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Minimal combinatorial models for maps of an interval with a given set of periods

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Abstract. A combinatorial model for a property of continuous self-maps of a compact interval is a self-map π of a finite ordered set such that every continuous π-weakly monotone self-map of a compact interval has that property. We identify the minimal combinatorial models for the property 'the set of periods is a given set'. Here the word minimal refers to the number of points in the domain of the model. We also identify the minimal permutation models and, in appropriate cases, the minimal combinatorial models for properties involving 'horseshoes'.

0. *Introduction*

A *combinatorial model* for a property of continuous self-maps of a compact interval is a self-map π of a finite-ordered set such that every continuous π-weakly monotone self-map of a compact interval has the property. Not every dynamical property has a combinatorial model. For example, the property 'f has a dense orbit' does not. We also consider *permutation models*, where we require that π be a permutation.

We consider the following two properties:

- the set of periods of f is a given set of positive integers;
- the set of periods of f is a given set of positive integers and f^{2^k} does not have a horseshoe. (*Horseshoe* is defined in §3.)

We show that for every set which is the set of periods of some continuous self-map of a compact interval, other than the set of all powers of 2, both properties have combinatorial models. For the properties above, we characterize the *minimal combinatorial models* and the *minimal permutation models*, where *minimal* refers to the cardinality of the domain of π, denoted $\#\pi$.

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Suppose that $\pi : P \to P$ is a self-map of a finite-ordered set $P = \{p_1 < p_2 < \cdots < p_n\}$. A self-map $f : I \to I$ of a compact interval is *π-weakly monotone* if and only if there exist $z_1 < z_2 < \cdots < z_n$ in I such that $I = [z_1, z_n]$, $f(z_i) = z_j$ if $\pi(p_i) = p_j$, and f is weakly (i.e. not necessarily strictly) monotone on every subinterval $[z_i, z_{i+1}]$. We call these points *π-points* and the set $\{z_1 < z_2 < \cdots < z_n\}$ a *π-set*.

We show in Theorems 3.5 and 2.6 that for each of the properties that we are considering, whether a continuous, π-weakly monotone map has the property depends only on π and its *Markov graph* (defined in §1), which is the same for two self-maps of finite-ordered sets which are conjugate via a one-to-one, order-preserving map. Thus we may, and often do, assume that $P = \{1, 2, \ldots, n\}$ when $\#\pi = n$.

Let $\mathrm{Per}(f)$ denote the set of (least) periods of the periodic points in the domain of f and let $<_S$ (Sharkovsky-less than) be the Sharkovsky order on the positive integers, defined in §1. It follows from Sharkovsky's Theorem, stated in §1, that $\mathrm{Per}(f)$ is of one of the following:

(1) $\{t : t \leq_S 2^k\}$ for some $k \geq 0$;

(2) $\{t : t$ is a power of $2\}$;

(3) $\{t : t \leq_S 3 \cdot 2^k\}$ for some $k \geq 0$;

(4) $\{t : t \leq_S r \cdot 2^k\}$ for some $k \geq 0$, and some odd $r \geq 5$.

The main results of this paper are Theorems 3.3, 3.6, 6.2–4 and 7.1. These say that sets of the forms (1), (3) and (4), but not (2), have combinatorial models and characterize the minimal combinatorial models for those that do. The characterizations are in terms of *simple cycles*, also known as *Štefan cycles*, which are defined in §3 for cycles of length a power of 2 and in §6 for cycles of other lengths.

THEOREM 3.3. *There are no combinatorial models for* '$\mathrm{Per}(f) = \{2^k : k = 0, 1, \ldots\}$'.

THEOREM 3.6. *The minimal combinatorial models for* '$\mathrm{Per}(f) = \{t : t \leq_S 2^k\}$' *are the simple cycles of length* 2^k.

THEOREM 6.2. *The minimal combinatorial models for* '$\mathrm{Per}(f) = \{t : t \leq_S 3 \cdot 2^k\}$ *and* f^{2^k} *does not have a horseshoe' are the simple cycles of length* $3 \cdot 2^k$.

Theorems 6.3 and 6.4 characterize the minimal permutation and minimal combinatorial models for '$\mathrm{Per}(f) = \{t : t \leq_S 3 \cdot 2^k\}$'. In the first case, $\#\pi = 3 \cdot 2^k$ but π need not be a cycle; in the second case, if $k \geq 1$, then $\#\pi = 3 \cdot 2^{k-1} + 1$ and π is not a permutation.

THEOREM 7.1. *The minimal combinatorial models for* '$\mathrm{Per}(f) = \{t : t \leq_S r \cdot 2^k\}$, $r \geq 5$, *r odd' are the simple cycles of length* $r \cdot 2^k$.

This implies, for example, that if f is continuous, π-weakly monotone and $\mathrm{Per}(f) = \{t : t \leq_S 1\,000\,000\}$, then $\#\pi \geq 1\,000\,000$.

We conclude the paper with a brief discussion of unimodal minimal combinatorial models in §8.

An informal way to summarize some of the conclusions of this paper is the following. For the property that the set of periods is a given set, the only way to obtain a minimal combinatorial model which is not a simple cycle is to create an appropriate horseshoe.

In particular, this is possible if and only if the Sharkovsky-largest member of the given set is $3 \cdot 2^k$ for some $k \geq 0$. The 'turbulence stratification' [**BCop**, p. 34] is relevant here.

1. *Three basic concepts*

Throughout this paper, $f : I \to I$ is a continuous self-map of a compact interval and $\pi : P \to P$ a self-map of a finite-ordered set. We denote the cardinality of P by $\#\pi$. We use exponentiation of maps to denote iterated composition. Thus $\pi^2 = \pi \circ \pi$, $f^2 = f \circ f$, etc. For $f : I \to I$, $x \in I$ and $n \geq 1$, we say that $x \in I$ is *periodic*, or *f-periodic*, of *period t* if and only if $f^t(x) = x$, and $f^s(x) \neq x$ for $1 \leq s < t$. We let $\mathrm{Per}(f)$ denote the set of $t \geq 1$ such that there exists $x \in I$ of period t and call it the *set of periods* of f.

The study of the periods of the periodic points of continuous self-maps of compact intervals goes back to Sharkovsky's Theorem [**Sh**], which is stated in terms of the *Sharkovsky order*, $<_S$, on the positive integers $\{1, 2, \dots\}$:

$$1 <_S 2 <_S 2^2 <_S 2^3 <_S \cdots <_S 2^2 \cdot 7 <_S 2^2 \cdot 5 <_S 2^2 \cdot 3 <_S \cdots$$
$$<_S 2 \cdot 7 <_S 2 \cdot 5 <_S 2 \cdot 3 <_S \cdots <_S 7 <_S 5 <_S 3.$$

Formally, if r and s are odd, then $2^k r <_S 2^\ell s$ if and only if either: (1) $k < \ell$ and $r = s = 1$; (2) $r = 1$ and $s > 1$; (3) $k = \ell$ and $1 < s < r$; or (4) $k > \ell$ and $r, s > 1$.

Sharkovsky's Theorem states that for every continuous self-map f of a compact interval, if s is the period of some point in the domain of f and if $t <_S s$, then t is also the period of a point in the domain of f. In symbols, if $s \in \mathrm{Per}(f)$, then $\mathrm{Per}(f) \supseteq \{t : t \leq_S s\}$. Thus every set $\mathrm{Per}(f)$, except the set $\{2^k : k \geq 0\}$ of all powers of 2, which we denote 2^∞, is determined by its Sharkovsky-largest member, which we denote S-max(Per(f)).

Štefan [**Š**] showed that the converse is also true. For every positive integer s, there is a continuous self-map f of a compact interval such that $\mathrm{Per}(f) = \{t : t \leq_S s\}$ and, in addition, there is such a map for which $\mathrm{Per}(f) = 2^\infty$.

For modern proofs of Sharkovsky's Theorem in English, including the converse, see [**BGMY**], [**ALM**], or [**BCop**].

Let $\pi : P \to P$, where $P = \{p_1 < p_2 < \cdots < p_n\}$. The *canonical π-linear map* is $L_\pi : [1, n] \to [1, n]$, where $L_\pi(i) = j$ if $\pi(p_i) = p_j$ and L_π is linear on each interval $[i, i + 1]$.

The *Markov graph* of π is the directed graph M_π with $n - 1$ vertices, $V_1, V_2, \dots, V_{n-1}$ and an arc from V_i to V_j, written $V_i \to V_j$, if and only if either $\pi(p_i) \leq p_j$ and $\pi(p_{i+1}) \geq p_{j+1}$, or $\pi(p_i) \geq p_{j+1}$ and $\pi(p_{i+1}) \leq p_j$.

LEMMA 1.1. *The following statements are equivalent.*

(1) $V_i \to V_j$ *in* M_π.

(2) *For some continuous, π-weakly monotone map f with π-set Z_π,* $f[z_i, z_{i+1}] \supseteq [z_j, z_{j+1}]$.

(3) *For every continuous, π-weakly monotone map f with π-set Z_π,* $f[z_i, z_{i+1}] \supseteq [z_j, z_{j+1}]$.

(4) *For some continuous, π-weakly monotone map f with π-set Z_π, there exists* $x \in (z_i, z_{i+1})$ *such that* $f(x) \in (z_j, z_{j+1})$.

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(5) *For every continuous, π-weakly monotone map f with π-set Z_π, there exists $x \in (z_i, z_{i+1})$ such that $f(x) \in (z_j, z_{j+1})$.*

2. *Horseshoes*

$f : I \to I$ has a *horseshoe* if and only if there exist subintervals J and K of I, with at most one point in common, such that $f(J), f(K) \supseteq J \cup K$. In this case we say that J and K *exhibit* a horseshoe for f, and if $J = [a, b]$ and $K = [b, c]$, we say that $a < b < c$ *exhibit* a horseshoe for f. As in Li-Yorke's famous proof that 'Period three implies chaos' [**LY**], we have the following.

LEMMA 2.1. *If f has a horseshoe, then* $\mathrm{Per}(f) = \{1, 2, 3, \dots\}$.

From Sharkovsky's Theorem we then have the following.

LEMMA 2.2. *If some power of f has a horseshoe, then* $\mathrm{Per}(f) \not\subseteq 2^\infty$.

LEMMA 2.3. *If there exist $a < b \le c < d$ in I such that $f[a, b] \supseteq [c, d]$, $f[c, d] \supseteq [a, b]$, $f^k(a)$ is a fixed point of f for some $k \ge 0$, and $f^i(a) \notin (a, b)$ for all $i \ge 0$, then some power of f has a horseshoe.*

Proof. Without loss of generality k is even.

Suppose first that $f^k(a) \le a$. Since $k + 1$ is odd, $f^{k+1}[a, b] \supseteq [c, d]$. However, $f^{k+1}(a) = f^k(a) \le a$, so $f^{k+1}[a, b]$ contains $[c, d]$ and a point less than or equal to a. Thus $f^{k+1}[a, b] \supseteq [a, b] \cup [c, d]$, and hence $f^{k+2}[a, b] \supseteq [a, b] \cup [c, d]$. However, $f[c, d] \supseteq [a, b]$, so $f^{k+2}[c, d] \supseteq [a, b] \cup [c, d]$. Thus $[a, b]$ and $[c, d]$ exhibit a horseshoe for f^{k+2}.

Finally, suppose that $f^k(a) \ge b$. Note that $f^k(a) = f^k(f^k(a))$. Since $f^k[a, b] \supseteq [a, b]$, there exists $x \in (a, f^k(a))$ such that $f^k(x) = a$. Then $a < x < f^k(a)$ exhibit a horseshoe for f^k. □

LEMMA 2.4. *Suppose that f is continuous and π-weakly monotone. Then f has a horseshoe if and only if there exist $p_i < p_j < p_k$ in P such that $\pi(p_i), \pi(p_k) \le p_i$ and $\pi(p_j) \ge p_k$, or $\pi(p_i), \pi(p_k) \ge p_k$ and $\pi(p_j) \le p_i$.*

Proof. The conditions are clearly sufficient for f to have a horseshoe. To prove the converse, it suffices to show that one of the conditions holds if f has a horseshoe.

By [**BCop**, Lemma II.2] we may assume that there exist $a < b < c$ in I such that

$$f(a) = f(c) = a \quad \text{and} \quad f(b) = c,$$
$$f(x) > a \quad \text{for all } x, a < x < c,$$
$$x < f(x) < c \quad \text{for all } x, a < x < b.$$

Let z_i be largest π-point less than or equal to a, let z_j be the smallest π-point greater than or equal to b and let $z_k = f(z_j)$. Then $z_i \le a < b \le z_j < c \le z_k$.

If $z_i = a$, then $f(z_i) = z_i$. If $z_i < a$, then it follows from the facts that for small $\epsilon > 0$, f is non-decreasing on $[a, a + \epsilon]$ and there are no π-points in $(z_i, a + \epsilon)$, that $f(z_i) \le z_i$. Similarly, $f(z_k) \le z_i$.

It follows that $\pi(p_i), \pi(p_k) \le p_i$ and $\pi(p_j) \ge p_k$. □

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LEMMA 2.5. *Suppose that f is continuous and π-weakly monotone. Then for every $k \geq 1$, f^k has a horseshoe if and only if there exist π-points $a < b < c$ of f such that $f^k[a, b], f^k[b, c] \supseteq [a, c]$.*

Proof. One direction is trivial. Suppose then that f^k has a horseshoe. Let Z_π denote the set of π-points of f.

Let Q be the set of endpoints of the connected components of $\bigcup_{i=0}^{k-1} f^{-k}(Z_\pi)$, and let $\theta = f^k|_Q$. Then f^k is θ-weakly monotone and $f^k(Z_\theta) \subseteq Z_\pi$. By Lemma 2.4, there are θ-points $x < y < z$ such that, without loss of generality, $f^k(x) \leq x$, $f^k(z) \leq x$ and $f^k(y) \geq z$. In particular, $f^i(y) \neq f^i(x), f^i(z)$ for $0 \leq i \leq k$.

Let m be the smallest positive integer such that $f^m(y)$ is not between $f^m(x)$ and $f^m(z)$. Then $1 \leq m \leq k$. We may assume that $f^m(y) < f^m(x), f^m(z)$.

Let $v = \min\{f^{m-1}(x), f^{m-1}(z)\}$ and $w = \max\{f^{m-1}(x), f^{m-1}(z)\}$. Then $v < f^{m-1}(y) < w$ and $f^{m-1}[x, z] \supseteq [v, w]$. The minimum value of f on $[v, w]$ is less than both $f(v)$ and $f(w)$, so we may assume that it takes place at a π-point, b.

Now let a be the largest π-point less than or equal to v and let c be the smallest π-point greater than or equal to w. Since $f^k[v, b], f^k[b, w] \supseteq [v, w]$ and $f^k(Z_\theta) \subseteq Z_\pi$, it follows that $f^k[v, b], f^k[b, w] \supseteq [a, c]$. Thus $f^k[a, b], f^k[b, c] \supseteq [a, c]$. □

We now phrase Lemma 2.5 in terms of π and its Markov graph.

A directed graph, with ordered vertices and arcs $\cdot \to \cdot$, *has a horseshoe* if and only if there are disjoint, non-empty collections $\mathcal{V}_L$ and $\mathcal{V}_R$ of vertices such that $V' < V''$ for every $V' \in \mathcal{V}_L$ and every $V'' \in \mathcal{V}_R$, and such that for every $V \in \mathcal{V}_L \cup \mathcal{V}_R$, there exist $V' \in \mathcal{V}_L$ and $V'' \in \mathcal{V}_R$ and arcs $V' \to V$ and $V'' \to V$.

Let D be a directed graph and let $k \geq 1$. The *kth power* of D, denoted D^k, is the directed graph whose vertices are the vertices of D (with the same order if the vertices of D are ordered) and whose arcs are the k-tuples of arcs in D, each starting where the previous arc ends.

THEOREM 2.6. *Suppose that π is a self-map of a finite-ordered set and that $f : I \to I$ is continuous and π-weakly monotone. Then for every $k \geq 1$, f^k has a horseshoe if and only if M_π^k has a horseshoe.*

Proof. Let $\{z_i\}$ denote the set of π-points of f. Suppose that f^k has a horseshoe. By Lemma 2.5, there are π-points $a < b < c$ such that $f^k[a, b], f^k[b, c] \supseteq [a, c]$. Let $\mathcal{V}_L$ denote the set of vertices V_i in M_π such that $a \leq z_i < z_{i+1} \leq b$, and let $\mathcal{V}_R$ denote the set of vertices V_j in M_π such that $b \leq z_j < z_{j+1} \leq c$. Then $\mathcal{V}_L$ and $\mathcal{V}_R$ are well defined and satisfy the conditions of the definition.

Conversely, suppose that $\mathcal{V}_L$ and $\mathcal{V}_R$ are as in the definition. Let i and j be least such that $V_i \in \mathcal{V}_L$ and $V_j \in \mathcal{V}_R$. Similarly, let i' and j' be greatest such that $V_{i'} \in \mathcal{V}_L$ and $V_{j'} \in \mathcal{V}_R$. Then $[z_i, z_{i'+1}]$ and $[z_j, z_{j'+1}]$ exhibit a horseshoe for f^k. □

3. *Periods and primitive closed walks*

A continuous, piecewise weakly monotone self-map $f : I \to I$ of a compact interval has finitely many turning points and finitely many turning intervals. A *turning point* is a point $x \in I$ such that either $f(x - \epsilon), f(x + \epsilon) > f(x)$ for all small $\epsilon > 0$, or $f(x - \epsilon)$,

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$f(x+\epsilon) < f(x)$ for all small $\epsilon > 0$. A *turning interval* is a non-degenerate subinterval $J = [a, b]$ of I such that $f(J)$ is one point and either $f(a-\epsilon), f(b+\epsilon) > f(J)$ for all small $\epsilon > 0$ or $f(a-\epsilon), f(b+\epsilon) < f(J)$ for all small $\epsilon > 0$. p_k is a *turning point* of π if and only if $k \neq 1, n$ and either $\pi(p_k) < \pi(p_{k-1}), \pi(p_{k+1})$ or $\pi(p_k) > \pi(p_{k-1}), \pi(p_{k+1})$.

A *cycle* of π is the restriction of π to a subset on which π is a cyclic permutation. Thus the cycles of π can be thought of as periodic orbits of L_π.

We review some standard directed graph-theoretic terminology.

- A *walk* is a sequences of arcs, each starting at the vertex where the previous one ends.
- The *length* of a walk is the number of arcs in it (counting multiplicity).
- A walk is *closed* if and only if it starts and ends at the same vertex.
- A *cycle* is a closed walk which does not pass through the same vertex twice.
- A closed walk is *primitive* if and only if it is not a shorter closed walk traversed several times. For example, $V_1 \to V_1 \to V_2 \to V_1$ is primitive, whereas $V_1 \to V_2 \to V_1 \to V_2 \to V_1$ is not.
- A directed graph is *strongly connected* if and only if for every pair of vertices V_1 and V_2, there are walks from V_1 to V_2 and from V_2 to V_1.
- For $\mathcal{V}$, a subset of the vertices of D, the *subgraph* of D *induced* by $\mathcal{V}$ is the graph whose set of vertices is $\mathcal{V}$ and whose set of arcs is the set of arcs in D from vertices in $\mathcal{V}$ to vertices in $\mathcal{V}$.
- A *strongly connected component* of D is a maximal strongly connected induced subgraph of D.

Let $V_{i_0} \to V_{i_1} \to \cdots \to V_{i_{t-1}} \to V_{i_t}$ be a walk of length t in M_π, and let $f: I \to I$ be continuous and π-weakly monotone with π-set Z_π. A point $x \in I$ *follows the walk* if and only if $f^k(x) \in [z_{i_k}, z_{i_k+1}]$ for $k = 0, 1, \ldots, t$.

LEMMA 3.1. *Let f be continuous and π-weakly monotone, with π-set Z_π. Suppose that $x \notin Z_\pi$ is periodic of period t. Then x follows a closed walk of length t which is either primitive or a primitive closed walk traversed twice.*

LEMMA 3.2. *Let f be continuous and π-weakly monotone. Suppose that M_π contains a primitive closed walk of length t. Then f has a periodic point of period t.*

Proof. Let $Z_\pi = \{z_i\}$ be a π-set and let $V_{i_0} \to V_{i_1} \to \cdots \to V_{i_{t-1}} \to V_{i_0}$ be a primitive closed walk in M_π. By [**BCop**, Lemma I.4] there is a fixed point x of f^t which follows the walk.

Suppose that $x \in (z_{i_0}, z_{i_0+1})$. Then $f^k(x) \in (z_{i_k}, z_{i_k+1})$ for $0 \le k \le t-1$. If the period of x were less than t, the walk would not be primitive.

Suppose that $x = z_{i_0}$ or z_{i_0+1}. Then $[z_{i_1}, z_{i_1+1}]$ is the unique π-interval with one endpoint $f(x)$ such that $V_{i_0} \to V_{i_1}$. Similarly, for $k = 2, 3, \ldots, t-1$, $[z_{i_k}, z_{i_k+1}]$ is the unique π-interval with one endpoint $f^k(x)$ such that $V_{i_{k-1}} \to V_{i_k}$.

Suppose that the period of x is s. If $V_{i_s} = V_{i_0}$, then $V_{i_s+1} = V_{i_1}$, etc. Since the walk is primitive, $s = t$. If $V_{i_s} \neq V_{i_0}$, then $[z_{i_s}, z_{i_s+1}]$ and $[z_{i_0}, z_{i_0+1}]$ are the distinct π-intervals with common endpoint x. Then $V_{i_{2s}} = V_{i_0}$ and $s = t/2$, and so $t/2 \in \mathrm{Per}(f)$.

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If t is not a power of 2, then $t <_S t/2$, and so $t \in \text{Per}(f)$ by Sharkovsky's Theorem. So suppose that t is a power of 2. We may assume that S-max(Per(f)) $= s$. Consider the endpoints of $[z_{i_0}, z_{i_0+1}]$ and $[z_{i_s}, z_{i_s+1}]$. Call them $a < b \leq c < d$. Then $[a, b]$ and $[c, d]$ satisfy the conditions of Lemma 2.3 with f^s in place of f. Therefore f^s has a horseshoe and so by Lemma 2.2, $\text{Per}(f) \not\subseteq 2^\infty$. By Sharkovsky's Theorem, $2^\infty \subseteq \text{Per}(f)$. Therefore $t \in \text{Per}(f)$. □

Remark. If in Lemma 3.2, π is a cyclic permutation, then it follows from [**ALM**, Theorem 2.6.4] that f has a periodic point of period t which follows the walk. However, if π is not a cyclic permutation, there may not exist a periodic point of period t which follows the walk. Consider $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 3 & 2 & 1 \end{pmatrix}$ and the primitive closed walk $V_2 \to V_3 \to V_2$.

THEOREM 3.3. *There are no combinatorial models for* '$\text{Per}(f) = 2^\infty$'.

Proof. It suffices to show that every strongly connected component of M_π is a cycle. Then M_π consists of finitely many cycles. By Lemma 3.1, Per(f) is finite for every continuous, π-weakly monotone self-map of a compact interval.

Let D be a strongly connected component of M_π and let C be a cycle in D. The length of C is a power of 2, say 2^k. Suppose that $D \neq C$. Then there is a vertex V_1 in C which is connected by an arc A of D to a vertex V_2 not in C. Consider the closed walk W consisting of A followed by a walk in D from V_2 to V_1 which passes through V_2 and V_1 each only once. Then W is primitive, so its length is a power of 2, say 2^ℓ. Since A appears in W but not in C, the closed walk consisting of W followed by five repetitions of C is primitive. It has length $2^\ell + 5 \cdot 2^k$, which is not a power of 2. Therefore $D = C$. □

Remark. Theorem 3.3 follows from [**JS**], but the proof above is much simpler (as it proves less).

THEOREM 3.4. *Suppose that π is a self-map of a finite-ordered set and that f is a continuous and π-weakly monotone map. If* Per(f) *is a finite subset of* 2^∞*, then π has a cycle of length* S-max(Per(f)).

Proof. Suppose that $\pi : P \to P$. We prove by induction on k that if $f : I \to I$ is continuous and π-weakly monotone, and if S-max(Per(f)) $= 2^k$, then π has a cycle of length 2^k.

Suppose that $k = 0$. Since the length of every cycle of π is also the period of some periodic orbit of f, it follows that π has no cycles of length other than 1. However, it does have a cycle of length 1.

Suppose that $k = 1$, but that the length of every cycle in M_π is 1. Let $\{v, w\}$ be a periodic orbit of period 2. Without loss of generality, $v < w$. Then both v and w are in the interiors of π-intervals. Let a be the left endpoint of the π-interval containing v and let d be the right endpoint of the π-interval containing w. If there is at least one π-point between v and w, let b be the right endpoint of the π-interval containing v and let c be the left endpoint of the π-interval containing w. If not, let $b = c$ be the unique fixed point of f between v and w. Then by Lemmas 2.2 and 2.3, $\text{Per}(f) \not\subseteq 2^\infty$.

So suppose that $k \geq 1$ and that the result is true for k; we prove it for $k + 1$. Let Z_π be a π-set for f. Let Q be the union of Z_π and the set of endpoints of the connected

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components of $f^{-1}(Z_\pi)$, and let $\theta = f^2|_Q$. Then f^2 is θ-weakly monotone. We have $\mathrm{Per}(f) = \{1, 2, \dots, 2^{k+1}\}$ and $\mathrm{Per}(f^2) = \{1, 2, \dots, 2^k\}$. By the inductive hypothesis, θ has a cycle of length 2^k. Because $\theta(Q) \subseteq Z_\pi$ this cycle is contained in Z_π. Hence π^2 has a cycle of length 2^k. Since $k \geq 1$, 2^k is even and so π has a cycle of length 2^{k+1}. □

THEOREM 3.5. *Let π be a self-map of a finite-ordered set. Then every continuous, π-weakly monotone map has the same set of periods. This set is the union of the set of lengths of the cycles of π and the set of lengths of the primitive closed walks in M_π.*

Proof. To prove the first assertion, by Theorem 3.3 it suffices to show that the following hold for every continuous, π-weakly monotone map f:

(1) if every cycle of π and every primitive closed walk in the Markov graph M_π of π have lengths powers of 2, then S-max(Per(f)) is the Sharkovsky-largest length of the cycles of π;

(2) if some cycle of π or some primitive closed walk in M_π has a length not a power of 2, then S-max(Per(f)) is the Sharkovsky-largest of the following two numbers: the Sharkovsky-largest length of the cycles of π and the Sharkovsky-largest length of the primitive closed walks in M_π.

(1) By Lemma 3.1, S-max(Per(f)) is a power of 2. By Theorem 3.4, S-max(Per(f)) is the Sharkovsky-largest length of the cycles of π.

(2) Consider the Sharkovsky-largest length of the cycles of π and the Sharkovsky-largest length of the primitive closed walks in M_π. By Lemma 3.1, S-max(Per(f)) is Sharkovsky-less than or equal to one of these numbers. By Lemma 3.2, S-max(Per(f)) is Sharkovsky-greater than or equal to both of them.

To prove the second assertion, we first show that if $t \in \mathrm{Per}(L_\pi)$, but no cycle of π has length t (i.e. no π-point has period t), then there is a primitive closed walk of length t in M_π.

Suppose that x is L_π-periodic of period t. Then x follows a closed walk $V_{i_0} \to V_{i_1} \to \cdots \to V_{i_{t-1}} \to V_{i_0}$ of length t. By Lemma 3.1, the walk is primitive or it is $V_{i_0} \to V_{i_1} \to \cdots \to V_{i_{(t/2)-1}} \to V_{i_0}$ traversed twice. Suppose the latter holds. Consider the set of all points which follow the closed walk $V_{i_0} \to V_{i_1} \to \cdots \to V_{i_{(t/2)-1}} \to V_{i_0}$. This set is a closed interval J. $L_\pi^{t/2}$ maps J linearly onto V_{i_0} and interchanges x and $L_\pi^{t/2}(x)$. Therefore, the slope of $L_\pi^{t/2}$ on J is -1 and hence $J = V_{i_0}$.

Now i_0 and $i_0 + 1$ are fixed points of L_π^t and are in the same L_π-orbit. If i_0 had period $s < t$, then x would be a fixed point of L_π^s. Since this is not the case, i_0 has period t.

This proves one containment of the second assertion. The other containment follows from Lemma 3.2. □

In light of Theorem 3.5, for π a self-map of a finite-ordered set, we let Per(π) denote the set of periods of every continuous, π-weakly monotone map. We call it the *set of periods* of π.

Let $\pi : P \to P$ be a cyclic permutation of a finite-ordered set, $P = \{p_1 < p_2 < \cdots < p_{2^k}\}$, of cardinality 2^k, $k \geq 0$. If $k = 0$, we say that π is a *simple cycle*. If $k > 0$, set $L = \{p_1 < p_2 < \cdots < p_{2^{k-1}}\}$ and $R = \{p_{2^{k-1}+1} < p_{2^{k-1}+2} < \cdots < p_{2^k}\}$. We say that π is a *simple cycle* if and only if $\pi(L) = R$, $\pi(R) = L$ and both $\pi^2|_L$ and $\pi^2|_R$ are simple cycles of length 2^{k-1}.

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Remark. Simple cycles of length a power of 2 were introduced by Block [**B**].

Suppose, for example, that $\pi : P \to P$ a simple cycle of length 8, where $P = \{p_1 < p_2 < \cdots < p_8\}$. Then $\pi\{p_1, p_2, p_3, p_4\} = \{p_5, p_6, p_7, p_8\}$ and $\pi\{p_5, p_6, p_7, p_8\} = \{p_1, p_2, p_3, p_4\}$. Furthermore, π maps each of the sets $\{p_1, p_2\}$ and $\{p_3, p_4\}$ onto one of the sets $\{p_5, p_6\}$ and $\{p_7, p_8\}$ and *vice versa*. Therefore, there are exactly three primitive closed walks in M_π, namely $V_4 \to V_4$, $V_2 \to V_6 \to V_2$ and a walk of length 4 passing through the vertices V_1, V_3, V_5, V_7.

Using the same analysis for arbitrary $k > 0$, we have the following.

Remark. Suppose that $\pi : P \to P$ is a simple cycle of length 2^k, $k > 0$ and let t be a positive integer. Then there is a primitive closed walk of length t in M_π if and only if $t = 2^j$ for some $j < k$. In particular, there are no primitive closed walks of length 2^k in M_π.

THEOREM 3.6. *For every $k \geq 0$, the minimal combinatorial models for* '$\mathrm{Per}(f) = \{t : t \leq_S 2^k\}$' *are the simple cycles of length 2^k.*

Proof. It suffices to consider only $k > 0$ and to prove:

(1) every simple cycle π of length 2^k satisfies $\mathrm{Per}(\pi) = \{t : t \leq_S 2^k\}$;

(2) every minimal combinatorial model for '$\mathrm{Per}(f) = \{t : t \leq_S 2^k\}$' is a simple cycle of length 2^k.

Statement (1) follows from Theorem 3.5 and the Remark immediately preceding the statement of Theorem 3.6. To prove (2), suppose that π is a minimal combinatorial model for '$\mathrm{Per}(f) = \{t : t \leq_S 2^k\}$'. Then by (1), $\#\pi \leq 2^k$, and so by Theorem 3.4, π is a cycle of length 2^k. By [**BCop**, Theorem VII.24] (see also [**B**]), if a continuous self-map of a compact interval has a periodic point of period 2^k whose orbit is not simple, then $\mathrm{Per}(f) \nsubseteq 2^\infty$. It follows that π is a simple cycle of length 2^k. □

4. *Reduction- and restriction-minimality*

A *block* of P is a subset B of consecutive members (possibly only one) of P. A block of P is *flat* if and only if it contains at least two points and π is constant on the block. For blocks B and B', write $B < B'$ if and only if $b < b'$ for every $b \in B, b' \in B'$. A *block structure* for π is a partition $\mathcal{B} = \{B_1 < B_2 < \cdots < B_m\}$ of P into disjoint blocks, such that if b and b' belong to the same block, then so do $\pi(b)$ and $\pi(b')$. The *reduction* of π corresponding to $\mathcal{B}$ is $\theta : \{q_1 < q_2 < \cdots < q_m\} \to \{q_1 < q_2 < \cdots < q_m\}$, defined by $\theta(q_i) = q_j$ if $\pi(B_i) \subseteq B_j$. The vertex $V_k = [p_k, p_{k+1}]$ in M_π is a *gap vertex* if p_k and p_{k+1} are in different blocks and a *block vertex* if they are in the same block.

LEMMA 4.1. *Suppose that $\mathcal{B}$ is a block structure for π and that θ is the corresponding reduction:*

(1) *every closed walk in M_π passes through only block vertices or through only gap vertices;*

(2) *M_θ is isomorphic, via an increasing map of the vertices, to the subgraph of M_π induced by the gap vertices [**MN**, Theorem 4.1].*

$\theta : Q \to Q$ is a *restriction* of $\pi : P \to P$ if and only if $Q \subseteq P$ and $\theta = \pi|_Q$.

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LEMMA 4.2. *Suppose that θ is a reduction of or a restriction of π:*

(1) *if π is a permutation, then so is θ;*

(2) *if M_θ^k has a horseshoe, then so does M_π^k.*

Proof of Lemma 4.2(2). For θ a reduction of π, the result follows from Lemma 4.1.

Suppose that $\theta : Q \to Q$ is a restriction of $\pi : P \to P$ and that M_θ^k has a horseshoe. Let f be continuous, π-weakly monotone, and suppose, without loss of generality, the set of π-points of f is P. Similarly, let g be continuous, θ-weakly monotone and suppose the set of θ-points of g is Q. Since M_θ^k has a horseshoe, it follows from Theorem 2.6 that g^k has a horseshoe. By Lemma 2.5, there are θ-points $a < b < c$ such that $g^k[a, b], g^k[b, c] \supseteq [a, c]$. Since $Q \subseteq P$, a, b and c are π-points. However, $f(J) \supseteq g(J)$ for every interval J whose endpoints are θ-points and, hence, $f^k(J) \supseteq g^k(J)$ for every such interval. Therefore $f^k[a, b], f^k[b, c] \supseteq [a, c]$. By Theorem 2.6, M_π^k has a horseshoe. □

π is *reduction- and restriction-minimal*, which we abbreviate to r&r-minimal, if and only if there is no proper reduction of π with the same set of periods as π and there is no proper restriction of π with the same set of periods as π.

The following theorem is immediate from Lemma 4.2.

THEOREM 4.3. *Suppose that π is a minimal combinatorial model or a minimal permutation model for either of the following properties:*

(1) $\mathrm{Per}(f) = \{t : t \leq_S s\}$, $s \geq 1$;

(2) $\mathrm{Per}(f) = \{t : t \leq_S 3 \cdot 2^k\}$, $k \geq 0$, *and f^{2^k} does not have a horseshoe.*

Then π is r&r-minimal.

In the remainder of this section, we establish several properties of r&r-minimal maps which will be needed later in the paper.

$f : I \to I$ *exhibits* π if and only if there exist $x_1 < x_2 < \cdots < x_n$ in I such that if $\pi(p_i) = p_j$, then $f(x_i) = x_j$. Every π-weakly monotone map exhibits π.

LEMMA 4.4. *If θ is a reduction of or a restriction of π, then* $\mathrm{Per}(\theta) \subseteq \mathrm{Per}(\pi)$.

Proof. If θ is a reduction of π, then by Theorem 3.5, the result follows from Lemma 4.1.

Suppose then that θ is a restriction of π. We show that if f is continuous and exhibits θ, then $\mathrm{Per}(\theta) \subseteq \mathrm{Per}(f)$. Then, since θ is a restriction of π, any continuous, π-weakly monotone map exhibits θ. By [**MN**, Corollary 1.15], there is a continuous, θ-weakly monotone map f_θ, such that for every map η of a finite-ordered set which has no flat blocks, if f_θ exhibits η, then any continuous f which exhibits θ also exhibits η. So let $k \in \mathrm{Per}(\theta) = \mathrm{Per}(f_\theta)$, and let f be a continuous map which exhibits θ. There is a cyclic permutation η of a finite-ordered set containing k points which is exhibited by f_θ. Since permutations have no flat blocks, f also exhibits η. Thus $k \in \mathrm{Per}(f)$. □

LEMMA 4.5. *r&r-minimal maps have no flat blocks.*

Proof. Suppose that $\pi : P \to P$ is r&r-minimal, but that $\pi(p_k) = \pi(p_{k+1})$. Let $\{p_k, p_{k+1}\}$ be a block and call the associated reduction θ. It follows from Theorem 3.5 that $\mathrm{Per}(\theta) = \mathrm{Per}(\pi)$. Therefore π has no flat blocks. □

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LEMMA 4.6. *Suppose that π is r&r-minimal and that $\{B_1, B_2, \dots, B_k\}$ is a collection of disjoint blocks, each containing at least two points, such that $\pi(B_i) \subseteq B_{i+1}$, $i = 1, 2, \dots, k$, where $B_{k+1} = B_1$. Then $B_1 \cup B_2 \cup \cdots \cup B_k = P$.*

Proof. Suppose that the result is false. Consider the block structure with blocks $B_1, B_2, \dots, B_k$ and the singletons in $P \setminus (B_1 \cup B_2 \cup \cdots \cup B_k)$. Let $t = \text{S-max}(\text{Per}(\pi))$.

There are three possibilities:

(1) π has a cycle of length t;

(2) M_π has a primitive closed walk of length t passing through only block vertices;

(3) M_π has a primitive closed walk of length t passing through only gap vertices.

In case (1), restrict to the cycle of length t. In case (2), restrict to $B_1 \cup B_2 \cup \cdots \cup B_k$. In case (3), reduce by collapsing each B_i to a point. The resulting restriction or reduction has the same set of periods as does π. □

LEMMA 4.7. *Suppose that π is r&r-minimal and that* $\text{Per}(\pi) \not\subseteq 2^\infty$. *Then there is no collection $\{B_1, B_2, \dots, B_k\}$ of disjoint blocks, each containing at least two points, such that π maps B_i monotonically onto B_{i+1}, $i = 1, 2, \dots, k$, where $B_{k+1} = B_1$.*

Proof. If there is such a collection, let θ be the reduction obtained by collapsing the blocks to points. Then $k \in \text{Per}(\theta)$, since θ contains a cycle of length k. Now $\text{Per}(\pi)$ consists of k, possibly $2k$, and the lengths of the primitive closed walks passing through gap vertices in M_π. However, the gap vertices in M_π and M_θ are the same, so $\text{Per}(\pi) \setminus \text{Per}(\theta)$ can contain only $2k$. In particular, $\text{Per}(\theta) \not\subseteq 2^\infty$. So $2k \neq \text{S-max}(\text{Per}(\pi))$ and hence $\text{Per}(\pi) = \text{Per}(\theta)$. □

We call a collection of blocks as in Lemma 4.7 a *monotone cycle of blocks*. In [**BCov**], an interval J with endpoints in P is called periodic if and only if there is a positive integer t such that L_π^t is the identity on J, where L_π is the canonical π-linear map. It follows from [**BCov**, Lemmas 2.6 and 2.4] that $\{p_i, p_{i+1}, \dots, p_{i+k}\}$ is in a monotone cycle of blocks if and only if $[i, i+k]$ is a periodic interval.

Recall that p_k is a *turning point* of π if and only if $k \neq 1, n$ and either $\pi(p_{k-1}), \pi(p_{k+1}) < \pi(p_k)$ or $\pi(p_{k-1}), \pi(p_{k+1}) > \pi(p_k)$.

LEMMA 4.8. *Suppose that π is r&r-minimal and that* $\text{Per}(\pi) \not\subseteq 2^\infty$. *Then every point in P is in the orbit of a turning point of π.*

Proof. Let θ be the restriction of π to Q, where Q is the set of points in P each of which is in the orbit of at least one of the following:

- the largest point in P;
- the smallest point in P;
- a turning point of π;
- an endpoint of a maximal flat block;
- an endpoint of a block in a maximal monotone cycle of blocks.

By [**BCov**, Theorem 2.6], L_θ and L_π are topologically conjugate; in particular they have the same set of periods. It follows from Theorem 3.5 that $\text{Per}(\theta) = \text{Per}(\pi)$. Since π is r&r-minimal, $\theta = \pi$.

Since π has no flat blocks and no monotone cycle of blocks, it suffices to show that the smallest and largest points in P, say p and p', are in the orbits of turning points of π. If $\pi^{-1}(p) \subseteq \{p\}$, $\pi^{-1}(p') \subseteq \{p'\}$ or $\pi^{-1}\{p, p'\} \subseteq \{p, p'\}$, then there is a restriction of π with the same set of periods as π. If none of these conditions hold, then p and p' are in the orbits of turning points of π. □

5. *Block structure over a simple cycle*

LEMMA 5.1. *Suppose $f : I \to I$ is continuous and that there is no non-degenerate, closed subinterval $J \neq I$ of I such that $f(J) \subseteq J$.*

(1) *If f has more than one fixed point, then f has a horseshoe.*

(2) *If some fixed point of f has more than one preimage, then f^2 has a horseshoe.*

Proof. Without loss of generality $I = [0, 1]$.

(1) It suffices to find $a < b < c$ in $[0, 1]$, such that $f(a) = f(c) = a$ and $f(b) = c$ (or $f(a) = f(c) = c$ and $f(b) = a$).

The set of non-fixed points of f is open and dense in $[0, 1]$. Since f has more than one fixed point, there is a connected component C of that set such that both endpoints $a < a'$ are fixed points. Without loss of generality $f(x) > x$ for every x, $a < x < a'$. Then $a' \neq 1$. (Otherwise let $J = [a + \epsilon, 1]$.)

Let $c > a'$ be least such that $f(c) = a$. (If no such c exists, again let $J = [a + \epsilon, 1]$.) There exists b, $a < b < c$, such that $f(b) = c$. (If no such b exists, let $J = [a, c - \epsilon]$.)

(2) We may assume that f has exactly one fixed point a; otherwise, by (1), f has a horseshoe and then so does f^2. Since f is onto, $a \neq 0, 1$. Suppose that $f(c) = a$, where without loss of generality, $c > a$.

Let $u = \min\{f(x) : a < x < c\}$. Then $u < a$. (If not, let $J = [a, c]$.) There exists b', $u \leq b' < a$, such that $f(b') = c$. (If not, let $J = f[u, c]$.) There exists b, $a < b < c$, such that $f(b) = b'$. Then $f^2(a) = f^2(c) = a$ and $f^2(b) = c$. □

LEMMA 5.2. *Suppose that $\pi : P \to P$ is r&r-minimal, $P = \{1, 2, \ldots, n\}$, and* $\mathrm{Per}(\pi) \nsubseteq 2^\infty$. *If there is a non-degenerate, closed interval $J \subseteq [1, n]$ and a positive integer t such that $J, L_\pi(J), \ldots, L_\pi^{t-1}(J)$ are pairwise disjoint and $L_\pi^t(J) \subseteq J$, then $J \cup L_\pi(J) \cup \cdots \cup L_\pi^{t-1}(J) \supseteq \{1, 2, \ldots, n\}$.*

Proof. Since π has no flat blocks, the slope of L_π on any π-interval $[i, i + 1]$ has absolute value at least 1. Let J be an interval satisfying the conditions of the lemma.

Suppose first that $J \cap P = \varnothing$. Then for $i = 0, 1, \ldots, t - 1$, $L_\pi^i(J) \cap P = \varnothing$. Therefore L_π^t is linear on J. Since $L_\pi^t(J) \subseteq J$ and the slope of L_π^t on J has absolute value at least 1, L_π^t maps J linearly onto J. Therefore L_π^{2t} is the identity on J, and so J is contained in a periodic (in the sense of [**BCov**]—see §4) interval whose endpoints belong to P. This contradicts Lemma 4.7. Therefore $J \cap P \neq \varnothing$.

Next suppose that $J \cap P$ contains exactly one point, say d. Since π has no flat blocks, each $L_\pi^i(J) \cap P$ contains exactly one point. Define a subinterval K of $J = [a, b]$, as follows. If $d = a$ or b, let $K = J$. Suppose that $a < d < b$. At least one of the following holds:

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(1) $L_\pi^t[a, d] \subseteq [a, d]$;
(2) $L_\pi^t[d, b] \subseteq [d, b]$;
(3) $L_\pi^t[a, d] \subseteq [d, b]$ and $L_\pi^t[d, b] \subseteq [a, d]$.

If (1) holds, set $K = [a, d]$; otherwise set $K = [d, b]$. Then $L_\pi^{2t}(K) \subseteq K$ and for each $i = 0, 1, \ldots, 2t - 1$, $L_\pi^i(K)$ is contained in a unique π-interval. As above, this leads to a contradiction of Lemma 4.7.

Therefore $J \cap P$ contains at least two points. For each $i = 1, 2, \ldots, t$, let $B'_i = L_\pi^{i-1}(J) \cap P$. Each B'_i is a block containing at least two points and $\pi(B'_i) \subseteq B'_{i+1}$ for $i = 1, 2, \ldots, t$, where $B'_{t+1} = B'_1$. It follows from Lemma 4.6 that $B'_1 \cup B'_2 \cup \cdots \cup B'_t = P$. Therefore $J \cup L_\pi(J) \cup \cdots \cup L_\pi^{t-1}(J) \supseteq P$. □

THEOREM 5.3. *Let $k \geq 0$. Suppose that $\pi : P \to P$ is an r&r-minimal self-map of a finite-ordered set such that* $\mathrm{Per}(\pi) \nsubseteq 2^\infty$ *and such that $M_\pi^{2^k}$ does not have a horseshoe. Then π has a block structure over a simple cycle of length 2^k.*

Proof. We prove, by induction on j, that for every $j = 0, 1, \ldots, k$, π has a block structure over a simple cycle of length 2^j.

There is nothing to prove for $j = 0$. Suppose then that $0 \leq j \leq k - 1$, and that π has a block structure over a simple cycle θ of length 2^j. Let $B''_1, B''_2, \ldots, B''_{2^j}$ be the blocks of this block structure, labelled so that $\pi(B''_i) \subseteq B''_{i+1}$ for $i = 1, 2, \ldots, 2^j$, where $B''_{2^j+1} = B''_1$. Since π has no flat blocks and $\mathrm{Per}(\pi) \nsubseteq 2^\infty$, each block contains at least two points of P.

Suppose that $\#\pi = n$. Without loss of generality, $P = \{1, 2, \ldots, n\}$. Let $L_\pi : [1, n] \to [1, n]$ be the canonical π-linear map. For $i = 1, 2, \ldots, 2^j$, let J_i be the convex hull, in $[1, n]$, of the block B''_i, where B''_i is as in the preceding paragraph, and let $g_i = L_\pi^{2^j}|_{J_i}$. Then g_i maps J_i to itself. Since $j \leq k - 1$, g_i^2 does not have a horseshoe.

Now fix i, $1 \leq i \leq 2^j$. We claim that there is no non-degenerate, proper, closed subinterval J of J_i such that $g_i(J) \subseteq J$. Suppose that J is such an interval. Then the intervals $J, L_\pi(J), \ldots, L_\pi^{2^j-1}(J)$ are pairwise disjoint. So by Lemma 5.2, $J \cup L_\pi(J) \cup \cdots \cup L_\pi^{2^j-1}(J) \supseteq P$. Since the endpoints of J_i are in P, it follows that $J = J_i$. This establishes the claim.

By Lemma 5.1, g_i has a unique fixed point w_i and $g_i^{-1}(w_i) = \{w_i\}$.

We claim that every L_π-periodic point not in $J_1 \cup J_2 \cup \cdots \cup J_{2^j}$ has period a power of 2. If not, then by Lemma 3.1, M_π has a primitive closed walk of length not a power of 2 which passes through only gap vertices. By Lemma 4.1, M_θ has a primitive closed walk of length not a power of 2. This contradicts the Remark immediately preceding Theorem 3.6.

Since $\mathrm{Per}(L_\pi) \nsubseteq 2^\infty$, g_i has a periodic orbit with more than one point. There exist adjacent points $y < z$ in this periodic orbit such that $g_i(y) > y$ and $g_i(z) < z$. Therefore $y < w_i < z$, $g_i(y) > w_i$ and $g_i(z) < w_i$. Since $g_i^{-1}(w_i) = \{w_i\}$, it follows that:

(∗) $g_i(x) > w_i$ for every $x \in J_i$ such that $x < w_i$ and $g_i(x) < w_i$ for every $x \in J_i$ such that $x > w_i$.

Now $\{w_1, w_2, \ldots, w_{2^j}\}$ is a periodic orbit of L_π. It follows from (∗) that if $x_1 \in J_1$ and $L_\pi(x_1) = w_2$, then $x_1 = w_1$. Therefore we have:

(∗∗) $L_\pi^{-1}(w_1) \cap J_{2^j} = \{w_{2^j}\}$ and for $i \neq 1$, $L_\pi^{-1}(w_i) \cap J_{i-1} = \{w_{i-1}\}$.

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It also follows from $(*)$ that none of the points w_i are turning points of L_π. Since $J_1 \cup J_2 \cup \cdots \cup J_t$ contains all the turning points of L_π, it follows from $(**)$ that none of the points w_i is in the orbit of a turning point of L_π. Then by Lemma 4.8, none of the points w_i is in P. This together with $(*)$ implies that π has a block structure over a simple cycle of length 2^{j+1}. □

6. *Sharkovsky-largest period* $3 \cdot 2^k$

This section contains the most surprising and technically difficult results to prove, the characterizations of the minimal combinatorial models and the minimal permutation models for 'Per$(f) = \{t : t \leq_S 3 \cdot 2^k\}$'. It is the only case for which the minimal combinatorial models and the minimal permutation models are not the same.

Let $\pi : P \to P$ be a cyclic permutation of a finite-ordered set of cardinality $r \cdot 2^k$, where $r \geq 3$ is odd. π is a *simple cycle* if and only if:

(1) π has a block structure over a simple cycle of length 2^k;
(2) π is monotone on all blocks but one;
(3) for each block B, letting p be the central point of B and $\theta = \pi^{2^k}|_B$, either

$$\theta^{r-1}(p) < \theta^{r-3}(p) < \cdots < \theta^2(p) < p < \theta(p) < \cdots < \theta^{r-4}(p) < \theta^{r-2}(p)$$

or

$$\theta^{r-1}(p) > \theta^{r-3}(p) > \cdots > \theta^2(p) > p > \theta(p) > \cdots > \theta^{r-4}(p) > \theta^{r-2}(p).$$

Remark. Simple cycles of odd length greater than 1 were introduced by Štefan [**Š**]. Simple cycles of length $r \cdot 2^k$, r odd, $r \geq 3$, were studied by Coppel [**C**] and by Alsedà *et al* [**ALS**]. In [**BCop**] simple cycles were called *strongly simple*. Our usage of the term corresponds to its usage in [**ALM**].

LEMMA 6.1. *Let* $\pi : P \to P$ *be r&r-minimal such that* Per$(\pi) = \{t : t \leq_S 3 \cdot 2^k\}$ *and* $M_\pi^{2^k}$ *does not have a horseshoe. If* $\#\pi \leq 3 \cdot 2^k$*, then* π *is a simple cycle of length* $3 \cdot 2^k$.

Proof. Consider the canonical π-linear map $L_\pi : [1, n] \to [1, n]$. By Theorem 5.3, π has a block structure over a simple cycle of length 2^k. Let $B_1, B_2, \ldots, B_{2^k}$ be the blocks of this structure, labelled so that $\pi(B_i) \subseteq B_{i+1}$ for $i = 1, 2, \ldots, 2^k$, where $B_{2^k+1} = B_1$. Let $C_i = [\min B_i, \max B_i]$ be the convex hull of B_i (in $[1, n]$). Since Per$(\pi) \not\subseteq 2^\infty$, at least one block contains more than one point. Since π has no flat blocks, every block contains at least two points. If every block contained exactly two points, then by Theorem 3.5, Per$(\pi) \subseteq 2^\infty$. Therefore, one of the blocks contains at least three points.

If one of the blocks contains exactly two points, then some block B_i contains more than two points, while B_{i+1} contains exactly two points. Since π has no flat blocks, any two consecutive members of B_i must map onto the two members of B_{i+1}. Therefore there are non-overlapping subintervals J' and J'' of C_i such that $L_\pi(J') = L_\pi(J'') = C_{i+1}$. It follows that, then $L_\pi^{2^k}$ has a horseshoe. Therefore, each block contains at least three points, hence exactly three points and $\#\pi = 3 \cdot 2^k$.

Let $\mathcal{J}$ be the set of π-intervals contained in $C_1 \cup C_2 \cup \cdots \cup C_{2^k}$, i.e. $\mathcal{J} = \{[j, j+1] : V_j$ is a block vertex$\}$. Note that each C_i contains exactly two intervals in $\mathcal{J}$. If, for each interval J in $\mathcal{J}$, $L_\pi(J)$ contains only one interval in $\mathcal{J}$, then every $L_\pi^{2^k}|_{C_i}$ is monotone.

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This contradicts the fact that $3 \in \mathrm{Per}(L_\pi^{2^k}|_{C_i})$ for some i. Therefore, there is at least one interval J_0 in $\mathcal{J}$ such that $L_\pi(J_0)$ contains two intervals in $\mathcal{J}$.

By relabelling, we may assume that $J_0 \subseteq C_{2^k}$. Let $I_{2^k,1} = J_0$ and let $I_{2^k,2}$ be the other interval in $\mathcal{J}$ which is a subset of C_{2^k}. Then $L_\pi(I_{2^k,1}) = C_1$. Now it follows from Lemma 4.6 that $L_\pi(C_i) = C_{i+1}$ for $i = 1, 2, \ldots, 2^k - 1$. In particular $L_\pi^{2^k}(I_{2^k,1}) = C_{2^k}$. Since $L_\pi^{2^k}$ does not have a horseshoe, $L_\pi(I_{2^k,2})$ is a proper subset of C_1. Denote the intervals in $\mathcal{J}$ which are contained in C_1 by $I_{1,1}$ and $I_{1,2}$, where $L_\pi(I_{2^k,2}) = I_{1,2}$. Similarly, for $i = 2, 3, \ldots, 2^k - 1$, denote the two members of $\mathcal{J}$ which are contained in C_i by $I_{i,1}$ and $I_{i,2}$, where $L_\pi(I_{i-1,2}) = I_{i,2}$.

Since $3 \in \mathrm{Per}(L_\pi^{2^k})$, we must have $L_\pi(I_{2^k-1,2}) \supseteq I_{2^k,1}$. If $L_\pi(I_{2^k-1,2}) \supseteq I_{2^k,2}$ as well, then the intervals $I_{2^k,1}$ and $I_{2^k,2}$ would exhibit a horseshoe for $L_\pi^{2^k}$. Thus $L_\pi(I_{2^k-1,2}) = I_{2^k,1}$.

If $L_\pi(I_{1,1}) \supseteq I_{2,2}$, then the intervals $I_{1,1}$ and $I_{1,2}$ would exhibit a horseshoe for $L_\pi^{2^k}$. Thus $L_\pi(I_{1,1}) = I_{2,1}$. Similarly $L_\pi(I_{i,1}) = I_{i+1,1}$ for $i = 2, 3, \ldots, 2^k - 2$ and $L_\pi(I_{2^k-1,1}) = I_{2^k,2}$.

We have determined all the arcs emanating from block vertices in M_π. In particular, π maps B_i monotonically onto B_{i+1} for $i = 1, 2, \ldots, 2^k - 1$. Furthermore, if x is the point in B_1 which is an endpoint of $I_{1,1}$ but not an endpoint of $I_{1,2}$, if y is the common endpoint of $I_{1,1}$ and $I_{1,2}$, and if z is the point in B_1 which is an endpoint of $I_{1,2}$ but not an endpoint of $I_{1,1}$, then it is easy to check that $\pi^{2^k}(x) = y$, $\pi^{2^k}(y) = z$, and $\pi^{2^k}(z) = x$. Therefore π is a simple cycle. □

THEOREM 6.2. *The minimal combinatorial models for* 'Per$(f) = \{t : t \leq_S 3 \cdot 2^k\}$ *and* f^{2^k} *does not have a horseshoe' are the simple cycles of length* $3 \cdot 2^k$.

Proof. First suppose that π is a simple cycle of length $3 \cdot 2^k$. Then π has a block structure over a simple cycle θ of length 2^k. Since the length of any primitive closed walk in M_θ is a power of 2, it follows from Lemma 4.1(2) that the length of any primitive closed walk in M_π which passes through only gap vertices is also a power of 2. On the other hand, any primitive closed walk in M_π which passes through only block vertices has length a multiple of 2^k. M_π has a primitive closed walk of length $3 \cdot 2^k$, the Sharkovsky-largest multiple of 2^k. By Theorem 3.5, S-max(Per$(\pi)) = 3 \cdot 2^k$.

Let f be continuous and π-weakly monotone. The monotonicity condition on π implies that $M_\pi^{2^k}$ does not have a horseshoe. By Theorem 2.6 neither does f^{2^k}. Therefore, π is a combinatorial model for 'Per$(f) = \{t : t \leq_S 3 \cdot 2^k\}$ and f^{2^k} does not have a horseshoe'.

To complete the proof, suppose that π is a minimal combinatorial model for 'Per$(f) = \{t : t \leq_S 3 \cdot 2^k\}$ and f^{2^k} does not have a horseshoe'. Then by the first part of the proof, $\#\pi \leq 3 \cdot 2^k$ and by Theorem 4.3, π is r&r-minimal. So by Lemma 6.1, π is a simple cycle of length $3 \cdot 2^k$. □

THEOREM 6.3. *The minimal permutation models for* 'Per$(f) = \{t : t \leq_S 3 \cdot 2^k\}$' *are the permutations of cardinality* $3 \cdot 2^k$ *which satisfy:*

(1) *π has a block structure over a simple cycle of length 2^k with each block having exactly three points;*

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(2) *for any pair of adjacent points, p_j and p_{j+1}, in the same block, $\pi^i(p_j)$ and $\pi^i(p_{j+1})$ are not adjacent points for some i, $1 \le i \le 2^{k+1}$.*

Proof. First, suppose that $\pi : P \to P$ is a permutation which satisfies (1) and (2). There is a block $B = \{p_j, p_{j+1}, p_{j+2}\}$ such that the restriction of π to B is not monotone. Consider V_j and V_{j+1}, the two block vertices associated with B in M_π. Without loss of generality, there are arcs from V_j to two distinct block vertices. Since π maps blocks onto blocks, it follows that there are walks of length 2^k from V_j to V_j and from V_j to V_{j+1}.

It follows from (2) that there is a closed walk of length 2^k from V_{j+1} to V_j. By following, in succession, the walks from V_j to V_j, from V_j to V_{j+1}, and from V_{j+1} to V_j, we obtain a primitive closed walk of length $3 \cdot 2^k$. By Theorem 3.5, $3 \cdot 2^k \in \mathrm{Per}(\pi)$.

On the other hand, it follows from (1) that the length of any cycle of π is a multiple of 2^k. Now suppose that W is a primitive closed walk. By Lemma 4.1(1), W passes through only block vertices or through only gap vertices. In the first case, the length of W is a multiple of 2^k. In the second case, it follows from Lemma 4.1(2) and the remark immediately preceding Theorem 3.6 that the length of W is 2^j for some $j < k$. Therefore, the length of W is Sharkovsky-less than or equal to $3 \cdot 2^k$. Thus by Theorem 3.5, $3 \cdot 2^k =$ S-max(Per(π)) and so π is a permutation model for ‘$\mathrm{Per}(f) = \{t : t \le_S 3 \cdot 2^k\}$’.

To complete the proof, suppose that $\pi : P \to P$ is a minimal permutation model for ‘$\mathrm{Per}(f) = \{t : t \le_S 3 \cdot 2^k\}$’ and consider the canonical π-linear map $L_\pi : [1, n] \to [1, n]$. The case $k = 0$ is straightforward, so suppose that $k \ge 1$.

By Theorem 4.3, π is r&r-minimal. Since $3 \cdot 2^{k-1} \notin \mathrm{Per}(\pi)$, $M_\pi^{2^{k-1}}$ does not have a horseshoe. Hence, by Theorem 5.3, π has a block structure over a simple cycle of length 2^{k-1}.

As in the proof of Theorem 5.3, let B be one of the blocks and let $C = [\min B, \max B]$ be its convex hull. Since π maps blocks onto blocks, L_π maps convex hulls of blocks onto convex hulls of blocks. Set $g = L_\pi^{2^{k-1}}|_C$. Then g maps C onto itself. We will show that the following holds.

($*$) There is a point $z \in C$, $z \notin B$, such that $g(p) > z$ for every $p \in B$ with $p < z$, and $g(p) < z$ for every $p \in B$ with $p > z$.

As in the proof of Theorem 5.3, g has no proper, non-degenerate, invariant subinterval. Since g does not have a horseshoe, by Lemma 5.1, it has a unique fixed point, call this z. Since g maps C onto itself, z is not an endpoint of C.

We first claim that z is not a turning point of g. Note that since π is r&r-minimal, L_π and hence g have no flat intervals. If z is a turning point of g, let Q be B union the set of endpoints of maximal monotone pieces of g. Let a be the largest point in Q less than z and let b be the smallest point in Q greater than z. Then since g maps endpoints of maximal monotone pieces of g into B, $g(a), g(b) \in B$. Since the turning point z is the unique fixed point of g, either $g(a) < a$ or $g(b) > b$. Without loss of generality assume the former. Then $g(x) < x$ for all $x \in C$ such that $x \le a$. However, $g(a) < a$ so $g^2(a) < g(a)$. Continuing, we have $\cdots < g^3(a) < g^2(a) < g(a)$, which contradicts the fact that $g(a)$ is g-periodic. Therefore, z is not a turning point of g and hence not a turning point of L_π.

Similarly no point in the L_π-orbit of z is a turning point of L_π. Since π is a permutation, z is not in the orbit of a turning point of L_π. Then by Lemma 4.8, $z \notin P$.

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Let $p \in B$. Then p is L_π-periodic and hence g-periodic. Since $p \neq z$ and $\mathrm{Per}(g)$ contains no odd number other than 1, it follows from [**ALM**, Lemma 2.1.6] that there exists $z' = z'(p) \in C$, z' not in the g-orbit of p, such that $g(p) > z'$ if $p < z'$ and $g(p) < z'$ if $p > z'$. Clearly z' may be chosen to be a fixed point of g. Therefore, we may choose $z'(p) = z$ for all $p \in B$, and $(*)$ is established.

It follows from $(*)$ that π has a block structure over a simple cycle of length 2^k. Since π is a permutation, the blocks all have the same number of points. If this number were one or two, then by Theorem 3.5, S-max(Per(π)) $= 2^k$ or 2^{k+1}. Since this is not the case, every block contains three points, i.e. (1) holds.

Finally, we show that (2) holds. Proceeding by contradiction, suppose that there is a pair of adjacent points, p_j and p_{j+1}, in the same block, such that for every i, $1 \le i \le 2^{k+1}$, $\pi^i(p_j)$ and $\pi^i(p_{j+1})$ are adjacent points. Without loss of generality, we may assume that the block which contains p_j and p_{j+1} is $\{p_j, p_{j+1}, p_{j+2}\}$. It follows from Lemma 4.6 that $\{\pi^{2^k}(p_j), \pi^{2^k}(p_{j+1})\} = \{p_{j+1}, p_{j+2}\}$ and hence that $\{\pi^{2^k}(p_{j+1}), \pi^{2^k}(p_{j+2})\} = \{p_j, p_{j+1}\}$. Therefore, π is monotone on every block, $\pi^{2^k}(p_j) = p_{j+2}$, $\pi^{2^k}(p_{j+1}) = p_{j+1}$ and $\pi^{2^k}(p_{j+2}) = p_j$. Then, by Theorem 3.5, S-max(Per(π)) $= 2^k$. This is a contradiction and so (2) holds. □

It follows from Theorem 6.3 that every cyclic permutation which satisfies (1) is a minimal permutation model for '$\mathrm{Per}(f) = \{t : t \le_S 3 \cdot 2^k\}$'. On the other hand, there are non-cyclic minimal permutation models. For example, the permutation $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 6 & 2 & 3 & 1 \end{smallmatrix}\right)$ is a minimal permutation model for '$\mathrm{Per}(f) = \{t : t \le_S 6\}$'.

THEOREM 6.4. *The minimal combinatorial models for* '$\mathrm{Per}(f) = \{t : t \le_S 3 \cdot 2^k\}$, $k \ge 1$' *are the self-maps π of finite-ordered sets of cardinality* $3 \cdot 2^{k-1} + 1$ *which satisfy:*

(1) *π has a block structure over a simple cycle of length 2^{k-1};*
(2) *there is a block B with four points and every other block has three points;*
(3) *π is strictly monotone on every block other than B;*
(4) $\pi^{2^{k-1}}|_B = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 3 & 1 \end{smallmatrix}\right)$ *or* $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 2 \end{smallmatrix}\right)$.

Proof. First, suppose that $\pi : P \to P$, $\#\pi = 3 \cdot 2^{k-1} + 1$ and π satisfies (1), (2), (3) and (4). Let $P = \{p_1 < p_2 < \cdots < p_{3 \cdot 2^{k-1}+1}\}$, the exceptional block be $B = \{p_m, p_{m+1}, p_{m+2}, p_{m+3}\}$ and V_m, V_{m+1}, V_{m+2} be the corresponding block vertices in M_π.

Number the blocks $B_1, B_2, \ldots, B_{2^{k-1}}$, so that $B_1 = B$ and each $\pi(B_i) \subseteq B_{i+1}$, where $B_{2^{k-1}+1} = B_1$. Without loss of generality, $\pi^{2^{k-1}}|_{B_1}$ is equivalent to $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 3 & 1 \end{smallmatrix}\right)$. Then it follows from (3) that the endpoints of B_2 are $\pi(p_{m+3})$ and $\pi(p_{m+1})$ and that $\pi(p_m) = \pi(p_{m+2})$ is the midpoint of B_2. Moreover, π maps the endpoints of $B_{2^{k-1}}$ onto $\{p_m, p_{m+3}\}$ and maps the midpoint of $B_{2^{k-1}}$ to p_{m+2}.

It follows that in M_π there are walks of length 2^{k-1} from V_m to V_{m+2}, from V_{m+1} to V_{m+2}, from V_{m+2} to V_m and from V_{m+2} to V_{m+1}, and no other walks of length 2^{k-1} between these vertices. Since by (1), every primitive closed walk which passes through only gap vertices has length a power of 2, it follows that the Sharkovsky-largest length of a primitive closed walk is $3 \cdot 2^k$. As the only cycle of π has length 2^{k-1}, it follows from Theorem 3.5 that $\mathrm{Per}(\pi) = \{t : t \le_S 3 \cdot 2^k\}$.

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To complete the proof, suppose that π is a minimal combinatorial model for 'Per$(f) = \{t : t \leq_S 3\cdot 2^k\}, k \geq 1$'. By Theorem 4.3, π is r&r-minimal. Also, since $3\cdot 2^{k-1} \notin \text{Per}(\pi)$, $M_\pi^{2^{k-1}}$ does not have a horseshoe. Hence, by Theorem 5.3, (1) holds.

To show that (2), (3) and (4) hold, first consider the case $k \geq 2$. By Lemma 4.5, π has no flat blocks. In particular, each block has at least two points. Suppose that some block has exactly two points and let V denote the corresponding block vertex in M_π. At least one block has more than two points, otherwise $\text{Per}(\pi) \subseteq 2^\infty$. It follows from Lemma 4.6 that there are two distinct primitive closed walks of length 2^{k-1} from V to itself. By Theorem 3.5, $3 \cdot 2^{k-1} \in \text{Per}(\pi)$. Since this is a contradiction, each block contains at least three points.

By the first part of the proof, $\#\pi \leq 3 \cdot 2^{k-1} + 1$. Since $k \geq 2$, there is a block B with exactly three points. Let V' and V'' be the corresponding block vertices in M_π. It follows from Lemma 4.6 that there are walks of length 2^{k-1} from V' to V'', and from V'' to V'. Since $3 \cdot 2^{k-1} \notin \text{Per}(\pi)$, it follows from Theorem 3.5 that:

$(*)$ there are no closed walks of length 2^{k-1} from V' to itself or from V'' to itself.

In light of $(*)$ and the fact that π has no flat blocks, we see that π cannot map the midpoint of B to an endpoint of another block. From this and Lemma 4.6, we have:

$(**)$ $\pi|_B$ is strictly monotonic and maps the endpoints of B to the endpoints of another block.

Since $3 \cdot 2^k \in \text{Per}(\pi)$, it follows from $(**)$ and Theorem 3.5 that at least one block has more than three points. On the other hand, since π is a minimal combinatorial model, it follows from the first part of the proof that $\#\pi \leq 3 \cdot 2^{k-1} + 1$. Hence, there is exactly one block with four points, i.e. (2) holds. Since (3) follows from $(**)$, it remains to prove (4).

Number the blocks $B_1, B_2, \ldots, B_{2^{k-1}}$, so that B_1 has four points, and $\pi(B_i) \subseteq B_{i+1}$, where $B_{2^{k-1}+1} = B_1$. It follows from $(*)$ and Lemma 4.6 that π maps adjacent points of B_1 to adjacent points of B_2. So π maps one endpoint of B_1 to an endpoint of B_2 and the other endpoint of B_1 to the midpoint of B_2. Write $B_1 = \{a < b < c < d\}$. We may assume that $\pi(d)$ is an endpoint of B_2. Write $B_2 = \{u, v, w\}$, where $w = \pi(d)$ and u is the other endpoint of B_2. Then using Lemma 4.6, we have $\pi(a) = \pi(c) = v$ and $\pi(b) = u$. Write $B_{2^{k-1}} = \{x, y, z\}$, where $\pi^{2^{k-1}-2}(u) = x$, $\pi^{2^{k-1}-2}(v) = y$ and $\pi^{2^{k-1}-2}(w) = z$. By $(**)$, $\pi\{x, z\} = \{a, d\}$. If $\pi(x) = a$ and hence $\pi(z) = d$, we obtain a contradiction to Lemma 4.6. Therefore, $\pi(x) = d$ and $\pi(z) = a$.

Suppose that $\pi(y) = b$. If V is the block vertex corresponding to the adjacent points x and y in $B_{2^{k-1}}$, there is a closed walk of length 2^{k-1} from V to itself. This contradicts $(*)$. Hence, $\pi(y) = c$. Therefore, $\pi^{2^{k-1}}(a) = c$, $\pi^{2^{k-1}}(b) = d$, $\pi^{2^{k-1}}(c) = c$ and $\pi^{2^{k-1}}(d) = a$, i.e. (4) holds.

Finally, we consider the case $k = 1$. It suffices to only show that (4) holds, i.e. that $\pi|_B = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 3 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 2 \end{smallmatrix}\right)$. Since π is a minimal combinatorial model, it follows from the first part of the proof that $\#\pi \leq 4$. On the other hand, it is easy to see, using Theorem 3.5, that there are no combinatorial models for 'Per$(f) = \{t : t \leq_S 6\}$' with fewer than four points. Therefore, $\#\pi = 4$. Write $P = \{p_1 < p_2 < p_3 < p_4\}$. By Theorem 6.3, π is not a permutation and by Lemma 4.6, $p_1, p_4 \in \pi(P)$. So we may assume that $p_2 \notin \pi(P)$. Now by Lemma 4.5, π has no flat blocks. So if $\pi(P) = \{p_1, p_4\}$, then M_π would have a horseshoe, implying that $3 \in \text{Per}(\pi)$. Therefore, $\pi(P) = \{p_1, p_3, p_4\}$.

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We claim that $\pi(p_1) \neq p_1$. If $\pi(p_1) = p_1$, then $\pi(p_2) = p_3$ or p_4. First suppose that $\pi(p_2) = p_3$. Since p_1 is in the orbit of a turning point of π, either $\pi(p_3) = p_1$ or $\pi(p_3) = p_4$ and $\pi(p_4) = p_1$. In either case there is a primitive closed walk of length 3 in M_π. This is a contradiction. By a similar argument, $\pi(p_2) = p_4$ leads to a contradiction. Therefore $\pi(p_1) \neq p_1$, i.e. $\pi(p_1) = p_3$ or p_4. Arguments similar to the ones above show that $\pi(p_1) \neq p_4$. Therefore, $\pi(p_1) = p_3$.

Now, if $\pi(p_2) = p_1$ then $V_1 \to V_1 \to V_2 \to V_1$ is a primitive closed walk of length 3, where V_1 is the block vertex corresponding to $\{p_1, p_2\}$ and V_2 is the block vertex corresponding to $\{p_2, p_3\}$. Therefore, $\pi(p_2) = p_4$. It follows similarly that $\pi(p_3) = p_3$ and $\pi(p_4) = p_1$. □

7. *Sharkovsky-largest period $r \cdot 2^k$, $r \geq 5$*

THEOREM 7.1. *The minimal combinatorial models for* 'Per$(f) = \{t : t \leq_S r \cdot 2^k\}$, $r \geq 5$, *r odd' are the simple cycles of length $r \cdot 2^k$.*

COROLLARY 7.2. *The minimal permutation models for* 'Per$(f) = \{t : t \leq_S r \cdot 2^k\}$, $r \geq 5$, *r odd' are the simple cycles of length $r \cdot 2^k$.*

Proof of Theorem 7.1. Let $r \geq 5$, r odd. Suppose first that π is a simple cycle of length $r \cdot 2^k$. Then by the remark following [**ALM**, Corollary 2.11.2], Per$(\pi) = \{t : t \leq_S r \cdot 2^k\}$. So to prove the theorem it suffices to show that if π is a minimal combinatorial model for 'Per$(f) = \{t : t \leq_S r \cdot 2^k\}$', then π is a simple cycle of length $r \cdot 2^k$.

Suppose then that π is a minimal combinatorial model for 'Per$(f) = \{t : t \leq_S r \cdot 2^k\}$'. It follows from the preceding paragraph that $\#\pi \leq r \cdot 2^k$. By Theorem 4.3, π is r&r-minimal. Since $3 \cdot 2^k \notin$ Per(π) and since a continuous self-map of a compact interval which has a horseshoe must have a periodic point of period 3, it follows from Theorem 2.6 that $M_\pi^{2^k}$ does not have a horseshoe. Hence, by Theorem 5.3, π has a block structure over a simple cycle of length 2^k.

By Theorem 3.5, either π is a cycle of length $r \cdot 2^k$ or there is a primitive closed walk of length $r \cdot 2^k$ in M_π. In the first case, it follows from [**BCop**, Theorem VII.11] that π is a simple cycle of length $r \cdot 2^k$. Hence, we may assume that there is a primitive closed walk W of length $r \cdot 2^k$ in M_π.

In light of Theorem 3.5 we have the following:

(∗) there is no primitive closed walk of length $s \cdot 2^k$, $1 < s < r$, s odd, in M_π.

We claim that every block in the block structure contains exactly r points. To prove the claim it suffices to show that if a block B has at most r points, then it has exactly r points. Let D_B be the subgraph of $M_\pi^{2^k}$ induced by the block vertices associated with the block B.

As in the proof of Theorem 6.3, the walk W passes through only block vertices. So from W, we obtain a closed walk W_B of length r in D_B.

If W_B contained only one vertex, then $M_\pi^{2^k}$ would have a horseshoe. So W_B contains at least two vertices. On the other hand, since there are at most $r - 1$ vertices in D_B, W_B passes through some vertex twice. It follows that W_B is the concatenation of two shorter closed walks, one of which has odd length. However, it follows from (∗) that if s is odd and $1 < s < r$, then there is no closed walk of length s in D_B which passes through two

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distinct vertices. Thus W_B is the concatenation of a closed walk of length 1 and a closed walk of length $r-1$, i.e. W_B is of the form

$$J_1 \to J_1 \to J_2 \to \cdots \to J_{r-1} \to J_1.$$

It now follows that $J_1, J_2, \ldots, J_{r-1}$ are distinct—otherwise there would be a closed walk of odd length s, $1 < s < r$, in D_B passing through two distinct vertices. In particular, there are at least $r-1$ distinct vertices in D_B. Hence, B contains at least r points and the claim is proved.

It follows from the preceding paragraph and (∗) that the blocks may be numbered $B_1, B_2, \ldots, B_{2^k}$, where $\pi(B_i) \subseteq B_{i+1}$ and $B_{2^k+1} = B_1$, in such a way that the closed walk W has the form

$$\begin{gathered} J_{1,1} \to J_{2,1} \to \cdots \to J_{2^k,1} \to \\ J_{1,1} \to J_{2,1} \to \cdots \to J_{2^k,1} \to \\ J_{1,2} \to J_{2,2} \to \cdots \to J_{2^k,2} \to \\ \cdots \to \cdots \\ J_{1,r-2} \to J_{2,r-2} \to \cdots \to J_{2^k,r-2} \to \\ J_{1,r-1} \to J_{2,r-1} \to \cdots \to J_{2^k,r-1} \to J_{1,1}, \end{gathered}$$

where for each $i = 1, 2, \ldots, 2^k$, the distinct block vertices associated with the block B_i are $J_{i,1}, J_{i,2}, \ldots, J_{i,r-1}$.

Recall that if $P = \{p_1 < p_2 < \cdots < p_{r\cdot 2^k}\}$, then the vertices V_i of M_π are labelled so that $V_i \to V_j$ if and only if either $\pi(p_i) \le p_j$ and $\pi(p_{i+1}) \ge p_{j+1}$ or $\pi(p_i) \ge p_{j+1}$ and $\pi(p_{i+1}) \le p_j$.

Using (∗) we have that arcs in W are the only arcs in M_π emanating from $J_{1,1}, J_{2,1}, \ldots, J_{2^k,1}$. In particular, $\{J_{1,1}, J_{1,2}\} = \{V_i, V_{i+1}\}$ for some i. We may assume that $J_{1,1} = V_i$ and $J_{1,2} = V_{i+1}$. It follows that $\pi^{2^k}\{p_i, p_{i+1}\} = \{p_i, p_{i+2}\}$.

We claim that $\pi^{2^k}(p_i) = p_{i+2}$ and $\pi^{2^k}(p_{i+1}) = p_i$. Suppose not, i.e. $\pi^{2^k}(p_i) = p_i$ and $\pi^{2^k}(p_{i+1}) = p_{i+2}$. Then $\pi^{2^k}(p_{i+2}) > p_i$, otherwise $M_\pi^{2^k}$ would have a horseshoe. Since there is a walk of length 2^k in M_π from $J_{1,2}$ to $J_{1,3}$, but no such walk from $J_{1,2}$ to $J_{1,4}, J_{1,5}, \ldots$ or $J_{1,r-1}$, we have $J_{1,3} = V_{i+2}$ and $\pi^{2^k}(p_{i+2}) = p_{i+3}$. In the same way, it follows that $B_1 = \{p_i, p_{i+1}, \ldots, p_{i+r-1}\}$, $J_{1,s} = V_{i+s-1}$ for $s = 1, 2, \ldots, r-1$ and $\pi^{2^k}(p_j) = p_{j+1}$ for $j = i, i+1, \ldots, i+r-2$. Since $\pi^{2^k}(p_{i+r-2}) = p_{i+r-1}$ and there is a walk of length 2^k from $J_{1,r-1} = V_{i+r-2}$ to $J_{1,1} = V_i$, it follows that there are walks of length 2^k from $J_{1,r-1}$ to each of the vertices $J_{1,1}, J_{1,2}, \ldots, J_{1,r-1}$. In particular, there is a primitive closed walk of length $3 \cdot 2^k$ in M_π. This establishes our claim that $\pi^{2^k}(p_i) = p_{i+2}$ and $\pi^{2^k}(p_{i+1}) = p_i$.

We claim next that if $k \ne 0$, then the arc $J_{1,2} \to J_{2,2}$ is the only arc in M_π emanating from $J_{1,2}$. If there were an arc $J_{1,2} \to J_{2,1}$, then $M_\pi^{2^k}$ would have a horseshoe. If there were any arcs $J_{1,2} \to J_{2,s}$, $s = 3, 4, \ldots, r-1$, it would contradict (∗). Similarly there is only one arc emanating from each of the vertices $J_{2,2}, J_{3,2}, \ldots, J_{2^k-1,2}$.

Now, each of the maps $\pi, \pi^2, \ldots, \pi^{2^k-1}$ is monotone on $\{p_i, p_{i+1}, p_{i+2}\}$, $\pi^{2^k}(p_i) = p_{i+2}$, and $\pi^{2^k}(p_{i+1}) = p_i$. By (∗) there cannot be a closed walk of length 2^k from $J_{1,2}$ to any of the vertices $J_{1,1}, J_{1,4}, J_{1,5}, \ldots, J_{1,r-1}$, so we must have $\pi^{2^k}(p_{i+2}) = p_{i-1}$.

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Similarly, we see that π is monotone on every block except B_{2^k}, p_{i+1} is the central point of B_1 and if $\theta = \pi^{2^k}|_{B_1}$, then

$$\theta^{r-2}(p_{i+1}) < \theta^{r-4}(p_{i+1}) < \cdots < \theta^3(p_{i+1}) < \theta(p_{i+1}) \\ < p_{i+1} < \theta^2(p_{i+1}) < \cdots < \theta^{r-3}(p_{i+1}) < \theta^{r-1}(p_{i+1}).$$

Finally, since there is a walk of length 2^k from $J_{1,r-1}$ to $J_{1,1}$ but by $(*)$ no such walk from $J_{1,r-1}$ to $J_{1,2}$, we must have $\theta^r(p_{i+1}) = p_{i+1}$.

Therefore, π is a simple cycle of length $r \cdot 2^k$. □

8. *Unimodal minimal combinatorial models*

Recall that a self-map π of a finite-ordered set $\{p_1 < p_2 < \cdots < p_n\}$ is *unimodal* if and only if for some m, $1 < m < n$, π is (not necessarily strictly) increasing on $\{p_1, p_2, \ldots, p_m\}$, (not necessarily strictly) decreasing on $\{p_m, p_{m+1}, \ldots, p_n\}$, but not constant on either set. Note that π is unimodal if and only if every π-weakly monotone map is unimodal.

For all the properties considered in this paper, except 'Per$(f) = \{t : t \leq_S 3 \cdot 2^k\}$, $k \geq 1$', the minimal combinatorial models are the simple cycles of the appropriate length. It is well known that there is a unique unimodal simple cycle of every length greater than 2. Using Theorem 6.4, it can be checked that there is a unique unimodal minimal combinatorial model for 'Per$(f) = \{t : t \leq_S 3 \cdot 2^k\}$, $k \geq 1$'.

These may be constructed inductively as follows. Let $\pi_1 = \begin{pmatrix} 1&2&3&4 \\ 3&4&3&1 \end{pmatrix}$. Assume that π_k is the unique unimodal minimal combinatorial model for 'Per$(f) = \{t : t \leq_S 3 \cdot 2^k\}$'. Let θ_{k+1} be the 'unimodal Štefan square root' of π_k. (See [**BkhC**] for how to take unimodal square roots of unimodal self-maps of finite-ordered sets.) Then $\#\theta_{k+1} = 3 \cdot 2^k + 2$, the turning point of θ_{k+1} is pre-periodic, but not periodic, and it has a unique pre-image under θ_{k+1}. The pre-image has no pre-images. Restricting θ_{k+1} to its domain with the pre-image deleted produces a unimodal map π_{k+1} with $\#\pi_{k+1} = 3 \cdot 2^k + 1$ and Per$(\pi_{k+1}) = \{t : t \leq_S 3 \cdot 2^{k+1}\}$. By Theorem 6.4, π_{k+1} is a minimal combinatorial model.

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