Bull. Austral. Math. Soc. **77** (2008), 485–494
doi:10.1017/S0004972708000427

STURM–LIOUVILLE PROBLEMS WITH DISCONTINUOUS POTENTIAL

Z. AKDOĞAN ✉ and Z. ŞAŞMAZ

(Received 23 August 2007)

Abstract

We consider a discontinuous Sturm–Liouville equation together with two supplementary transmission conditions at the point of discontinuity. We suggest our own approach for finding asymptotic approximation formulas for the eigenvalues of such discontinuous problems.

2000 *Mathematics subject classification*: 34L20.

Keywords and phrases: Sturm–Liouville problems, transmission conditions, asymptotic of eigenvalues and eigenfunctions, discontinuous boundary-value problems.

1. Introduction

The Sturmian theory is an important aid in solving many problems in mathematical physics. Therefore this theory is one of the most current and extensively developing fields in the spectral analysis of boundary-value problems of Sturm–Liouville type (for various physical applications, see for example [2, 3, 5, 6]). There is a quite substantial literature on such problems. Here we mention the results of [1, 2, 4, 8, 13] and references cited therein.

Basically boundary-value problems that consist of ordinary differential equations with continuous coefficients and endpoint boundary conditions have been investigated. But in this study we shall consider one discontinuous eigenvalue problem that consists of the Sturm–Liouville equation

$$\tau u := -a(x)u'' + q(x)u = \lambda u, \qquad x \in [-1, 0) \cup (0, 1], \tag{1.1}$$

with boundary conditions at the endpoints

$$l_1(u) := \cos \alpha u(-1) + \sin \alpha u'(-1) = 0, \tag{1.2}$$
$$l_2(u) := \cos \beta u(1) + \sin \beta u'(1) = 0, \tag{1.3}$$

and transmission conditions at the point of discontinuity

$$l_3(u) := u(-0) - u(+0) = 0, \tag{1.4}$$

$$l_4(u) := u'(-0) - u'(+0) = 0, \tag{1.5}$$

where $a(x) = a_1^2$ for $x \in [-1, 0)$ and $a(x) = a_2^2$ for $x \in (0, 1]$; a_1, a_2 are positive real numbers.

Boundary-value problems with transmission condition arise, as a rule, in the theory of heat and mass transfer and in a varied assortment of physical transfer problems (see for example [6]).

Note that some discontinuous problems with transmission conditions have been investigated in [5, 7–10, 13].

2. An operator formulation in the adequate Hilbert space

We introduce the special inner product in the Hilbert space $L_2(-1, 0) \oplus L_2(0, 1)$ and a symmetric linear operator A defined on this Hilbert space such that (1.1)–(1.5) can be considered as the eigenvalue problem of this operator.

Let us introduce a new equivalent inner product on $H := L_2(-1, 0) \oplus L_2(0, 1)$ by

$$\langle u, v\rangle_H := \frac{1}{a_1^2}\int_{-1}^{0} u(x)\overline{v(x)}\,dx + \frac{1}{a_2^2}\int_{0}^{1} u(x)\overline{v(x)}\,dx.$$

In the Hilbert space H we define a linear operator A with domain of definition

$$\begin{aligned} D(A) := \{&u(x) \mid u(x) \text{ and } u'(x) \text{ are absolutely continuous on } [-1, 0) \text{ and } (0, 1],\\ &\text{and have finite limits } u(\pm 0) \text{ and } u'(\pm 0),\\ &-a(x)u'' + q(x)u \in L_2(-1, 0) \oplus L_2(0, 1),\ l_i(u) = 0,\ (i = 1, 2, 3, 4)\},\\ Au := &-a(x)u'' + q(x)u. \end{aligned}$$

Now we can rewrite the considered problem (1.1)–(1.5) in operator form as

$$Au = \lambda u.$$

The eigenvalues and eigenfunctions of the problem (1.1)–(1.5) are defined as the eigenvalue and the first components of the corresponding eigenelements of the operator A respectively.

THEOREM 2.1. *The operator A is symmetric.*

PROOF. Let $u, v \in D(A)$. By two partial integrations we obtain

$$\begin{aligned} \langle Au, v\rangle_H - \langle u, Av\rangle_H = {}& W(u, \overline{v}; -0) - W(u, \overline{v}; -1)\\ &+ W(u, \overline{v}; 1) - W(u, \overline{v}; +0), \end{aligned} \tag{2.1}$$

https://doi.org/10.1017/S0004972708000427 Published online by Cambridge University Press

where, as usual, by $W(u, v; x)$ we denote the Wronskian of the functions u and v:

$$W(u, v; x) = u(x)v'(x) - v(x)u'(x).$$

Since u and $\overline{v}$ satisfy the boundary conditions (1.2)–(1.3) and transmission conditions (1.4) and (1.5) we get

$$\begin{aligned} W(u, \overline{v}; -0) &= u(-0)\overline{v'(-0)} - u'(-0)\overline{v(-0)} \\ &= u(+0)\overline{v'(+0)} - u'(+0)\overline{v(+0)} \\ &= W(u, \overline{v}; +0), \\ W(u, \overline{v}; -1) &= 0, \\ W(u, \overline{v}; 1) &= 0. \end{aligned} \tag{2.2}$$

Finally substituting (2.2) in (2.1),

$$\langle Au, v\rangle_H = \langle u, Av\rangle_H \quad (u, v \in D(A)), \tag{2.3}$$

so A is symmetric. □

COROLLARY 2.2. *All eigenvalues of the problem (1.1)–(1.5) are real.*

We can now assume that all eigenfunctions of the problem (1.1)–(1.5) are real valued.

3. Asymptotic approximations of fundamental solutions

First we shall construct a special fundamental system of solutions of (1.1) for λ not an eigenvalue.

Let us consider the following initial-value problem:

$$-a_1^2 u''(x) + q(x)u(x) = \lambda u(x), \quad x \in [-1, 0], \tag{3.1}$$

$$u(-1) = \sin\alpha, \tag{3.2}$$

$$u'(-1) = -\cos\alpha. \tag{3.3}$$

By virtue of [12, Theorem 1.5] this problem has a unique solution

$$u = \phi_1(x) \equiv \phi_1(x, \lambda),$$

which is an entire function of the parameter $\lambda \in \mathcal{C}$ for each fixed $x \in [-1, 0]$. Similarly employing the same method as in the proof of [12, Theorem 1.5], we see that the problem

$$-a_2^2 u''(x) + q(x)u(x) = \lambda u(x), \quad x \in [0, 1], \tag{3.4}$$

$$u(1) = -\sin\beta, \tag{3.5}$$

$$u'(1) = \cos\beta, \tag{3.6}$$

has a unique solution $u = \chi_2(x) \equiv \chi_2(x, \lambda)$, which is an entire function of the parameter $\lambda \in \mathcal{C}$ for each fixed $x \in [0, 1]$.

We shall define the functions $\phi_2(x, \lambda)$ and $\chi_1(x, \lambda)$ using $\phi_1(x, \lambda)$ and $\chi_2(x, \lambda)$, respectively. Modifying the method of the proof of [12, Theorem 1.5], we can prove that the next special type initial-value problem,

$$-a_2^2 u''(x) + q(x)u(x) = \lambda u(x), \quad x \in [0, 1], \tag{3.7}$$

$$u(0) = \phi_1(0, \lambda), \tag{3.8}$$

$$u'(0) = \phi_1'(0, \lambda), \tag{3.9}$$

has a unique solution $u = \phi_2(x) \equiv \phi_2(x, \lambda)$, which is an entire function of the parameter $\lambda \in \mathcal{C}$ for each fixed $x \in [0, 1]$. Similarly, the following problem also has a unique solution $u = \chi_1(x) \equiv \chi_1(x, \lambda)$:

$$-a_1^2 u''(x) + q(x)u(x) = \lambda u(x), \quad x \in [-1, 0], \tag{3.10}$$

$$u(0) = \chi_2(0, \lambda), \tag{3.11}$$

$$u'(0) = \chi_2'(0, \lambda). \tag{3.12}$$

The Wronskians $W(\phi_1(x, \lambda), \chi_1(x, \lambda))$ and $W(\phi_2(x, \lambda), \chi_2(x, \lambda))$ are independent of the variable x and $\phi_i(x, \lambda)$ and $\chi_i(x, \lambda)$ $(i = 1, 2)$ are entire functions of the parameter λ for each x.

Let us consider the Wronskians

$$w_1(\lambda) := W(\phi_1(x, \lambda), \chi_1(x, \lambda)), \quad w_2(\lambda) := W(\phi_2(x, \lambda), \chi_2(x, \lambda)),$$

which are entire functions of the parameter λ and are independent of x. It is clear from (3.8), (3.9), (3.11) and (3.12), because of

$$\begin{aligned} w_1(\lambda) &:= W(\phi_1(x, \lambda), \chi_1(x, \lambda)) \\ &= \phi_1(0, \lambda)\chi_1'(0, \lambda) - \phi_1'(0, \lambda)\chi_1(0, \lambda) \\ &= \phi_2(0, \lambda)\chi_2'(0, \lambda) - \phi_2'(0, \lambda)\chi_2(0, \lambda) \\ &= W(\phi_2(x, \lambda), \chi_2(x, \lambda))\,|_{x=0} \\ &= w_2(\lambda). \end{aligned}$$

Let us construct two basic solutions of (1.1) as

$$\phi(x, \lambda) = \begin{cases} \phi_1(x, \lambda), & x \in [-1, 0) \\ \phi_2(x, \lambda), & x \in (0, 1] \end{cases} \quad \text{and} \quad \chi(x, \lambda) = \begin{cases} \chi_1(x, \lambda), & x \in [-1, 0) \\ \chi_2(x, \lambda), & x \in (0, 1]. \end{cases}$$

By virtue of (3.8), (3.9), (3.11) and (3.12) these solutions satisfy both transmission conditions (1.4) and (1.5).

NOTE. Below we shall denote by $w(\lambda)$ the Wronskian of the functions $\phi(x, \lambda)$ and $\chi(x, \lambda)$,

$$w(\lambda) := W(\phi(x, \lambda), \chi(x, \lambda)).$$

THEOREM 3.1. *The eigenvalues of the problem (1.1)–(1.5) consist of the zeros of the functions* $w(\lambda)$.

PROOF. Let $w(\lambda_0) = 0$. We shall show that $\chi(x, \lambda_0)$ is an eigenfunction. By definition of this solution, $\chi(x, \lambda_0)$ satisfies the boundary condition (1.3). Further, since

$$W(\phi_1(x, \lambda_0), \chi_1(x, \lambda_0)) = w(\lambda_0) = 0,$$

the functions $\phi_1(x, \lambda_0)$ and $\chi_1(x, \lambda_0)$ are linearly dependent, that is,

$$\phi_1(x, \lambda_0) = k_1 \chi_1(x, \lambda_0), \quad x \in [-1, 0),$$

for some $k_1 \neq 0$. Consequently, the function $\chi(x, \lambda_0)$ also satisfies the boundary condition (1.2). Recalling that the solution $\chi(x, \lambda_0)$ satisfies both transmission conditions (1.4) and (1.5), we have that $\chi(x, \lambda_0)$ is an eigenfunction of the problem (1.1)–(1.5) corresponding to the eigenvalue λ_0. Thus, each zero of $w(\lambda)$ is an eigenvalue.

Now let λ_0 be an eigenvalue and $u_0(x)$ be any eigenfunction corresponding to this eigenvalue. Suppose, for a moment, that $w(\lambda_0) \neq 0$. Thus

$$W(\phi_1(x, \lambda), \chi_1(x, \lambda)) \neq 0 \quad \text{and} \quad W(\phi_2(x, \lambda), \chi_2(x, \lambda)) \neq 0.$$

From this, by virtue of well-known properties of Wronskians, it follows that each of the pairs $\phi_1(x, \lambda)$, $\chi_1(x, \lambda)$ and $\phi_2(x, \lambda)$, $\chi_2(x, \lambda)$ are linearly independent. Therefore the solution $u_0(x)$ of (1.1) may be represented in the form

$$u_0(x) = \begin{cases} c_1\phi_1(x, \lambda_0) + c_2\chi_1(x, \lambda_0), & x \in [-1, 0), \\ c_3\phi_2(x, \lambda_0) + c_4\chi_2(x, \lambda_0), & x \in (0, 1], \end{cases}$$

where at least one of the constants c_1, c_2, c_3 and c_4 is not zero. Considering the true equalities

$$\begin{aligned} l_k(u_0(x)) = c_1 l_k(\phi_1(x, \lambda_0)) + c_2 l_k(\chi_1(x, \lambda_0)) \\ + c_3 l_k(\phi_2(x, \lambda_0)) + c_4 l_k(\chi_2(x, \lambda_0)), \quad k = 1, 2, 3, 4, \end{aligned} \tag{3.13}$$

as the homogenous system of linear equations of the variables c_1, c_2, c_3 and c_4, and taking into account (3.8), (3.9), (3.11) and (3.12) it follows that the determinant of this system is equal to

$$\begin{vmatrix} 0 & w(\lambda_0) & 0 & 0 \\ 0 & 0 & w(\lambda_0) & 0 \\ \phi_1(-0, \lambda_0) & \chi_1(-0, \lambda_0) & -\phi_2(+0, \lambda_0) & -\chi_2(+0, \lambda_0) \\ \phi_1'(-0, \lambda_0) & \chi_1'(-0, \lambda_0) & -\phi_2'(+0, \lambda_0) & -\chi_2'(+0, \lambda_0) \end{vmatrix} = -w^3(\lambda_0),$$

and therefore is not equal to zero by assumption. Consequently this homogenous system of linear equations has only the trivial solution $(c_1, c_2, c_3, c_4) = (0, 0, 0, 0)$. Thus we get contradiction, which completes the proof. □

https://doi.org/10.1017/S0004972708000427 Published online by Cambridge University Press

LEMMA 3.2. *Let* $\lambda = s^2$. *Then the following integral equations hold for* $k = 0$ *and* $k = 1$.

$$\frac{d^k}{dx^k}\phi_{1\lambda}(x) = \sin\alpha \frac{d^k}{dx^k}\cos\frac{s(x+1)}{a_1} - \frac{a_1}{s}\cos\alpha \frac{d^k}{dx^k}\sin\frac{s(x+1)}{a_1} + \frac{1}{sa_1}\int_{-1}^{x}\frac{d^k}{dx^k}\sin\frac{s(x-y)}{a_1}q(y)\phi_{1\lambda}(y)\,dy, \tag{3.14}$$

$$\frac{d^k}{dx^k}\phi_{2\lambda}(x) = \phi_{1\lambda}(-0)\frac{d^k}{dx^k}\cos\frac{sx}{a_2} + \frac{a_2}{s}\phi'_{1\lambda}(-0)\frac{d^k}{dx^k}\sin\frac{sx}{a_2} + \frac{1}{a_2 s}\int_{0}^{x}\frac{d^k}{dx^k}\sin\frac{s(x-y)}{a_2}q(y)\phi_{2\lambda}(y)\,dy. \tag{3.15}$$

PROOF. For the proof it is enough to substitute

$$s^2\phi_{1\lambda}(y) + a(y)\phi''_{1\lambda}(y) \quad \text{and} \quad s^2\phi_{2\lambda}(y) + a(y)\phi''_{2\lambda}(y),$$

instead of $q(y)\phi_{1\lambda}(y)$ and $q(y)\phi_{2\lambda}(y)$, in the integral terms of (3.14) and (3.15) respectively and integrate by parts twice. □

LEMMA 3.3. *Let* $\lambda = s^2$, $\operatorname{Im} s = t$. *Then for* $\sin\alpha \neq 0$,

$$\frac{d^k}{dx^k}\phi_{1\lambda}(x) = \sin\alpha \frac{d^k}{dx^k}\cos\frac{s(x+1)}{a_1} + O(|s|^{k-1}e^{(|t|(x+1))/a_1}), \tag{3.16}$$

$$\frac{d^k}{dx^k}\phi_{2\lambda}(x) = a_2\sin\alpha \frac{d^k}{dx^k}\left(\frac{1}{a_2}\cos\frac{sx}{a_2}\cos\frac{s}{a_1} - \frac{1}{a_1}\sin\frac{sx}{a_2}\sin\frac{s}{a_1}\right) + O(|s|^{k-1}e^{(|t|(a_1x+a_2))/(a_1a_2)}), \tag{3.17}$$

as $|\lambda| \to \infty$*; while if* $\sin\alpha = 0$,

$$\frac{d^k}{dx^k}\phi_{1\lambda}(x) = -\frac{a_1}{s}\cos\alpha \frac{d^k}{dx^k}\sin\frac{s(x+1)}{a_1} + O(|s|^{k-2}e^{(|t|(x+1))/a_1}), \tag{3.18}$$

$$\frac{d^k}{dx^k}\phi_{2\lambda}(x) = -\frac{\cos\alpha}{s}\frac{d^k}{dx^k}\left(a_1\cos\frac{sx}{a_2}\sin\frac{s}{a_1} + a_2\sin\frac{sx}{a_2}\cos\frac{s}{a_1}\right) + O(|s|^{k-2}e^{(|t|(a_1x+a_2))/(a_1a_2)}), \tag{3.19}$$

as $|\lambda| \to \infty$ $(k = 0, 1)$. *Each of these asymptotic equalities holds uniformly for* x.

PROOF. The asymptotic formulas for $\phi_1(x, \lambda)$ can be found in same way as in [2]. Let us prove the formula (3.17) for $\phi_2(x, \lambda)$.

Let $\sin\alpha \neq 0$. Substituting (3.16) in (3.15) (for $k = 0$),

$$\phi_{2\lambda}(x) = \sin\alpha\cos\frac{s}{a_1}\cos\frac{sx}{a_2} - \frac{a_2}{a_1}\sin\alpha\sin\frac{s}{a_1}\sin\frac{sx}{a_2} + \frac{1}{a_2 s}\int_0^x \sin\frac{s(x-y)}{a_2}q(y)\phi_{2\lambda}(y)\,dy + O(|s|^{-1}e^{(|t|(a_1x+a_2))/(a_1a_2)}). \tag{3.20}$$

https://doi.org/10.1017/S0004972708000427 Published online by Cambridge University Press

Multiplying by $e^{(-|t|(a_1x+a_2))/(a_1a_2)}$ and denoting

$$F(x, \lambda) = e^{(-|t|(a_1x+a_2))/(a_1a_2)}\phi_{2\lambda}(x),$$

we have the next 'asymptotic integral equation'

$$\begin{aligned}F(x, \lambda) = e^{(-|t|(a_1x+a_2))/(a_1a_2)}&\left\{\sin\alpha\cos\frac{s}{a_1}\cos\frac{sx}{a_2} - \frac{a_2}{a_1}\sin\alpha\sin\frac{s}{a_1}\sin\frac{sx}{a_2}\right\}\\ &+ \frac{1}{a_2 s}\int_0^x \sin\frac{s(x-y)}{a_2} q(y) e^{(-|t|(x-y))/a_2} F(y, \lambda)\, dy + O\left(\frac{1}{s}\right).\end{aligned}$$

Denoting $M(\lambda) := \max_{x\in[0,1]} |F(x, \lambda)|$ from the last equation we derive that

$$M(\lambda) \le \left|\sin\alpha\left(1 - \frac{a_2}{a_1}\right)\right| + \frac{M_0}{|s|},$$

for some $M_0 > 0$. Consequently $M(\lambda) = O(1)$ as $|\lambda| \to \infty$, so

$$\phi_{2\lambda}(x) = O(e^{(|t|(a_1x+a_2))/(a_1a_2)}) \quad \text{as } |\lambda| \to \infty. \tag{3.21}$$

Substituting in (3.20) gives (3.17) for the case $k = 0$. The case $k = 1$ of (3.17) follows by applying the same procedure as in the case $k = 0$.

The proof of (3.19) is similar to that of (3.17) and hence is omitted. □

Similarly one can establish the following lemma for $\chi_i(x, \lambda)$ $(i = 1, 2)$.

LEMMA 3.4. *Let* $\lambda = s^2$, $\operatorname{Im} s = t$. *Then for* $\sin\beta \neq 0$,

$$\begin{aligned}\frac{d^k}{dx^k}\chi_{1\lambda}(x) &= -a_1\sin\beta\frac{d^k}{dx^k}\left(\frac{1}{a_1}\cos\frac{sx}{a_1}\cos\frac{s}{a_2} + \frac{1}{a_2}\sin\frac{sx}{a_1}\sin\frac{s}{a_2}\right)\\ &\quad + O(|s|^{-1}e^{(|t|(a_1-a_2x))/(a_1a_2)}),\end{aligned}\tag{3.22}$$

$$\frac{d^k}{dx^k}\chi_{2\lambda}(x) = -\sin\beta\frac{d^k}{dx^k}\cos\frac{s(x-1)}{a_2} + O(|s|^{-1}e^{(|t|(1-x))/a_2}), \tag{3.23}$$

as $|\lambda| \to \infty$; *while if* $\sin\beta = 0$,

$$\begin{aligned}\frac{d^k}{dx^k}\chi_{1\lambda}(x) &= \frac{\cos\beta}{s}\frac{d^k}{dx^k}\left(-a_2\cos\frac{sx}{a_1}\sin\frac{s}{a_2} + a_1\sin\frac{sx}{a_1}\cos\frac{s}{a_2}\right)\\ &\quad + O(|s|^{-2}e^{(|t|(a_1-a_2x))/(a_1a_2)}),\end{aligned}\tag{3.24}$$

$$\frac{d^k}{dx^k}\chi_{2\lambda}(x) = \frac{a_2}{s}\cos\beta\frac{d^k}{dx^k}\sin\frac{s(x-1)}{a_2} + O(|s|^{-2}e^{(|t|(1-x))/a_2}), \tag{3.25}$$

as $|\lambda| \to \infty$ $(k = 0, 1)$. *Each of these asymptotic equalities holds uniformly for* x.

THEOREM 3.5. *Let* $\lambda = s^2$, $\operatorname{Im} s = t$. *Then the characteristic function* $w(\lambda)$ *has the following asymptotic representations.*

https://doi.org/10.1017/S0004972708000427 Published online by Cambridge University Press

CASE 1. *If* $\sin\beta \neq 0$ *and* $\sin\alpha \neq 0$, *then*

$$w(\lambda) = -\sin\alpha\sin\beta s\left(\frac{1}{a_2}\sin\frac{s}{a_2}\cos\frac{s}{a_1} + \frac{1}{a_1}\cos\frac{s}{a_2}\sin\frac{s}{a_1}\right) + O(e^{(|t|(a_1+a_2))/(a_1a_2)}). \tag{3.26}$$

CASE 2. *If* $\sin\beta \neq 0$ *and* $\sin\alpha = 0$, *then*

$$w(\lambda) = a_1\sin\beta\cos\alpha\left(\frac{1}{a_2}\sin\frac{s}{a_2}\sin\frac{s}{a_1} - \frac{1}{a_1}\cos\frac{s}{a_2}\cos\frac{s}{a_1}\right) + O(|s|^{-1}e^{(|t|(a_1+a_2))/(a_1a_2)}). \tag{3.27}$$

CASE 3. *If* $\sin\beta = 0$ *and* $\sin\alpha \neq 0$, *then*

$$w(\lambda) = a_2\cos\beta\sin\alpha\left(\frac{1}{a_2}\cos\frac{s}{a_2}\cos\frac{s}{a_1} - \frac{1}{a_1}\sin\frac{s}{a_2}\sin\frac{s}{a_1}\right) + O(|s|^{-1}e^{(|t|(a_1+a_2))/(a_1a_2)}). \tag{3.28}$$

CASE 4. *If* $\sin\beta = 0$ *and* $\sin\alpha = 0$, *then*

$$w(\lambda) = -\frac{\cos\beta\cos\alpha}{s}\left(a_1\cos\frac{s}{a_2}\sin\frac{s}{a_1} + a_2\sin\frac{s}{a_2}\cos\frac{s}{a_1}\right) + O(|s|^{-2}e^{(|t|(a_1+a_2))/(a_1a_2)}). \tag{3.29}$$

COROLLARY 3.6. *The eigenvalues of problem (1.1)–(1.5) are bounded below.*

PROOF. Putting $s = it$ $(t > 0)$ in the above formulas it follows that $w(-t^2) \to \infty$ as $t \to \infty$. Consequently, $w(\lambda) \neq 0$ for λ negative and sufficiently large in moduli. □

THEOREM 3.7. *Let* $\lambda = s^2$, $\operatorname{Im} s = t$. *Then, the following asymptotic formula holds for the eigenvalues of the boundary-value-transmission problem (1.1)–(1.5):*

$$s_n = \frac{a_1a_2}{a_1+a_2}\pi n + O(1). \tag{3.30}$$

PROOF. Let $\sin\beta \neq 0$ and $\sin\alpha \neq 0$ (Case 1). By putting

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

in (3.26) we derive that

$$\begin{aligned} w(\lambda) = {} & -\sin\alpha\sin\beta s \\ & \times\left\{\frac{1}{a_2}\frac{e^{is((a_1+a_2)/(a_1a_2))} + e^{is((a_1-a_2)/(a_1a_2))} - e^{is((a_2-a_1)/(a_1a_2))} - e^{-is((a_1+a_2)/(a_1a_2))}}{4i}\right. \\ & \left. + \frac{1}{a_1}\frac{e^{is((a_1+a_2)/(a_1a_2))} - e^{is((a_1-a_2)/(a_1a_2))} + e^{is((a_2-a_1)/(a_1a_2))} - e^{-is((a_1+a_2)/(a_1a_2))}}{4i}\right\} \\ & + O\left(\exp\frac{|t|(a_1+a_2)}{a_1a_2}\right). \end{aligned}$$

https://doi.org/10.1017/S0004972708000427 Published online by Cambridge University Press

Denoting $\widetilde{w}(\lambda) = (1/s)w(\lambda)$,

$$\begin{aligned}\widetilde{w}(\lambda) = &\left[-\frac{\sin\alpha\,\sin\beta}{4i}\frac{a_1+a_2}{a_1a_2}\right]e^{is((a_1+a_2)/(a_1a_2))}\\ &+\left[\frac{\sin\alpha\,\sin\beta}{4i}\frac{a_2-a_1}{a_1a_2}\right]e^{is((a_1-a_2)/(a_1a_2))}\\ &+\left[\frac{\sin\alpha\,\sin\beta}{4i}\frac{a_1-a_2}{a_1a_2}\right]e^{is((a_2-a_1)/(a_1a_2))}\\ &+\left[\frac{\sin\alpha\,\sin\beta}{4i}\frac{a_1+a_2}{a_1a_2}\right]e^{is((-(a_1+a_2))/(a_1a_2))}\\ &+O\left(\exp\frac{|t|(a_1+a_2)}{a_1a_2}\right).\end{aligned}$$

Now denoting by $\widetilde{w}_0(\lambda)$ and $\widetilde{w}_1(\lambda)$ the first and O-term we shall represented the function $\widetilde{w}(\lambda)$ in the form

$$\widetilde{w}_0(s) = M_1e^{m_1\rho} + M_2e^{m_2\rho} + M_3e^{m_3\rho} + M_4e^{m_4\rho}, \tag{3.31}$$

where

$$\rho = is, \quad m_1 = -\frac{a_1+a_2}{a_1a_2}, \ldots, m_4 = \frac{a_1+a_2}{a_1a_2}, \quad m_4 > m_2 > m_3 > m_1,$$

and

$$M_1 = \frac{\sin\alpha\,\sin\beta}{4i}s\frac{a_1+a_2}{a_1a_2}, \ldots, M_4 = -\frac{\sin\alpha\,\sin\beta}{4i}s\frac{a_1+a_2}{a_1a_2}(M_1 \neq 0,\ M_4 \neq 0).$$

By virtue of [11, Page 100, Lemma 1] the function (3.31) has an infinite number of roots $\widetilde{s}_n$ with asymptotic expression

$$\begin{aligned}|\widetilde{s}_n| &= \frac{2\pi n}{m_4-m_1}\left(1+O\left(\frac{1}{n}\right)\right)\\ &= \frac{a_1a_2}{a_1+a_2}\pi n + \eta_n,\end{aligned}$$

where $\eta_n = O(1/n)$. By applying the well-known Rouche's theorem, which asserts that if $f(s)$ and $g(s)$ are analytic inside and on a closed contour Γ, and $|g(s)| < |f(s)|$ on Γ, then $f(s)$ and $f(s)+g(s)$ have the same number of zeros inside Γ provided that each zero is counted according to its multiplicity, then

$$\begin{aligned}s_n &= \widetilde{s}_n + O(1),\\ s_n &= \frac{a_1a_2}{a_1+a_2}\pi n + O(1).\end{aligned}$$

This completes the proof for the case $\sin\beta \neq 0$ and $\sin\alpha \neq 0$. □

https://doi.org/10.1017/S0004972708000427 Published online by Cambridge University Press

References

[1] P. A. Binding and P. J. Browne, 'Oscillation theory for indefinite Sturm–Liouville problems with eigenparameter dependent boundary conditions', *Proc. Roy. Soc. Edinburgh Sect. A* **127**(6) (1997), 1123–1136.

[2] C. T. Fulton, 'Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions', *Proc. Roy. Soc. Edinburgh Sect. A* **77** (1977), 293–308.

[3] A. A. Shkalikov, 'Boundary value problems for ordinary differential equations with a parameter in boundary conditions', *Tr. Semin. im. I. G. Petrovskogo* **9** (1983), 190–229.

[4] N. B. Kerimov and Kh. K. Memedov, 'On a boundary value problem with a spectral parameter in the boundary conditions', *Sibirsk. Mat. Zh.* **40**(2) (1999), 325–335 (English translation: *Siberian Math. J.* **40**(2) (1999), 281–290).

[5] M. Kobayashi, 'Eigenvalues of discontinuous Sturm–Liouville problems with symmetric potentials', *Comput. Math. Appl.* **18**(4) (1989), 357–364.

[6] A. V. Likov and Yu. A. Mikhailov, *The Theory of Heat and Mass Transfer* (Qosenergoizdat, 1963), (in Russian).

[7] O. Sh. Mukhtarov, M. Kandemir and N. Kuruoğlu, 'Distribution of eigenvalues for the discontinunous boundary value problem with functional many point conditions', *Israel J. Math.* **129** (2002), 143–156.

[8] O. Sh. Mukhtarov and M. Kandemir, 'Asymptotic behaviour of eigenvalues for the discontinuous boundary-value problem with functional-transmission conditions', *Acta Math. Sci. B* **22**(3) (2002), 335–345.

[9] O. Sh. Mukhtarov, M. Kadalak and N. Altınışık, 'Eigenvalues and eigenfunctions of discontinuous Sturm–Liouville problems with eigenparameter in the boundary conditions', *Indian J. Pure Appl. Math.* **34**(3) (2003).

[10] O. Sh. Mukhtarov and S. Yakubov, 'Problems for ordinary differential equations with transmission conditions', *Appl. Anal.* **81** (2002), 1033–1064.

[11] M. L. Rasulov, *Methods of Contour Integratian* (North-Holland, Amsterdam, 1967).

[12] E. C. Titchmarsh, *Eigenfunction Expansions Associated with Second Order Differential Equations I*, 2nd edn (Oxford University Press, London, 1962).

[13] S. Yakubov and Y. Yakubov, 'Abel basis of root functions of regular boundary value problems', *Math. Nachr.* **197** (1999), 157–187.

Z. AKDOĞAN, Gaziosmanpasa University, Faculty of Science-Art,
Department of Mathematics, 60240 Tokat, Turkey
e-mail: zakdogan@gop.edu.tr

Z. ŞAŞMAZ, Şehit Vural ArıcıAnadolu Lisesi, 06920 Nallıhan/Ankara, Turkey
e-mail: zekimat@hotmail.com

https://doi.org/10.1017/S0004972708000427 Published online by Cambridge University Press