C\|u_x\|_{H^m}N_m \leqslant CN_m^2. \tag{3.6}$$

From (2.45) and (3.3), it follows that $\langle P_1(u)U_m, U_m\rangle \leqslant CN_m^2$. By using this and by noting that $\Lambda(u)U_m$ includes at most $(m-2)$th derivatives of u_x, we get

$$\langle P_1(u)U_m, V_m\rangle = \langle P_1(u)U_m, U_m\rangle + \langle P_1(u)U_m, \Lambda(u)U_m\rangle \leqslant CN_m^2. \tag{3.7}$$

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In this section, any term whose L^2-norm is bounded by CN_m, such as $\tilde{R}_{(m)}$, will be denoted by R_{ij} subscripted by some $(i,j) \in (\mathbb{N}\cup\{0\})^2$, and any term including $\partial_x U_m$ (without the symmetric structure such as $T_3(u)\partial_x U_m$ or $T_4(u)\partial_x U_m$) whose inner product with V_m can be bounded by CN_m^2, such as $P_1(u)\partial_x U_m$, will be denoted by $\tilde{P}_{ij}U_m$ subscripted with some $(i,j) \in (\mathbb{N}\cup\{0\})^2$.

We now look at $P_4(u)(\Lambda(u)U_m) - \partial_t(\Lambda(u)U_m)$. We write

$$P_4(u)(\Lambda(u)U_m) - \partial_t(\Lambda(u)U_m) = \sum_{i=1}^{2}\{P_4(u)(\Lambda_i(u)U_m) - \partial_t(\Lambda_i(u)U_m)\}, \tag{3.8}$$

and consider each $P_4(u)(\Lambda_i(u)U_m) - \partial_t(\Lambda_i(u)U_m)$ with $i = 1, 2$ separately.

First, we consider $P_4(u)(\Lambda_1(u)U_m) - \partial_t(\Lambda_1(u)U_m)$. Roughly speaking, $\Lambda_1(u)$ acts on U_m as a pseudo-differential operator of order -2. In other words, up to second-order partial differential operators in $P_4(u)$ are negligible, since $\Lambda_1(u)U_m$ includes at most $(m-2)$th derivatives of u_x. Indeed, a simple computation yields

$$\begin{aligned}
&P_4(\Lambda_1(u)U_m)\\
&\quad = \alpha\partial_x^2\{u \wedge \partial_x^2(\Lambda_1(u)U_m)\} + \alpha(m-1)(\partial_x^3(\Lambda_1(u)U_m), u \wedge u_x)u + R_{10}\\
&\quad = \alpha\partial_x^2\{u \wedge \partial_x^2(\Lambda_1(u)U_m)\} - \tfrac{1}{2}d_1(m-1)((\partial_x U_m, u \wedge u_x)u \wedge u_x, u \wedge u_x)u + R_{11}\\
&\quad = \alpha\partial_x^2\{u \wedge \partial_x^2(\Lambda_1(u)U_m)\} - \tfrac{1}{2}d_1(m-1)|u_x|^2(\partial_x U_m, u \wedge u_x)u + R_{11}.
\end{aligned}$$

Moreover, (2.31) and (2.47) with $k = m-2$ and the assumption that $m \geqslant 4$ imply that

$$\begin{aligned}
\partial_t(\Lambda_1(u)U_m) &= -\frac{d_1}{2\alpha}(\partial_t\partial_x^{m-2}u_x, u \wedge u_x)u \wedge u_x + R_{12}\\
&= \frac{d_1\varepsilon}{2\alpha}(\partial_x^{m+2}u_x - N(u,\dots,\partial_x^{m+1}u_x), u \wedge u_x)u \wedge u_x\\
&\qquad - \tfrac{1}{2}d_1(\partial_x^2\{u \wedge \partial_x^2(\partial_x^{m-2}u_x)\}, u \wedge u_x)u \wedge u_x\\
&\qquad - \tfrac{1}{2}d_1(m-3)((\partial_x^3(\partial_x^{m-2}u_x), u \wedge u_x)u, u \wedge u_x)u \wedge u_x + R_{13}\\
&= -\varepsilon\tilde{R}_1 - \tfrac{1}{2}d_1(u \wedge \partial_x^2 U_m, u \wedge u_x)u \wedge u_x\\
&\qquad - d_1(u_x \wedge \partial_x U_m, u \wedge u_x)u \wedge u_x + R_{14}\\
&= -\varepsilon\tilde{R}_1 + \Lambda_1(u)\alpha\partial_x^2\{u \wedge \partial_x^2 U_m\} + d_1|u_x|^2(\partial_x U_m, u)u \wedge u_x + R_{14},
\end{aligned}$$

where $\tilde{R}_1 = -(d_1/2\alpha)(\partial_x^{m+2}u_x - N(u,\dots,\partial_x^{m+1}u_x), u \wedge u_x)u \wedge u_x$. Collecting these relations, we have

$$\begin{aligned}
&P_4(\Lambda_1(u)U_m) - \partial_t(\Lambda_1(u)U_m)\\
&\qquad = \varepsilon\tilde{R}_1 + [\alpha\partial_x^2\{u \wedge \partial_x^2\}, \Lambda_1(u)]U_m + \tilde{P}_{1,4}U_m + R_{11} - R_{14},
\end{aligned} \tag{3.9}$$

where

$$\tilde{P}_{1,4}U_m = -\tfrac{1}{2}d_1(m-1)|u_x|^2(\partial_x U_m, u \wedge u_x)u - d_1|u_x|^2(\partial_x U_m, u)u \wedge u_x.$$

To deal with the second term of the right-hand side of (3.9), we set $\Lambda_1(u) = -(d_1/2\alpha)B_1(u)\partial_x^{-2}$ with $B_1(u) = (\cdot, u \wedge u_x)u \wedge u_x$ and write

$$[\alpha\partial_x^2\{u \wedge \partial_x^2\}, \Lambda_1(u)]U_m = -\tfrac{1}{2}d_1[\partial_x^2\{u \wedge \partial_x^2\}, B_1(u)\partial_x^{-2}]U_m. \tag{3.10}$$

The formal expression makes sense since both $\partial_x^2\{u\wedge\partial_x^2 U_m\}$ and $U_m=\partial_x^2(\partial_x^{m-2}u_x)$ are regarded as images of ∂_x^2. Bearing the right-hand side of (3.10) in mind, we deduce that

$$
\begin{aligned}
&[\partial_x^2\{u\wedge\partial_x^2\}, B_1(u)\partial_x^{-2}]U_m \\
&\quad= \partial_x^2\{u\wedge\partial_x^2(B_1(u)\partial_x^{-2}U_m)\} - B_1(u)\partial_x^{-2}\partial_x^2\{u\wedge\partial_x^2 U_m\} \\
&\quad= \partial_x^2\{u\wedge(B_1(u)U_m + 2(\partial_x B_1)(u)\partial_x^{-1}U_m + (\partial_x^2 B_1)(u)\partial_x^{-2}U_m)\} \\
&\qquad - B_1(u)\{u\wedge\partial_x^2 U_m\} \\
&\quad= u\wedge B_1(u)\partial_x^2 U_m - B_1(u)(u\wedge\partial_x^2 U_m) \\
&\qquad + 2u_x\wedge B_1(u)\partial_x U_m + 4u\wedge(\partial_x B_1)(u)\partial_x U_m + R_{15} \\
&\quad= -2B_1(u)(u\wedge\partial_x^2 U_m) + (B_1(u)u\wedge\cdot + u\wedge B_1(u))\partial_x^2 U_m \\
&\qquad + 2u_x\wedge B_1(u)\partial_x U_m + 4u\wedge(\partial_x B_1)(u)\partial_x U_m + R_{15},
\end{aligned}
\tag{3.11}
$$

where

$$(\partial_x B_1)(u)\partial_x U_m = \partial_x\{B_1(u)\partial_x U_m\} - B_1(u)\partial_x\{\partial_x U_m\}$$

is given by

$$(\partial_x B_1)(u)Y = (\partial_x U_m, u\wedge\partial_x u_x)u\wedge u_x + (\partial_x U_m, u\wedge u_x)u\wedge\partial_x u_x.$$

The terms of the right-hand side of (3.11) are expressed as follows:

$$
\begin{aligned}
&B_1(u)(u\wedge\partial_x^2 U_m) \\
&\quad= (u\wedge\partial_x^2 U_m, u\wedge u_x)u\wedge u_x = (\partial_x^2 U_m, u_x)u\wedge u_x = T_1(u)\partial_x^2 U_m, \\
&(B_1(u)u\wedge\cdot + u\wedge B_1(u))\partial_x^2 U_m \\
&\quad= (u\wedge\partial_x^2 U_m, u\wedge u_x)u\wedge u_x + u\wedge(\partial_x^2 U_m, u\wedge u_x)u\wedge u_x \\
&\quad= u\wedge\{(\partial_x^2 U_m, u_x)u_x + (\partial_x^2 U_m, u\wedge u_x)u\wedge u_x\} \\
&\quad= u\wedge\{|u_x|^2\partial_x^2 U_m - |u_x|^2(\partial_x^2 U_m, u)u\} \quad \text{(due to (2.24))} \\
&\quad= |u_x|^2 u\wedge\partial_x^2 U_m \\
&\quad= \partial_x\{|u_x|^2 u\wedge\partial_x U_m\} - 2(\partial_x u_x, u_x)u\wedge\partial_x U_m - |u_x|^2 u_x\wedge\partial_x U_m \\
&\quad= \partial_x\{|u_x|^2 u\wedge\partial_x U_m\} - 2(\partial_x T_2)(u)\partial_x U_m \quad \text{(due to (2.16))} \\
&u_x\wedge B_1(u)\partial_x U_m \\
&\quad= u_x\wedge(\partial_x U_m, u\wedge u_x)u\wedge u_x = |u_x|^2(\partial_x U_m, u\wedge u_x)u, \\
&u\wedge(\partial_x B_1)(u)\partial_x U_m \\
&\quad= u\wedge\{(\partial_x U_m, u\wedge\partial_x u_x)u\wedge u_x + (\partial_x U_m, u\wedge u_x)u\wedge\partial_x u_x\} \\
&\quad= -(\partial_x U_m, u\wedge\partial_x u_x)u_x + (\partial_x U_m, u\wedge u_x)\{(u,\partial_x u_x)u - \partial_x u_x\} \\
&\quad= -(\partial_x U_m, u\wedge\partial_x u_x)u_x - (\partial_x U_m, u\wedge u_x)\partial_x u_x - |u_x|^2(\partial_x U_m, u\wedge u_x)u \\
&\quad= (\partial_x T_2)(u)\partial_x U_m - T_3(u)\partial_x U_m - |u_x|^2(\partial_x U_m, u\wedge u_x)u \\
&\qquad\qquad\qquad\qquad \text{(due to (2.19)–(2.20))}.
\end{aligned}
$$

Collecting the information, we have

$$
\begin{aligned}
&[\partial_x^2\{u\wedge\partial_x^2\}, B_1(u)\partial_x^{-2}]U_m\\
&\qquad = -2T_1(u)\partial_x^2 U_m + \partial_x\{|u_x|^2 u\wedge\partial_x U_m\} + 2(\partial_x T_2)(u)\partial_x U_m\\
&\qquad\quad - 4T_3(u)\partial_x U_m - 2|u_x|^2(\partial_x U_m, u\wedge u_x)u + R_{15}.
\end{aligned}
\tag{3.12}
$$

From (3.9), (3.10) and (3.12), it follows that

$$
\begin{aligned}
&P_4(u)(\Lambda_1(u)U_m) - \partial_t(\Lambda_1(u)U_m)\\
&\quad = \varepsilon\tilde{R}_1 + d_1 T_1(u)\partial_x^2 U_m - \tfrac{1}{2}d_1\partial_x\{|u_x|^2 u\wedge\partial_x U_m\} - d_1(\partial_x T_2)(u)\partial_x U_m\\
&\qquad + 2d_1 T_3(u)\partial_x U_m + \tilde{P}_{1,5}U_m + R_1,
\end{aligned}
\tag{3.13}
$$

where

$$
\tilde{P}_{1,5}U_m = \tilde{P}_{1,4}U_m + d_1|u_x|^2(\partial_x u_m, u\wedge u_x)u \quad\text{and}\quad R_1 = R_{11} - R_{14} - \tfrac{1}{2}d_1 R_{15}.
$$

Next, we observe that $P_4(u)(\Lambda_2(u)U_m) - \partial_t(\Lambda_2(u)U_m)$. Though $\Lambda_2(u)$ acts as a pseudo-differential operator of order -2, second-order derivatives of U_m do not appear in the computation, since $B_2(u) = |u_x|^2\,\mathrm{Id}$ commutes with the action $u\wedge$. A simple computation yields

$$
\begin{aligned}
&P_4(u)(\Lambda_2(u)U_m)\\
&\quad = \alpha\partial_x^2\{u\wedge\partial_x^2(\Lambda_2(u)U_m)\} + \alpha(m-1)(\partial_x^3(\Lambda_2(u)U_m), u\wedge u_x)u + R_{20}\\
&\quad = \alpha\partial_x^2\{u\wedge\partial_x^2(\Lambda_2(u)U_m)\} - \tfrac{1}{8}d_2(m-1)|u_x|^2(\partial_x U_m, u\wedge u_x)u + R_{21}.
\end{aligned}
$$

Moreover, (2.31) and (2.47) with $k = m-2$ and the assumption that $m \geqslant 4$ imply that

$$
\begin{aligned}
\partial_t(\Lambda_2(u)U_m) &= \frac{d_2}{8\alpha}|u_x|^2\partial_t\partial_x^{m-2}u_x + R_{22}\\
&= -\frac{d_2\varepsilon}{8\alpha}|u_x|^2(\partial_x^{m+2}u_x - N(u,\dots,\partial_x^{m+1}u_x))\\
&\quad + \tfrac{1}{8}d_2|u_x|^2\partial_x^2\{u\wedge\partial_x^2(\partial_x^{m-2}u_x)\}\\
&\quad + \tfrac{1}{8}d_2(m-3)|u_x|^2(\partial_x^3(\partial_x^{m-2}u_x), u\wedge u_x)u + R_{23}\\
&= -\varepsilon\tilde{R}_2 + \tfrac{1}{8}d_2|u_x|^2\{u\wedge\partial_x^2 U_m + 2u_x\wedge\partial_x U_m + \partial_x u_x\wedge U_m\}\\
&\quad + \tfrac{1}{8}d_2(m-3)|u_x|^2(\partial_x U_m, u\wedge u_x)u + R_{23}\\
&= -\varepsilon\tilde{R}_2 + \tfrac{1}{8}d_2|u_x|^2 u\wedge\partial_x^2 U_m + \tfrac{1}{4}d_2|u_x|^2 u_x\wedge\partial_x U_m\\
&\quad + \tfrac{1}{8}d_2(m-3)|u_x|^2(\partial_x U_m, u\wedge u_x)u + R_{24}\\
&= -\varepsilon\tilde{R}_2 + \Lambda_2(u)\alpha\partial_x^2\{u\wedge\partial_x^2 U_m\} + \tfrac{1}{4}d_2|u_x|^2 u_x\wedge\partial_x U_m\\
&\quad + \tfrac{1}{8}d_2(m-3)|u_x|^2(\partial_x U_m, u\wedge u_x)u + R_{24},
\end{aligned}
$$

where $\tilde{R}_2 = (d_2/8\alpha)|u_x|^2(\partial_x^{m+2}u_x - N(u,\dots,\partial_x^{m+1}u_x))$. Collecting these relations, we have

$$
\begin{aligned}
&P_4(\Lambda_2(u)U_m) - \partial_t(\Lambda_2(u)U_m)\\
&\qquad = \varepsilon\tilde{R}_2 + [\alpha\partial_x^2\{u\wedge\partial_x^2\}, \Lambda_2(u)]U_m + \tilde{P}_{1,6}U_m + R_{21} - R_{24},
\end{aligned}
\tag{3.14}
$$

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where

$$\tilde{P}_{1,6}U_m = -\tfrac{1}{4}d_2|u_x|^2 u_x \wedge \partial_x U_m - \tfrac{1}{4}d_2(m-2)|u_x|^2(\partial_x U_m, u \wedge u_x)u.$$

To deal with the second term of the right-hand side of (3.14), we set $\Lambda_2(u) = (d_2/8\alpha)B_2(u)\partial_x^{-2}$ with $B_2(u) = |u_x|^2\,\mathrm{Id}$ and write

$$[\alpha\partial_x^2\{u \wedge \partial_x^2\}, \Lambda_2(u)]U_m = \tfrac{1}{8}d_2[\partial_x^2\{u \wedge \partial_x^2\}, B_2(u)\partial_x^{-2}]U_m. \tag{3.15}$$

By the same simple computation as in the second step, we see that

$$\begin{aligned}[\partial_x^2\{u \wedge \partial_x^2\}, B_2(u)\partial_x^{-2}]U_m &= (u \wedge B_2(u) - B_2(u)u \wedge \cdot)\partial_x^2 U_m + 2u_x \wedge B_2(u)\partial_x U_m \\ &\quad + 4u \wedge (\partial_x B_2)(u)\partial_x U_m + R_{25}.\end{aligned} \tag{3.16}$$

The terms of the right-hand side become

$$\begin{aligned}(u \wedge B_2(u) - B_2(u)u \wedge \cdot)\partial_x^2 U_m &= u \wedge |u_x|^2\partial_x^2 U_m - |u_x|^2 u \wedge \partial_x^2 U_m \\ &= 0, \\ u_x \wedge B_2(u)\partial_x U_m &= |u_x|^2 u_x \wedge \partial_x U_m, \\ u \wedge (\partial_x B_2)(u)\partial_x U_m &= 2u \wedge (\partial_x u_x, u_x)\partial_x U_m \\ &= 2(\partial_x T_2)(u)\partial_x U_m - |u_x|^2 u_x \wedge \partial_x U_m.\end{aligned}$$

This shows that

$$[\partial_x^2\{u \wedge \partial_x^2\}, B_2(u)\partial_x^{-2}]U_m = 8(\partial_x T_2)(u)\partial_x U_m - 2|u_x|^2 u_x \wedge \partial_x U_m + R_{25}. \tag{3.17}$$

From (3.14), (3.15) and (3.17), it follows that

$$P_4(u)(\Lambda_2(u)U_m) - \partial_t(\Lambda_2(u)U_m) = \varepsilon\tilde{R}_2 + d_2(\partial_x T_2)(u)\partial_x U_m + \tilde{P}_{1,7}U_m + R_2, \tag{3.18}$$

where

$$\tilde{P}_{1,7}U_m = \tilde{P}_{1,6}U_m - (d_2/4)|u_x|^2 u_x \wedge \partial_x U_m \quad \text{and} \quad R_2 = R_{21} - R_{24} + \tfrac{1}{8}d_2 R_{25}.$$

We now recall (3.8) and combine (3.13) and (3.18) to obtain

$$\begin{aligned}&P_4(u)(\Lambda(u)U_m) - \partial_t(\Lambda(u)U_m) \\ &\quad = \varepsilon(\tilde{R}_1 + \tilde{R}_2) + d_1 T_1(u)\partial_x^2 U_m - \tfrac{1}{2}d_1\partial_x\{|u_x|^2 u \wedge \partial_x U_m\} \\ &\qquad + (-d_1 + d_2)(\partial_x T_2)(u)\partial_x U_m + 2d_1 T_3(u)\partial_x U_m \\ &\qquad + \tilde{P}_{1,5}U_m + \tilde{P}_{1,7}U_m + R_1 + R_2.\end{aligned} \tag{3.19}$$

Furthermore, by substituting $U_m = V_m - \Lambda(u)U_m$ into some (not necessarily all) terms of the right-hand side of (3.19), and by noting that $\Lambda(u)U_m$ includes at most $(m-2)$th derivatives of u_x, we obtain

$$\begin{aligned}&P_4(u)(\Lambda(u)U_m) - \partial_t(\Lambda(u)U_m) \\ &\quad = \varepsilon(\tilde{R}_1 + \tilde{R}_2) + d_1 T_1(u)\partial_x^2 V_m - \tfrac{1}{2}d_1\partial_x\{|u_x|^2 u \wedge \partial_x V_m\} \\ &\qquad + (-d_1 + d_2)(\partial_x T_2)(u)\partial_x V_m + 2d_1 T_3(u)\partial_x V_m \\ &\qquad + \tilde{P}_{1,5}U_m + \tilde{P}_{1,7}U_m + R_1 + R_2 + R_3.\end{aligned} \tag{3.20}$$

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Then (2.32) with $k = m$ and (3.20) imply that (3.5) becomes

$$\partial_t V_m = -\varepsilon\partial_x^4 V_m + \varepsilon\tilde{Q} + \tilde{P}_4(u)V_m + \tilde{P}_1(u)U_m + \tilde{R}, \tag{3.21}$$

where

$$\begin{aligned}\tilde{P}_4(u)V_m &= \alpha\partial_x^2\{u \wedge \partial_x^2 V_m\} + \alpha(m-1)(\partial_x^3 V_m, u \wedge u_x)u \\ &\quad + c_{2,m}\partial_x\{\partial_x u_x \wedge \partial_x V_m\} + \partial_x[\{1 + (\gamma + \tfrac{1}{2}d_1)|u_x|^2\}u \wedge \partial_x V_m] \\ &\quad + (A_{1,m} - d_1)T_1(u)\partial_x^2 V_m + (A_{2,m} + d_1 - d_2)(\partial_x T_2)(u)\partial_x V_m \\ &\quad + (c_{3,m} - 2d_1)T_3(u)\partial_x V_m + c_{4,m}T_4(u)\partial_x V_m, \\ \tilde{P}_1(u)U_m &= P_1(u)U_m - \tilde{P}_{1,5}U_m - \tilde{P}_{1,7}U_m, \\ \tilde{Q} &= \partial_x^4(\Lambda(u)U_m) + N(u, u_x, \dots, \partial_x^{m+3}u_x) - \tilde{R}_1 - \tilde{R}_2, \\ \tilde{R} &= \tilde{R}_{(m)} - R_1 - R_2 - R_3.\end{aligned}$$

By using (3.21), we see

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|V_m\|_{L^2}^2 &= -\varepsilon\langle\partial_x^4 V_m, V_m\rangle + \varepsilon\langle\tilde{Q}, V_m\rangle + \langle\tilde{P}_4(u)V_m, V_m\rangle \\ &\quad + \langle\tilde{P}_1(u)U_m, V_m\rangle + \langle\tilde{R}, V_m\rangle.\end{aligned} \tag{3.22}$$

We evaluate the right-hand side of (3.22). In the same way as that used to show (2.48), we use integration by parts, the Sobolev embedding and Young's inequality to show that

$$-\varepsilon\langle\partial_x^4 V_m, V_m\rangle + \varepsilon\langle\tilde{Q}, V_m\rangle \leqslant -\tfrac{1}{2}\varepsilon\|\partial_x^2 V_m\|_{L^2}^2 + CN_m^2,$$

where the constant C is independent of ε. Furthermore, the terms $\partial_x U_m$ included in $\tilde{P}_1 U_m$ cause no trouble when we take the inner product with V_m in L^2, that is,

$$\langle\tilde{P}_1(u)U_m, V_m\rangle \leqslant CN_m^2$$

holds. In addition, it is easy to see that $\tilde{R} \leqslant CN_m$ and $\langle\tilde{R}, V_m\rangle \leqslant CN_m^2$. Moreover,

$$\begin{aligned}\langle\tilde{P}_4(u)V_m, V_m\rangle &\leqslant \alpha(m-1)\langle(\partial_x^3 V_m, u \wedge u_x)u, V_m\rangle + (A_{1,m} - d_1)\langle T_1(u)\partial_x^2 V_m, V_m\rangle \\ &\quad + (A_{2,m} + d_1 - d_2)\langle(\partial_x T_2)(u)\partial_x V_m, V_m\rangle + CN_m^2\end{aligned} \tag{3.23}$$

follows from integration by parts. Let us look at the first term of the right-hand side of (3.23). A simple computation shows that

$$\begin{aligned}\langle(\partial_x^3 V_m, u \wedge u_x)u, V_m\rangle &= \langle(\partial_x^3 U_m, u \wedge u_x)u, U_m\rangle + \langle(\partial_x^3 U_m, u \wedge u_x)u, \Lambda(u)U_m\rangle \\ &\quad + \langle(\partial_x^3(\Lambda(u)U_m), u \wedge u_x)u, V_m\rangle \\ &=: E_4 + E_5 + E_6.\end{aligned}$$

We recall here that

$$\begin{aligned}E_4 &\leqslant (m+1)\langle T_1(u)\partial_x^2 U_m, U_m\rangle + \tfrac{1}{4}(m+1)(m+12)\langle(\partial_x T_2)(u)\partial_x U_m, U_m\rangle \\ &\quad + C\|u_x\|_{H^m}^2\end{aligned}$$

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follows from (2.53) with $k = m$. Since $U_m = V_m - \Lambda(u)U_m$ and $\Lambda(u)U_m$ includes at most $(m-2)$th derivatives of u_x, we obtain

$$E_4 \leqslant (m+1)\langle T_1(u)\partial_x^2 V_m, V_m\rangle + \tfrac{1}{4}(m+1)(m+12)\langle(\partial_x T_2)(u)\partial_x V_m, V_m\rangle + CN_m^2.$$

For E_5, noting that $\Lambda(u)U_m$ includes at most $(m-2)$th derivatives of u_x, we integrate by parts and use (2.25) with $k = m$ to deduce that

$$\begin{aligned} E_5 &\leqslant -\langle (U_m, u\wedge u_x)u, \partial_x^3(\Lambda(u)U_m)\rangle + CN_m^2 \\ &= \frac{d_1}{2\alpha}\langle (U_m, u\wedge u_x)u, (\partial_x U_m, u\wedge u_x)u\wedge u_x\rangle - \frac{d_2}{8\alpha}\langle (U_m, u\wedge u_x)u, |u_x|^2\partial_x U_m\rangle \\ &\quad + CN_m^2 \\ &= -\frac{d_2}{8\alpha}\langle (U_m, u\wedge u_x)u, |u_x|^2\partial_x U_m\rangle + CN_m^2 \\ &\leqslant CN_m^2. \end{aligned}$$

For E_6, by integrating by parts, we have

$$E_6 \leqslant -\langle (\partial_x^2(\Lambda(u)U_m), u\wedge u_x)u, \partial_x V_m\rangle + CN_m^2.$$

By (2.25) with $k = m$, we note that both $\partial_x^2(\Lambda(u)U_m)$ and $(u, \partial_x V_m) = (u, \partial_x U_m) + (u, \partial_x(\Lambda(u)U_m))$ include at most mth derivatives of u_x. By noting this, we find that $E_6 \leqslant CN_m^2$ holds. Collecting the estimates for E_4, E_5 and E_6, we get

$$\begin{aligned} &\langle (\partial_x^3 V_m, u\wedge u_x)u, V_m\rangle \\ &\quad \leqslant (m+1)\langle T_1(u)\partial_x^2 V_m, V_m\rangle + \tfrac{1}{4}(m+1)(m+12)\langle(\partial_x T_2)(u)\partial_x V_m, V_m\rangle \\ &\qquad + CN_m^2. \end{aligned} \tag{3.24}$$

From (3.23) and (3.24) it follows that

$$\begin{aligned} &\langle \tilde{P}_4(u)V_m, V_m\rangle \\ &\quad \leqslant \{\alpha(m^2-1) + A_{1,m} - d_1\}\langle T_1(u)\partial_x^2 V_m, V_m\rangle \\ &\qquad + \{\tfrac{1}{4}\alpha(m^2-1)(m+12) + A_{2,m} + d_1 - d_2\}\langle(\partial_x T_2)(u)\partial_x V_m, V_m\rangle + CN_m^2. \end{aligned}$$

We now choose d_1 and d_2 so that

$$\begin{aligned} \alpha(m^2-1) + A_{1,m} - d_1 &= 0, \\ \tfrac{1}{4}\alpha(m^2-1)(m+12) + A_{2,m} + d_1 - d_2 &= 0, \end{aligned}$$

where $A_{1,m}$, $A_{2,m}$ are given by (2.34). Then $\langle \tilde{P}_4(u)V_m, V_m\rangle \leqslant CN_m^2$ holds.

Consequently, we derive

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|V_m\|_{L^2}^2 \leqslant -\frac{\varepsilon}{2}\|\partial_x^2 V_m\|_{L^2}^2 + CN_m^2. \tag{3.25}$$

Finally, in view of (3.4) and (3.25), we come to the conclusion that there exists a positive constant C depending on α, β, γ, m, $\|u_{0x}\|_{H^4}$ and not on ε such that

$$\frac{\mathrm{d}}{\mathrm{d}t}N_m(u^\varepsilon(t))^2 \leqslant CN_m(u^\varepsilon(t))^2 \tag{3.26}$$

for all $t \in [0, T_\varepsilon^\star]$. This shows that

$$N_m(u^\varepsilon(t))^2 \leqslant N_m(u_0)^2 \exp\left(C(\|u_{0x}\|_{H^4})t\right) \tag{3.27}$$

for all $t \in [0, T_\varepsilon^\star]$ and $m \geqslant 4$. Then it follows from the definition of $T_\varepsilon^\star$ that $4N_4(u_0)^2 = N_4(u(T_\varepsilon^\star))^2 \leqslant N_4(u_0)^2 \exp\left(C(\|u_{0x}\|_{H^4})T_\varepsilon^\star\right)$. By solving the inequality, we can conclude that $T_\varepsilon^\star \geqslant T := \log 4/C(\|u_{0x}\|_{H^4})$ for any $\varepsilon \in (0,1]$ and $\{N_m(u^\varepsilon)\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T)$. This completes the proof. □

4. Proof of theorems 1.1 and 1.2

This section will be devoted to the proof of theorems 1.1 and 1.2.

Proof of theorem 1.1. Assume that $m \geqslant 6$. The existence of a $u \in C([0,T]\times\mathbb{T};\mathbb{S}^2)$ that satisfies $u_x \in L^\infty(0,T;H^m(\mathbb{T};\mathbb{R}^3)) \cap C([0,T];H^{m-1}(\mathbb{T};\mathbb{R}^3))$ and solves (1.1), (1.2) has been established in theorem 3.1. Therefore, to complete the proof of theorem 1.1, we first show the uniqueness of the solution and then show the L^2-valued continuity of $\partial_x^m u_x$ in time, that is, $\partial_x^m u_x \in C([0,T];L^2(\mathbb{T};\mathbb{R}^3))$.

(i) *Uniqueness of the solution.* Let $u, v \in C([0,T]\times\mathbb{T};\mathbb{S}^2)$ be two solutions to (1.1), (1.2) satisfying $u_x, v_x \in L^\infty(0,T;H^m(\mathbb{T};\mathbb{R}^3)) \cap C([0,T];H^{m-1}(\mathbb{T};\mathbb{R}^3))$. Set $z = u - v$. We shall show that $z = 0$. More precisely, we can bound the energy estimate for z within H^2 by evaluating the L^2-norm of a gauged function of $\partial_x z_x$. For this purpose, set $U = \partial_x u_x$, $V = \partial_x v_x$, and set $W = U - V = \partial_x z_x$. To investigate the energy estimate for $\|W\|_{L^2}^2$, we begin with the study of the PDE for U. After lengthy calculations, we obtain

$$\begin{aligned}\partial_t U = {}&\alpha\partial_x^2(u\wedge\partial_x^2 U) + \partial_x\{(1+\gamma|u_x|^2)u\wedge\partial_x U\} + \beta T_1(u)\partial_x^2 U\\ &+ (\tfrac{7}{2}\beta + 3\gamma)(\partial_x T_2)(u)\partial_x U + (\tfrac{5}{2}\beta+\gamma)T_3(u)\partial_x U\\ &+ (\tfrac{1}{2}\beta-\gamma)T_4(u)\partial_x U + (1-\tfrac{1}{2}\beta|u_x|^2)u_x\wedge\partial_x U\\ &+ (-\tfrac{1}{2}\beta+\gamma)|u_x|^2(\partial_x U, u)u\wedge u_x\\ &+ \mathcal{O}(|U|^2|u_x|^2 + |U|^3|u|).\end{aligned} \tag{4.1}$$

The computation actually agrees with the previous results obtained from (2.4) and (2.29) with $k=1$, $U=U_1$, $\varepsilon=0$. Moreover, we do not need to deal with the third-order term of the form $(\partial_x^3 U, u\wedge u_x)u$, because the coefficient $\alpha(k-1)$ vanishes when $k=1$ in (2.29). Noting also that $(\partial_x U, u) = (\partial_x^2 u_x, u) = -3(\partial_x u_x, u_x) = -3(U, u_x)$ follows from $|u|^2 = 1$, we see that

$$|u_x|^2(\partial_x U, u)u\wedge u_x = \mathcal{O}(|U|\,|u_x|^4|u|).$$

Thus, we get

$$\begin{aligned}\partial_t U = {}&\alpha\partial_x^2(u\wedge\partial_x^2 U) + \partial_x\{(1+\gamma|u_x|^2)u\wedge\partial_x U\} + \beta T_1(u)\partial_x^2 U\\ &+ (\tfrac{7}{2}\beta + 3\gamma)(\partial_x T_2)(u)\partial_x U + (\tfrac{5}{2}\beta+\gamma)T_3(u)\partial_x U\\ &+ (\tfrac{1}{2}\beta-\gamma)T_4(u)\partial_x U + (1-\tfrac{1}{2}\beta|u_x|^2)u_x\wedge\partial_x U\\ &+ \mathcal{O}(|U|^2|u_x|^2 + |U|^3|u| + |U|\,|u_x|^4|u|).\end{aligned} \tag{4.2}$$

The same PDE as (4.2) is satisfied by V. Then, by taking the difference between $\partial_t U$ and $\partial_t V$, we obtain

$$\partial_t W = P(u)W + (1 - \tfrac{1}{2}\beta|u_x|^2)u_x \wedge \partial_x W + R, \tag{4.3}$$

where

$$\begin{aligned} P(u)W &= \alpha\partial_x^2(u \wedge \partial_x^2 W) + \partial_x\{(1+\gamma|u_x|^2)u \wedge \partial_x W\} \\ &\quad + \beta T_1(u)\partial_x^2 W + (\tfrac{7}{2}\beta + 3\gamma)(\partial_x T_2)(u)\partial_x W \\ &\quad + (\tfrac{5}{2}\beta + \gamma)T_3(u)\partial_x W + (\tfrac{1}{2}\beta - \gamma)T_4(u)\partial_x W, \end{aligned} \tag{4.4}$$

$$\begin{aligned} R &= \alpha\partial_x^2(z \wedge \partial_x^2 V) + \partial_x[\{(1+\gamma|u_x|^2)u - (1+\gamma|v_x|^2)v\} \wedge \partial_x V] \\ &\quad + \beta\{T_1(u) - T_1(v)\}\partial_x^2 V + (\tfrac{7}{2}\beta + 3\gamma)\{(\partial_x T_2)(u) - (\partial_x T_2)(v)\}\partial_x V \\ &\quad + (\tfrac{5}{2}\beta + \gamma)\{T_3(u) - T_3(v)\}\partial_x V + (\tfrac{1}{2}\beta - \gamma)\{T_4(u) - T_4(v)\}\partial_x V \\ &\quad + \{(1 - \tfrac{1}{2}\beta|u_x|^2)u_x - (1 - \tfrac{1}{2}\beta|v_x|^2)v_x\} \wedge \partial_x V \\ &\quad + \{\mathcal{O}(|U|^2|u_x|^2 + |U|^3|u| + |U||u_x|^4|u|) \\ &\quad - \mathcal{O}(|V|^2|v_x|^2 + |V|^3|v| + |V||v_x|^4|v|)\}. \end{aligned} \tag{4.5}$$

Hereafter in the proof, all positive constants depending on $\|u_x\|_{L^\infty(0,T;H^6)}$ and $\|v_x\|_{L^\infty(0,T;H^6)}$ will be denoted by the same C without further comment. In particular,

$$\|\partial_x^k u_x\|_{L^\infty((0,T)\times\mathbb{T})} \leqslant C \quad \text{and} \quad \|\partial_x^k v_x\|_{L^\infty((0,T)\times\mathbb{T})} \leqslant C \quad \text{for any } k = 0, 1, \ldots, 5$$

follow from the Sobolev embedding.

In this setting, we evaluate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|W\|_{L^2}^2 = \langle \partial_t W, W\rangle$$

by applying (4.4), (4.5). The linear combination of $T_1(u)\partial_x^2 W$ and $(\partial_x T_2)(u)\partial_x W$ in $P(u)W$ causes loss of derivatives in the classical energy estimate. To avoid this, we introduce the same type of gauged function as that used in the proof of theorem 3.1. Set

$$\tilde{W} = W + \tilde{\Lambda}(u)W, \tag{4.6}$$

$$\tilde{\Lambda}(u)W = -\frac{e_1}{2a}(z, u \wedge u_x)u \wedge u_x + \frac{e_2}{8a}|u_x|^2 z, \tag{4.7}$$

where e_1 and e_2 are real constants that will be decided later. Instead of $\|z(t)\|_{H^2}$, we consider the estimate for $D(z(t))$ defined by

$$D(z(t)) = \{\|z(t)\|_{L^2}^2 + \|z_x(t)\|_{L^2}^2 + \|\tilde{W}(t)\|_{L^2}^2\}^{1/2}. \tag{4.8}$$

We shall show that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}D(z(t))^2 \leqslant CD(z(t))^2$$

for all $t \in [0, T]$. If this is true, we have $0 \leqslant D(z(t)) \leqslant D(z(0))\mathrm{e}^{2Ct}$, which combined with $D(z(0)) = 0$ implies that $z = 0$.

In this connection, it is easy to show that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{\|z(t)\|_{L^2}^2 + \|z_x(t)\|_{L^2}^2\} \leqslant CD(z(t))^2 \tag{4.9}$$

for all $t \in [0, T]$. Thus, it suffices to evaluate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{W}(t)\|_{L^2}^2 = \langle \partial_t \tilde{W}(t), \tilde{W}(t)\rangle.$$

For this purpose, we consider the PDE for $\tilde{W}$. From (4.3) and (4.6), we have

$$\begin{aligned}\partial_t \tilde{W} &= \partial_t W + \partial_t(\tilde{\Lambda}(u)W) \\ &= P(u)\tilde{W} - \{P(\tilde{\Lambda}(u)W) - \partial_t(\tilde{\Lambda}(u)W)\} + (1 - \tfrac{1}{2}\beta|u_x|^2)u_x \wedge \partial_x W + R.\end{aligned} \tag{4.10}$$

Observing the form of R, we see that the mean value theorem and $m \geqslant 6$ yield $\|R\|_{L^2} \leqslant C\|z\|_{H^2}$. This implies that $\langle R, W\rangle \leqslant C\|z\|_{H^2}^2$, and thus we have $\langle R, W\rangle \leqslant CD(z)^2$. The above estimate shows that the L^2-norm of R is bounded by $C\|z\|_{H^2}$. In what follows, any term whose L^2-norm is bounded by $C\|z\|_{H^2}$ will be denoted by R_i, with some $i \in \mathbb{N}$, without further mention.

Next, we look at $P(u)(\tilde{\Lambda}(u)W) - \partial_t(\tilde{\Lambda}(u)W)$. By noting that $\tilde{\Lambda}(u)W = \mathcal{O}(|z|)$, we see that

$$P(u)(\tilde{\Lambda}(u)W) = \alpha\partial_x^2\{u \wedge \partial_x^2(\tilde{\Lambda}(u)W)\} + R_1.$$

On the other hand, it follows that

$$\partial_t(\tilde{\Lambda}(u)W) = -\frac{e_1}{2\alpha}(z_t, u \wedge u_x)u \wedge u_x + \frac{e_2}{8\alpha}|u_x|^2 z_t + R_2.$$

Observing that $z_t = \alpha u \wedge \partial_x^2 W + R_3$ follows from a simple calculation, we have

$$\begin{aligned}\partial_t(\tilde{\Lambda}(u)W) &= -\frac{e_1}{2\alpha}(\alpha u \wedge \partial_x^2 W, u \wedge u_x)u \wedge u_x + \frac{e_2}{8\alpha}|u_x|^2\alpha u \wedge \partial_x^2 W + R_4 \\ &= \tilde{\Lambda}(u)\alpha\partial_x^2\{u \wedge \partial_x^2 W\} + R_4,\end{aligned}$$

where $\tilde{\Lambda}(u) = -(e_1/2\alpha)B_1(u)\partial_x^{-2} + (e_2/8\alpha)B_2(u)\partial_x^{-2}$, and $B_1(u)$ and $B_2(u)$ have been defined in the previous section.

Collecting these relations, we have

$$\begin{aligned}&P(u)(\tilde{\Lambda}(u)W) - \partial_t(\tilde{\Lambda}(u)W) \\ &\quad= [\alpha\partial_x^2\{u \wedge \partial_x^2\}, \tilde{\Lambda}(u)]W + R_1 - R_4 \\ &\quad= -\tfrac{1}{2}e_1[\partial_x^2\{u \wedge \partial_x^2\}, B_1(u)\partial_x^{-2}]W + \tfrac{1}{8}e_2[\partial_x^2\{u \wedge \partial_x^2\}, B_1(u)\partial_x^{-2}]W \\ &\qquad + R_1 - R_4.\end{aligned} \tag{4.11}$$

In view of (3.12), (3.17) and $W = \tilde{W} + \mathcal{O}(|z|)$, we derive that

$$\begin{aligned}&P(u)(\tilde{\Lambda}(u)W) - \partial_t(\tilde{\Lambda}(u)W) \\ &\quad= e_1 T_1(u)\partial_x^2\tilde{W} - \tfrac{1}{2}e_1\partial_x\{|u_x|^2 u \wedge \partial_x\tilde{W}\} + (-e_1 + e_2)(\partial_x T_2)(u)\partial_x\tilde{W} \\ &\qquad + 2e_1 T_3(u)\partial_x\tilde{W} - \tfrac{1}{4}e_2|u_x|^2 u_x \wedge \partial_x W + e_1|u_x|^2(\partial_x W, u \wedge u_x)u \\ &\qquad + R_1 - R_4 + R_5.\end{aligned} \tag{4.12}$$

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Observing (4.4), (4.10) and (4.12), we set $e_1=\beta$ and $e_2=9\beta/2+3\gamma$ to obtain

$$\begin{aligned}\partial_t\tilde{W} &= \alpha\partial_x^2(u\wedge\partial_x^2\tilde{W})+\partial_x[\{1+(\gamma+\tfrac{1}{2}e_1)|u_x|^2\}u\wedge\partial_x\tilde{W}]\\ &\quad+(\tfrac{5}{2}\beta+\gamma-2e_1)T_3(u)\partial_x\tilde{W}+(\tfrac{1}{2}\beta-\gamma)T_4(u)\partial_x\tilde{W}\\ &\quad+\{1-(\tfrac{1}{2}\beta-\tfrac{1}{4}e_2)|u_x|^2\}u_x\wedge\partial_xW-e_1|u_x|^2(\partial_xW,u\wedge u_x)u\\ &\quad+R-R_1+R_4-R_5. \end{aligned}\tag{4.13}$$

In view of (4.13), integration by parts yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{W}\|_{L^2}^2 &\leqslant CD(z)^2+\langle\{1-(\tfrac{1}{2}\beta-\tfrac{1}{4}e_2)|u_x|^2\}u_x\wedge\partial_xW,\tilde{W}\rangle\\ &\quad-e_1\langle|u_x|^2(\partial_xW,u\wedge u_x)u,\tilde{W}\rangle.\end{aligned}$$

We look at the second and the third terms of the right-hand side. By noting that $|u|^2=1$, we see that

$$\begin{aligned}(W,u)&=-|u_x|^2+|v_x|^2-(\partial_xv_x,z)\\ &=\mathcal{O}(|z|+|z_x|),\end{aligned}\tag{4.14}$$

$$(\partial_xW,u)=\mathcal{O}(|z|+|z_x|+|W|).\tag{4.15}$$

By using integration by parts and (4.15), we have $\langle(\partial_xW,u\wedge u_x)u,W\rangle\leqslant C\|z\|_{H^2}^2$, which combined with $\tilde{W}=W+\mathcal{O}(|z|)$ implies that $\langle(\partial_xW,u\wedge u_x)u,\tilde{W}\rangle\leqslant CD(z)^2$. On the other hand, a simple computation and (2.21) yield

$$\begin{aligned}u_x\wedge\partial_xW &= u_x\wedge\partial_xU-v_x\wedge\partial_xV-z_x\wedge\partial_xV\\ &=(\partial_xU,u\wedge u_x)u-(\partial_xU,u)u\wedge u_x-\{(\partial_xV,v\wedge v_x)v-(\partial_xV,v)v\wedge v_x\}\\ &\quad-z_x\wedge\partial_xV\\ &=(\partial_xW,u\wedge u_x)u-(\partial_xW,u)u\wedge u_x+\mathcal{O}(|z|+|z_x|).\end{aligned}$$

Therefore, by using (4.14) and (4.15), integration by parts and $W=\tilde{W}+\mathcal{O}(|z|)$, we have $\langle u_x\wedge\partial_xW,\tilde{W}\rangle\leqslant CD(z)^2$. Collecting the information, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{W}(t)\|_{L^2}^2\leqslant CD(z(t))^2,$$

which combined with (4.9) implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}D(z(t))^2\leqslant CD(z(t))^2$$

for all $t\in[0,T]$. As is stated above, this completes the proof of the uniqueness.

(ii) *Proof of the L^2-valued continuity of $\partial_x^m u_x$ in time.* Let u be the unique solution to (1.1), (1.2) satisfying $u_x\in L^\infty(0,T;H^m(\mathbb{T};\mathbb{R}^3))\cap C([0,T];H^{m-1}(\mathbb{T};\mathbb{R}^3))$. We shall prove that $V_m\in C([0,T];L^2(\mathbb{T};\mathbb{R}^3))$. If this is true, then the fact that $\partial_x^m u_x\in C([0,T];L^2(\mathbb{T};\mathbb{R}^3))$ follows from $u_x\in C([0,T];H^{m-1}(\mathbb{T};\mathbb{R}^3))$. This implies that $u_x\in C([0,T];H^m(\mathbb{T};\mathbb{R}^3))$. For this purpose, it suffices to prove that

$$\lim_{t\downarrow 0}V_m(t)=V_m(0)\quad\text{in } L^2(\mathbb{T};\mathbb{R}^3),\tag{4.16}$$

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since the continuity at other times can be proved in the same way using the uniqueness of the solution. To prove (4.16), the estimate (3.27) plays a crucial role. Indeed, if we let $\varepsilon \downarrow 0$ in (3.27), the lower semi-continuity of L^2-norm implies that

$$\|V_m(t)\|_{L^2}^2 + \|u_x(t)\|_{H^{m-1}}^2 \leqslant (\|V_m(0)\|_{L^2}^2 + \|u_x(0)\|_{H^{m-1}}^2)\mathrm{e}^{Ct}$$

for $t \in [0, T]$. By using this and $u_x \in C([0, T]; H^{m-1}(\mathbb{T}; \mathbb{R}^3))$, we see that

$$\limsup_{t\downarrow 0} \|V_m(t)\|_{L^2}^2 \leqslant \|V_m(0)\|_{L^2}^2. \tag{4.17}$$

On the other hand, we find that u_x is weakly H^m-valued continuous in time, by noting that $u_x \in L^\infty(0, T; H^m(\mathbb{T}; \mathbb{R}^3)) \cap C([0, T]; H^{m-1}(\mathbb{T}; \mathbb{R}^3))$. Combining the weak continuity at $t = 0$ and (4.17), we obtain (4.16). □

Proof of theorem 1.2. Equation (1.1) with $\alpha = \beta = \gamma = 0$ possesses a recursion operator to generate a hierarchy of completely integrable equations (see, for example, [1, 2, 7] and references therein). Among them, we point out that Anco and Myrzakulov [1] derived a hierarchy of integrable $\mathbb{S}^2$-valued models of the form

$$u_t = (u \wedge \partial_x - u_x \partial_x^{-1}\{(u \wedge u_x, \cdot)\})^n u_x =: f^{(n)}, \quad n = 0, 1, 2, \ldots, \tag{4.18}$$

and a set of conserved quantities $I_n = \int_X H^{(n)}\,\mathrm{d}x$, $n = 0, 1, 2, \ldots$, for each model (4.18), where $X = \mathbb{R}$ or $\mathbb{T}$. They provided the following explicit expression of $H^{(n)}$:

$$H^{(n)} = \frac{1}{1+n}\partial_x^{-1}\{(u_x, \partial_x f^{(n)})\}. \tag{4.19}$$

When $\alpha \neq 0$, $\beta = 2\gamma = 5\alpha$ our (1.1) has the structure

$$u_t = f^{(1)} - \alpha f^{(3)}.$$

This means that each I_n, $n = 0, 1, 2, \ldots$, is also a conserved quantity for (1.1).

Let $T > 0$ be the maximal existence time of the solution u to (1.1), (1.2). Assume that $T < \infty$. To complete the proof of theorem 1.2, it suffices to show that $\|u_x(t)\|_{H^m}$ is bounded on the time interval $[0, T)$. Indeed, if this is true, then we can extend the solution beyond T, which implies that $T = \infty$.

We now turn our attention to the *a priori* estimate for $\|u_x(t)\|_{H^m}$. In view of (4.18) and (4.19), we find that conserved quantities I_{2n} for $n = 0, 1, \ldots$ are expressed as

$$I_0 = \tfrac{1}{2}\|u_x(t)\|_{L^2}^2, \qquad I_2 = \frac{1}{2}\left\{\|\partial_x u_x\|_{L^2}^2 - \frac{5}{4}\int_{\mathbb{T}} |u_x|^4\,\mathrm{d}x\right\},$$
$$I_4 = \frac{1}{2}\left\{\|\partial_x^2 u_x\|_{L^2}^2 - 14\int_{\mathbb{T}} (u_x, \partial_x u_x)^2\,\mathrm{d}x - \frac{7}{2}\int_{\mathbb{T}} |u_x|^2|\partial_x u_x|^2\,\mathrm{d}x + \frac{21}{8}\int_{\mathbb{T}} |u_x|^6\,\mathrm{d}x\right\},$$

and inductively

$$I_{2n} = \tfrac{1}{2}\|\partial_x^n u_x\|_{L^2}^2 + \int_{\mathbb{T}} \mathcal{P}(u_x, \partial_x u_x, \ldots, \partial_x^{n-1} u_x)\,\mathrm{d}x, \quad n = 3, 4, \ldots, \tag{4.20}$$

where $\mathcal{P}(u_x, \partial_x u_x, \dots, \partial_x^{n-1} u_x)$ is a polynomial of $u_x, \partial_x u_x, \dots, \partial_x^{n-1} u_x$ and satisfies

$$|\mathcal{P}(u_x, \partial_x u_x, \dots, \partial_x^{n-1} u_x)| = \sum_{\substack{(j_1+1)+\cdots+(j_\ell+1)=2n+2,\\ 0\leqslant j_1,\dots,j_\ell\leqslant n-1}} \mathcal{O}(|\partial_x^{j_1} u_x| \cdots |\partial_x^{j_\ell} u_x|). \quad (4.21)$$

From (4.21) and the Gagliardo–Nirenberg inequality, it follows that there exists a positive constant C_n depending on n such that

$$\int_{\mathbb{T}} |\mathcal{P}(u_x, \partial_x u_x, \dots, \partial_x^{n-1} u_x)| \, \mathrm{d}x \leqslant C_n \sum_{\substack{(j_1+1)+\cdots+(j_\ell+1)=2n+2,\\ 0\leqslant j_1,\dots,j_\ell\leqslant n-1}} \|\partial_x^n u_x\|_{L^2}^p \|u_x\|_{L^2}^{\ell-p},$$

where $p = (2n+1-\ell/2)/n$. Noting that ℓ is required to satisfy $3 \leqslant \ell \leqslant 2n+2$ in the summation above, we see that $1 \leqslant p < 2$ and $\ell - p > 0$. Thus, by using the Young inequality and the conservation law for I_0, that is, $\|u_x(t)\|_{L^2} = \|u_{0x}\|_{L^2}$, we get

$$\int_{\mathbb{T}} |\mathcal{P}(u_x, \partial_x u_x, \dots, \partial_x^{n-1} u_x)| \, \mathrm{d}x \leqslant \rho \|\partial_x^n u_x\|_{L^2}^2 + C(\rho, \|u_{0x}\|_{L^2}), \quad (4.22)$$

where ρ is arbitrary positive constant, $C(\rho, \|u_{0x}\|_{L^2})$ is some positive constant depending on ρ and $\|u_{0x}\|_{L^2}$. By applying (4.22) with $\rho = 1/4$ to the conservation law $I_8(t) = I_8(0)$, we show that

$$\begin{aligned} \tfrac{1}{2}\|\partial_x^4 u_x(t)\|_{L^2}^2 &\leqslant \tfrac{1}{2}\|\partial_x^4 u_{0x}\|_{L^2}^2 + \int_{\mathbb{T}} |\mathcal{P}(u_x, \partial_x u_x, \dots, \partial_x^3 u_x)(t)| \, \mathrm{d}x \\ &\qquad + \int_{\mathbb{T}} |\mathcal{P}(u_{0x}, \partial_x u_{0x}, \dots, \partial_x^3 u_{0x})| \, \mathrm{d}x \\ &\leqslant \tfrac{1}{2}\|\partial_x^4 u_{0x}\|_{L^2}^2 + \tfrac{1}{4}\|\partial_x^4 u_x\|_{L^2}^2 + \tfrac{1}{4}\|\partial_x^4 u_{0x}\|_{L^2}^2 + C(\|u_{0x}\|_{L^2}), \end{aligned}$$

which implies that $\|\partial_x^4 u_x(t)\|_{L^2}^2 \leqslant C(\|u_{0x}\|_{H^4})$ on $[0, T)$. By interpolating this and $\|u_x(t)\|_{L^2} = \|u_{0x}\|_{L^2}$, we see that $\sup_{t\in[0,T)} \|u_x(t)\|_{H^4} \leqslant C(\|u_{0x}\|_{H^4})$. On the other hand, in the same way as before, we can show that

$$\frac{\mathrm{d}}{\mathrm{d}t} N_m(u(t))^2 \leqslant C(\|u_x(t)\|_{H^4}) N_m(u(t))^2, \quad t \in [0, T),$$

where $C(\cdot)$ denotes a non-negative non-decreasing function on $[0, \infty)$. Collecting the information, we see that $N_m(u(t))^2 \leqslant N_m(u(0))^2 \exp(C(\|u_{0x}\|_{H^4})T) \leqslant C(\|u_{0x}\|_{H^m}, T)$ holds for any $t \in [0, T)$. Thus, the equivalence of $\|u_x\|_{H^m}$ and $N_m(u)$ implies that $\|u_x(t)\|_{H^m} \leqslant C(\|u_{0x}\|_{H^m}, T)$ for any $t \in [0, T)$. This completes the proof. □

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Appendix A. Proof of lemma 2.1

Our proof consists of the combination of a sixth-order parabolic regularization and the classical energy method for $\|u_x\|^2_{H^m}$.

A sixth-order parabolic regularization

Fix $\varepsilon \in (0,1]$ and $\delta \in (0,1]$ independently. We consider the IVP for a sixth-order parabolic PDE of the form

$$u_t = \delta F_6(u, u_x, \dots, \partial_x^5 u_x) - \varepsilon F_4(u, u_x, \dots, \partial_x^3 u_x) \\ + \alpha u \wedge \partial_x^3 u_x + \beta(\partial_x u_x, u_x) u \wedge u_x + \gamma |u_x|^2 u \wedge \partial_x u_x + u \wedge \partial_x u_x, \tag{A 1}$$

$$u(0,x) = u_0(x), \tag{A 2}$$

where $u = u(t,x)\colon [0,\infty) \times \mathbb{T} \to \mathbb{S}^2$ is an unknown function, $u_0 = u_0(x)\colon \mathbb{T} \to \mathbb{S}^2$ is the same initial function as that in the original problem (1.1), (1.2), and

$$F_6(u, u_x, \dots, \partial_x^5 u_x) = \partial_x^5 u_x + 6(\partial_x^4 u_x, u_x)u + 15(\partial_x^3 u_x, \partial_x u_x)u + 10|\partial_x^2 u_x|^2 u \tag{A 3}$$

forms the sixth-order parabolic term $\delta F_6(u, u_x, \dots, \partial_x^5 u_x)$ added to (2.1). Similarly to $F_4(u, u_x, \dots, \partial_x^3 u_x)$, the choice of $F_6(u, u_x, \dots, \partial_x^5 u_x)$ comes from the observation that $F_6(u, u_x, \dots, \partial_x^5 u_x) = \partial_x^5 u_x - (\partial_x^5 u_x, u)u$ if $|u|^2 = 1$.

We shall show the following.

Lemma A.1. *Let $\varepsilon, \delta \in (0,1]$, let m be a positive integer satisfying $m \geqslant 4$, and let $u_0 \in C(\mathbb{T}; \mathbb{S}^2)$ satisfy $u_{0x} \in H^m(\mathbb{T}; \mathbb{R}^3)$. Then there exists a positive constant $T_{\varepsilon,\delta} = T(\varepsilon, \delta, \|u_{0x}\|_{H^m}) > 0$ depending on ε, δ, α, β, γ, and on $\|u_{0x}\|_{H^m}$ such that (A 1), (A 2) admits a unique solution $u = u^{\varepsilon,\delta} \in C([0, T_{\varepsilon,\delta}] \times \mathbb{T}; \mathbb{S}^2)$ that satisfies $u_x^{\varepsilon,\delta} \in C([0, T_{\varepsilon,\delta}]; H^m(\mathbb{T}; \mathbb{R}^3))$.*

The proof of lemma A.1 is divided into the following two propositions.

Proposition A.2. *Under the same assumptions as in lemma A.1, there exist a positive constant $T_{\varepsilon,\delta} = T(\varepsilon, \delta, \|u_{0x}\|_{H^m}) > 0$ and a unique $u = u^{\varepsilon,\delta} \in C([0, T_{\varepsilon,\delta}] \times \mathbb{T}; \mathbb{R}^3)$ that satisfies $u_x^{\varepsilon,\delta} \in C([0, T_{\varepsilon,\delta}]; H^m(\mathbb{T}; \mathbb{R}^3))$ and (A 1), (A 2).*

Proposition A.3. *Under the same assumptions as in lemma A.1, assume that $u \in C([0, T_{\varepsilon,\delta}]; H^{m+1}(\mathbb{T}; \mathbb{R}^3))$ satisfies (A 1), (A 2). Then $|u| = 1$ holds on $[0, T_{\varepsilon,\delta}] \times \mathbb{T}$.*

Noting that (A 1) is a semilinear sixth-order parabolic equation with leading-order term $\delta \partial_x^5 u_x$, we can prove proposition A.2 by using the contraction mapping argument, where the parabolic smoothing effect coming from the estimate

$$|(2\pi \mathrm{i} n)^j \mathrm{e}^{-\delta(2\pi n)^6 t}| \leqslant 6(\delta t)^{-6}$$

for all $n \in \mathbb{Z}$, $t > 0$, $j = 0, 1, \dots, 5$ plays the crucial part. The argument is standard and hence we omit the detail. Note also that the constraint $|u|^2 = 1$ is ensured by proposition A.3, where the form of $F_6(u, u_x, \dots, \partial_x^5 u_x)$ and $F_4(u, u_x, \dots, \partial_x^3 u_x)$ play the crucial part.

Proof of proposition A.3. We define a function $h = h(t,x)\colon [0,T_{\varepsilon,\delta}] \times \mathbb{T} \to \mathbb{R}$ by

$$h(t,x) = |u(t,x)|^2 - 1.$$

It suffices to show that $h = 0$. The idea to evaluate h comes from the argument by Nishiyama and Tani in [16, 20]. A simple computation shows that

$$h_x = 2(u,u_x), \tag{A 4}$$

$$\partial_x h_x = 2\{(u,\partial_x u_x) + |u_x|^2\}, \tag{A 5}$$

$$\partial_x^2 h_x = 2\{(u,\partial_x^2 u_x) + 3(u_x,\partial_x u_x)\}, \tag{A 6}$$

$$\partial_x^3 h_x = 2\{(u,\partial_x^3 u_x) + 4(u_x,\partial_x^2 u_x) + 3|\partial_x u_x|^2\}, \tag{A 7}$$

$$\partial_x^4 h_x = 2\{(u,\partial_x^4 u_x) + 5(u_x,\partial_x^3 u_x) + 10(\partial_x u_x,\partial_x^2 u_x)\}, \tag{A 8}$$

$$\partial_x^5 h_x = 2\{(u,\partial_x^5 u_x) + 6(u_x,\partial_x^4 u_x) + 15(\partial_x u_x,\partial_x^3 u_x) + 10|\partial_x^2 u_x|^2\}. \tag{A 9}$$

As u satisfies (A 1), we have

$$\tfrac{1}{2}h_t = (u,u_t) = \delta(u, F_6(u,u_x,\dots,\partial_x^5 u_x)) - \varepsilon(u, F_4(u,u_x,\dots,\partial_x^3 u_x)). \tag{A 10}$$

It follows from (A 4), (A 5) and (A 7) that

$$\begin{aligned}(u, F_4(u,u_x,\dots,\partial_x^3 u_x)) &= (u,\partial_x^3 u_x) + 4(\partial_x^2 u_x, u_x)|u|^2 + 3|\partial_x u_x|^2|u|^2 \\ &= \tfrac{1}{2}\partial_x^3 h_x + 4(\partial_x^2 u_x,u_x)(|u|^2-1) + 3|\partial_x u_x|^2(|u|^2-1) \\ &= \tfrac{1}{2}\partial_x^3 h_x + \{4(\partial_x^2 u_x,u_x) + 3|\partial_x u_x|^2\}h. \end{aligned} \tag{A 11}$$

In the same way, it follows from (A 4)–(A 7) and (A 9) that

$$\begin{aligned}&(u, F_6(u,u_x,\dots,\partial_x^5 u_x)) \\ &\quad = (u,\partial_x^5 u_x) + 6(\partial_x^4 u_x,u_x)|u|^2 + 15(\partial_x^3 u_x,\partial_x u_x)|u|^2 + 10|\partial_x^2 u_x|^2|u|^2 \\ &\quad = \tfrac{1}{2}\partial_x^5 h_x + \{6(\partial_x^4 u_x,u_x) + 15(\partial_x^3 u_x,\partial_x u_x) + 10|\partial_x^2 u_x|^2\}h. \end{aligned} \tag{A 12}$$

Substituting (A 11) and (A 12) into (A 10), we deduce that

$$\begin{aligned} h_t &= \delta[\partial_x^5 h_x + \{12(\partial_x^4 u_x,u_x) + 30(\partial_x^3 u_x,\partial_x u_x) + 20|\partial_x^2 u_x|^2\}h] \\ &\quad - \varepsilon[\partial_x^3 h_x + \{8(\partial_x^2 u_x,u_x) + 6|\partial_x u_x|^2\}h]. \end{aligned} \tag{A 13}$$

Applying (A 13), the Sobolev embedding and integration by parts, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h\|_{L^2}^2 = \langle h,h_t\rangle \leqslant -\delta\|\partial_x^2 h_x\|_{L^2}^2 + \delta C_1\|h\|_{L^2}^2 - \varepsilon\|\partial_x h_x\|_{L^2}^2 + \varepsilon C_2\|h\|_{L^2}^2, \tag{A 14}$$

where C_1 (respectively, C_2) is a positive constant that depends on α, β, γ and $\|u_x\|_{L^\infty(0,T_{\varepsilon,\delta};H^4)}$ (respectively, $\|u_x\|_{L^\infty(0,T_{\varepsilon,\delta};H^3)}$). From $\varepsilon,\delta \in (0,1]$, it follows that there exists a constant $C > 0$ that depends on α, β, γ, $\|u_x\|_{L^\infty(0,T_{\varepsilon,\delta};H^4)}$ and not on ε and δ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|h(t)\|_{L^2}^2 \leqslant C\|h(t)\|_{L^2}^2,$$

which implies that

$$\|h(t)\|_{L^2}^2 \leqslant \|h(0)\|_{L^2}^2 \exp(Ct) \quad \text{on } [0,T_{\varepsilon,\delta}].$$

Since $h(0) = |u(0,x)|^2 - 1 = |u_0(x)|^2 - 1 = 0$, we see that $h(t) = 0$ in L^2 on $[0,T_{\varepsilon,\delta}]$, which completes the proof. □

Energy estimate to prove lemma 2.1

For fixed $\varepsilon \in (0,1]$, let $\{u^{\varepsilon,\delta}\}_{\delta\in(0,1]}$ be a family of solutions to (A 1), (A 2) constructed in lemma A.1. We shall show that there exists a $T_\varepsilon = T(\varepsilon, \|u_{0x}\|_{H^4}) > 0$ that is independent of $\delta \in (0,1]$ such that $\{u_x^{\varepsilon,\delta}\}_{\delta\in(0,1]}$ is uniformly bounded in $L^\infty(0,T_\varepsilon; H^m(\mathbb{T};\mathbb{R}^3))$. Set $u = u^{\varepsilon,\delta}$ and $U_k = \partial_x^k u_x$ for $k \leqslant m$ as before. We take the $(k+1)$th derivative of (A 1) in x to get the same expression as in (2.4) with an additional term, that is,

$$\partial_t U_k = \delta\partial_x^{k+1}\{F_6(u,\dots,\partial_x^5 u_x)\} - \varepsilon\partial_x^{k+1}\{F_4(u,\dots,\partial_x^3 u_x)\} + P(u,\dots,\partial_x^{k+4}u_x). \tag{A 15}$$

It follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_x\|_{H^m}^2 = \sum_{k=0}^m \langle \partial_t U_k, U_k\rangle = \delta E_6 - \varepsilon E_4 + \sum_{k=0}^m \langle P(u,\dots,\partial_x^{k+4}u_x), U_k\rangle, \tag{A 16}$$

where E_6 is defined by $E_6 = \sum_{k=0}^m \langle \partial_x^{k+1}\{F_6(u,\dots,\partial_x^5 u_x)\}, U_k\rangle$ and E_4 is defined by $E_4 = \sum_{k=0}^m \langle \partial_x^{k+1}\{F_4(u,\dots,\partial_x^3 u_x)\}, U_k\rangle$. We have already evaluated the second and the third terms of the right-hand side of (A 16) to obtain (2.55). This shows that

$$\sum_{k=0}^m \langle P(u,\dots,\partial_x^{k+4}u_x), U_k\rangle \leqslant C(\|u_x\|_{H^4})\|u_x\|_{H^{m+1}}^2, \tag{A 17}$$

$$-\varepsilon E_4 \leqslant -\frac{\varepsilon}{2}\sum_{k=0}^m \|\partial_x^2 U_k\|_{L^2}^2 + C(\|u_x\|_{H^4})\|u_x\|_{H^m}^2. \tag{A 18}$$

See (2.55) with $\varepsilon = 0$ for (A 17), and see (2.48) for (A 18). Hereafter in this section, any non-negative monotonically non-decreasing function in A is denoted by the same $C(A)$, which may depend also on α, β, γ, m but is independent of ε and δ. We next look at the first term of the right-hand side of (A 16). After lengthy computations similar to those used to obtain (2.48), we obtain

$$\langle \delta\partial_x^{k+1}\{F_6(u, u_x\dots,\partial_x^5 u_x)\}, U_k\rangle \leqslant -\tfrac{1}{2}\delta\|\partial_x^3 U_k\|_{L^2}^2 + C(\|u_x\|_{H^4})\|u_x\|_{H^k}^2,$$

which implies that

$$\delta E_6 \leqslant -\frac{\delta}{2}\sum_{k=0}^m \|\partial_x^3 U_k\|_{L^2}^2 + C(\|u_x\|_{H^4})\|u_x\|_{H^m}^2. \tag{A 19}$$

Combining (A 17)–(A 19), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_x\|_{H^m}^2 \leqslant -\delta\sum_{k=0}^m \|\partial_x^3 U_k\|_{L^2}^2 - \varepsilon\sum_{k=0}^m \|\partial_x^2 U_k\|_{L^2}^2 + C(\|u_x\|_{H^4})\|u_x\|_{H^{m+1}}^2. \tag{A 20}$$

The loss of derivatives caused by the third term of the right-hand side of (A 20) can be absorbed by the second term. Indeed, by the Young inequality of the form $ab \leqslant \varepsilon a^2/2 + b^2/\varepsilon$ for $a, b \geqslant 0$ and for $\varepsilon > 0$ and integration by parts, we see that

$$\begin{aligned} C\|u_x\|_{H^{m+1}}^2 &= C\|u_x\|_{H^m}^2 + C\|\partial_x U_m\|_{L^2}^2 = C\|u_x\|_{H^m}^2 - C\langle \partial_x^2 U_m, U_m\rangle \\ &\leqslant \tfrac{1}{2}\varepsilon\|\partial_x^2 U_m\|_{L^2}^2 + (C + C^2/\varepsilon)\|u_x\|_{H^m}^2. \end{aligned}$$

Collecting the above information, we deduce that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u_x\|_{H^m}^2 &\leqslant -\delta\sum_{k=0}^{m}\|\partial_x^3U_k\|_{L^2}^2-\frac{\varepsilon}{2}\sum_{k=0}^{m}\|\partial_x^2U_k\|_{L^2}^2+(1+1/\varepsilon)C(\|u_x\|_{H^4})\|u_x\|_{H^m}^2\\ &\leqslant (1+1/\varepsilon)C(\|u_x\|_{H^4})\|u_x\|_{H^m}^2.\end{aligned}$$

This implies that there exists a $T_\varepsilon>0$ depending on ε and $\|u_{0x}\|_{H^4}$ and not on $\delta\in(0,1]$ such that $\{u_x^{\varepsilon,\delta}\}_{\delta\in(0,1]}$ is bounded uniformly in $L^\infty(0,T_\varepsilon;H^m(\mathbb{T};\mathbb{R}^3))$. Then the standard compactness argument yields the existence of a u that solves (2.1), (2.2) on the time interval $[0,T_\varepsilon]$ and satisfies $u_x\in L^\infty(0,T_\varepsilon;H^m(\mathbb{T};\mathbb{R}^3))\cap C([0,T_\varepsilon];H^{m-1}(\mathbb{T};\mathbb{R}^3))$. The uniqueness of the solution and the H^m-valued continuity of u_x in time follow from the similar classical energy estimate, where the smoothing effect coming from the added fourth-order parabolic term again plays the crucial part. We omit the detail.

Appendix B. The detail of the calculations

We describe the detail of the calculations carried out to obtain (2.6)–(2.7). We recall (2.5) and write $P(u,u_x,\dots,\partial_x^{k+4}u_x)=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}$, where

$$\begin{aligned}\mathrm{I}&=\alpha\partial_x^{k+1}(u\wedge\partial_x^3u_x), & \mathrm{II}&=\beta\partial_x^{k+1}\{(\partial_xu_x,u_x)u\wedge u_x\},\\ \mathrm{III}&=\gamma\partial_x^{k+1}\{|u_x|^2u\wedge\partial_xu_x\}, & \mathrm{IV}&=\partial_x^{k+1}(u\wedge\partial_xu_x).\end{aligned}$$

Each term is calculated separately by the product formula:

$$\begin{aligned}\mathrm{I}&=\alpha\sum_{j=0}^{k+1}{}_{k+1}\mathrm{C}_j\partial_x^{k+1-j}u\wedge\partial_x^{j+3}u_x\\ &=\alpha\Big\{u\wedge\partial_x^{k+4}u_x+{}_{k+1}\mathrm{C}_ku_x\wedge\partial_x^{k+3}u_x+{}_{k+1}\mathrm{C}_{k-1}\partial_xu_x\wedge\partial_x^{k+2}u_x\\ &\qquad+{}_{k+1}\mathrm{C}_{k-2}\partial_x^2u_x\wedge\partial_x^{k+1}u_x+\sum_{j=0}^{k-3}{}_{k+1}\mathrm{C}_j\partial_x^{k-j}u_x\wedge\partial_x^{j+3}u_x\Big\}\\ &=\alpha\Big\{u\wedge\partial_x^4U_k+{}_{k+1}\mathrm{C}_ku_x\wedge\partial_x^3U_k+{}_{k+1}\mathrm{C}_{k-1}\partial_xu_x\wedge\partial_x^2U_k\\ &\qquad+{}_{k+1}\mathrm{C}_{k-2}\partial_x^2u_x\wedge\partial_xU_k+\sum_{j=0}^{k-3}{}_{k+1}\mathrm{C}_j\partial_x^{k-j}u_x\wedge\partial_x^{j+3}u_x\Big\}\\ &=\alpha\Big\{\partial_x^2(u\wedge\partial_x^2U_k)+({}_{k+1}\mathrm{C}_k-2)u_x\wedge\partial_x^3U_k+({}_{k+1}\mathrm{C}_{k-1}-1)\partial_x(\partial_xu_x\wedge\partial_xU_k)\\ &\qquad+({}_{k+1}\mathrm{C}_{k-2}-{}_{k+1}\mathrm{C}_{k-1}+1)\partial_x^2u_x\wedge\partial_xU_k+\sum_{j=0}^{k-3}{}_{k+1}\mathrm{C}_j\partial_x^{k-j}u_x\wedge\partial_x^{j+3}u_x\Big\};\end{aligned}\tag{B 1}$$

$$\mathrm{II}=\beta\sum_{\substack{p+q+r+s=k+1,\\0\leqslant p,q,r,s\leqslant k+1}}\frac{(k+1)!}{p!q!r!s!}(\partial_x^{p+1}u_x,\partial_x^qu_x)\partial_x^ru\wedge\partial_x^su_x$$

$$
\begin{aligned}
&= \beta\bigg\{(\partial_x^{k+2}u_x, u_x)u \wedge u_x + \frac{(k+1)!}{k!}(\partial_x^{k+1}u_x, \partial_x u_x)u \wedge u_x \\
&\quad + \frac{(k+1)!}{k!}(\partial_x^{k+1}u_x, u_x)u_x \wedge u_x + \frac{(k+1)!}{k!}(\partial_x^{k+1}u_x, u_x)u \wedge \partial_x u_x \\
&\quad + (\partial_x u_x, \partial_x^{k+1}u_x)u \wedge u_x + (\partial_x u_x, u_x)u \wedge \partial_x^{k+1}u_x \\
&\quad + \mathcal{O}((|\partial_x^2 u_x|\,|u_x| + |\partial_x u_x|^2 + |\partial_x u_x|\,|u_x|^2)|\partial_x^k u_x|) \\
&\qquad + \sum_{p_1,p_2,p_3,p_4} \mathcal{O}(|\partial_x^{p_1}u_x|\,|\partial_x^{p_2}u_x|\,|\partial_x^{p_3}u|\,|\partial_x^{p_4}u_x|)\bigg\} \\
&= \beta\bigg\{(\partial_x^2 U_k, u_x)u \wedge u_x + (k+2)(\partial_x U_k, \partial_x u_x)u \wedge u_x \\
&\quad + (k+1)(\partial_x U_k, u_x)u \wedge \partial_x u_x + (\partial_x u_x, u_x)u \wedge \partial_x U_k \\
&\quad + \mathcal{O}((|\partial_x^2 u_x|\,|u_x| + |\partial_x u_x|^2 + |\partial_x u_x|\,|u_x|^2)|U_k|) \\
&\qquad + \sum_{p_1,p_2,p_3,p_4} \mathcal{O}(|\partial_x^{p_1}u_x|\,|\partial_x^{p_2}u_x|\,|\partial_x^{p_3}u|\,|\partial_x^{p_4}u_x|)\bigg\},
\end{aligned} \tag{B 2}
$$

where the summation in the final line of (B 2) is over all (p_1, p_2, p_3, p_4) satisfying $1 \leqslant p_1 \leqslant k-1$, $0 \leqslant p_2, p_4 \leqslant k-1$, $-1 \leqslant p_3 - 1 \leqslant k-1$, and $p_1+p_2+p_3+p_4 = k+2$;

$$
\begin{aligned}
\mathrm{III} &= \gamma \sum_{\substack{p+q+r+s=k+1,\\ 0\leqslant p,q,r,s\leqslant k+1}} \frac{(k+1)!}{p!q!r!s!}(\partial_x^p u_x, \partial_x^q u_x)\partial_x^r u \wedge \partial_x^{s+1}u_x \\
&= \gamma\bigg\{(u_x, u_x)u \wedge \partial_x^{k+2}u_x + \frac{(k+1)!}{k!}(\partial_x u_x, u_x)u \wedge \partial_x^{k+1}u_x \\
&\quad + \frac{(k+1)!}{k!}(u_x, \partial_x u_x)u \wedge \partial_x^{k+1}u_x + \frac{(k+1)!}{k!}(u_x, u_x)u_x \wedge \partial_x^{k+1}u_x \\
&\quad + (\partial_x^{k+1}u_x, u_x)u \wedge \partial_x u_x + (u_x, \partial_x^{k+1}u_x)u \wedge \partial_x u_x \\
&\quad + \mathcal{O}((|\partial_x^2 u_x|\,|u_x| + |\partial_x u_x|^2 + |\partial_x u_x|\,|u_x|^2)|\partial_x^k u_x|) \\
&\qquad + \sum_{p_1,p_2,p_3,p_4} \mathcal{O}(|\partial_x^{p_1}u_x|\,|\partial_x^{p_2}u_x|\,|\partial_x^{p_3}u|\,|\partial_x^{p_4}u_x|)\bigg\} \\
&= \gamma\bigg\{|u_x|^2 u \wedge \partial_x^2 U_k + 2(k+1)(\partial_x u_x, u_x)u \wedge \partial_x U_k + (k+1)|u_x|^2 u_x \wedge \partial_x U_k \\
&\quad + 2(\partial_x U_k, u_x)u \wedge \partial_x u_x + \mathcal{O}((|\partial_x^2 u_x|\,|u_x| + |\partial_x u_x|^2 + |\partial_x u_x|\,|u_x|^2)|U_k|) \\
&\qquad + \sum_{p_1,p_2,p_3,p_4} \mathcal{O}(|\partial_x^{p_1}u_x|\,|\partial_x^{p_2}u_x|\,|\partial_x^{p_3}u|\,|\partial_x^{p_4}u_x|)\bigg\} \\
&= \gamma\bigg\{\partial_x(|u_x|^2 u \wedge \partial_x U_k) + 2k(\partial_x u_x, u_x)u \wedge \partial_x U_k + k|u_x|^2 u_x \wedge \partial_x U_k \\
&\quad + 2(\partial_x U_k, u_x)u \wedge \partial_x u_x + \mathcal{O}((|\partial_x^2 u_x|\,|u_x| + |\partial_x u_x|^2 + |\partial_x u_x|\,|u_x|^2)|U_k|) \\
&\qquad + \sum_{p_1,p_2,p_3,p_4} \mathcal{O}(|\partial_x^{p_1}u_x|\,|\partial_x^{p_2}u_x|\,|\partial_x^{p_3}u|\,|\partial_x^{p_4}u_x|)\bigg\},
\end{aligned} \tag{B 3}
$$

where the summation in the final line of (B 3) is over all (p_1, p_2, p_3, p_4) satisfying $0 \leqslant p_1, p_2 \leqslant k-1$, $-1 \leqslant p_3 - 1 \leqslant k-1$, $1 \leqslant p_4 \leqslant k-1$ and $p_1+p_2+p_3+p_4 = k+2$;

$$\begin{aligned}
\mathrm{IV} &= \sum_{j=0}^{k+1} {}_{k+1}\mathrm{C}_j \partial_x^{k+1-j} u \wedge \partial_x^{j+1} u_x \\
&= u \wedge \partial_x^{k+2} u_x + {}_{k+1}\mathrm{C}_k u_x \wedge \partial_x^{k+1} u_x + {}_{k+1}\mathrm{C}_{k-1} \partial_x u_x \wedge \partial_x^k u_x \\
&\quad + \partial_x^k u_x \wedge \partial_x u_x + \sum_{j=1}^{k-2} {}_{k+1}\mathrm{C}_j \partial_x^{k+1-j} u \wedge \partial_x^{j+1} u_x \\
&= u \wedge \partial_x^2 U_k + {}_{k+1}\mathrm{C}_k u_x \wedge \partial_x U_k + {}_{k+1}\mathrm{C}_{k-1} \partial_x u_x \wedge U_k \\
&\quad + U_k \wedge \partial_x u_x + \sum_{j=1}^{k-2} {}_{k+1}\mathrm{C}_j \partial_x^{k-j} u_x \wedge \partial_x^{j+1} u_x \\
&= \partial_x (u \wedge \partial_x U_k) + ({}_{k+1}\mathrm{C}_k - 1) u_x \wedge \partial_x U_k \\
&\quad + ({}_{k+1}\mathrm{C}_{k-1} - 1) \partial_x u_x \wedge U_k + \sum_{j=1}^{k-2} {}_{k+1}\mathrm{C}_j \partial_x^{k-j} u_x \wedge \partial_x^{j+1} u_x. \qquad \text{(B 4)}
\end{aligned}$$

By combining (B 1)–(B 4), we deduce (2.6) and (2.7).

Appendix C. Proof of propositions 2.2–2.4

Proof of proposition 2.2. Since (2.14) and (2.16) easily follow from definitions (2.11) and (2.13), we omit the detail.

We show (2.15). If $u_x(x) = 0$ at $x \in \mathbb{T}$, (2.15) obviously holds as $(T_4(u)Y_1, Y_2) = (Y_1, T_4(u)Y_2) = 0$. If $u_x(x) \neq 0$ at $x \in \mathbb{T}$, it suffices to show that

$$\begin{aligned}
|u_x|^4 (T_4(u)Y_1, Y_2) = (T_4(u)|u_x|^2 Y_1, |u_x|^2 Y_2) &= (|u_x|^2 Y_1, T_4(u)|u_x|^2 Y_2) \\
&= |u_x|^4 (Y_1, T_4(u) Y_2). \qquad \text{(C 1)}
\end{aligned}$$

The first and the third equalities of (C 1) are obvious. The second equality of (C 1) follows from (2.24) and $|u|^2 = 1$. To see this, we use (2.24) to find that

$$\begin{aligned}
T_4(u)|u_x|^2 Y_1 &= (|u_x|^2 Y_1, \partial_x u_x + |u_x|^2 u) u \wedge u_x - (|u_x|^2 Y_1, u_x) u \wedge \partial_x u_x \\
&= (\partial_x u_x, u_x)(Y_1, u_x) u \wedge u_x + (\partial_x u_x, u \wedge u_x)(Y_1, u \wedge u_x) u \wedge u_x \\
&\quad - |u_x|^2 (Y_1, u_x) u \wedge \partial_x u_x. \qquad \text{(C 2)}
\end{aligned}$$

In the above computation note that $(u, \partial_x u_x + |u_x|^2 u) = -|u_x|^2 + |u_x|^2 = 0$ holds as $|u|^2 = 1$. By taking the inner product of the right-hand side of (C 2) and $|u_x|^2 Y_2$, we have

$$\begin{aligned}
(T_4(u)|u_x|^2 Y_1, |u_x|^2 Y_2) &= |u_x|^2 (\partial_x u_x, u_x)(Y_1, u_x)(Y_2, u \wedge u_x) \\
&\quad + |u_x|^2 (\partial_x u_x, u \wedge u_x)(Y_1, u \wedge u_x)(Y_2, u \wedge u_x) \\
&\quad - |u_x|^2 (Y_1, u_x)(|u_x|^2 Y_2, u \wedge \partial_x u_x). \qquad \text{(C 3)}
\end{aligned}$$

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By using (2.24) with $Y = Y_2$ again, we see that

$$\begin{aligned} - |u_x|^2(Y_1, u_x)(|u_x|^2 Y_2, u \wedge \partial_x u_x) &= -|u_x|^2(u_x, u \wedge \partial_x u_x)(Y_1, u_x)(Y_2, u_x) \\ &\quad - |u_x|^2(u \wedge u_x, u \wedge \partial_x u_x)(Y_1, u_x)(Y_2, u \wedge u_x) \\ &= |u_x|^2(\partial_x u_x, u \wedge u_x)(Y_1, u_x)(Y_2, u_x) \\ &\quad - |u_x|^2(\partial_x u_x, u_x)(Y_1, u_x)(Y_2, u \wedge u_x). \qquad \text{(C 4)} \end{aligned}$$

From (C 3) and (C 4) it follows that

$$\begin{aligned} &(T_4(u)|u_x|^2 Y_1, |u_x|^2 Y_2) \\ &\quad = |u_x|^2(\partial_x u_x, u \wedge u_x)\{(Y_1, u_x)(Y_2, u_x) + (Y_1, u \wedge u_x)(Y_2, u \wedge u_x)\}. \end{aligned}$$

The right-hand side of the above is obviously symmetric with respect to Y_1 and Y_2, which implies the desired equality $(T_4(u)|u_x|^2 Y_1, |u_x|^2 Y_2) = (|u_x|^2 Y_1, T_4(u)|u_x|^2 Y_2)$. □

Proof of proposition 2.3. To begin with, we define $T_5(u)$ by

$$\begin{aligned} T_5(u)Y = \tfrac{1}{2}\{&(Y, \partial_x u_x) u \wedge u_x + (Y, u_x) u \wedge \partial_x u_x \\ &- (Y, u \wedge \partial_x u_x) u_x - (Y, u \wedge u_x)\partial_x u_x\}. \qquad \text{(C 5)} \end{aligned}$$

We also note the relation

$$(\partial_x T_2)(u)Y = T_5(u)Y. \qquad \text{(C 6)}$$

Indeed, from (2.24) and $|u|^2 = 1$ it follows that

$$\begin{aligned} T_2(u)Y &= \tfrac{1}{2} u \wedge \{|u_x|^2(Y, u)u + (Y, u_x)u_x + (Y, u \wedge u_x) u \wedge u_x\} \\ &= \tfrac{1}{2}\{(Y, u_x) u \wedge u_x - (Y, u \wedge u_x) u_x\}, \end{aligned}$$

and hence (C 6) follows from the definition (2.13).

From (2.9)–(2.13) and (C 5) and (C 6) it follows that

$$\begin{aligned} (Y, \partial_x u_x) u \wedge u_x &= \tfrac{1}{2}\{(Y, \partial_x u_x + |u_x|^2 u) u \wedge u_x - (Y, u_x) u \wedge \partial_x u_x\} \\ &\quad - \tfrac{1}{2}|u_x|^2 (Y, u) u \wedge u_x + \tfrac{1}{2}\{(Y, \partial_x u_x) u \wedge u_x + (Y, u_x) u \wedge \partial_x u_x\} \\ &= \tfrac{1}{2} T_4(u)Y - \tfrac{1}{2}|u_x|^2(Y, u) u \wedge u_x + \tfrac{1}{2}\{T_3(u)Y + T_5(u)Y\} \\ &= \tfrac{1}{2}(\partial_x T_2)(u)Y + \tfrac{1}{2} T_3(u)Y + \tfrac{1}{2} T_4(u)Y - \tfrac{1}{2}|u_x|^2(Y, u) u \wedge u_x \end{aligned}$$

and

$$\begin{aligned} (Y, u_x) u \wedge \partial_x u_x &= \tfrac{1}{2}\{(Y, u_x) u \wedge \partial_x u_x - (Y, \partial_x u_x + |u_x|^2 u) u \wedge u_x\} \\ &\quad + \tfrac{1}{2}|u_x|^2 (Y, u) u \wedge u_x + \tfrac{1}{2}\{(Y, u_x) u \wedge \partial_x u_x + (Y, \partial_x u_x) u \wedge u_x\} \\ &= -\tfrac{1}{2} T_4(u)Y + \tfrac{1}{2}|u_x|^2(Y, u) u \wedge u_x + \tfrac{1}{2}\{T_3(u)Y + T_5(u)Y\} \\ &= \tfrac{1}{2}(\partial_x T_2)(u)Y + \tfrac{1}{2} T_3(u)Y - \tfrac{1}{2} T_4(u)Y + \tfrac{1}{2}|u_x|^2(Y, u) u \wedge u_x. \end{aligned}$$

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In order to observe (2.19) and (2.20), we set $T_6(u)Y = (Y, u\wedge u_x)\partial_x u_x$ and $T_7(u)Y = (Y, u \wedge \partial_x u_x)u_x$, and write

$$(Y, u \wedge u_x)\partial_x u_x = \tfrac{1}{2}(T_6(u) + T_7(u))Y + \tfrac{1}{2}(T_6(u) - T_7(u))Y, \tag{C 7}$$
$$(Y, u \wedge \partial_x u_x)u_x = \tfrac{1}{2}(T_6(u) + T_7(u))Y - \tfrac{1}{2}(T_6(u) - T_7(u))Y. \tag{C 8}$$

On the other hand, we can show that

$$(T_6(u) + T_7(u))Y = T_3(u)Y - (\partial_x T_2)(u)Y, \tag{C 9}$$
$$(T_6(u) - T_7(u))Y = T_4(u)Y - |u_x|^2(Y, u \wedge u_x)u. \tag{C 10}$$

Indeed, (C 9) follows from

$$\begin{aligned}
&((T_6(u) + T_7(u))Y_1, Y_2)\\
&\quad = (Y_1, u \wedge u_x)(\partial_x u_x, Y_2) + (Y_1, u \wedge \partial_x u_x)(u_x, Y_2)\\
&\quad = (Y_1, (Y_2, \partial_x u_x)u \wedge u_x + (Y_2, u_x)u \wedge \partial_x u_x)\\
&\quad = (Y_1, T_3(u)Y_2 + (\partial_x T_2)(u)Y_2) \quad \text{(due to (2.17), (2.18))}\\
&\quad = (T_3(u)Y_1 - (\partial_x T_2)(u)Y_1, Y_2) \quad \text{(due to (2.14), (2.16))},
\end{aligned}$$

and (C 10) follows from

$$\begin{aligned}
&((T_6(u) - T_7(u))Y_1, Y_2)\\
&\quad = (Y_1, (Y_2, \partial_x u_x)u \wedge u_x - (Y_2, u_x)u \wedge \partial_x u_x)\\
&\quad = (Y_1, T_4(u)Y_2 - |u_x|^2(Y_2, u)u \wedge u_x) \quad \text{(due to (2.17), (2.18))}\\
&\quad = (T_4(u)Y_1 - |u_x|^2(Y_1, u \wedge u_x)u, Y_2) \quad \text{(due to (2.15))}.
\end{aligned}$$

Thus, by substituting (C 9) and (C 10) into (C 7) and (C 8), we obtain (2.19) and (2.20). □

Proof of proposition 2.4. It follows from (2.24) that

$$\begin{aligned}
|u_x|^2 u_x \wedge Y &= u_x \wedge \{|u_x|^2(Y, u)u + (Y, u_x)u_x + (Y, u \wedge u_x)u \wedge u_x\}\\
&= |u_x|^2(Y, u)u_x \wedge u + (Y, u \wedge u_x)u_x \wedge (u \wedge u_x)\\
&= -|u_x|^2(Y, u)u \wedge u_x + (Y, u \wedge u_x)|u_x|^2 u,
\end{aligned}$$

which implies (2.21). By differentiating both sides of (2.21), (2.22) is obtained. In the same way, by differentiating both sides of (2.22), we see that

$$\begin{aligned}
\partial_x^2 u_x \wedge Y = (Y, u \wedge \partial_x^2 u_x)u &- (Y, u)u \wedge \partial_x^2 u_x + (Y, u_x \wedge \partial_x u_x)u - (Y, u)u_x \wedge \partial_x u_x\\
&+ 2(Y, u \wedge \partial_x u_x)u_x - 2(Y, u_x)u \wedge \partial_x u_x + (Y, u \wedge u_x)\partial_x u_x\\
&- (Y, \partial_x u_x)u \wedge u_x.
\end{aligned} \tag{C 11}$$

Here, (2.21) with $Y = \partial_x u_x$ yields

$$u_x \wedge \partial_x u_x = (\partial_x u_x, u \wedge u_x)u - (\partial_x u_x, u)u \wedge u_x = (\partial_x u_x, u \wedge u_x)u + |u_x|^2 u \wedge u_x.$$

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By substituting this into the third and the fourth terms of the right-hand side of (C 11), we have

$$\begin{aligned}\partial_x^2 u_x \wedge Y = {} & (Y, u \wedge \partial_x^2 u_x)u - (Y, u)u \wedge \partial_x^2 u_x + |u_x|^2 (Y, u \wedge u_x)u \\ & - |u_x|^2 (Y, u)u \wedge u_x + 2(Y, u \wedge \partial_x u_x)u_x - 2(Y, u_x)u \wedge \partial_x u_x \\ & + (Y, u \wedge u_x)\partial_x u_x - (Y, \partial_x u_x)u \wedge u_x,\end{aligned}$$

which can be expressed as (2.23) by using (2.17)–(2.21). □

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