Probability in the Engineering and Informational Sciences, **35**, 2021, 451–464.
doi:10.1017/S0269964820000042

PRESERVATION OF LOG-CONCAVITY AND LOG-CONVEXITY UNDER OPERATORS

Wanwan Xia
Department of Statistics and Finance, School of Management, University of Science and Technology of China, Hefei, Anhui 230026, China;
School of Physical and Mathematical Sciences, Nanjing Tech University, Nanjing, Jiangsu 211816, China
E-mail: xiaww@mail.ustc.edu.cn

Tiantian Mao and Taizhong Hu
Department of Statistics and Finance, School of Management, University of Science and Technology of China, Hefei, Anhui 230026, China
E-mail: tmao@ustc.edu.cn; thu@ustc.edu.cn

Log-concavity [log-convexity] and their various properties play an increasingly important role in probability, statistics, operations research and other fields. In this paper, we first establish general preservation theorems of log-concavity and log-convexity under operator $\phi \longmapsto T(\phi,\theta) = \mathbb{E}[\phi(X_\theta)]$, $\theta \in \Theta$, where Θ is an interval of real numbers or an interval of integers, and the random variable X_θ has a distribution function belonging to the family $\{F_\theta, \theta \in \Theta\}$ possessing the semi-group property. The proofs are based on the theory of stochastic comparisons and weighted distributions. The main results are applied to some special operators, for example, operators occurring in reliability, Bernstein-type operators and Beta-type operators. Several known results in the literature are recovered.

Keywords: bernstein-type operator, beta-type operator, gamma process, poisson process, reliability properties, renewal process, semi-group property, stochastic order, weighted distribution

1. INTRODUCTION

A nonnegative function ϕ defined on an interval $D \subseteq \mathbb{R}$ is said to be log-concave on D if

$$\phi(\alpha x + (1-\alpha)y) \geq [\phi(x)]^\alpha [\phi(y)]^{1-\alpha} \quad \text{for all } x, y \in D \text{ and } \alpha \in (0,1).$$

If $\phi(x) > 0$ for all $x \in D$, then an equivalent condition is that $\log \phi(x)$ is concave on D. ϕ is said to be log-convex if the above inequality is reversed. A log-concave or log-convex function ϕ on D does not have internal zeros, that is, there does not exist three points $x, y, z \in D$ such that $x < y < z$, $\phi(y) = 0$ and $\phi(x)\phi(z) > 0$. Similarly, a sequence $\{a_n, n \in \mathbb{N}\}$ is said to be log-concave, if $a_n \geq 0$ for $n \in \mathbb{N} \equiv \{0, 1, \ldots\}$, and

$$a_n^2 \geq a_{n+1} a_{n-1} \quad \text{for all } n \in \mathbb{N}_+ \equiv \{1, 2, \ldots\}.$$

$\{a_n\}$ is said to be log-convex on $\mathbb{N}$ if the above inequality is reversed. A log-concave or log-convex sequence $\{a_n\}$ does not have internal zeros.

 0269-9648/19 $25.00

Log-concave [log-convex] functions have many nice analytical properties and play an important role in statistics, probability, operations research, reliability, economics and other fields. Saumard & Wellner [30] is a comprehensive review of log-concavity in the statistics literature, which also includes some connections between log-concavity and other areas of mathematics and statistics. An [2] and Bagnoli & Bergstrom [7] are two other reviews of log-concavity in econometrics. For more on log-concavity, we refer the reader to Dharmadhikari & Joag-dev [14], Finner & Roters [15], Finner & Roters [16], Liggett [21], Sengupta & Nanda [31], Wang & Yeh [34], Kahn & Neiman [19], Yu [35], Bobkov & Madiman [9], Badía & Sangüesa [5,6] and references therein.

Let $\mathscr{C}$ be a class of functions $\phi : D \to \mathbb{R}$, satisfying certain Property-P, and define an operator:

$$\phi \longmapsto T_t(\phi, \theta), \quad \theta \in \Theta,$$

where t is a (possible) parameter, and Θ is an interval of real numbers or an interval of integers. What conditions must the function ϕ and the operator T_t satisfy in order that $T_t(\phi, \cdot)$ has Property-P? This is a preservation problem, that is, the operator T_t as a function of θ possesses or inherits Property-P of the function ϕ. Property-P can, for example, be increasing, concave, Schur-concave and so on. There is a long history of preservation theorems in the literature (see, e.g., [22] and references therein). Preservation theorems have generally enabled one to understand the property preserved and to generate other functions with the same property.

In this paper, we consider preservation properties of log-concavity and log-convexity of function ϕ under the operator T_t. The semi-group property plays an important role in the whole paper. Let Θ be an interval of real numbers or an interval of integers. A family of distribution functions $\{F_\theta, \theta \in \Theta\}$ is said to possess the semi-group property (see [28]) if $F_{\theta_1} * F_{\theta_2} = F_{\theta_1+\theta_2}$ whenever $\theta_1, \theta_2 \in \Theta$ and $\theta_1 + \theta_2 \in \Theta$, where * denotes convolution; that is,

$$F_{\theta_1+\theta_2}(x) = \int_{\mathbb{R}} F_{\theta_1}(x-t)\,\mathrm{d}F_{\theta_2}(t), \quad x \in \mathbb{R}.$$

If $\{F_\theta, \theta \in \Theta\}$ has the semi-group property, there exists a stochastic process $\{X_\theta, \theta \in \Theta\}$ with independent increments and X_θ has the distribution F_θ for all $\theta \in \Theta$.

The rest of the paper is organized as follows. In Section 2, we present general preservation theorems of log-concavity and log-convexity under the operator

$$\phi \longmapsto T(\phi, \theta) = \mathbb{E}[\phi(X_\theta)], \quad \theta \in \Theta,$$

where the random variable X_θ has a distribution function belonging to the family $\{F_\theta, \theta \in \Theta\}$ possessing the semi-group property. The proofs are based on the theory of stochastic comparisons and weighted distributions. In Section 3, the main results of Section 2 are applied to some special operators, for example, operators occurring in reliability, Bernstein-type operators and Beta-type operators. Several known results in the literature are recovered.

Throughout, the terms "increasing" and "decreasing" mean "nondecreasing" and "nonincreasing", respectively. $a/0$ is understood to be ∞ whenever $a > 0$, and $0/0$ is not defined. All ratios are well defined. All expectations and integrals are implicitly assumed to exist whenever they are written.

2. GENERAL PRESERVATION THEOREMS

First, we recall the definitions of some stochastic orders from Shaked & Shanthikumar [32]. Let X and Y be two random variables with respective survival functions $\overline{F}$ and $\overline{G}$. Then, X is said to be smaller than Y

- in the *usual stochastic order*, denoted by $X \le_{\mathrm{st}} Y$, if $\overline{F}(t) \le \overline{G}(t)$ for all t;
- in the *hazard rate order*, denoted by $X \le_{\mathrm{hr}} Y$, if $\overline{G}(t)/\overline{F}(t)$ is increasing in t;
- in the *reversed hazard rate order*, denoted by $X \le_{\mathrm{rh}} Y$, if $G(t)/F(t)$ is increasing in t;
- in the *likelihood ratio order*, denoted by $X \le_{\mathrm{lr}} Y$, if X and Y have respective density (mass) functions f and g, and if $g(t)/f(t)$ is increasing in t.

The relationships among these orders are shown in the following diagram:

$$\begin{array}{ccc} X \le_{\mathrm{lr}} Y & \Rightarrow & X \le_{\mathrm{hr}} Y \\ \Downarrow & & \Downarrow \\ X \le_{\mathrm{rh}} Y & \Rightarrow & X \le_{\mathrm{st}} Y \end{array}$$

To prove the main results, Theorems 2.4 and 2.5, we need three useful lemmas.

LEMMA 2.1 ([17,24]): *Let Θ be a subset of the real line $\mathbb{R}$, and let X_θ be a nonnegative random variable having a distribution function F_θ belonging to the family $\mathcal{P} = \{F_\theta, \theta \in \Theta\}$, which satisfies that $F_{\theta_1} \le_{\mathrm{st}} F_{\theta_2}$ whenever $\theta_1, \theta_2 \in \Theta$ and $\theta_1 < \theta_2$. Let $\Psi(x, \theta)$ be a real-valued function defined on $\mathbb{R} \times \Theta$, and be measurable in x for each θ. Then,*

(i) $\mathbb{E}[\Psi(X_\theta, \theta)]$ *is increasing in θ, if $\Psi(x, \theta)$ is increasing in θ and increasing in x;*
(ii) $\mathbb{E}[\Psi(X_\theta, \theta)]$ *is decreasing in θ, if $\Psi(x, \theta)$ is decreasing in θ and decreasing in x.*

Let $\phi : \mathbb{R} \to \mathbb{R}_+ \equiv [0, \infty)$ be a measurable function. For a random variable X_θ with distribution function F_θ, $\theta \in \Theta$, we define the *weighted distribution* corresponding to F_θ as

$$G_\phi(x \mid \theta) = \int_{-\infty}^{x} \frac{\phi(u)}{\mathbb{E}[\phi(X_\theta)]} \, \mathrm{d}F_\theta(u), \quad \forall\, x \in \mathbb{R}. \tag{2.1}$$

We use the notation $H_\phi[X_\theta]$ to denote a random variable having distribution function $G_\phi(x|\theta)$. The weighted distributions have been applied in several areas, such as reliability, renewal theory, biometry, ecology and wildlife population studies. We refer to Patil [27] and Patil & Rao [26] for a survey of weighted distributions. Under the assumption that the underlying random variables are independent and have marginal density or mass functions, Li *et al.* [20] in their Theorem 2.3 established the subadditivity [superadditivity] property of weighted distributions for log-concave [log-convex] weight functions in the sense of the usual stochastic order. However, this subadditivity [superadditivity] property also holds when the underlying random variables do not have density or mass functions. We state it here as a lemma.

LEMMA 2.2: *Let $X_1, X_2, \ldots, X_n$ be independent and nonnegative random variables, and let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function. Assume that $H_\phi[X_1], H_\phi[X_2], \ldots, H_\phi[X_n]$ are independent.*

(i) *If ϕ is log-concave, then $H_\phi\left[\sum_{i=1}^n X_i\right] \le_{\mathrm{st}} \sum_{i=1}^n H_\phi[X_i]$;*
(ii) *If ϕ is log-convex, then $H_\phi\left[\sum_{i=1}^n X_i\right] \ge_{\mathrm{st}} \sum_{i=1}^n H_\phi[X_i]$ and, hence, $H_\phi[X_1 + X_2] \ge_{\mathrm{st}} H_\phi[X_1]$.*

PROOF: We use an idea similar to that in the proof of Theorem 2.3 in Li *et al.* [20]. We give the proof of part (i); the proof of part (ii) is similar. First, assume $n = 2$, suppose that ϕ is log-concave on $\mathbb{R}_+$and denote by F_i the distribution function of X_i for each i. It can be checked that there exist nonnegative random variables Y_1 and Y_2 with the joint distribution function given by

$$G(y_1, y_2) = \int_0^{y_1} \int_0^{y_2} \frac{\phi(x_1 + x_2)}{\mathbb{E}[\phi(X_1 + X_2)]} \, \mathrm{d}F_1(x_1) \, \mathrm{d}F_2(x_2), \quad \forall \, (y_1, y_2) \in \mathbb{R}_+^2, \tag{2.2}$$

such that

$$H_\phi[X_1 + X_2] =_{\mathrm{st}} Y_1 + Y_2.$$

From (2.2) and the independence of X_1 and X_2, it follows that Y_1 has the marginal distribution function

$$G_1(y_1) = \int_0^{y_1} \int_0^{\infty} \frac{\phi(x_1 + x_2)}{\mathbb{E}[\phi(X_1 + X_2)]} \, \mathrm{d}F_1(x_1) \, \mathrm{d}F_2(x_2) = \int_0^{y_1} \frac{\mathbb{E}[\phi(x_1 + X_2)]}{\mathbb{E}[\phi(X_1 + X_2)]} \, \mathrm{d}F_1(x_1).$$

Note that $H_\phi[X_i]$ has the distribution function

$$G_\phi(y|i) = \int_0^y \frac{\phi(u)}{\mathbb{E}[\phi(X_i)]} \, \mathrm{d}F_i(u), \quad \forall \, y \in \mathbb{R}_+, \; i = 1, 2.$$

The remaining proof involves the following two steps.

(a) We first show $Y_1 \le_{\mathrm{rh}} H_\phi[X_1]$, which implies

$$Y_1 \le_{\mathrm{st}} H_\phi[X_1]. \tag{2.3}$$

To show it, it suffices to verify that $G_1(y)/G_\phi(y|1)$ is decreasing in $y \in \mathbb{R}_+$, that is, for $0 \le y_1 < y_2$,

$$\int_0^{y_1} \mathbb{E}[\phi(x + X_2)] \mathrm{d}F_1(x) \cdot \int_0^{y_2} \phi(u) \, \mathrm{d}F_1(u) \ge \int_0^{y_2} \mathbb{E}[\phi(x + X_2)] \, \mathrm{d}F_1(x) \times \int_0^{y_1} \phi(u) \, \mathrm{d}F_1(u)$$

or, equivalently,

$$\begin{aligned} &\int_0^{y_1} \mathbb{E}[\phi(x + X_2)] \, \mathrm{d}F_1(x) \cdot \int_{y_1}^{y_2} \phi(u) \, \mathrm{d}F_1(u) \\ &\quad \ge \int_{y_1}^{y_2} \mathbb{E}[\phi(u + X_2)] \, \mathrm{d}F_1(u) \cdot \int_0^{y_1} \phi(x) \, \mathrm{d}F_1(x) \\ &\quad \Longleftrightarrow \int_0^{y_1} \int_{y_1}^{y_2} \mathbb{E}\Big[\phi(x + X_2)\phi(u) - \phi(x)\phi(u + X_2)\Big] \, \mathrm{d}F_1(x) \, \mathrm{d}F_1(u) \ge 0. \end{aligned} \tag{2.4}$$

This is true since the integrand in (2.4) is nonnegative by using the log-concavity of ϕ. Hence, (2.3) follows.

https://doi.org/10.1017/S0269964820000042 Published online by Cambridge University Press

(b) For any $\Delta > 0$ and $y_1 \in \mathbb{R}_+$, denote $A = [y_1, y_1 + \Delta)$. We then aim to show

$$\big[Y_2 | Y_1 \in A\big] \le_{\rm rh} H_\phi[X_2]. \tag{2.5}$$

Note that the conditional distribution function of Y_2 given $Y_1 \in A$ is

$$G_{2|1}(y|A) = \mathbb{P}(Y_2 \le y | Y_1 \in A) = \frac{\int_{y_1}^{y_1+\Delta} \int_0^y \phi(x_1 + x_2)\,\mathrm{d}F_1(x_1)\,\mathrm{d}F_2(x_2)}{\int_{y_1}^{y_1+\Delta} \int_0^\infty \phi(x_1 + x_2)\,\mathrm{d}F_1(x_1)\,\mathrm{d}F_2(x_2)}.$$

It requires to prove that $G_{2|1}(y|A)/G_\phi(y|2)$ is decreasing in $y \in \mathbb{R}_+$. Note that

$$\begin{aligned} \frac{G_{2|1}(y|A)}{G_\phi(y|2)} &\propto \frac{\int_{y_1}^{y_1+\Delta} \int_0^y \phi(x_1 + x_2)\,\mathrm{d}F_1(x_1)\,\mathrm{d}F_2(x_2)}{\int_0^y \phi(x)\,\mathrm{d}F_2(x)} \\ &= [F_1(y_1 + \Delta) - F_1(y_1)] \times \frac{\int_0^y \mathbb{E}[\phi(x + X_1^*)]\,\mathrm{d}F_2(x)}{\int_0^y \phi(x)\,\mathrm{d}F_2(x)}, \end{aligned}$$

where X_1^* has the same distribution as that of X_1 given $X_1 \in A$. A similar argument to that of Case (a) yields that $G_{2|1}(y|A)/G_\phi(y|2)$ is decreasing in $y \in \mathbb{R}_+$ and, hence, (2.5) follows.

Observing (2.3) and (2.5), by Theorem 6.B.3 in Shaked & Shanthikumar [32], we have $(Y_1, Y_2) \le_{\mathbf{st}} (H_\phi[X_1], H_\phi[X_2])$. Then by Theorem 6.B.16 (a) in Shaked & Shanthikumar [32], we obtain

$$H_\phi[X_1 + X_2] =_{\rm st} Y_1 + Y_2 \le H_\phi[X_1] + H_\phi[X_2].$$

For $n > 2$, the general result follows by induction. This completes the proof of the lemma. ■

Based on part (ii) of Lemma 2.2, we have the following result for a family of distribution functions with the semi-group property.

LEMMA 2.3: *Let X_θ be a nonnegative random variable having a distribution function belonging to the family $\{F_\theta, \theta \in \Theta\}$, which possesses the semigroup property. If $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is log-convex, then*

$$H_\phi[X_{\theta_1}] \le_{\rm st} H_\phi[X_{\theta_2}]$$

whenever $\theta_1, \theta_2 \in \Theta$ and $\theta_1 < \theta_2$.

By Lemma 2.3, we can establish the following preservation property of log-convexity.

THEOREM 2.4: *Let X_θ be a nonnegative random variable having a distribution function belonging to the family $\{F_\theta, \theta \in \Theta\}$, which possesses the semi-group property. Then, for a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, $T(\phi, \theta) = \mathbb{E}[\phi(X_\theta)]$ is log-convex in $\theta \in \Theta$ if ϕ is log-convex on $\mathbb{R}_+$.*

PROOF: Assume that ϕ is log-convex. To prove the desired result, it suffices to prove that

$$\frac{T(\phi,\theta_1+\delta)}{T(\phi,\theta_1)} \le \frac{T(\phi,\theta_2+\delta)}{T(\phi,\theta_2)} \tag{2.6}$$

whenever $\theta_1 < \theta_2$ and $\delta > 0$ such that $\theta_i, \theta_i+\delta \in \Theta$. Denote by $W_i = X_{\theta_i+\delta} - X_{\theta_i}$ for $i =$ 1,2. Then $W_1 =_{\rm st} W_2$ and W_2 is independent of X_{θ_i}. First, note that

$$\begin{aligned}\frac{T(\phi,\theta_i+\delta)}{T(\phi,\theta_i)} &= \frac{\mathbb{E}[\phi(X_{\theta_i}+W_2)]}{\mathbb{E}[\phi(X_{\theta_i})]} = \mathbb{E}\left\{\mathbb{E}\left[\left.\frac{\phi(X_{\theta_i}+W_2)}{\mathbb{E}[\phi(X_{\theta_i})]}\right| W_2\right]\right\} \\ &= \mathbb{E}\left[\int_{\mathbb{R}} \frac{\phi(x+W_2)}{\phi(x)}\, \mathrm{d}G_\phi(x|\theta_i)\right] \\ &= \mathbb{E}\big\{\mathbb{E}[\Psi(U_i,W_2)|W_2]\big\},\end{aligned} \tag{2.7}$$

where $\Psi(u,w) = \phi(u+w)/\phi(u)$ for $(u,w) \in \mathbb{R}_+^2$, and U_i is a nonnegative random variable, independent of W_2, which has a distribution function belonging to the family $\mathcal{P} = \{G_\phi(\cdot|\theta_i), i = 1,2\}$ with $G_\phi(\cdot|\theta)$ defined by (2.1).

Observe that the following two facts hold.

- $\Psi(u,w) = \phi(u+w)/\phi(u)$ is increasing in $u \in \mathbb{R}_+$ for each $w \in \mathbb{R}_+$ since ϕ is log-convex;
- By Lemma 2.3, we have $U_1 \le_{\rm st} U_2$ since $\theta_1 < \theta_2$.

Applying Lemma 2.1 yields that $\mathbb{E}[\Psi(U_1,w)] \le \mathbb{E}[\Psi(U_2,w)]$ for all $w \in \mathbb{R}_+$. Therefore, it follows from (2.7) that

$$\frac{T(\phi,\theta_1+\delta)}{T(\phi,\theta_1)} = \mathbb{E}\{\mathbb{E}[\Psi(U_1,W_2)|W_2]\} \le \mathbb{E}\{\mathbb{E}[\Psi(U_2,W_2)|W_2]\} = \frac{T(\phi,\theta_2+\delta)}{T(\phi,\theta_2)}.$$

This proves the log-convexity of $T(\phi,\theta)$ with respect to θ. ■

Next, we investigate the preservation property of log-concavity.

THEOREM 2.5: *Let X_θ be a nonnegative random variable having a distribution function belonging to the family $\{F_\theta, \theta \in \Theta\}$, which possesses the semi-group property. For a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, define an operator $T(\phi,\theta) = \mathbb{E}[\phi(X_\theta)]$.*

(i) *If ϕ is log-concave, and if*

$$X_{\theta_1} \le_{\rm lr} X_{\theta_2} \quad \text{whenever } \theta_1,\theta_2 \in \Theta \text{ and } \theta_1 < \theta_2, \tag{2.8}$$

then $T(\phi,\theta)$ is log-concave in $\theta \in \Theta$;

(ii) *If ϕ is increasing and log-concave, and if*

$$X_{\theta_1} \le_{\rm hr} X_{\theta_2} \quad \text{whenever } \theta_1,\theta_2 \in \Theta \text{ and } \theta_1 < \theta_2, \tag{2.9}$$

then $T(\phi,\theta)$ is increasing and log-concave in $\theta \in \Theta$;

(iii) *If ϕ is decreasing and log-concave, and if*

$$X_{\theta_1} \le_{\rm rh} X_{\theta_2} \quad \text{whenever } \theta_1,\theta_2 \in \Theta \text{ and } \theta_1 < \theta_2, \tag{2.10}$$

then $T(\phi,\theta)$ is decreasing and log-concave in $\theta \in \Theta$.

https://doi.org/10.1017/S0269964820000042 Published online by Cambridge University Press

PROOF: We proceed the proof by using a similar argument to that in the proof of Theorem 2.4. Assume that ϕ is log-concave. To prove the desired result, it suffices to prove that

$$\frac{T(\phi,\theta_1+\delta)}{T(\phi,\theta_1)} \geq \frac{T(\phi,\theta_2+\delta)}{T(\phi,\theta_2)} \tag{2.11}$$

whenever $\theta_1 < \theta_2$ and $\delta > 0$ such that $\theta_i, \theta_i + \delta \in \Theta$. Let W_1, W_2 and U_1, U_2 be as defined in the proof of Theorem 2.4. Now we consider three cases as follows.

(i) Suppose that ϕ does not possess the monotonicity. We consider the subcase that X_θ has a density function f_θ (the proof of the discrete case is similar). In this case, $G_\phi(\cdot|\theta)$ has a density function given by

$$g_\phi(u|\theta) = \frac{\phi(u) f_\theta(u)}{\mathbb{E}[\phi(X_\theta)]}, \quad \forall\, u \in \mathbb{R}_+. \tag{2.12}$$

Since $X_{\theta_1} \leq_{\text{lr}} X_{\theta_2}$, it follows that $U_1 \leq_{\text{lr}} U_2$ and, hence, $U_1 \leq_{\text{st}} U_2$.

(ii) Suppose that ϕ is increasing. Then, by the characterizations of the hazard rate order (see Theorem 3.1 of [18]), (2.9) implies that $G_\phi(\cdot|\theta_1) \leq_{\text{st}} G_\phi(\cdot|\theta_2)$, that is, $U_1 \leq_{\text{st}} U_2$.

(iii) Suppose that ϕ is decreasing. Then, we know from Capéraá [11] that (2.10) also implies $U_1 \leq_{\text{st}} U_2$.

Thus, for all three cases, we have that $U_1 \leq_{\text{st}} U_2$ and $\Psi(u,w) = \phi(u+w)/\phi(u)$ is decreasing in $u \in \mathbb{R}_+$ for each $w \in \mathbb{R}_+$. By Lemma 2.1, we get that $\mathbb{E}[\Psi(U_1,w)] \geq \mathbb{E}[\Psi(U_2,w)]$ for all $w \in \mathbb{R}_+$. Therefore, from (2.7), it follows that (2.11). Thus, we complete the proof. ■

Remark 2.6: It is worthnoting that Theorem 3.7 of Badía & Sangüesa [5] established similar results to Theorem 2.5. We point out the difference between Theorem 2.5 and Theorem 3.7 of Badía & Sangüesa [5]. First, our proof is different from theirs. Second, Badía & Sangüesa [5] constrains the operator $T(\phi,\theta)$ to be continuous in $\theta \in \mathbb{R}_+$ and the parameter space to be $\Theta = \mathbb{R}_+$. In order to ensure that $T(\phi,\theta)$ is continuous in $\theta \in \mathbb{R}_+$, they imposed the assumption of continuity in probability of the process $\{X_\theta, \theta \in \mathbb{R}_+\}$ as follows:

$$\lim_{s\to t} \mathbb{P}(|X_t - X_s| > \epsilon) = 0, \quad \text{for all } \epsilon > 0 \text{ and } t > 0.$$

Note that the semi-group property of $\{X_\theta, \theta \in \Theta\}$ is equivalent to the property of independent and stationary increments. Hence, Theorem 2.5(i) generalizes Theorem 3.7(c) of Badía & Sangüesa [5] by dropping off the assumption that $T(\phi,\theta)$ is continuous in $\theta \in \mathbb{R}_+$. However, the results of Theorem 2.5(ii) and (iii) and those of Theorem 3.7(a) and (b) of Badía & Sangüesa [5] do not include each other as they consider a slightly more general class of distributions called the class IPII [IPDI] (independent positive increasing [decreasing] increments) under the additional assumption that $T(\phi,\theta)$ is continuous in $\theta \in \mathbb{R}_+$.

An immediate consequence of Theorems 2.4 and 2.5 is the following proposition, which is interesting in itself. Let Z be a random variable with distribution function H. Recall that Z is said to be ILR (*increasing likelihood ratio*) if Z has a log-concave density or mass function; Z is IFR (*increasing failure rate*) if $\overline{H}(x)$ is log-concave in $x \in \mathbb{R}$, and Z is DRHR (*decreasing reversed hazard rate*) if $H(x)$ is log-concave in $x \in \mathbb{R}$. It is well known (see [8]) that ILR implies both IFR and DRHR.

Proposition 2.7: *Let $\{V_n, n \in \mathbb{N}_+\}$ be a sequence of independent and identically distributed nonnegative random variables, and let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$. Denote $S_n = \sum_{i=1}^n V_i$ for $n \in \mathbb{N}_+$, and set $S_0 = 0$.*

(i) *If ϕ is log-concave and V_1 is ILR, then $\mathbb{E}[\phi(S_n)]$ is log-concave on $\mathbb{N}$;*
(ii) *If ϕ is increasing and log-concave, and V_1 is IFR, then $\mathbb{E}[\phi(S_n)]$ is log-concave on $\mathbb{N}$;*
(iii) *If ϕ is decreasing and log-concave, and V_1 is DRHR, then $\mathbb{E}[\phi(S_n)]$ is log-concave on $\mathbb{N}$;*
(iv) *If ϕ is log-convex, then $\mathbb{E}[\phi(S_n)]$ is log-convex on $\mathbb{N}$.*

Proof: (i) Since $0 \le_{\rm lr} X_{n+1}$, it follows from Theorem 1.C.9 of Shaked & Shanthikumar [32] that $S_n \le_{\rm lr} S_{n+1}$ for $n \in \mathbb{N}$. Thus, the desired result follows from Theorem 2.5(i).

(ii) By Theorem 1.B.4 of Shaked & Shanthikumar [32], the IFR property of V_1 implies $S_n \le_{\rm hr} S_{n+1}$ for $n \in \mathbb{N}$. The rest proof follows from Theorem 2.5(ii).

(iii) By Theorem 1.B.45 of Shaked & Shanthikumar [32], the DRHR property of V_1 implies $S_n \le_{\rm rh} S_{n+1}$ for $n \in \mathbb{N}$. The rest proof follows from Theorem 2.5(iii).

(iv) It trivially follows from Theorem 2.4. This completes the proof of the proposition. ■

Badía [3] proved part (iv) of Proposition 2.7 by using a different method. However, his method does not apply to the case of log-concavity.

For any interval $A \subset \mathbb{R}_+$, the indicator function 1_A is log-concave. Thus, under the condition of Proposition 2.7, if V_1 is ILR, then $\mathbb{P}(S_n \in A)$ is log-concave on $\mathbb{N}$; if V_1 is IFR, then $\mathbb{P}(S_n > b)$ is log-concave on $\mathbb{N}$ for any $b \in \mathbb{R}_+$; and if V_1 is DRHR, then $\mathbb{P}(S_n \le b)$ is log-concave on $\mathbb{N}$ for any $b \in \mathbb{R}_+$.

Choose the special parameter space $\Theta = \mathbb{N}_+$, and let X_n denote the partial sum S_n of random variables $\{V_n, n \in \mathbb{N}_+\}$. Checking the proofs of Theorems 2.4 and 2.5 carefully, we know that Proposition 2.7 can be generalized by dropping off the assumption that V_1 has the same distribution as V_2. We state it in the following proposition without the proof.

Proposition 2.8: *Let $\{V_n, n \in \mathbb{N}_+\}$ be a sequence of independent nonnegative random variables such that $V_k \sim F$ for $k \ge 2$(V_1 may have a different distribution from V_2), and let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$. Denote $S_n = \sum_{i=1}^n V_i$ for $n \in \mathbb{N}_+$, and set $S_0 = 0$.*

(i) *If ϕ is log-concave, and V_1 and V_2 are ILR, then $\mathbb{E}[\phi(S_n)]$ is log-concave on $\mathbb{N}$;*
(ii) *If ϕ is increasing and log-concave, and V_1 and V_2 are IFR, then $\mathbb{E}[\phi(S_n)]$ is log-concave on $\mathbb{N}$;*
(iii) *If ϕ is decreasing and log-concave, and V_1 and V_2 are DRHR, then $\mathbb{E}[\phi(S_n)]$ is log-concave on $\mathbb{N}$;*
(iv) *If ϕ is log-convex, then $\mathbb{E}[\phi(S_n)]$ is log-convex on $\mathbb{N}$.*

We end this section with a remark concerning the preservation of convexity and concavity for some operators.

Remark 2.9: Let X_θ be a nonnegative random variable, on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, having a distribution function belonging to the family $\{F_\theta, \theta \in \mathbb{R}_+\}$, which possesses the semi-group property. It was shown in Adell *et al.* [1] that if ϕ is convex on $\mathbb{R}_+$, then the operator $T(\phi,\theta)=\mathbb{E}[\phi(X_\theta)]$ is also convex in $\theta\in\mathbb{R}_+$, and the operator

$$T_t(\phi,s)=\mathbb{E}\left[\phi\left(\frac{X_{st}}{X_t}\right)\right] \tag{2.13}$$

is convex in $s\in[0,1]$ for any $t>0$. Clearly, these two operators also preserve concavity. The proof is based on the construction of reverse martingale,

$$\left(\frac{X_t-X_s}{t-s},\mathscr{F}_s^t\right),\quad 0\le s<t<\infty,$$

where $\mathscr{F}_s^t$ is a σ-field generated by $\{X_u:\forall\,u\in[0,s]\cup[t,\infty)\}$ with respect to $(\Omega,\mathscr{F},\mathbb{P})$.

3. APPLICATIONS

3.1. Preservation of Reliability Properties under Renewal Processes

Consider a renewal process $\{N(t),t\in\mathbb{R}_+\}$ with independent interarrival times $\{X_n\}$ with a common distribution function F, that is,

$$N(t)=\max\{n:S_n\le t,n\in\mathbb{N}\},\quad t\in\mathbb{R}_+,$$

where $S_0=0$, $S_n=\sum_{i=1}^n X_i$ for $n\in\mathbb{N}_+$, and $F(0)<1$. Let T be another nonnegative random variable, independent of $\{X_n,n\in\mathbb{N}_+\}$, with distribution function G. Denote by $\overline{G}_*=1-G_*$ and $G_*(x)=\lim_{t\uparrow x}G(t)$ for all $x\in\mathbb{R}$. Then

$$\mathbb{P}(N(T)\ge n)=\mathbb{P}(S_n\le T)=\mathbb{E}\big[\overline{G}_*(S_n)\big],\quad n\in\mathbb{N}, \tag{3.1}$$

and

$$\mathbb{P}(N(T)\le n)=\mathbb{P}(S_{n+1}>T)=\mathbb{E}\big[G_*(S_{n+1})\big],\quad n\in\mathbb{N}. \tag{3.2}$$

It is easy to see that if G [resp. $\overline{G}$] is log-concave on $\mathbb{R}_+$, then so is G_* [resp. $\overline{G}_*$], and that if $\overline{G}$ is log-convex on $\mathbb{R}_+$, then so is $\overline{G}_*$.

An immediate consequence of Proposition 2.7 is the following corollary.

Corollary 3.1: *Let $\{N(t),t\in\mathbb{R}_+\}$ and T be described as above.*

(i) *If T is IFR, and if F is DRHR, then $\mathbb{P}(N(T)\ge n)$ is log-concave in $n\in\mathbb{N}$;*
(ii) *If T is DRHR, and if F is IFR, then $\mathbb{P}(N(T)\le n)$ is log-concave in $n\in\mathbb{N}$;*
(iii) *If T is DFR, then $\mathbb{P}(N(T)\ge n)$ is log-convex in $n\in\mathbb{N}$.*

Badía & Sangüesa [4] proved Corollary 3.1 by using a different method under the assumption that the distribution function of T has no common discontinuity points with the distribution functions corresponding to S_n for all n. Corollary 3.1(iii) was also recovered by Badía [3] under the same assumption. Corollary 3.1(i) can also be derived from Ross *et al.* [29]. Note that T is IFR and DRHR if T is a constant. From Corollary 3.1, it follows that if F is DRHR, then $\mathbb{P}(N(t)\ge n)$ is log-concave in $n\in\mathbb{N}$, and if F is IFR, then $\mathbb{P}(N(t)\le n)$ is log-concave in $n\in\mathbb{N}$.

By Proposition 2.8, Corollary 3.1 can be generalized to the delayed renewal process.

3.2. Preservation under Bernstein-Type Operators

Throughout this subsection, let $\{N(t), t \in \mathbb{R}_+\}$ be a standard Poisson process, that is, a counting process with independent and stationary increments such that $N(0) = 0$ and $N(t)$ follows the Poisson distribution with parameter t. Let $\{M(t), t \in \mathbb{R}_+\}$ represent a standard Gamma process, that is, a process with independent and stationary increments such that $M(0) = 0$ and $M(t)$ follows the Gamma distribution with shape parameter t. Also, we assume that $\{N(t), t \in \mathbb{R}_+\}$ and $\{M(t), t \in \mathbb{R}_+\}$ are independent.

Consider the following Bernstein-type operators [3,5,12,13]:

- Bernstein operator

$$B_n(\phi, x) = \sum_{k=0}^{n} \phi\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0,1], \tag{3.3}$$

with $\phi : [0,1] \to \mathbb{R}$. Denote by $J_x \sim B(n, x)$, where $B(n, x)$ is the binomial distribution with parameters (n, x). Then, B_n has the stochastic representation given by

$$B_n(\phi, x) = \mathbb{E}\left[\phi\left(\frac{J_x}{n}\right)\right] = \mathbb{E}\left[\phi\left(\frac{N(tx)}{N(t)}\right) \Big| N(t) = n\right], \quad \forall x \in [0,1],\ t > 0.$$

- Szász–Mirakyan operator

$$S_t(\phi, x) = \sum_{k=0}^{\infty} \phi\left(\frac{k}{t}\right) e^{-tx} \frac{(tx)^k}{k!}, \quad x \geq 0,\ t > 0,$$

which has the stochastic representation given by

$$S_t(\phi, x) = \mathbb{E}\left[\phi\left(\frac{N(tx)}{t}\right)\right], \quad x \geq 0,\ t > 0. \tag{3.4}$$

The generalized phase-type distribution introduced in Shanthikumar [33] can be derived via Szász–Mirakyan operator.

- Szász–Mirakyan–Kantorovich operator

$$S_t^*(\phi, x) = \sum_{k=0}^{\infty} t e^{-tx} \frac{(tx)^k}{k!} \int_{k/t}^{(k+1)/t} \phi(u)\, \mathrm{d}u, \quad x \geq 0,\ t > 0,$$

which has the stochastic representation given by

$$S_t^*(\phi, x) = \mathbb{E}\left[\int_{N(tx)}^{N(tx)+1} \phi\left(\frac{u}{t}\right) \mathrm{d}u\right] = \mathbb{E}[\phi_t^*(N(tx))], \quad x \geq 0,\ t > 0, \tag{3.5}$$

with

$$\phi_t^*(x) = \int_x^{x+1} \phi\left(\frac{u}{t}\right) \mathrm{d}u = \int_0^1 \phi\left(\frac{x+u}{t}\right) \mathrm{d}u, \quad x \geq 0. \tag{3.6}$$

- Gamma-star operator

$$G_t(\phi, x) = \int_0^{\infty} \phi\left(\frac{\theta}{t}\right) e^{-\theta} \frac{\theta^{tx-1}}{\Gamma(tx)}\, \mathrm{d}\theta, \quad x \geq 0,\ t > 0, \tag{3.7}$$

which has the stochastic representation given by

$$G_t(\phi, x) = \mathbb{E}\left[\phi\left(\frac{M(tx)}{t}\right)\right], \quad x \geq 0,\ t > 0. \tag{3.8}$$

A counterexample was given in Badía [3] to illustrate that Bernstein operator doses not preserve the log-convexity. Badía [3] gave a wrong counterexample with $\phi(x) = (x - 1/2)^2$ to show that B_n does not preserve the log-concavity because ϕ is not log-concave on $[0, 1]$. This was also pointed out in Badía & Sangüesa [5]. A log-concave function on $\mathbb{R}_+$ can not have any internal zero. In fact, it was shown by Mu [25] that Bernstein operator preserves the log-concavity.

By Remark 2.9, Szász–Mirakyan and Gamma-star operators preserve the convexity and concavity. Note that $N(t_1) \le_{\rm lr} N(t_2)$ and $M(t_1) \le_{\rm lr} M(t_2)$ for all $t_2 > t_1 > 0$. Applying Theorems 2.4 and 2.5 to (3.4), (3.5) and (3.8), we have the following corollary.

Corollary 3.2: *The Szász–Mirakyan, Szász–Mirakyan–Kantorovich and Gamma-star operators preserve the log-concavity and log-convexity. That is, if ϕ is log-concave [log-convex] on $\mathbb{R}_+$, then $S_t(\phi, x)$, $S_t^*(\phi, x)$ and $G_t(\phi, x)$ are log-concave [log-convex] in $x \in \mathbb{R}_+$.*

Proof: The proof of $S_t(\phi, x)$ and $G_t(\phi, x)$ are trivial. We only show the result for $S_t^*(\phi, x)$. It suffices to show that ϕ^* defined by (3.6) is log-concave [log-convex] when ϕ is log-concave [log-convex]. The log-convexity follows immediately from Theorem 9.3 in Saumard & Wellner [30]. Here, we give a proof for log-concavity. Since t is fixed, it suffices to show that

$$\phi(x)\,\text{is log}-\text{concave} \Longrightarrow \phi^*(x) = \int_0^1 \phi(x+u)\,{\rm d}u, \quad \text{is log}-\text{concave}. \tag{3.9}$$

It is trivial to see that the log-concavity of ϕ implies that $\phi(x + u)$ is log-concave in $(x, u) \in \mathbb{R}_+ \times [0, 1]$. Then, by Prékopa's Theorem (see also Theorem 3.3 in Saumard & Wellner [30]), which says log-concavity is preserved under marginalization, we get that (3.9) holds. ■

The preservation of log-convexity in Corollary 3.2 was also established by Badía [3]. Badía [3] gave a counterexample with $\phi(x) = (x - d)^2$ (d is a positive constant) to illustrate that Szász–Mirakyan and Gamma-star operators do not preserve the log-concavity. However, this counterexample is wrong because $\phi(x) = (x - d)^2$ is not log-concave on $\mathbb{R}_+$.

In fact, the preservation of log-concavity for Szász–Mirakyan operator can be proved directly as follows. Assume that ϕ is log-concave on $\mathbb{R}_+$. Then, $\alpha_k = \phi(k/n)$ is log-concave in $k \in \mathbb{N}$. To prove that $S_t(\phi, x)$ is log-concave in $x \in \mathbb{R}_+$, it suffices to verify that $S(x) = \sum_{k=0}^{\infty} \alpha_k x^k/k!$ is log-concave on $\mathbb{R}_+$. To this end, note that

$$S'(x) = \sum_{k=0}^{\infty} \alpha_{k+1} \frac{x^k}{k!}, \quad S''(x) = \sum_{k=0}^{\infty} \alpha_{k+2} \frac{x^k}{k!}.$$

Then, for any $x \in \mathbb{R}_+$,

$$\begin{aligned}
[S'(x)]^2 - S(x)S''(x) &= \left(\sum_{k=0}^{\infty} \alpha_{k+1} \frac{x^k}{k!}\right)^2 - \left(\sum_{k=0}^{\infty} \alpha_k \frac{x^k}{k!}\right)\left(\sum_{k=0}^{\infty} \alpha_{k+2} \frac{x^k}{k!}\right) \\
&= \sum_{\ell=0}^{\infty} \frac{x^\ell}{i!(\ell-i)!} \cdot [\alpha_{i+1}\alpha_{\ell-i+1} - \alpha_i \alpha_{\ell-i+2}] \ge 0.
\end{aligned}$$

This means that $S(x)$ is log-concave on $\mathbb{R}_+$.

A special consequence of Corollary 3.2 concerning Gamma-star operator is the next proposition, which is well known in convex geometry. Marsiglietti & Kostina [23] exploited

Proposition 3.3 to derive a lower bound on the differential entropy of log-concave random variables.

Proposition 3.3 ([10]): *The function*

$$H(r) = \frac{1}{\Gamma(r)} \int_0^\infty x^{r-1} \psi(x) \, \mathrm{d}x \tag{3.10}$$

is log-concave [log-convex] on $\mathbb{R}_+$*, whenever* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *is log-concave [log-convex].*

Proof: The proof follows by observing $H(r) = G_1(\phi, r)$, where $G_t(\phi, r)$ is given by (3.7) with $\phi(x) = \psi(x)e^x$. ■

3.3. Preservation under Beta-Type Operators

Beta-type operators are associated with beta-type probability distributions, two of which are as follows ([1]):

- Beta operator

$$A_t(\phi, x) = \int_0^1 \phi(\theta) \frac{\theta^{tx-1}(1-\theta)^{t(1-x)-1}}{B(tx, t(1-x))} \, \mathrm{d}\theta, \quad t > 0, \; x \in (0,1),$$

  which has the following stochastic representation

$$A_t(\phi, x) = \mathbb{E}\left[\phi\left(\frac{M(tx)}{M(t)}\right)\right], \tag{3.11}$$

  where $\{M(t), t \in \mathbb{R}_+\}$ is a standard gamma process.

- Inverse Beta operator

$$D_t(\phi, x) = \int_0^\infty \phi(\theta) \frac{1}{B(tx, t)} \cdot \frac{\theta^{tx-1}}{(1+\theta)^{tx+t}} \, \mathrm{d}\theta, \quad t > 0, \; x > 0,$$

  which has the following stochastic representation

$$D_t(\phi, x) = \mathbb{E}\left[\phi\left(\frac{M(tx)}{V(t)}\right)\right], \tag{3.12}$$

  where $\{M(t), t \in \mathbb{R}_+\}$ and $\{V(t), t \in \mathbb{R}_+\}$ are two independent standard gamma processes. Here,

$$f(\theta | tx, t) = \frac{1}{B(tx, t)} \cdot \frac{\theta^{tx-1}}{(1+\theta)^{tx+t}}, \quad \theta \in \mathbb{R}_+,$$

  is the density of an inverse Beta distribution with parameters tx and t.

Both Beta and inverse Beta operators preserve the convexity and concavity (see Remark 2.9 or [1]). Since log-convexity is closed under mixture, it follows from (3.12) and Theorem 2.4 that the inverse Beta operator preserves the log-convexity.

Table 1. Preservation of log-concavity and log-convexity under operators

Operators	Log-concavity	Log-convexity
Bernstein operator B_n	Preserved	Not preserved
Szász–Mirakyan operator S_t	Preserved	Preserved
Szász–Mirakyan–Kantorovich operator S_t^*	Preserved	Preserved
Gamma-star operator G_t	Preserved	Preserved
Beta operator A_t	?	?
Inverse Beta operator D_t	Not preserved	Preserved

Counterexample 3.4 (The inverse Beta operator does not preserve the log-concavity): *Choose $\phi(x) = e^{-x}$, which is both log-concave and log-convex. Define $Z_t = 1 + 1/V(t)$. Then, $D_t(\phi, x) = \mathbb{E}[Z_t]^{-tx}$, and*

$$\frac{\partial}{\partial x} D_t(\phi, x) = -t\mathbb{E}[(Z_t)^{-tx} \log Z_t], \quad \frac{\partial^2}{\partial x^2} D_t(\phi, x) = t^2 \mathbb{E}[(Z_t)^{-tx} \log^2 Z_t].$$

By Hölder's inequality, it follows that

$$\left[\frac{\partial}{\partial x} D_t(\phi, x)\right]^2 - D_t(\phi, x) \frac{\partial^2}{\partial x^2} D_t(\phi, x) < 0, \quad \forall\, x > 0.$$

Therefore, $D_t(\phi, x)$ is not log-concave.

It is unknown whether Beta operator preserves the log-concavity or log-convexity. More generally, it is still unknown whether $T_t(\phi, s)$, defined by (2.13), is log-concave or log-convex in $s \in [0, 1]$ for each $t > 0$ under the assumptions in Remark 2.9.

We may summarize in Table 1 the results obtained concerning preservation of log-concavity and log-convexity under different operators in subsections 3.2 and 3.3.

Acknowledgments

The authors are grateful to the Editor and one anonymous referee for their comprehensive reviews of an earlier version of this paper. T. Mao was supported by the NNSF of China (No. 71671176). T. Hu, the corresponding author, was supported by the NNSF of China (No. 71871208).

References

1. Adell, J.A., Badía, F.G., & de la Cal, J. (1993). Beta-type operators preserve shape properties. *Stochastic Processes and Their Applications* 48: 1–8.
2. An, M.Y. (1998). Log-concavity versus log-convexity: a complete characterization. *Journal of Economic Theory* 80: 350–369.
3. Badía, F.G. (2009). On the preservation of log convexity and log concavity under some classical Bernstein-type operators. *Journal of Mathematical Analysis and Applications* 360: 485–490.
4. Badía, F.G. & Sangüesa, C. (2008). Preservation of reliability classes under mixtures of renewal processes. *Probability in the Engineering and Informational Sciences* 22: 1–17.
5. Badía, F.G. & Sangüesa, C. (2014). Log-concavity for Bernstein-type operators using stochastic orders. *Journal of Mathematical Analysis and Applications* 413: 953–962.
6. Badía, F.G. & Sangüesa, C. (2015). Inventory models with nonlinear shortage costs and stochastic lead times: applications of shape properties of randomly stopped counting processes. *Naval Research Logistics* 62: 345–356.
7. Bagnoli, M. & Bergstrom, T. (2005). Log-concave probability and its applications. *Economic Theory* 26: 445–469.

8. Barlow, R.E. & Proschan, F. (1981). *Statistical theory of reliability and life testing.* Silver Spring, MD: To Begin With.
9. Bobkov, S. & Madiman, M. (2011). Concentration of the information in data with log-concave distributions. *Annals of Probability* 39: 1528–1543.
10. Borell, C. (1973). Complements of Lyapunov's inequality. *Mathematische Annalen* 205: 323–331.
11. Capéraá, P. (1988). Tail ordering and asymptotic efficiency of rank tests. *Annals of Statistics* 3: 1045–1069.
12. de la Cal, J. & Cárcamo, J. (2003). On uniform approximation by some classical Bernstein-type operators. *Journal of Mathematical Analysis and Applications* 279: 625–638.
13. de la Cal, J. & Cárcamo, J. (2007). On the approximation of convex functions by Bernstein-type operators. *Journal of Mathematical Analysis and Applications* 334: 1106–1115.
14. Dharmadhikari, S. & Joag-dev, K (1988). *Unimodality, convexity and applications.* New York: Academic Press.
15. Finner, H. & Roters, M. (1993). Distribution functions and log-concavity. *Communications in Statistics: Theory and Methods* 22: 2381–2396.
16. Finner, H. & Roters, M. (1997). Log-concavity and inequalities for chi-square, F and Beta distributions with applications in multiple comparisons. *Statistica Sinica* 7: 771–787.
17. Hu, T. & Zhuang, W. (2006). Stochastic comparisons of m-spacings. *Journal of Statistical Planning and Inference* 136: 33–42.
18. Joag-dev, K., Kochar, S., & Proschan, F. (1995). A general composition theorem and its applications to certain partial orderings of distributions. *Statistics and Probability Letters* 22: 111–119.
19. Kahn, J. & Neiman, M. (2010). Negative correlation and log-concavity. *Random Structures and Algorithms* 37: 271–406.
20. Li, Y., Yu, L., & Hu, T. (2012). Probability inequalities for weighted distributions. *Journal of Statistical Planning and Inference* 142: 1272–1278.
21. Liggett, T.M. (1997). Ultra log-concave sequence and negative dependence. *Journal of Combinatorial Theory, Series A* 79: 315–325.
22. Marshall, A.W., Olkin, I., & Arnold, B.C. (2011). Inequalities: theory of majorization and its applications, 2nd ed. New York: Springer.
23. Marsiglietti, A. & Kostina, V. (2018). A lower bound on the differential entropy of log-concave random vectors with applications. *Entropy* 115: article number 185, 24 pages.
24. Misra, N. & van der Meulen, E.C. (2003). On stochastic properties of m-spacings. *Journal of Statistical Planning and Inference* 115: 683–697.
25. Mu, X. (2015). Log-concavity of a mixture of beta distributions. *Statistics and Probability Letters* 99: 125–130.
26. Patil, G.P. & Rao, C.R. (2002). The weighted distributions: a survey and their applications. In P.R. Krishnaiah (ed.), *Applications of statistic.* Amsterdam: North-Holland, pp. 385–405.
27. Patil, G.P. (2002). Weighted distributions. In A.H. El-Shaarawi & W.W. Piegorsch (eds), *Encyclopedia of Environmetrics*, vol. 4. Chichester: John Wiley & Sons, Ltd., pp. 2369–2377.
28. Proschan, F. & Sethuraman, J. (1977). Schur functions in statistics I. The preservation theorem. *Annals of Statistics* 5: 256–262.
29. Ross, S.M., Shanthikumar, J.G., & Zhu, Z. (2005). On increasing-failure-rate random variables. *Journal of Applied Probability* 42: 797–809.
30. Saumard, A. & Wellner, J.A. (2014). Log-concavity and strong log-concavity: a review. *Statistics Surveys* 8: 45–114.
31. Sengupta, D. & Nanda, A.K. (1999). Log-concave and concave distributions in reliability. *Naval Research Logistics* 46: 419–433.
32. Shaked, M. & Shanthikumar, J.G (2007). *Stochastic orders.* New York: Springer.
33. Shanthikumar, J.G. (1985). Bilateral phase-type distributions. *Naval Research Logistics Quarterly* 32: 119–136.
34. Wang, Y. & Yeh, Y.-N. (2007). Log-concavity and LC-positivity. *Journal of Combinatorial Theory, Series A* 114: 195–210.
35. Yu, Y. (2010). Relative log-concavity and a pair of triangle inequalities. *Bernoulli* 16: 459–470.