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ON SEMIGROUP ALGEBRAS

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1. *Introduction.* In the classical theory of representations of a finite group by matrices over a field $\mathfrak{F}$, the concept of the group algebra (group ring) over $\mathfrak{F}$ is of fundamental importance. The chief property of such an algebra is that it is semi-simple, provided that the characteristic of $\mathfrak{F}$ is zero or a prime not dividing the order of the group. As a consequence of this, the representations of the algebra, and hence of the group, are completely reducible.

In the present paper we discuss a more general concept, the algebra of a finite semi-group over a given field. Our main task is to find necessary and sufficient conditions for such an algebra to be semisimple, and to interpret some of the results of this investigation in terms of representation theory.

Since we shall be concerned mainly with so-called 'semisimple' semigroups, we give a brief account of these in § 2; there we do not restrict ourselves to finite semi-groups, but we do assume the existence of a 'principal series'. In § 3 we give the formal definition of the algebra of a finite semigroup S over a field $\mathfrak{F}$. In the case where S has a zero, we usually find it convenient to identify this element with the zero of the algebra, thus forming the 'contracted' algebra of S over $\mathfrak{F}$. The problem of finding necessary and sufficient conditions for the semisimplicity of the algebra of an arbitrary semigroup is then reduced to that of finding these conditions for the contracted algebra of a simple semigroup.

A new class of algebras is defined in § 4. An algebra of this class consists of all rectangular matrices of given dimensions with entries from an algebra $\mathfrak{A}$ with an identity; multiplication is defined by means of a fixed 'sandwich' matrix P. In particular the contracted algebra of a simple semigroup has this structure. Necessary and sufficient conditions are found for the semisimplicity of such an algebra in 4·7; these are that $\mathfrak{A}$ is semisimple and P non-singular. Tests for the non-singularity of P are given in § 5.

In § 6 we combine the results of the previous sections. The notion of a 'c-non-singular' simple semigroup is introduced, and is used in the formulation of the main result (6·4). § 7 is devoted to a discussion of the simplicity of a semigroup algebra, while in § 8 we outline Clifford's representation theory for a simple semigroup, and show how it links up with the results of § 6 when the semigroup algebra is semisimple.

Finally, in § 9 we discuss semigroups of an important type to which our results may readily be applied, namely, those which admit relative inverses. These semigroups are first characterized in terms of their principal series (9·2). The algebra of a finite semigroup S of this type over a field $\mathfrak{F}$ is shown in 9·5 to be semisimple if and only if all the idempotents of S commute (with a restriction on the characteristic of $\mathfrak{F}$). The structure of such an S has been determined by Clifford, and in 9·6 we obtain a complete set of inequivalent irreducible representations of S over $\mathfrak{F}$.

2. *Semisimple semigroups.* A semigroup is a set which is closed with respect to a uniquely defined associative binary operation. For the definitions of 'ideal' (left, right, and two-sided), 'zero', 'idempotent', and 'difference semigroup' we refer the reader to (9).

Let S be a semigroup which possesses a minimal ideal K (an ideal which contains no ideal of S other than itself). Then K is readily seen to be unique, and, following Clifford (4), we call it the *kernel* of S. K may consist of a single element, which is the case if and only if S has a zero. A subsemigroup T of S is termed *K-potent* if $T^r \subseteq K$ for some r. In particular, if S has a zero z and if $S^2 = (z)$ then S is termed a *zero semigroup*.

Rees (9) has called a semigroup *simple* if (i) it has no ideals except itself, and possibly the ideal consisting of zero alone, and (ii) it is not the zero semigroup of order 2. In particular, a semigroup is called *simple without zero* if its only ideal is itself. From this definition it follows that a semigroup of order 1 must be regarded as being simple without zero. As has been pointed out by Clifford ((4), Theorem 1·1), the kernel of a semigroup is simple without zero.

A series for a semigroup S is a finite descending sequence of subsemigroups

$$S = S_1 \supseteq S_2 \supseteq \ldots \supseteq S_n \supset S_{n+1} = \emptyset$$

(where $\emptyset$ is the empty set), such that S_{i+1} is an ideal of S_i $(i = 1, \ldots, n-1)$. The series will be called proper if each inclusion is strict; the symbol $\supset$ will always be used to denote strict inclusion. A refinement of a series is a series containing every term of the given series, and a proper refinement of a proper series is a refinement which is a proper series containing strictly more terms than the given series.

Now consider the proper series

$$S = S_1 \supset S_2 \supset \ldots \supset S_n \supset S_{n+1} = \emptyset. \tag{2·1}$$

The semigroups $S_i - S_{i+1}$ $(i = 1, \ldots, n)$ are called the factors of the series (with the convention that $S_n - S_{n+1} = S_n$). If (2·1) has no proper refinement it is called a *composition series* for S. Rees (9) has shown that any two composition series for a semigroup are isomorphic; that is, the factors of the two series may be put in 1-1 correspondence in such a way that corresponding factors are isomorphic. The factors of a composition series are called the *composition factors* of S. A composition factor is either simple or is a zero semigroup of order 2.

If in (2·1) each S_i is an ideal of S, and if the series has no proper refinement each term of which is an ideal of S, then (2·1) is called a *principal series* for S. It may be shown that any two principal series for S are isomorphic. The factors of a principal series are called the *principal factors* of S, and, as remarked by Green ((7), pp. 168–169), a principal factor† is either simple or is a zero semigroup. Clearly if a semigroup

† Green has defined a principal factor of a semigroup without reference to a principal series. The principal factors defined above are also principal factors in this more general sense.

has a principal series, then it has a kernel, and this is the last non-empty term in every such series.

As examples, consider the semigroups with the multiplication tables shown (the

	z	a	b	c	e
z	z	z	z	z	z
a	z	z	z	b	a
b	z	z	z	a	b
c	z	b	a	e	c
e	z	a	b	c	e

(i)

	z	a	b	c
z	z	z	z	z
a	z	z	z	z
b	z	z	z	a
c	z	z	a	z

(ii)

associative law is readily verified in both cases). In case (i) we have the principal series $S \supset (z, a, b) \supset (z)$, so that this semigroup has a zero principal factor of order 3. On the other hand, $S \supset (z, a, b) \supset (z, a) \supset (z)$ and $S \supset (z, a, b) \supset (z, b) \supset (z)$ are composition series. In case (ii) $S \supset (z, a, b) \supset (z, a) \supset (z)$ is both a principal series and a composition series, while $S \supset (z, a, b) \supset (z, b) \supset (z)$ is a composition series but not a principal series. This shows that a composition series need not be a refinement of some principal series.

Whereas the existence of a composition series for a semigroup implies the existence of a principal series, the converse is not true.

A semigroup S will be called *idempotent* if $S^2 = S$.

2·2. LEMMA. *Let* (2·1) *be a composition series for S. Then if S_i is idempotent it is an ideal of S.*

Proof. The result is certainly true if $i = 1, 2$: hence assume that $i > 2$. Suppose we have proved that S_i is an ideal of S_{i-j} for some $j \leqslant i-2$. Then $S_{i-j} S_i S_{i-j} \subseteq S_i$. But $S_{i-j} S_i S_{i-j} \supseteq S_i^3 = S_i$ by hypothesis, so that $S_i = S_{i-j} S_i S_{i-j}$, which is an ideal of S_{i-j-1}. Now S_i is an ideal of S_{i-1}; hence the result follows by induction on j.

In particular, since S_n^2 is an ideal of S_n, we have $S_n^2 = S_n$ and so S_n is an ideal of S. It is minimal, and hence is the kernel of S.

A semigroup is said to be *semisimple* if it has a principal series all of whose factors are simple†.

2·3. THEOREM. *Every principal series of a semisimple semigroup is also a composition series, and conversely.*

Proof. Let (2·1) be a principal series for a semisimple semigroup S. Suppose that T_i is an ideal of S_i such that $S_i \supset T_i \supseteq S_{i+1}$ for some i. Then since $S_i - S_{i+1}$ is simple it follows that $T_i = S_{i+1}$. Hence (2·1) is also a composition series for S, and in particular the composition factors of S coincide with the principal factors, and are all simple.

Conversely, let (2·1) be a composition series for S. By 2·2, to prove that it is a principal series it is sufficient to show that $S_i^2 = S_i$ for $i = 1, \ldots, n$. Suppose we have proved that $S_{i+1}^2 = S_{i+1}$ for some i. Then $S_i \supseteq S_i^2 \supseteq S_{i+1}^2 = S_{i+1}$, and so since S_i^2 is an ideal of S_i we have either $S_i^2 = S_i$ or $S_i^2 = S_{i+1}$. But the latter contradicts the simplicity of the factor $S_i - S_{i+1}$. Hence $S_i^2 = S_i$, and since $S_n^2 = S_n$ the result follows by induction on i.

† Green uses his generalized definition of a principal factor to define a semisimple semigroup whether or not a principal series exists.

The second part of the proof shows that we could equally well have defined a semisimple semigroup to be a semigroup with a composition series, all of whose factors are simple.

We mention two further results:

2·4. LEMMA. *Let S be a semigroup, and M an ideal of S. Then S is semisimple if and only if M and $S-M$ are semisimple.*

In the proof of this, which is omitted, it is convenient to use our second definition of semisimplicity.

2·5. THEOREM. *Let S be a semigroup with a principal series. Then the set of all ideals M of S such that $S-M$ is semisimple has a unique minimal member.*

Proof. It suffices to prove that if M, N belong to the set, then so does $M \cap N$. Let $A = M \cup N$, $B = M \cap N$ ($\neq \emptyset$, since $M \cap N \supseteq MN$). $A-M$ is an ideal of $S-M$, and so is semisimple by 2·4. But $A-M \cong N-B$, and $S-N$ is semisimple; hence since $(S-B)-(N-B) \cong S-N$ it follows from 2·4 that $S-B$ is semisimple.

We call the unique ideal U whose existence is established in 2·5 the *upper radical* of S.

If K is the kernel of S, then we call the union L of all the K-potent left, right, and two-sided ideals of S the *lower radical* of S. This radical has been discussed by Clifford (5).

Clearly $U \supseteq L$. However we need not have $U = L$, as the following trivial example shows: let S have the multiplication table given below; then $U = S$ and $L = (z)$.

	z	a	b
z	z	z	z
a	z	a	a
b	z	a	a

3. *Semigroup algebras.* We shall assume for the remainder of the paper, with the exception of the first part of § 9, that the semigroups with which we are dealing are finite.

Let S be a semigroup and $\mathfrak{F}$ a field. We define *the algebra* $\mathfrak{A}_{\mathfrak{F}}(S)$ *of S over $\mathfrak{F}$* as follows: The vector space of $\mathfrak{A}_{\mathfrak{F}}(S)$ is the space whose elements are the formal sums $\sum_{i=1}^{n} \lambda_i s_i$, where $s_1, \ldots, s_n$ are the elements of S, and $\lambda_i \in \mathfrak{F}$ ($i = 1, \ldots, n$). Multiplication is then defined by the rule

$$(\sum_i \lambda_i s_i)(\sum_j \mu_j s_j) = \sum_{i,j} (\lambda_i \mu_j) s_{k(i,j)},$$

where $s_{k(i,j)}$ is the product $s_i s_j$ in S. The associative law in $\mathfrak{A}_{\mathfrak{F}}(S)$ follows from that in S. We identify s_i with $1s_i$, where 1 is the identity of $\mathfrak{F}$, so that S is embedded in the multiplicative semigroup of $\mathfrak{A}_{\mathfrak{F}}(S)$.

When there is no risk of ambiguity we shall write $\mathfrak{A}(S)$ for $\mathfrak{A}_{\mathfrak{F}}(S)$. If T is any subset of S, then $\mathfrak{A}(T)$ will denote that subspace of $\mathfrak{A}(S)$ which has T as basis. In particular, if M is an ideal of S then $\mathfrak{A}(M)$ is an ideal of $\mathfrak{A}(S)$.

3·1. If S has a zero z, then it is usually convenient to discuss $\mathfrak{A}(S)-\mathfrak{A}(z)$† instead of $\mathfrak{A}(S)$, and we call this algebra *the contracted algebra of S over $\mathfrak{F}$*. If M is an ideal of

† We write $\mathfrak{A}(z)$ in place of $\mathfrak{A}((z))$.

a semigroup S, then there is a natural isomorphism between $\mathfrak{A}(S)-\mathfrak{A}(M)$ and the contracted algebra of $S-M$ over $\mathfrak{F}$.

The term 'algebra' will always be used to mean 'associative linear algebra of finite order'. We require the following result, which is the analogue of 2·4:

3·2. LEMMA. *Let $\mathfrak{B}$ be an algebra over a field $\mathfrak{F}$, and let $\mathfrak{N}$ be any ideal of $\mathfrak{B}$. Then $\mathfrak{B}$ is semisimple if and only if $\mathfrak{N}$ and $\mathfrak{B}-\mathfrak{N}$ are semisimple.*

Proof. The result is trivial if $\mathfrak{N}=\mathfrak{B}$ or (0); hence suppose $\mathfrak{N}$ is a proper ideal. First let $\mathfrak{N}$ and $\mathfrak{B}-\mathfrak{N}$ be semisimple, and let a be any properly nilpotent element of $\mathfrak{B}$. Let $x \to \bar{x}$ denote the natural homomorphism of $\mathfrak{B}$ onto $\mathfrak{B}-\mathfrak{N}$. For any given $x \in \mathfrak{B}$ we can find r such that $(xa)^r = 0$. Hence $(\overline{xa})^r = \bar{0}$, and so $\bar{a}$ is a properly nilpotent element of $\mathfrak{B}-\mathfrak{N}$. Thus by hypothesis $\bar{a}=\bar{0}$, and hence $a \in \mathfrak{N}$. But then a is *a fortiori* properly nilpotent in $\mathfrak{N}$, and so $a = 0$. Hence $\mathfrak{B}$ is semisimple.

The converse is an immediate consequence of Wedderburn's Theorem. Let

$$\mathfrak{B} = \mathfrak{B}_1 \oplus \ldots \oplus \mathfrak{B}_k = \bigoplus_{i=1}^{k} \mathfrak{B}_i$$

be the expression for $\mathfrak{B}$ as a direct sum of simple algebras. This decomposition is unique apart from the numbering of the components, and this can be done in such a way that $\mathfrak{N} = \bigoplus_{i=1}^{r} \mathfrak{B}_i$ where $1 \leqslant r \leqslant k-1$. Hence we also have $\mathfrak{B}-\mathfrak{N} \cong \bigoplus_{i=r+1}^{k} \mathfrak{B}_i$. Thus by the converse of Wedderburn's Theorem both $\mathfrak{N}$ and $\mathfrak{B}-\mathfrak{N}$ are semisimple.

The number of simple components in a semisimple algebra $\mathfrak{B}$ will be called its *class number*, and will be denoted by $\mathrm{Cl}\,\mathfrak{B}$. This is also the number of inequivalent irreducible representations of $\mathfrak{B}$ over its ground field. From the above proof we see that $\mathrm{Cl}\,\mathfrak{B} = \mathrm{Cl}\,\mathfrak{N} + \mathrm{Cl}\,(\mathfrak{B}-\mathfrak{N})$.

As a corollary of 3·2 we see that if S is a semigroup with a zero z, and if $\mathfrak{F}$ is any field, then $\mathfrak{A}(S)$ is semisimple if and only if $\mathfrak{A}(S)-\mathfrak{A}(z)$ is semisimple; for $\mathfrak{A}(z) \cong \mathfrak{F}$, which is semisimple *qua* algebra over $\mathfrak{F}$.

3·3. LEMMA. *Let $\mathfrak{A}(S)$ be the algebra of a semigroup S over a field $\mathfrak{F}$. Then $\mathfrak{A}(S)$ is semisimple if and only if the algebra of each of the principal factors of S over $\mathfrak{F}$ is semisimple.*

Proof. Let $S = S_1 \supset S_2 \supset \ldots \supset S_n \supset S_{n+1} = \emptyset$ be a principal series for S. Then we have a corresponding series of ideals of $\mathfrak{A}(S)$, namely

$$\mathfrak{A}(S) = \mathfrak{A}(S_1) \supset \mathfrak{A}(S_2) \supset \ldots \supset \mathfrak{A}(S_n) \supset (0).$$

We adopt the convention that $\mathfrak{A}(\emptyset) = (0)$. As remarked in 3·1 $\mathfrak{A}(S_i)-\mathfrak{A}(S_{i+1})$ is isomorphic with the contracted algebra of $S_i - S_{i+1}$ over $\mathfrak{F}$ $(i = 1, \ldots, n-1)$, and it is enough to show that $\mathfrak{A}(S)$ is semisimple if and only if $\mathfrak{A}(S_i)-\mathfrak{A}(S_{i+1})$ is semisimple for $i = 1, \ldots, n$. But this is a consequence of 3·2.

3·4. COROLLARY. *If $\mathfrak{A}(S)$ is semisimple, then S is semisimple.*

Proof. Each principal factor of S is either simple or is a zero semigroup. Suppose one of the factors is a zero semigroup. Then its contracted algebra over $\mathfrak{F}$ is a zero algebra, and so by 3·3 $\mathfrak{A}(S)$ cannot be semisimple.

3·5. COROLLARY. *If* $\mathfrak{A}(S)$ *is semisimple, then*

$$\mathrm{Cl}\,\mathfrak{A}(S) = \sum_{i=1}^{n} \mathrm{Cl}\,\{\mathfrak{A}(S_i) - \mathfrak{A}(S_{i+1})\}.$$

As a result of 3·3 and 3·4, to find necessary and sufficient conditions for $\mathfrak{A}(S)$ to be semisimple we need only consider the case in which S is simple.

3·6. Let S be a simple semigroup with a zero. Rees (9) has shown that the structure of S is completely determined to within isomorphism by a certain (finite) group G, two integers m and n, and an $n \times m$ array P of elements $p_{jr} \in G(z)$ (the group-with-zero formed by adjoining a zero element z to G). P has the property that at least one element in each row and column is not z. The elements of S may then be regarded as the $m \times n$ matrices $(x)_{ij}$, where $(x)_{ij}$ has $x \in G(z)$ in the (i, j)th position and z elsewhere. Multiplication in S, which we denote by $\circ$, is according to the rule

$$(x)_{ij} \circ (y)_{rs} = (x)_{ij} P (y)_{rs},$$

where the product on the right is calculated by the ordinary rules of matrix multiplication, assuming that z has the properties of an additive zero. Thus

$$(x)_{ij} \circ (y)_{rs} = (x p_{jr} y)_{is}.$$

The zero of S is $(z)_{ij}$ (all i, j).

The semigroup described above is called a *regular matrix semigroup over a group-with-zero*, and will be denoted by $S_{mn}[G, P]$. G is called its *basic group*, and P its *matrix*.

3·7. Rees has also shown that $S_{mn}[G, P] \cong S_{mn}[G, P^*]$ if and only if there exist 'monomial' matrices A and B over $G(z)$ (matrices with one and only one entry $\neq z$ in each row and column), of types $n \times n$ and $m \times m$ respectively, such that $P^* = AP^\theta B$, where θ is an automorphism of $G(z)$ and P^θ is that $n \times m$ matrix whose (i, j)th entry is p_{ij}^θ.

3·8. From the description of S in 3·6 we see that *the contracted algebra of* S *over* $\mathfrak{F}$ *may be regarded as the vector space* $\mathfrak{S}_{mn}(\mathfrak{A})$ *of all* $m \times n$ *matrices over the group algebra* $\mathfrak{A} = \mathfrak{A}(G)$† *of* G *over* $\mathfrak{F}$, *with multiplication* ($\circ$) *defined by* $A \circ B = APB$ ($A, B \in \mathfrak{S}_{mn}(\mathfrak{A})$), *where if* z *occurs in* P *it is taken as the zero of* $\mathfrak{A}$.

4. *The algebra* $M_{mn}[\mathfrak{A}, P]$. It is convenient to discuss a somewhat more general algebra than that described in 3·8. Before doing so, however, we state some results which we shall presently require.

Let $\mathfrak{B}$ be an algebra over $\mathfrak{F}$ with an identity f. Then the element $a \in \mathfrak{B}$ is termed *non-singular* if there exists $b \in \mathfrak{B}$ such that either $ab = f$ or $ba = f$. We first note the following well-known result (see, for example, (1), chapter 1, Theorem 4):

4·1. LEMMA. (i) *The element* a *is non-singular if and only if the constant term of its minimum function is non-zero.* (ii) *If* $ab = f$ *then* b *is expressible as a polynomial in* a *over* $\mathfrak{F}$, *and hence* (iii) *if* $ab = f$, *then* $ba = f$.

The element b is unique, and we write $b = a^{-1}$. We note that every subalgebra of $\mathfrak{B}$ which contains a and f also contains a^{-1} in virtue of (ii).

† This is in accord with our notational convention, for G may be regarded as a subsemigroup of S.

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4·2. COROLLARY. *Let $\mathfrak{B}^*$ be an algebra with an identity f^*, and let θ be an isomorphism of $\mathfrak{B}$ into $\mathfrak{B}^*$ such that $f^\theta = f^*$. Then a is non-singular in $\mathfrak{B}$ if and only if a^θ is non-singular in $\mathfrak{B}^*$.*

Proof. Let a^θ be non-singular. Then by 4·1 (ii) there exists a polynomial $h(a^\theta)$ in a^θ over $\mathfrak{F}$ such that $a^\theta . h(a^\theta) = f^*$. Hence $[a.h(a)]^\theta = f^\theta$, and so $a.h(a) = f$, since θ is 1-1. The converse is trivial.

4·3. COROLLARY. *Let $\mathfrak{F}^*$ be an extension of $\mathfrak{F}$ and let $\mathfrak{B}^*$ be the corresponding scalar extension*† *of $\mathfrak{B}$. Then a is non-singular in $\mathfrak{B}$ if and only if it is non-singular in $\mathfrak{B}^*$.*

Proof. The minimum function of a is the same whether we consider a as an element of $\mathfrak{B}$ or of $\mathfrak{B}^*$. The result then follows from 4·1 (i).

Now let $\mathfrak{A}$ be an algebra over $\mathfrak{F}$ with an identity e. Let $\mathfrak{S}_{mn}(\mathfrak{A})$ denote the vector space of all $m \times n$ matrices with entries from $\mathfrak{A}$. Let P be any fixed $n \times m$ matrix over $\mathfrak{A}$. Then we turn $\mathfrak{S}_{mn}(\mathfrak{A})$ into an algebra by defining multiplication ($\circ$) by the rule $A \circ B = APB$ $(A, B \in \mathfrak{S}_{mn}(\mathfrak{A}))$; the distributive and associative laws are readily verified. This algebra will be denoted by $M_{mn}[\mathfrak{A}, P]$. The algebra of 3·8 is a special case.

Let U_n denote the $n \times n$ 'unit' matrix over $\mathfrak{A}$. We write $M_n(\mathfrak{A}) = M_{nn}[\mathfrak{A}, U_n]$.

Let θ be any homomorphism of $\mathfrak{A}$ into an algebra $\mathfrak{A}^*$, and let $A = (a_{ij}) \in \mathfrak{S}_{rs}(\mathfrak{A})$. Then A^θ will denote that element of $\mathfrak{S}_{rs}(\mathfrak{A}^*)$ whose (i, j)th entry is a_{ij}^θ. If θ is a matrix representation we shall use the more customary 'functional' notation θ: $a \to \theta(a)$, and accordingly write $\theta(A)$ in place of A^θ.

4·4. LEMMA. *Let $\mathfrak{B} = M_{mn}[\mathfrak{A}, P]$, $\mathfrak{B}^* = M_{mn}[\mathfrak{A}, P^*]$. If there exist non-singular matrices A, B in $M_n(\mathfrak{A})$, $M_m(\mathfrak{A})$ respectively such that $P^* = AP^\theta B$, where θ is an automorphism of $\mathfrak{A}$, then $\mathfrak{B} \simeq \mathfrak{B}^*$.*

Proof. The mapping ϕ: $X \to B^{-1}X^\theta A^{-1}$ is a non-singular linear transformation of $\mathfrak{S}_{mn}(\mathfrak{A})$. Furthermore, if X, $Y \in \mathfrak{S}_{mn}(\mathfrak{A})$ then

$$(XPY)^\phi = B^{-1}X^\theta P^\theta Y^\theta A^{-1} = B^{-1}X^\theta A^{-1}P^*B^{-1}Y^\theta A^{-1} = X^\phi P^* Y^\phi.$$

Hence ϕ is an isomorphism of $\mathfrak{B}$ onto $\mathfrak{B}^*$.

4·5. LEMMA. *$M_n(\mathfrak{A})$ is semisimple if and only if $\mathfrak{A}$ is semisimple.*

For the proof we refer the reader to (8), chapter 5, Theorem 8. ($M_n(\mathfrak{A})$ may be regarded as the direct product $M_n(\mathfrak{F}) \times \mathfrak{A}$). In fact if $\mathfrak{A} \simeq \bigoplus_{i=1}^{k} M_{r_i}(\mathfrak{D}_i)$, where each $\mathfrak{D}_i$ is a division algebra, then it can be shown that $M_n(\mathfrak{A}) \simeq \bigoplus_{i=1}^{k} M_{nr_i}(\mathfrak{D}_i)$; hence we have

4·6. COROLLARY. *If $\mathfrak{A}$ is semisimple, then* $\mathrm{Cl}\, M_n(\mathfrak{A}) = \mathrm{Cl}\, \mathfrak{A}$.

An $r \times s$ matrix A over $\mathfrak{A}$ will be termed *non-singular* if $r = s$ and A is a non-singular element of $M_r(\mathfrak{A})$. Otherwise A is termed *singular*.

We now prove the main result of this section.

4·7. THEOREM. *$M_{mn}[\mathfrak{A}, P]$ is semisimple if and only if* (i) *$\mathfrak{A}$ is semisimple, and* (ii) *P is non-singular.*

Proof. First suppose that $M_{mn}[\mathfrak{A}, P]$ is semisimple. Then it must have an identity; let this be E. We have $EPX = X$, $XPE = X$ for all $X \in \mathfrak{S}_{mn}(\mathfrak{A})$. It is then easily verified that we must have $EP = U_m$ and $PE = U_n$; hence to prove (ii) we need only

† See (1), chap. 1, §12.

show that $m = n$. Assume without loss of generality that $m \geqslant n$, and suppose that $m > n$. Write $E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$ and $P = (P_1\ P_2)$, where $E_1, P_1 \in M_n(\mathfrak{A})$. Then

$$EP = \begin{pmatrix} E_1 P_1 & E_1 P_2 \\ E_2 P_1 & E_2 P_2 \end{pmatrix} = \begin{pmatrix} U_n & 0 \\ 0 & U_{m-n} \end{pmatrix}.$$

In particular $E_1 P_1 = U_n$, whence $P_1 E_1 = U_n$ (4·1 (iii)). But $E_2 P_1 = 0$; hence $E_2 = E_2 P_1 E_1 = 0$, contradicting $E_2 P_2 = U_{m-n}$. Thus we have $m = n$, and so P is non-singular (showing, incidentally, that $E = P^{-1}$). Since $P = PU_n U_n$, we have

$$M_{nn}[\mathfrak{A}, P] \cong M_{nn}[\mathfrak{A}, U_n] = M_n(\mathfrak{A}) \tag{4·8}$$

by 4·4; hence $M_n(\mathfrak{A})$ is semisimple, and so $\mathfrak{A}$ is semisimple, by 4·5.

Conversely, suppose that (i) and (ii) hold. From (ii) $m = n$, and again (4·8) holds. From (i) and 4·5 $M_n(\mathfrak{A})$ is semisimple, and the result follows.

4·9. Corollary. *If* $M_{nn}[\mathfrak{A}, P]$ *is semisimple, then* $\mathrm{Cl}\, M_{nn}[\mathfrak{A}, P] = \mathrm{Cl}\, \mathfrak{A}$.

This follows at once from 4·6 and (4·8).

A discussion of the radical of $M_{mn}[\mathfrak{A}, P]$ in the general case will be given in a later paper.

5. *Tests for the non-singularity of a matrix over* $\mathfrak{A}$. In view of 4·7 it is of some importance to find criteria for the non-singularity of a matrix over $\mathfrak{A}$.

Let $\mathfrak{A}^*$ be any subalgebra of $\mathfrak{A}$ which contains e, the identity of $\mathfrak{A}$; we can of course choose $\mathfrak{A}^* = \mathfrak{A}$. Let Γ^* denote the regular representation of $\mathfrak{A}^*$. If $\mathfrak{A}^*$ has order r over $\mathfrak{F}$, then Γ^* is a faithful representation of degree r, and $\Gamma^*(e) = I_r$.

5·1. Lemma. *Let* $A \in M_n(\mathfrak{A}^*)$. *Then* A *is non-singular in* $M_n(\mathfrak{A})$ *if and only if* $\Gamma^*(A)$ *is a non-singular* $nr \times nr$ *matrix over* $\mathfrak{F}$.

Proof. Consider the mapping $X \to \Gamma^*(X)$ of $M_n(\mathfrak{A}^*)$ into $M_{nr}(\mathfrak{F})$. By linearity, and the rule for block multiplication, this is a representation of $M_n(\mathfrak{A}^*)$, and it is readily seen to be faithful. We also have $\Gamma^*(U_n) = I_{nr}$. The result then follows by 4·2.

5·2. Lemma. *Let* $\mathfrak{A}^*$ *be semisimple, and let* $\{\Gamma_i^*;\ i = 1, \ldots, k^*\}$ *be a complete set of inequivalent irreducible representations of* $\mathfrak{A}^*$ *over* $\mathfrak{F}$. *Then* $A \in M_n(\mathfrak{A}^*)$ *is non-singular in* $M_n(\mathfrak{A})$ *if and only if each matrix* $\Gamma_i^*(A)$ $(i = 1, \ldots, k^*)$ *is non-singular*.

Proof. Let Γ^* be the regular representation of $\mathfrak{A}^*$ as before. Then $\Gamma^*(A)$ can be transformed to a diagonal sum of matrices $\bigoplus_{i=1}^{k^*} m_i \Gamma_i^*(A)$, where each m_i is a positive integer. Thus $\Gamma^*(A)$ is non-singular if and only if each matrix $\Gamma_i^*(A)$ is non-singular, and the result follows from 5·1.

Finally, we mention a useful test for the non-singularity of a special type of matrix over $\mathfrak{A}$.

5·3. Lemma. *Let*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \in M_n(\mathfrak{A}), \quad \textit{where} \quad A_{11} \in M_r(\mathfrak{A}) \quad (1 \leqslant r < n).$$

Then A *is non-singular if and only if both* A_{11} *and* A_{22} *are non-singular*.

The proof, which we omit, again depends essentially on 4·1.

6. *Semigroup algebras* (*continued*). We now apply the results of §§ 4, 5 to semigroup algebras.

6·1. Theorem. *Let $S = S_{mn}[G, P]$, and let $\mathfrak{F}$ be a field of characteristic zero or a prime not dividing the order of G. Let G^* be any subgroup of G containing all the non-zero entries of P, and let $\mathfrak{A}^* = \mathfrak{A}(G^*)$ be the algebra of G^* over $\mathfrak{F}$. Then $\mathfrak{A}^*$ is semisimple. Let $\{\Gamma_i^*; i = 1, \ldots, k^*\}$ be a complete set of inequivalent irreducible representations of G^* over $\mathfrak{F}$. Then the algebra of S over $\mathfrak{F}$ is semisimple if and only if each of the matrices $\Gamma_i^*(P)$ is non-singular ($i = 1, \ldots, k^*$).*

(Note that if $\Gamma_i^*(P)$ is non-singular then P must be square.)

Proof. If $\mathfrak{F}$ has characteristic p then p cannot divide the order of G^*; hence $\mathfrak{A}^*$ is semisimple. Let $\mathfrak{A} = \mathfrak{A}(G)$; then $\mathfrak{A}$ is also semisimple. But the contracted algebra of S over $\mathfrak{F}$ is $M_{mn}[\mathfrak{A}, P]$ by 3·8, and so by 4·7 this is semisimple if and only if P is non-singular. The result then follows from 5·2.

It is usually convenient to take G^* to be the group generated by the non-zero entries of P, but we may take $G^* = G$.

6·2. Corollary. *Let S be a simple semigroup without zero. Then if $\mathfrak{A}(S)$ is semisimple, S is a group.*

Proof. S may be regarded as the set of non-zero elements of $S_{mn}[G, P]$, say, where P has no zero entries. Let Γ_1 denote the identical representation of G; then every entry of the $n \times m$ matrix $\Gamma_1(P)$ is 1. Thus $\Gamma_1(P)$ is non-singular only if $m = n = 1$, in which case S reduces to a group (isomorphic with G).

This result was previously obtained by M. Teissier (**10**).

The non-singularity of the matrix P has so far been discussed with reference to a particular field $\mathfrak{F}$. We now show that it depends only on the characteristic of $\mathfrak{F}$.

6·3. Lemma. *Let P be an $n \times n$ matrix over a group-with-zero $G(z)$, and let $\mathfrak{F}_1$, $\mathfrak{F}_2$ be two fields having the same characteristic. Let $\mathfrak{A}_1$, $\mathfrak{A}_2$ be the algebras of G over $\mathfrak{F}_1$, $\mathfrak{F}_2$ respectively. Then P is non-singular qua matrix over $\mathfrak{A}_1$ if and only if it is non-singular qua matrix over $\mathfrak{A}_2$.*

Proof. Let

$$\mathfrak{F}_0 = \begin{cases} \text{rational field} & \text{if } \mathfrak{F}_1, \mathfrak{F}_2 \text{ have characteristic } 0, \\ GF(p) & \text{if } \mathfrak{F}_1, \mathfrak{F}_2 \text{ have characteristic } p. \end{cases}$$

Let $\mathfrak{A}_0$ be the algebra of G over $\mathfrak{F}_0$. It is sufficient to prove the result for $\mathfrak{F}_2 = \mathfrak{F}_0$. We may regard $\mathfrak{F}_0$ as a subfield of $\mathfrak{F}_1$. Thus $\mathfrak{A}_1$ is a scalar extension of $\mathfrak{A}_0$, and so $M_n(\mathfrak{A}_1)$ is a scalar extension of $M_n(\mathfrak{A}_0)$. But $P \in M_n(\mathfrak{A}_0)$, and we then apply 4·3.

As an example to show that the non-singularity of P does in fact depend on the characteristic of $\mathfrak{F}$, let P be the $n \times n$ matrix whose (i, j)th entry is the identity e of G if $i \neq j$, and is 0 if $i = j$. Take $G^* = (e)$. Then by 5·2, P is non-singular if and only if $\Gamma_1^*(P)$ is a non-singular matrix over $\mathfrak{F}$ (Γ_1^* the identical representation). Now it can readily be shown that $\det \Gamma_1^*(P) = (-1)^{n-1}(n-1)$ if $\mathfrak{F}$ has characteristic 0, whence it follows that when $n = p + 1$ the matrix P is singular for fields of characteristic p and is non-singular for all other fields.

A simple semigroup S will be termed *c-non-singular* if $S \cong S_{nn}[G, P]$, where P is non-singular *qua* matrix over $\mathfrak{A}$, the algebra of G over any field of characteristic c. Otherwise S will be termed *c-singular*. (Note that if $S_{nn}[G, P] \cong S_{nn}[G, P^*]$, and if P is c-non-singular, then so is P^* by 3·7.)

Combining the results of 3·3, 3·4 and 4·7 we obtain

6·4. Theorem. *Let $\mathfrak{A}(S)$ be the algebra of a semigroup S over a field of characteristic c. Then $\mathfrak{A}(S)$ is semisimple if and only if each of the principal factors of S is a c-non-singular simple semigroup, provided that c, if non-zero, does not divide the orders of any of the basic groups of the principal factors.*

6·5. Corollary. *If $\mathfrak{A}(S)$ is semisimple, then the kernel of S is a group.*

Proof. Let the kernel be K; then K is simple without zero. But if $\mathfrak{A}(S)$ is semisimple then so is $\mathfrak{A}(K)$, and the result follows from 6·2.

This means that if $\mathfrak{A}(S)$ is semisimple, then S possesses zeroid elements (see (6)).

7. *Simple semigroup algebras.*

7·1. Theorem. *A simple algebra $\mathfrak{M}$ over a field $\mathfrak{F}$ is a contracted semigroup algebra over $\mathfrak{F}$ if and only if it is isomorphic with a complete matrix algebra over $\mathfrak{F}$.*

Proof. Let $\mathfrak{M} \cong M_n(\mathfrak{F})$ for some n. Then as basis for $M_n(\mathfrak{F})$ we can choose the set of 'matrix units' E_{ij} whose multiplication is given by $E_{ij}E_{rs} = \delta_{jr}E_{is}$, whence $\mathfrak{M}$ is a contracted semigroup algebra over $\mathfrak{F}$.

Conversely, let S be a semigroup with zero, and let $\mathfrak{B}$, the contracted algebra of S over $\mathfrak{F}$, be simple. Then S is simple. Let $S \cong S_{mn}[G, P]$, so that $\mathfrak{B} \cong M_{mn}[\mathfrak{A}, P]$ where $\mathfrak{A} = \mathfrak{A}(G)$. Now $\mathfrak{B}$ is *a fortiori* semisimple, and so by 4·7 $\mathfrak{A}$ is semisimple and P is non-singular (implying that $m = n$). By 4·9 $\mathrm{Cl}\,\mathfrak{A} = \mathrm{Cl}\,\mathfrak{B} = 1$, from which it follows that G consists of a single element. Thus $\mathfrak{U} \cong \mathfrak{F}$, and so $\mathfrak{B} \cong M_n(\mathfrak{F})$ from (4·8).

We note in passing that $\mathfrak{A}(S)$, the non-contracted algebra of a semigroup S over a field $\mathfrak{F}$, is simple if and only if S consists of a single element.

Consider the semigroup $S_{nn}[G, P]$. If G consists of one element, we identify this element with the identity 1 of $\mathfrak{F}$. Then P may be regarded as an element of $M_n(\mathfrak{F})$ whose entries are either 0 or 1. Let $\mathscr{S}_n = \mathscr{S}_n(\mathfrak{F})$ denote the set of all non-singular $n \times n$ matrices whose entries are 0 or 1, and let $P \in \mathscr{S}_n$. Write $P^* \sim P$ if there exist $n \times n$ permutation matrices A, B such that $P^* = APB$. This defines an equivalence on $\mathscr{S}_n$, and by 3·7 $S_{nn}[(1), P] \cong S_{nn}[(1), P^*]$ if and only if $P^* \sim P$. Let s_n denote the number of equivalence classes; then s_n is the number of non-isomorphic simple semigroups whose contracted algebras over $\mathfrak{F}$ are isomorphic with $M_n(\mathfrak{F})$. It is easily seen that $s_2 = 2$ for all fields $\mathfrak{F}$, and it may be verified that $s_3 = 7$ or 8 according as the characteristic of $\mathfrak{F}$ is or is not 2.

8. *Representations of simple semigroups.* The representations of a simple semigroup S by matrices over a field $\mathfrak{F}$ have been discussed by Clifford (3). These were obtained as 'extensions' of the representations of the basic group of S. We give a summary of the results, and show how they are related to those of § 6 when the algebra of S over $\mathfrak{F}$ is semisimple.

Let $S = S_{mn}[G, P]$; we shall use the notation of 3·6 throughout. Without loss of generality we can assume that $p_{11} = e$, the identity of G (this is a consequence of 3·7). Let z' be the zero of S; thus $z' = (z)_{ij}$ for all i, j. Now let Γ' be a representation of S by $s \times s$ matrices over $\mathfrak{F}$. We suppose that $\Gamma'(z') = 0_s$. Γ' induces a representation of the subsemigroup $G_{11} = \{(x)_{11};\, x \in G(z)\} \cong G(z)$, and we may suppose that Γ' is transformed so that

$$\Gamma'\{(x)_{11}\} = \begin{pmatrix} \Gamma(x) & 0 \\ 0 & 0 \end{pmatrix}. \tag{8·1}$$

Γ is thus a representation of $G(z)$ such that $\Gamma(z) = 0$. We regard z as the zero of $\mathfrak{A}$, the algebra of G over $\mathfrak{F}$, so that Γ may be considered as a representation of $\mathfrak{A}$. Let t be the degree of Γ; then there is no loss of generality if we assume that $\Gamma(e) = I_t$. In this case Γ is said to be *proper*.

We call Γ' an *extension* of Γ.

From the law of multiplication in S Clifford shows that

$$\Gamma'\{(x)_{ij}\} = \begin{pmatrix} \Gamma(p_{1i}xp_{j1}) & \Gamma(p_{1i}x)\,Q_j \\ R_i\,\Gamma(xp_{j1}) & R_i\,\Gamma(x)\,Q_j \end{pmatrix}, \tag{8·2}$$

where Q_j, R_i are $t \times (s-t)$, $(s-t) \times t$ matrices respectively, such that

$$\left.\begin{aligned} Q_1 = 0, \quad R_1 = 0 \\ \text{and} \qquad Q_jR_i = \Gamma(p_{ji}) - \Gamma(p_{j1})\,\Gamma(p_{1i}) \quad (i = 2, \ldots, m;\, j = 2, \ldots, n). \end{aligned}\right\} \tag{8·3}$$

Conversely, if Γ is any proper representation of G, and if Q_j, R_i are matrices satisfying conditions (8·3), then the mapping Γ' defined on S by (8·2) is a representation of S such that (8·1) holds. We write

$$Q = \begin{pmatrix} Q_2 \\ Q_3 \\ \vdots \\ Q_n \end{pmatrix}; \quad R = (R_2 R_3 \ldots R_m).$$

Let $H = H(\Gamma)$ be that $(n-1)\,t \times (m-1)\,t$ matrix whose (j,i)th block is the $t \times t$ matrix $\Gamma(p_{ji}) - \Gamma(p_{j1})\,\Gamma(p_{1i})$ $(i = 2, \ldots, m;\, j = 2, \ldots, n)$. Then from (8·3) we have

$$QR = H. \tag{8·4}$$

Conversely, any proper representation Γ of G and any factorization (8·4) of H determine a unique representation Γ' of S. We call Q and R the *defining matrices* of Γ'.

If h is the rank of H, then $s \geqslant t+h$. In particular there are factorizations (8·4) in which Q, R are $(n-1)\,t \times h$, $h \times (m-1)\,t$ matrices respectively. Such factorizations are termed *basic*. The representation of S derived from a basic factorization is unique to within equivalence ((3), Theorem 6·1), and hence we call it *the basic extension* of Γ.

8·5. A representation of S is termed *proper* if it does not decompose into the sum of two representations, one of which is null. Clifford shows by an examination of matrix factorizations that if Γ' is any representation of S of degree s over $\mathfrak{F}$, extending a proper representation Γ of G of degree t, and if the matrices Q, R, H of (8·4) have ranks q, r, h respectively, then $s-t \geqslant q+r-h$, and Γ' is proper if and only if $s-t = q+r-h$ ((3), Theorem 5·1). In particular, the basic extension is proper.

Now let $\mathfrak{B}(S)$ denote the contracted algebra of S over $\mathfrak{F}$. The representation described in (8·2) can be extended to a representation of $\mathfrak{B}(S)$. With the usual notation we have

8·6. Theorem. *Let $\mathfrak{F}$ have characteristic zero or a prime not dividing the order of G. Then $\mathfrak{B}(S)$ is semisimple if and only if the only proper representation of S extending any given proper representation of G is its basic extension.*

Proof. Let

$$A = \begin{pmatrix} e & & & & \\ -p_{21} & e & & 0 & \\ -p_{31} & & e & & \\ \vdots & 0 & & \ddots & \\ -p_{n1} & & & & e \end{pmatrix} \in M_n(\mathfrak{A}).$$

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Then A is non-singular, by 5·3. Also

$$AP = \begin{pmatrix} e & p_{12} & \cdots & p_{1m} \\ 0 & (p_{22}-p_{21}p_{12}) & \cdots & (p_{2m}-p_{21}p_{1m}) \\ \vdots & \vdots & & \vdots \\ 0 & (p_{n2}-p_{n1}p_{12}) & \cdots & (p_{nm}-p_{n1}p_{1m}) \end{pmatrix}.$$

Let P_1 denote the submatrix of AP from rows $2, \ldots, n$ and columns $2, \ldots, m$. By 4·7 $\mathfrak{B}(S)$ is semisimple if and only if P is non-singular, and so by 5·3 $\mathfrak{B}(S)$ is semisimple if and only if P_1 is non-singular.

Let P_1 be non-singular, so that in particular $m = n$, and let Γ be a proper representation of G of degree t. Then $\Gamma(P_1)$ is a non-singular $(n-1)t \times (n-1)t$ matrix. Let Γ' be a proper representation of S of degree s with defining matrices Q, R. Then from (8·4) we have $QR = H = \Gamma(P_1)$. With the notation of 8·5 we have $q = r = h = (n-1)t$. Since Γ' is proper, 8·5 gives $s-t = q+r-h = h$, and so Γ' is the basic extension of Γ. We also note that $s = nt$.

Conversely, let S be such that the only proper representation of S extending any given proper representation of G over $\mathfrak{F}$ is the basic extension. Let Γ be a proper representation of G of degree t, and let h be the rank of $H = \Gamma(P_1)$ as before. Suppose that $h < (n-1)t$. Then for our factorization (8·4) we may take $Q = I_{(n-1)t}$, $R = H$. Let Γ' be the representation of degree s, say, obtained from this factorization. Then with the usual notation we have $q = (n-1)t$, $r = h$ and so $s-t = (n-1)t = q+r-h$. Thus by 8·5 Γ' is a proper representation. But it is not the basic extension of Γ, and this contradicts our hypothesis. Hence we must have $h = (n-1)t$. Similarly, we must have $h = (m-1)t$, and so $\Gamma(P_1)$ is non-singular. Now let Γ be chosen to be the regular representation of G. Then from 5·1 we see that P_1 is non-singular, and so $\mathfrak{B}(S)$ is semisimple.

8·7. Theorem. *Let $\{\Gamma_i;\ i = 1, \ldots, k\}$ be a complete set of inequivalent irreducible representations of G over $\mathfrak{F}$, and let $\mathfrak{B}(S)$ be semisimple. Then $\{\Gamma'_i;\ i = 1, \ldots, k\}$ is a complete set of inequivalent irreducible representations of S over $\mathfrak{F}$, where Γ'_i is the basic extension of Γ_i.*

Proof. Since Γ_i is irreducible, so also is Γ'_i ((3), Theorem 7·1). Further, the representations Γ'_i $(i = 1, \ldots, k)$ are inequivalent since they induce inequivalent representations of G. But by 4·9 $\mathrm{Cl}\,\mathfrak{B}(S) = \mathrm{Cl}\,\mathfrak{A} = k$, and the result follows.

9. *Semigroups which admit relative inverses.* In this final section we apply the results of § 6 to semigroups of a type first discussed by Clifford (2). A semigroup S (not necessarily finite) is said to *admit relative inverses* if for any a in S there exist elements e and a' in S such that $ea = a = ae$ and $a'a = e = aa'$.

By a 'semilattice' we mean a commutative semigroup, all of whose elements are idempotent. Let α, β belong to a semilattice Y; then if $\alpha\beta = \beta$ we write $\alpha \geqslant \beta$. This defines a partial ordering in Y with the property that any two elements ξ, η have a unique greatest lower bound, namely $\xi\eta$. Clifford's main result may then be stated thus:

9·1. Let S be a semigroup which admits relative inverses. Then S determines a semilattice Y such that to each $\alpha \in Y$ there corresponds a subsemigroup K_α of S. The K_α's have the following properties:

(i) they are mutually disjoint and their union is S,

(ii) each K_α is a completely simple semigroup† without zero,

(iii) $K_\alpha K_\beta \subseteq K_{\alpha\beta}$.

Conversely, every semigroup with this structure admits relative inverses.

Y will be called 'the semilattice of S', and the K_α's 'the components of S'.

9·2. LEMMA. *A semigroup S with a principal series admits relative inverses if and only if* (i) *its kernel is completely simple without zero, and* (ii) *its remaining principal factors, if any, are completely simple without divisors of zero.*

(A semigroup with these properties is *a fortiori* semisimple.)

We omit the formal proof. It can be verified that in this case the semilattice Y of S is finite, and if ω is the zero of Y, then the principal factors of S other than the kernel K_ω are the semigroups K_α^* ($\alpha \in Y$, $\alpha \neq \omega$), where K_α^* is formed from K_α by the adjunction of a zero.

We shall also make use of the following result due to Clifford:

9·3. Let S be a semigroup which admits relative inverses, and let all the idempotents of S commute. Let Y be the semilattice of S, and $\{K_\alpha; \alpha \in Y\}$ the set of components of S. Then each K_α is a group. Further, to every pair of elements α, $\beta \in Y$ such that $\alpha > \beta$ there corresponds a homomorphism $\phi_{\alpha\beta}$ of K_α into K_β, and these homomorphisms satisfy the transitivity relation

$$\phi_{\alpha\beta}\phi_{\beta\gamma} = \phi_{\alpha\gamma} \quad (\alpha > \beta > \gamma). \tag{9·3a}$$

If we take $\phi_{\alpha\alpha}$ to be the identity automorphism of K_α, then the structure of S is given in terms of the structures of the component groups by the relation

$$a_\alpha b_\beta = (a_\alpha \phi_{\alpha\gamma})(b_\beta \phi_{\beta\gamma}), \tag{9·3b}$$

where a_α, b_β are any elements of K_α, K_β respectively, and $\gamma = \alpha\beta$.

Conversely, any semigroup which is a union of disjoint groups and whose structure is defined as above admits relative inverses, and all its idempotents commute.

We now obtain an additional result in this connexion.

9·4. LEMMA.‡ *Let S be a semigroup which admits relative inverses. Then if each component of S is a group, all the idempotents of S commute.*

Proof. Let $\{K_\alpha; \alpha \in Y\}$ be the set of components of S, and let e_α be the identity of the group K_α. We wish to prove that $e_\alpha e_\beta = e_\beta e_\alpha$ for all $\alpha, \beta \in Y$. Let $\gamma = \alpha\beta$; then by 9·1 $e_\alpha e_\gamma, e_\gamma e_\alpha \in K_\gamma$, and so $e_\alpha e_\gamma = e_\gamma(e_\alpha e_\gamma) = (e_\gamma e_\alpha) e_\gamma = e_\gamma e_\alpha$. Thus $(e_\alpha e_\gamma)^2 = e_\alpha e_\gamma$, and so $e_\alpha e_\gamma = e_\gamma$. Similarly, $e_\beta e_\gamma = e_\gamma = e_\gamma e_\beta$. Hence since $e_\alpha e_\beta, e_\beta e_\alpha \in K_\gamma$ we have

$$e_\alpha e_\beta = (e_\alpha e_\beta) e_\gamma = e_\alpha(e_\beta e_\gamma) = e_\alpha e_\gamma = e_\gamma = e_\gamma e_\alpha = (e_\gamma e_\beta) e_\alpha = e_\gamma(e_\beta e_\alpha) = e_\beta e_\alpha.$$

We now let S be a finite semigroup which admits relative inverses, and consider the algebra $\mathfrak{A}(S)$ of S over a field $\mathfrak{F}$.

9·5. THEOREM. *Let $\mathfrak{F}$ have characteristic zero or a prime not dividing the orders of any of the basic groups of the principal factors of S. Then $\mathfrak{A}(S)$ is semisimple if and only if all the idempotents of S commute.*

† See (9). In particular, a finite simple semigroup is completely simple.

‡ (*Added in proof*). This result has also been obtained by A. H. Clifford ('Bands of semigroups,' *Proc. Amer. phil. Soc.* 5 (1954), 499–594 (Theorem 8)) and by R. Croisot ('Demi-groupes inversifs et demi-groupes réunions de demi-groupes simples', *Ann. Ec. Norm.* (3), 70 (1953), 361–79 (p. 375)).

Proof. Let $\mathfrak{A}(S)$ be semisimple. Then by 3·3 the algebra of each principal factor over $\mathfrak{F}$ is semisimple. In particular, by 6·5 the kernel of S is a group. Now by 9·2 the remaining principal factors of S are all simple semigroups without divisors of zero. By applying 6·2 to their contracted algebras, we see that these factors are groups-with-zero, and so the components of S are groups. Hence by 9·4 all the idempotents of S commute.

Conversely, let all the idempotents of S commute. Then by 9·3 all the components of S are groups; that is, the kernel of S is a group and the remaining principal factors are groups-with-zero. It then follows from 3·3 that $\mathfrak{A}(S)$ is semisimple.

Finally we discuss the matrix representations of a semigroup S which admits relative inverses, and whose idempotents commute. As a consequence of 9·5, if $\mathfrak{F}$ is a field of characteristic zero or a prime not dividing the orders of any of the components of S (groups), then the representations of S over $\mathfrak{F}$ are completely reducible. We use the notation of 9·3.

9·6. Theorem. *Let $\{\Gamma_{\eta i};\ i = 1, \ldots, k_\eta\}$ be a complete set of inequivalent irreducible representations of the group K_η $(\eta \in Y)$ over $\mathfrak{F}$. Define the extension $\Gamma'_{\eta i}$ of $\Gamma_{\eta i}$ to S as follows:*

$$\Gamma'_{\eta i}(a_\alpha) = \begin{cases} \Gamma_{\eta i}(a_\alpha \phi_{\alpha\eta}) & \text{for all} \quad a_\alpha \in K_\alpha,\ \alpha \geqslant \eta, \\ 0 & \text{for all} \quad a_\alpha \in K_\alpha,\ \alpha \not\geqslant \eta. \end{cases}$$

Then $\{\Gamma'_{\eta i};\ i = 1, \ldots, k_\eta;\ \eta \in Y\}$ is a complete set of inequivalent irreducible representations of S over $\mathfrak{F}$.

Proof. Let a_α, b_β be any elements of K_α, K_β respectively. If both $\alpha, \beta \geqslant \eta$ then we have

$$\begin{aligned} \Gamma'_{\eta i}(a_\alpha b_\beta) &= \Gamma'_{\eta i}\{(a_\alpha \phi_{\alpha\gamma})(b_\beta \phi_{\beta\gamma})\} && \text{by } (9\cdot 3b)\ (\gamma = \alpha\beta), \\ &= \Gamma_{\eta i}\{[(a_\alpha \phi_{\alpha\gamma})(b_\beta \phi_{\beta\gamma})]\, \phi_{\gamma\eta}\} && \text{since } \gamma \geqslant \eta, \\ &= \Gamma_{\eta i}\{(a_\alpha \phi_{\alpha\gamma} \phi_{\gamma\eta})(b_\beta \phi_{\beta\gamma} \phi_{\gamma\eta})\} && \text{since } \phi_{\gamma\eta} \text{ is a homomorphism}, \\ &= \Gamma_{\eta i}\{(a_\alpha \phi_{\alpha\eta})(b_\beta \phi_{\beta\eta})\} && \text{by } (9\cdot 3a), \\ &= \Gamma_{\eta i}(a_\alpha \phi_{\alpha\eta})\, \Gamma_{\eta i}(b_\beta \phi_{\beta\eta}) && \text{since } \Gamma_{\eta i} \text{ is a representation of } K_\eta, \\ &= \Gamma'_{\eta i}(a_\alpha)\, \Gamma'_{\eta i}(b_\beta). \end{aligned}$$

If not both $\alpha, \beta \geqslant \eta$, then $\alpha\beta \not\geqslant \eta$. But $a_\alpha b_\beta \in K_{\alpha\beta}$, and so from our definition we have

$$\Gamma'_{\eta i}(a_\alpha b_\beta) = 0 = \Gamma'_{\eta i}(a_\alpha)\, \Gamma'_{\eta i}(b_\beta).$$

Hence $\Gamma'_{\eta i}$ is a representation of S. It is irreducible since $\Gamma_{\eta i}$, its restriction to K_η, is irreducible. Similarly, the $\Gamma'_{\eta i}$'s $(i = 1, \ldots, k_\eta)$ are inequivalent, and clearly if $\eta \neq \zeta$ then we cannot have $\Gamma'_{\eta i}$ equivalent to $\Gamma'_{\zeta j}$. But the number of inequivalent irreducible representations of S over $\mathfrak{F} = \mathrm{Cl}\, \mathfrak{A}(S) = \sum_{\eta \in Y} k_\eta$, by 3·5, and the result follows.

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