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On Teter rings

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We give a new characterization, in the equicharacteristic case, of Teter rings by using Macaulay inverse systems. We extend the previous characterizations due to Teter, to Huneke and Vraciu and to Ananthnarayan *et al.*, to any characteristic of the ground field and remove the hypothesis on the socle ideal. We construct and describe the variety parametrizing Teter covers and we show how to check if an Artin ring is Teter. If this is the case, we show how to compute a Teter cover.

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1. Introduction

Let $\boldsymbol{k}$ be an arbitrary field. Let $R = \boldsymbol{k}[\![x_1, \dots, x_n]\!]$ be the ring of the formal series with maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$ and let $S = \boldsymbol{k}[y_1, \dots, y_n]$ be a polynomial ring. We denote by $\mathfrak{m} = (x_1, \dots, x_n)$ the homogeneous maximal ideal of S. We assume that $n \geqslant 1$. Given a local ring $A = R/I$ with maximal ideal $\mathfrak{n}$, we denote by $E_A(\boldsymbol{k})$ the injective hull of the residue field $\boldsymbol{k}$.

In [22, lemma 1.1 and theorem 2.3] Teter characterized the Artin local rings $A = R/I$ that are of the form $G/\mathrm{soc}(G)$, where G is a Gorenstein ring and $\mathrm{soc}(G)$ is its socle. From now on we will call such rings Teter rings [21]. Note that from the proof of [22, theorem 2.3] we may assume that G is a quotient of R. Teter characterized such rings as those for which there is an isomorphism of A-modules $\phi\colon \mathfrak{n} \to \mathrm{Hom}_A(\mathfrak{n}, E_A(\boldsymbol{k}))$ such that $\phi(x)(y) = \phi(y)(x)$. Huneke and Vraciu improved this result by proving that A is Teter if and only if there is an epimorphism $\varphi\colon \omega_A \to \mathfrak{n}$, provided $\mathrm{char}(\boldsymbol{k}) \neq 2$ and $\mathrm{soc}(A) \subset \mathfrak{n}^2$ [15, theorem 2.4]. Ananthnarayan *et al.* improved the Huneke–Vraciu characterization by removing the hypothesis $\mathrm{soc}(A) \subset \mathfrak{n}^2$ [3, theorem 4.7]. For more results on Teter rings and related problems, see, for example, [1, 2, 21].

The main aim of this paper is to give a characterization of Teter rings $A = R/I$ in terms of their Macaulay inverse system $I^{\perp}$ (see theorem 3.4). See § 2 for the basic facts on Macaulay inverse systems. Using a Macaulay inverse system device we generalize the characterization of Ananthnarayan *et al*. to any characteristic of the ground field, and we give an explicit description of the above isomorphism, ϕ. As a corollary of theorem 3.4, in proposition 3.6 we characterize the quotients $A = R/\mathfrak{m}^r$ that are Teter, simplifying and improving [22, corollary 2.2] and [3, corollary 4.4]. We show that a Gorenstein ring A is Teter if and only if its embedding dimension is 1 (proposition 3.7). In proposition 3.12 we prove that Teter rings with a compressed Teter cover are the level Artin rings of maximal Cohen–Macaulay type. In corollary 3.13 we study compressed Teter rings with a compressed Teter cover.

Section 4 is devoted to the construction of the Teter variety $\mathrm{TC}(A)$, defined as the variety parametrizing the set of Teter covers of a given Teter ring A. We prove that $\mathrm{TC}(A)$ is a non-empty open Zariski subset of a linear variety of a projective space over the ground field $\boldsymbol{k}$ (proposition 4.2). In proposition 4.5 we give a method for checking if an Artin ring is Teter and, if this is the case, to compute a Teter cover. This method is fully effective. In fact, in [8] we implemented such a method in a SINGULAR library [6].

We conclude in § 5 by considering some special classes of Artin rings for which we know a full classification: we study stretched Teter rings, and consider almost stretched and short Teter rings.

2. Preliminaries

Let $A = R/I$ be an Artin ring with maximal ideal $\mathfrak{n}$. The Hilbert function of A is the numerical function $\mathrm{HF}_A\colon \mathbb{N} \to \mathbb{N}$ defined by $\mathrm{HF}_A(i) = \dim_{\boldsymbol{k}}(\mathfrak{n}^i/\mathfrak{n}^{i+1})$, $i \geqslant 0$. The socle degree of A is the last integer s such that $\mathrm{HF}_A(s) \neq 0$. The socle of A is the $\boldsymbol{k}$-vector subspace of A $\mathrm{soc}(A) = (0 :_A \mathfrak{n})$, and the Cohen–Macaulay type of A is $\tau(A) = \dim_{\boldsymbol{k}}(\mathrm{soc}(A))$. Recall that A is Gorenstein if and only if $\tau(A) = 1$.

The polynomial ring S can be considered as an R-module with two linear structures by derivation and by contraction. If $\mathrm{char}(\boldsymbol{k}) = 0$, the R-module structure of S by derivation is defined by

$$R \times S \to S, \qquad (x^\alpha, y^\beta) \mapsto x^\alpha \circ y^\beta = \begin{cases} \dfrac{\beta!}{(\beta-\alpha)!} y^{\beta-\alpha}, & \beta \geqslant \alpha, \\ 0, & \text{otherwise,} \end{cases}$$

where, for all $\alpha, \beta \in \mathbb{N}^n$, $\alpha! = \prod_{i=1}^n \alpha_i!$. If $\mathrm{char}(\boldsymbol{k}) \geqslant 0$, the R-module structure of S by contraction is defined by

$$R \times S \to S, \qquad (x^\alpha, y^\beta) \mapsto x^\alpha \circ y^\beta = \begin{cases} y^{\beta-\alpha}, & \beta \geqslant \alpha, \\ 0, & \text{otherwise.} \end{cases}$$

From now on we write

$$x_i \circ F = \frac{\partial F}{\partial y_i} = \partial_{y_i} F, \quad i = 1, \dots, n.$$

For all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we denote by $|\alpha|$ the total degree of α, i.e. $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

It is easy to prove that for any field $\boldsymbol{k}$ there is an R-module homomorphism

$$\sigma\colon (S,\mathrm{der}) \to (S,\mathrm{cont})$$
$$y^\alpha \mapsto \alpha! y^\alpha.$$

If $\mathrm{char}(\boldsymbol{k}) = 0$, then σ is an isomorphism of R-modules.

The R-module S is the injective hull $E_R(\boldsymbol{k})$ of the R-module $\boldsymbol{k}$.

Theorem 2.1 (Gabriel [14]). *If $\boldsymbol{k}$ is of characteristic 0, then $E_R(\boldsymbol{k}) \cong (S,\mathrm{der}) \cong (S,\mathrm{cont})$. If $\boldsymbol{k}$ is of positive characteristic, then $E_R(\boldsymbol{k}) \cong (S,\mathrm{cont})$.*

Since the characteristic of the ground field $\boldsymbol{k}$ is arbitrary, from now on we will use the structure of S as an R-module defined by the contraction.

Given a commutative ring R, we denote by R_mod (respectively, R_mod.noeth, R_mod.Artin) the category of R-modules (respectively, the category of noetherian R-modules, the category of Artin R-modules).

Definition 2.2. Given an R-module M, the Matlis dual of M is

$$M^\vee = \mathrm{Hom}_R(M, E_R(\boldsymbol{k})).$$

We write $(\cdot)^\vee = \mathrm{Hom}_R(\cdot, E_R(\boldsymbol{k}))$.

Theorem 2.3 (Matlis duality). *The functor $(\cdot)^\vee$ is a contravariant, additive and exact and defines anti-equivalence between R_mod.noeth and R_mod.Artin (respectively, between R_mod.Artin and R_mod.noeth). It holds that $(\cdot)^\vee \circ (\cdot)^\vee$ is the identity functor of R_mod.noeth (respectively, R_mod.Artin). Furthermore, if M is an R-module of finite length, then $\mathrm{Length}_R(M^\vee) = \mathrm{Length}_R(M)$.*

From the previous result we can recover the classical result of Macaulay [18] for the power series ring (see [13, 17]). If $I \subset R$ is an ideal, then $(R/I)^\vee$ is the sub-R-module of S:

$$I^\perp = \{g \in S \mid I \circ g = 0\}.$$

This is the Macaulay inverse system of I. Given a sub-R-module M of S, the module $M^\vee$ is an ideal of R:

$$M^\perp = \{f \in R \mid f \circ g = 0 \text{ for all } g \in M\}.$$

Proposition 2.4 (Macaulay's duality). *There is an order-reversing bijection $\perp$ between the set of finitely generated sub-R-submodules of S and the set of $\mathfrak{m}$-primary ideals of R given by the condition that if M is a submodule of S, then $M^\perp = (0 :_R M)$, and $I^\perp = (0 :_S I)$ for an ideal $I \subset R$. Moreover, $A = R/I$ is Gorenstein of socle degree s if and only if $I^\perp$ is a cyclic R-module generated by a polynomial of degree s.*

We denote by $S_{\leqslant i}$ (respectively, $S_{<i}$, S_i), $i \in \mathbb{N}$, the $\boldsymbol{k}$-vector space of polynomials of S of degree less than or equal to (respectively, less than, equal to) i, and we consider the following $\boldsymbol{k}$-vector space:

$$(I^\perp)_i := \frac{I^\perp \cap S_{\leqslant i} + S_{<i}}{S_{<i}}.$$

Our next result is well known by the specialists. Here we present a new proof using Matlis duality.

PROPOSITION 2.5 (Elias [7]). *Let $A = R/I$ be a local ring with maximal ideal $\mathfrak{n}$. For all $i \geqslant 0$, it holds that*

$$\mathrm{HF}_A(i) = \dim_{\boldsymbol{k}}(I^{\perp})_i.$$

Proof. Let us consider the following natural exact sequence of R-modules:

$$0 \to \frac{\mathfrak{n}^i}{\mathfrak{n}^{i+1}} \to \frac{A}{\mathfrak{n}^{i+1}} \to \frac{A}{\mathfrak{n}^i} \to 0.$$

Dualizing this sequence, we get

$$0 \to (I + \mathfrak{m}^i)^{\perp} \to (I + \mathfrak{m}^{i+1})^{\perp} \to \left(\frac{\mathfrak{n}^i}{\mathfrak{n}^{i+1}}\right)^{\vee} \to 0$$

so we get the following sequence of $\boldsymbol{k}$-vector spaces:

$$\left(\frac{\mathfrak{n}^i}{\mathfrak{n}^{i+1}}\right)^{\vee} \cong \frac{(I + \mathfrak{m}^{i+1})^{\perp}}{(I + \mathfrak{m}^i)^{\perp}} = \frac{I^{\perp} \cap S_{\leqslant i}}{I^{\perp} \cap S_{\leqslant i-1}} \cong \frac{I^{\perp} \cap S_{\leqslant i} + S_{<i}}{S_{<i}}.$$

From theorem 2.3 we get the claim. □

Given an R-module M, we denote by $\mu_R(M)$ the minimal number of generators of M.

PROPOSITION 2.6 (Elias [7]). *Let $A = R/I$ be an Artin local ring. Then*

$$\mathrm{soc}(A)^{\vee} = \frac{I^{\perp}}{\mathfrak{m} \circ I^{\perp}}.$$

In particular,the Cohen–Macaulay type of A is

$$\tau(A) = \dim_{\boldsymbol{k}}(I^{\perp}/\mathfrak{m} \circ I^{\perp}) = \mu_R(I^{\perp}).$$

Proof. Let us consider the exact sequence of R-modules

$$0 \to \mathrm{soc}(A) = (0 :_A \mathfrak{n}) \to A \xrightarrow{(x_1,\dots,x_n)} A^n.$$

Dualizing this sequence, we get

$$(I^{\perp})^n \xrightarrow{\sigma} I^{\perp} \to \mathrm{soc}(A)^{\vee} \to 0,$$

where

$$\sigma(f_1, \dots, f_n) = \sum_{i=1}^{n} x_i \circ f_i.$$

Hence,

$$\mathrm{soc}(A)^{\vee} = \frac{I^{\perp}}{(x_1, \dots, x_n) \circ I^{\perp}} = \frac{I^{\perp}}{\mathfrak{m} \circ I^{\perp}}.$$

From this fact and theorem 2.3 we get $\tau(A) = \dim_{\boldsymbol{k}}(\mathrm{soc}(A)) = \dim_{\boldsymbol{k}}(\mathrm{soc}(A)^{\vee}) = \mu(I^{\perp})$. □

3. Teter rings

We know that any Artin ring A is quotient of an Artin Gorenstein ring G (see, for example, [1, proposition 2.1]). In fact, G can be taken as Nagata's idealization $G = A \ltimes \omega_A$ [4, theorem 3.3.6]. Note that if the embedding dimension of A is n, then the embedding dimension of G is $n + \tau(A)$. In the next result we prove that if A is the quotient of a Gorenstein ring by its socle, then we may assume that the embedding dimension of G equals the embedding dimension of A.

PROPOSITION 3.1. *Let $A = R/I$ be an Artin ring. Then*

(i) *there exists an Artin Gorenstein ring $G = R/J$ of embedding dimension n such that A is a quotient of G.*

Let G be an Artin Gorenstein ring such that $A \cong G/\mathrm{soc}(G)$. Then

(ii) *if $I \subset \mathfrak{m}^2$, then the embedding dimension of A and G is n.*

Moreover, if $G = R/J$ is an Artin Gorenstein ring such that $A \cong G/\mathrm{soc}(G)$, then

(iii) $I = (J :_R \mathfrak{m})$ *and* $I^{\perp} = \mathfrak{m} \circ J^{\perp}$.

Proof.
(i) If s is the socle degree of A, then $I^{\perp}$ is generated by polynomials of degree at most s (proposition 2.4). Hence, $I^{\perp} \subset S_{\leqslant s}$. Since $S_{\leqslant s} \subset \langle y_1^s \cdots y_n^s \rangle$, the ideal $J = \mathrm{Ann}_{\mathrm{R}} \langle y_1^s \cdots y_n^s \rangle$ satisfies the claim.

(ii) If $I \subset \mathfrak{m}^2$, then the embedding dimension of A is n. Hence, we only have to prove that $\mathrm{soc}(G) \subset \mathfrak{m}_G^2$. Here $\mathfrak{m}_G$ is the maximal ideal G. Assume that $\mathrm{soc}(G) \nsubseteq \mathfrak{m}_G^2$. Since G is Gorenstein, we deduce that its socle degree is 1. Then the Hilbert function of G is $\mathrm{HF}_G = \{1, 1\}$ and the Hilbert function of A is $\mathrm{HF}_A = \{1\}$. Therefore, $I = \mathfrak{m}$. This is not possible, since we assumed that $I \subset \mathfrak{m}^2$.

(iii) The first identity follows from a direct computation, so $I^{\perp} = (J :_R \mathfrak{m})^{\perp} = \mathfrak{m} \circ J^{\perp}$. □

In [21] Teter rings are defined without any restriction on the embedding dimension. From proposition 3.1 we may define Teter rings by assuming that the rings A and G have the same embedding dimension.

DEFINITION 3.2. An Artin ring $A = R/I$ is Teter if $I \subset \mathfrak{m}^2$ and there exists an Artin Gorenstein ring $G = R/J$ such that $A \cong G/\mathrm{soc}(G)$. We say that G is a Teter cover of A.

If A is a Teter ring and G is a Teter cover, from the definition of the Hilbert function we have that

$$\mathrm{HF}_A(i) = \begin{cases} \mathrm{HF}_G(i), & i = 0, \ldots, s-1, \\ 0, & i \geqslant s. \end{cases}$$

Note that if $G = R/J$ is an Artin Gorenstein ring, then $A = G/\mathrm{soc}(G)$ is trivially a Teter ring; here $A = R/I$ with $I = (J :_R \mathfrak{m})$ and $\mathrm{soc}(G) = I/J$. See example 4.6 and proposition 3.6 for examples of non-Teter rings.

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Example 3.3 (example of a Teter ring). The ring $A = R/\mathfrak{m}^2$ is Teter because it is a quotient of the Gorenstein ring $G = R/J$ by its socle, where $J = \mathrm{Ann}_{\mathrm{R}}\langle y_1^2 + \cdots + y_n^2\rangle$. In fact, G is a Gorenstein ring of socle degree 2 because $J^\perp = \langle y_1^2 + \cdots + y_n^2\rangle$ is cyclic and generated by a polynomial of degree 2. Moreover, J is generated by $x_i x_j$, $1 \leqslant i < j \leqslant n$, and $\lambda_1 x_1^2 + \cdots + \lambda_n x_n^2$ such that $\lambda_i \in \boldsymbol{k}$ and $\lambda_1 + \cdots + \lambda_n = 0$. Hence, the coset of x_1^2 in G generates the socle of G, and we get that $J + (x_1^2) = \mathfrak{m}^2$. Note that we can pick as the generator of $\mathrm{soc}(G)$ any square x_i^2, $i = 1, \dots, n$. See proposition 3.6 for a complete characterization of Teter rings $R/\mathfrak{m}^r$.

In the next result we present a characterization of Teter rings in terms of their inverse systems and we generalize the characterization due to Ananthnarayan *et al.* to any characteristic of the ground field.

Recall that if $A = R/I$ is an Artin ring, then A admits a canonical module ω_A that can be identified to $E_A(\boldsymbol{k}) = (0 :_{E_R(\boldsymbol{k})} I) = I^\perp$ [4, theorem 3.3.4 (a)].

Theorem 3.4. *Let $A = R/I$ be an Artin ring with maximal ideal $\mathfrak{n}$ and socle degree $s - 1 \geqslant 1$. Then the following conditions are equivalent:*

(i) *A is a Teter ring;*

(ii) *there exists a degree-s polynomial $F \in S$ such that $I^\perp = \langle \partial_{y_1} F, \dots, \partial_{y_n} F\rangle$;*

(iii) *there exists an epimorphism of A-modules $\omega_A \to \mathfrak{n}$.*

In particular, if A is a Teter ring, then the Cohen–Macaulay type of A is n.

Proof. Suppose that A is a Teter ring of socle degree $s - 1$. Then $A = G/\mathrm{soc}(G)$, where $G = R/J$ is an Artin Gorenstein ring of socle degree s. From Macaulay correspondence, $J^\perp$ is a cyclic R-module generated by a polynomial F of degree s: $J^\perp = \langle F\rangle$. From proposition 3.1(iii) we get (ii):

$$I^\perp = \mathfrak{m} \circ J^\perp = \mathfrak{m} \circ \langle F\rangle = \langle \partial_{y_1} F, \dots, \partial_{y_n} F\rangle.$$

Assume now that (ii) holds. Let $A = R/I$ be an Artin ring with

$$I^\perp = \langle \partial_{y_1} F, \dots, \partial_{y_n} F\rangle,$$

where $F \in S$ is a degree-s polynomial. Let $G = R/J$ be the Gorenstein ring defined by $J = \mathrm{Ann}_{\mathrm{R}}\langle F\rangle$. Hence, we have the following exact sequence of R-modules:

$$0 \to I^\perp \to J^\perp = \langle F\rangle \to \boldsymbol{k} \to 0.$$

From this we get that $A = G/\mathrm{soc}(G)$, i.e. A is a Teter ring.

Let A be a Teter ring with Teter cover G. Then we have an exact sequence of R-modules,

$$0 \to \boldsymbol{k} \to G \to A \to 0.$$

If we denote by $\sharp$ the functor $\mathrm{Hom}_G(\cdot, E_G(\boldsymbol{k}))$, we get the exact sequence of R-modules

$$0 \to A^\sharp = E_A(\boldsymbol{k}) \to G^\sharp \cong G \to \boldsymbol{k} \to 0.$$

From this we get that $\omega_A = E_A(\boldsymbol{k}) \cong \mathfrak{m}_G$, where $\mathfrak{m}_G$ is the maximal ideal of G. Composing this isomorphism with the natural projection $\mathfrak{m}_G \to \mathfrak{n}$ yields (iii).

Let us assume that there exists an epimorphism of A-modules $\phi\colon \omega_A \cong E_A(\boldsymbol{k}) \to \mathfrak{n}$. Since $\mathrm{Length}_A(E_A(\boldsymbol{k})) = \mathrm{Length}_A(A)$ and $\mathrm{Length}_A(\mathfrak{n}) = \mathrm{Length}_A(A) - 1$, we deduce that $\ker(\phi) \cong \boldsymbol{k}$. Hence, we have an exact sequence of A-modules

$$0 \to \boldsymbol{k} \to E_A(\boldsymbol{k}) \xrightarrow{\phi} \mathfrak{n} \to 0.$$

Dualizing this sequence, we get the exact sequence of A-modules

$$0 \to \mathfrak{n}^\vee \xrightarrow{\phi^\vee} E_A(\boldsymbol{k})^\vee = A \to \boldsymbol{k} \to 0.$$

In particular, we have an isomorphism of A-modules:

$$\varphi = (\phi^\vee)^{-1}\colon \mathfrak{n} \xrightarrow{\cong} \mathfrak{n}^\vee.$$

On the other hand, dualizing the natural exact sequence

$$0 \to \mathfrak{n} \to A \to \boldsymbol{k} \to 0,$$

we get the isomorphism of A-modules

$$\begin{aligned} \beta\colon \frac{I^\perp}{\boldsymbol{k}} &\to \mathfrak{n}^\vee \\ \bar{z} &\mapsto \beta(\bar{z})\colon \mathfrak{n} \to S \\ &\qquad\qquad x \mapsto x \circ z. \end{aligned}$$

Then φ can be factorized throughout β, i.e. $\varphi = \beta \circ \alpha$ with the isomorphism of A-modules

$$\alpha = \beta^{-1}\varphi\colon \mathfrak{n} \to \frac{I^\perp}{\boldsymbol{k}}.$$

Furthermore, if $x \in \mathfrak{n}$ and $y \in \mathfrak{n}$, then

$$\varphi(x)(y) = y \circ \alpha(x) = \alpha(xy) = x \circ \alpha(y) = \varphi(y)(x).$$

From the main result of [22] we get that A is Teter.

If A is a Teter ring, then from (ii) and proposition 2.6 we get that the Cohen–Macaulay type of A is at most n. By (iii) we get that the Cohen–Macaulay type is at least n, so the Cohen–Macaulay type of A is n. □

Remark 3.5. Teter [22] proves that a ring A is Teter if and only if there is an isomorphism of R-modules $\phi\colon \mathfrak{n} \to \mathfrak{n}^\vee$ such that $\phi(x)(y) = \phi(y)(x)$ for all $x, y \in \mathfrak{n}$. We can explicitly describe an isomorphism of A-modules $\phi\colon \mathfrak{n} \to \mathfrak{n}^\vee$. In fact, let G be a Teter cover of A and let F be a generator of the inverse system of G. The morphism

$$\begin{aligned} \phi\colon \mathfrak{n} &\to \mathfrak{n}^\vee \cong \frac{I^\perp}{\boldsymbol{k}} \\ a &\mapsto \overline{a \circ F} \end{aligned}$$

is an isomorphism such that $\phi(x)(y) = \phi(y)(x)$.

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In example 3.3 we prove that $R/\mathfrak{m}^2$ is Teter. In the next result we characterize the powers of the maximal ideal defining Teter rings, simplifying and improving [22, corollary 2.2] and [3, corollary 4.4].

Proposition 3.6. *Let r be a positive integer. Then $R/\mathfrak{m}^r$ is Teter if and only if $n = 1$ or $r \leqslant 2$.*

Proof. Let $R/\mathfrak{m}^r$ be a Teter ring with $r \geqslant 1$. Then $(\mathfrak{m}^r)^\perp = \langle \partial_{y_1} F, \ldots, \partial_{y_n} F \rangle$ for some degree-r polynomial F (theorem 3.4). On the other hand, we know that $(\mathfrak{m}^r)^\perp = S_{\leqslant r-1}$, so

$$\binom{r-1+n-1}{n-1} = \mu_R(S_{r-1}) = \mu_R((\mathfrak{m}^r)^\perp) \leqslant n.$$

From this inequality we get that $n = 1$ or $r \leqslant 2$.

If $n = 1$, then $R = \boldsymbol{k}[\![x_1]\!]$, $\mathfrak{m}^r = (x_1^r)$ and, trivially, $A = R/\mathfrak{m}^r$ is Teter. The case $r \leqslant 2$ is shown in example 3.3. □

A natural class of Teter rings could be the class of Gorenstein and level rings. In the next result we prove that Teter Gorenstein rings are of embedding dimension 1.

Proposition 3.7. *Let A be an Artin Gorenstein ring. Then A is a Teter ring if and only if $n = 1$.*

Proof. Assume that A is of socle degree t. If $n = 1$, then $A = \boldsymbol{k}[\![x_1]\!]/(x_1^{t+1})$. Hence, $G = \boldsymbol{k}[\![x_1]\!]/(x_1^{t+2})$ is a Gorenstein ring with socle generated by the coset of x_1^{t+1}. Since $A = G/(x_1^{t+1})$, we get that A is a Teter ring.

If A is a Teter ring, then its Cohen–Macaulay type is n (theorem 3.4). On the other hand, we assumed that A is Gorenstein, so its Cohen–Macaulay type is 1. Hence, we get that $n = 1$. □

Remark 3.8. Let A be a Teter ring such that its associated graded ring $\mathrm{gr}_{\mathfrak{n}}(A)$ is Gorenstein. In particular, if A is an Artin Gorenstein ring, then $n = 1$ (proposition 3.7). Hence, $\mathrm{gr}_{\mathfrak{n}}(A)$ is Gorenstein if and only if $n = 1$.

Next we will define level rings as a natural generalization of Gorenstein rings.

Definition 3.9. An Artin ring A with socle degree s is level if $\mathrm{soc}(A) = \mathfrak{n}^s$.

Given a polynomial $H \in S$ of degree l we denote by $\mathrm{top}(H)$ the degree-l form of H.

Proposition 3.10 (De Stefani [5, proposition 2.2]). *Let $A = R/I$ be an Artin ring of socle degree s and Cohen–Macaulay type t. Then A is level if and only if $I^\perp$ is generated by t polynomials $H_1, \ldots, H_t \in S$ such that $\deg(H_i) = s$ for $i = 1, \ldots, t$, and the homogeneous forms $\mathrm{top}(\mathrm{H}_1), \ldots, \mathrm{top}(\mathrm{H}_t)$ are $\boldsymbol{k}$-linear independent.*

Compressed Artin rings are those Artin rings with the maximal Hilbert function among the Artin quotients of R with a given socle degree and socle type. We avoid defining the socle type here because we will use the following equivalent definition of a compressed ring for level Artin rings [16, definition 2.4.B].

Definition 3.11. A local level Artin ring A of socle degree s and Cohen–Macaulay type τ is compressed if

$$\mathrm{HF}_A(i) = \min\{\dim_{\boldsymbol{k}}(S_i), \tau \dim_{\boldsymbol{k}}(S_{s-i})\}$$

for all $i = 0, \dots, s$.

Theorem 3.12. *Let $A = R/I$ be a Teter ring of socle degree $s-1$ and let $G = R/J$ be a Teter cover of $A = R/I$. Then A is a level Artin ring if and only if $\mathrm{HF}_A(s-1) = n$. In particular, if G is compressed, then A is a level Artin ring of Cohen–Macaulay type n.*

Proof. If A is a level Artin ring, then its Cohen–Macaulay type is n (theorem 3.4). Hence, $\mathrm{HF}_A(s-1) = n$.

Assume now that $\mathrm{HF}_A(s-1) = n$. Since G is of socle degree s, the inverse system $J^\perp$ is generated by a degree-s polynomial F and $I^\perp = \langle \partial_{y_1} F, \dots, \partial_{y_n} F\rangle$ (theorem 3.4). In particular, $\deg(\partial_{x_i} F) \leqslant s-1$, $i = 1, \dots, n$. Since $\mathrm{HF}_A(s-1) = n$ we have that

$$\begin{aligned} n = \mathrm{HF}_A(s-1) &= \dim_{\boldsymbol{k}}((I^\perp)_{s-1}) \\ &= \dim_{\boldsymbol{k}}\left(\frac{I^\perp \cap S_{\leqslant s-1} + S_{<s-1}}{S_{<s-1}}\right) \\ &= \dim_{\boldsymbol{k}}\left(\frac{I^\perp + S_{<s-1}}{S_{<s-1}}\right). \end{aligned}$$

Observe that $\{\partial_{y_1} F, \dots, \partial_{y_n} F\}$ generates $I^\perp$ as an R-module. Then $I^\perp$ is generated as $\boldsymbol{k}$-vector space by $\{x^L \circ \partial_{y_1} F, \dots, x^L \circ \partial_{y_n} F, L \in \mathbb{N}^n\}$. If $|L| \geqslant 1$, then $x^L \circ \partial_{x_i} F \in S_{<s-1}$ for $i = 1, \dots, n$. Then the cosets of $\{\partial_{y_1} F, \dots, \partial_{y_n} F\}$ are a system of generators of $I^\perp + S_{<s-1}/S_{<s-1}$ as a $\boldsymbol{k}$-vector space. Since this vector space is of dimension n, we get that the cosets of $\{\partial_{y_1} F, \dots, \partial_{y_n} F\}$ form a $\boldsymbol{k}$-basis of $I^\perp + S_{<s-1}/S_{<s-1}$, so $\mathrm{top}(\partial_{y_1} F), \dots, \mathrm{top}(\partial_{y_n} F)$ are linear independent forms of degree $s-1$. Then A is level of socle degree $s-1$ and maximal Cohen–Macaulay type n (proposition 3.10).

If G is compressed, then $n = \mathrm{HF}_G(s-1) = \mathrm{HF}_A(s-1)$. From the first part of the statement we get the claim. □

Corollary 3.13. *Let A be a Teter ring of socle degree $s-1$, and let G be a Teter cover of A. Assume that G is compressed. Then A is a level ring with Cohen–Macaulay type n and*

(i) *if $2 \leqslant s \leqslant 4$, then A is compressed,*

(ii) *if $s \geqslant 5$, then A is compressed if and only if $n = 1$.*

Proof. If G is compressed, then A is a level ring of maximal Cohen–Macaulay type n by theorem 3.12.

Assume that $2 \leqslant s \leqslant 4$. Then the Hilbert function of G is $\{1, n, 1\}$, $\{1, n, n, 1\}$ or $\{1, n, \binom{n+1}{2}, n, 1\}$. From this we can compute the Hilbert function of A and use it to prove that A is compressed.

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Assume now that A and G are compressed and $s \geqslant 5$. From the fact that G is compressed, $s \geqslant 5$ and that G is a Teter cover of A we get

$$\binom{n+1}{2} = \mathrm{HF}_G(s-2) = \mathrm{HF}_A(s-2).$$

Since A is compressed of Cohen–Macaulay type n and $s \geqslant 5$ we deduce

$$\mathrm{HF}_A(s-2) = \min\{\dim_{\boldsymbol{k}}(S_{s-2}), n\dim_{\boldsymbol{k}}(S_1)\} = n^2.$$

From the above identities we have $\binom{n+1}{2} = n^2$, so $n = 1$. If $n = 1$, the claim is trivial. □

Note that example 3.3 provides an example of a level Teter ring $A = R/\mathfrak{m}^2$ with a compressed Gorenstein Teter cover.

4. Teter variety

The aim of this section is to study the family of all Teter cover rings of a given Teter ring A. We also present a procedure for testing if an Artin ring is Teter and, if this is the case, to compute a Teter cover.

Let $A = R/I$ be a Teter ring of length e and socle degree $s-1$. Let $G = R/J$ be a Teter cover of A. Then $J^\perp = \langle F\rangle$ with F a degree-s polynomial and $I^\perp = \langle \partial_{y_1}F, \dots, \partial_{y_n}F_n\rangle$ (theorem 3.4). Since $\mathrm{Length}_R(G) = e+1$ we have that $\langle F\rangle/I^\perp$ is a one-dimensional subspace of $W_I := S_{\leqslant s}/I^\perp$. Therefore, G defines a closed point $p_G = [\langle F\rangle/I^\perp]$ of the projective space $\mathbb{P}^N_{\boldsymbol{k}} = \mathbb{P}_{\boldsymbol{k}}(W_I)$ with $N = \binom{n+s}{s} - e - 1$.

Definition 4.1. We denote by $\mathrm{TC}(A) \subset \mathbb{P}^N_{\boldsymbol{k}}$ the set of all closed points p_G, where G is a Teter cover of A; $\mathrm{TC}(A)$ is the Teter variety of A.

Proposition 4.2. *The Teter variety* $\mathrm{TC}(A)$ *of a Teter ring* A *is a non-empty Zariski open subset of a linear subvariety of* $\mathbb{P}^N_{\boldsymbol{k}}$. *In particular,* $\mathrm{TC}(A)$ *is an irreducible and non-singular variety of* $\mathbb{P}^N_{\boldsymbol{k}}$.

Proof. Assume that the socle degree of A is $s-1$. Let $p = [V/I^\perp]$ be a closed point of $\mathbb{P}^N_{\boldsymbol{k}}$ with V a $\boldsymbol{k}$-vector space of dimension $e+1$. Then $V = \langle H\rangle_{\boldsymbol{k}} + I^\perp$, where H is a degree at most s polynomial of S. From theorem 3.4 we deduce that the following conditions are equivalent to having $p = p_G \in \mathrm{TC}(A)$, with $G = R/\operatorname{Ann}_{\mathrm{R}}(H)$,

(i) $\deg(H) = s$, and

(ii) $\mathfrak{m} \circ H = I^\perp$.

Note that the above conditions are independent of the representative we choose for the base of the quotient $V/I^\perp$. In fact, let H' be a polynomial of S such that its coset in $V/I^\perp$ defines a $\boldsymbol{k}$-basis. Then $H' = \lambda H + \alpha$, where $\lambda \in \boldsymbol{k}\setminus\{0\}$ and $\alpha \in I^\perp$. Since $I^\perp \subset S_{\leqslant s-1}$ we get that $\deg(H') = \deg(H) = s$. From the identities

$$\mathfrak{m}\circ H' + \mathfrak{m}\circ I^\perp = \mathfrak{m}\circ H + \mathfrak{m}\circ I^\perp = I^\perp$$

and Nakayama's lemma, we deduce $\mathfrak{m}\circ H' = I^\perp$.

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Let U be the Zariski open set of points $p = [V/I^{\perp}]$ of $\mathbb{P}^N_{\boldsymbol{k}}$ such that H is of degree s, where $V = \langle H\rangle_{\boldsymbol{k}} + I^{\perp}$.

Let $a_1, \dots, a_e$ be a $\boldsymbol{k}$-basis of $I^{\perp}$. Then $\mathfrak{m} \circ H \subset I^{\perp}$ if and only if

$$\dim_{\boldsymbol{k}} \langle \partial_{x_i} H, a_1, \dots, a_e\rangle_{\boldsymbol{k}} = e$$

for $i = 1, \dots, n$. Since the above conditions can be expressed as the vanishing of all maximal minors of the matrix defined by the coordinates of $\partial_{x_i} H, a_1, \dots, a_e$, the above conditions define a linear subvariety L of $\mathbb{P}^N_{\boldsymbol{k}}$.

On the other hand, by the lower semi-continuity of the dimension of $\boldsymbol{k}$-vector spaces there is a Zariski open set $W \subset U \cap L$ such that for all points $p \in W$ it holds that $dim_{\boldsymbol{k}}(\mathfrak{m} \circ \langle H\rangle) \geqslant e$, and then $\mathfrak{m} \circ \langle H\rangle = I^{\perp}$. Therefore, since for all $p \in W$ we have that $\deg(H) = s$ and $\mathfrak{m} \circ H = I^{\perp}$, we get $\mathrm{TC}(A) = W$. □

Note that the proof of the last result shows how to compute Teter variety. In the next example we show how to compute $\mathrm{TC}(R/\mathfrak{m}^2)$.

Example 4.3 (example of a $\mathrm{TC}(A)$). Let us consider the Teter ring $A = R/\mathfrak{m}^2$ with $R = \boldsymbol{k}[\![x_1, x_2]\!]$ (example 3.3). Its inverse system is $(\mathfrak{m}^2)^{\perp} = \langle 1, y_1, y_2\rangle_{\boldsymbol{k}}$. Let

$$H = \sum_{0 \leqslant i+j \leqslant 2} a_{i,j} y_1^i y_2^j$$

be a general polynomial of degree at most 2, and consider the $\boldsymbol{k}$-vector space (see the proof of proposition 4.2)

$$\frac{\langle H\rangle_{\boldsymbol{k}} + I^{\perp}}{I^{\perp}} \subset W_I = \frac{S_{\leqslant 2}}{I^{\perp}}.$$

The closed point $p \in \mathbb{P}^2_{\boldsymbol{k}} = \mathbb{P}(W_I)$ belongs to $\mathrm{TC}(A)$ if and only if H satisfies conditions (i) and (ii) of the proof of proposition 4.2. If this is the case, then

$$\langle H\rangle_{\boldsymbol{k}} + I^{\perp} = \langle H_2\rangle_{\boldsymbol{k}} + I^{\perp},$$

where H_2 is the degree-2 homogeneous form of H. If we take as coordinates of $p \in \mathbb{P}^2_{\boldsymbol{k}}$ the 3-uple $\{a_{2,0}{:}a_{1,1}{:}a_{0,2}\}$, a simple computation shows that $\mathrm{TC}(A)$ is the open Zariski subset of the projective plane:

$$\mathrm{TC}(A) = \mathbb{P}^2_{\boldsymbol{k}} \setminus \{\Delta \equiv 0\},$$

where $\Delta = a_{2,0}a_{0,2} - a_{1,1}^2$.

Our next step is to present some effective criterion for testing if an Artin ring is Teter and, if this is the case, to compute a Teter cover.

In the following remark we explore a possible issue when we consider a given system of generators of $I^{\perp}$ in order to check if $A = R/I$ is Teter by using theorem 3.4.

Remark 4.4. Assume that $A = R/I$ is a Teter ring of socle degree $s - 1$. From theorem 3.4 we know that there exists a degree-s polynomial $F \in S$ such that $I^{\perp} = \langle \partial_{y_1} F, \dots, \partial_{y_n} F\rangle$. Not all systems of generators of $I^{\perp}$ are formed by the first derivatives of a polynomial. Let us consider $F = y_1^2 + y_2^2 \in S = \boldsymbol{k}[y_1, y_2]$; let $G = R/\operatorname{Ann_R}\langle F\rangle$ be the Gorenstein ring associated to F and let $A = G/\mathrm{soc}(G) =$

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$R/\mathfrak{m}^2$ be the attached Teter ring (see example 3.3). Then $I^\perp = \langle y_1, y_2\rangle$. The pair $3y_1+2y_2, y_1$ is a system of generators of the R-module $I^\perp$ but there is no polynomial G such that $\partial_{y_1}G = 3y_1+2y_2, \partial_{y_2}G = y_1$ or $\partial_{y_1}G = y_1, \partial_{y_2}G = 3y_1+2y_2$.

In the following result we give a method for checking whether an Artin ring is Teter and how to compute a Teter cover of it. This method has been implemented in the Singular library inverse-syst.lib [8].

Proposition 4.5. *Let $A = R/I$ be an Artin ring of socle degree $s-1$ and let $F_1, \dots, F_r$ be a minimal system of generators of $I^\perp$. The Artin ring A is Teter if and only if $n \geqslant r$ and there is an $n\times r$ matrix C of maximal rank r modulo the maximal ideal m with entries polynomials of R of degree at most $s-1$ such that the elements $H_1, \dots, H_n \in S$ defined by*

$$\begin{pmatrix} H_1 \\ \vdots \\ H_n \end{pmatrix} = C \circ \begin{pmatrix} F_1 \\ \vdots \\ F_r \end{pmatrix}$$

satisfy the Schwartz conditions $\partial_{y_i}H_j = \partial_{y_j}H_i$, $1 \leqslant i < j \leqslant n$.

Under these conditions there exists $H \in S$ such that $\partial_{y_i}H = H_i$, $i = 1, \dots, n$, and $G = R/\operatorname{Ann_R}\langle H\rangle$ is a Teter cover of A.

Proof. If A is a Teter ring with a Teter cover $G = R/J$ with $J^\perp = \langle H\rangle$, then $I^\perp = \langle \partial_{y_1}H, \dots, \partial_{y_n}H\rangle$. Hence, $n \geqslant r$ and there is an $n\times r$ matrix C with entries in R such that

$$\begin{pmatrix} \partial_{y_1}H \\ \vdots \\ \partial_{y_n}H \end{pmatrix} = C \circ \begin{pmatrix} F_1 \\ \vdots \\ F_r \end{pmatrix}.$$

Since the degree of F_i is at most $s-1$, $i = 1, \dots, n$, we may assume that the entries of C are polynomials of R of degree at most $s-1$. We may assume that $\partial_{y_{i_j}}H$, $j = 1, \dots, r$, a minimal system of generators of $I^\perp$. The $r\times r$ submatrix of C defined by the rows i_j, $j = 1, \dots, r$, is of maximal rank modulo the maximal ideal m.

Assume now that $n \geqslant r$ and that there is an $n\times r$ matrix C of maximal rank modulo the maximal ideal m with entries polynomials of R of degree at most $s-1$ such that the elements $H_1, \dots, H_n \in S$ defined by

$$\begin{pmatrix} H_1 \\ \vdots \\ H_n \end{pmatrix} = C \circ \begin{pmatrix} F_1 \\ \vdots \\ F_r \end{pmatrix}$$

satisfy the Schwartz conditions $\partial_{y_i}H_j = \partial_{y_j}H_i$, $1 \leqslant i < j \leqslant n$.

Then the R-module $L = \langle H_1, \dots, H_n\rangle$ is a submodule of $I^\perp$, and since C is of maximal rank modulo the maximal ideal m we get that $L = I^\perp$. On the other hand, since the family $H_1, \dots, H_n$ satisfies the Schwartz conditions there is a polynomial H such that $H_i = \partial_{y_i}H$, $i = 1, \dots, n$. Therefore, $I^\perp = \langle \partial_{y_1}H, \dots, \partial_{y_n}H\rangle$. By theorem 3.4 we get that $G = R/\operatorname{Ann_R}\langle H\rangle$ is a Teter cover of A. □

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EXAMPLE 4.6. Let $L \subset S$ be the sub-R-module of S generated by y_1^2, y_2y_3, y_1y_3. Note that this system of generators is minimal. The Artin ring $A = R/I$, with $I = \mathrm{Ann}_{\mathrm{R}}(L)$, is not a Teter ring. Assume that A is Teter. Then there is a degree 3 homogeneous form $F \in S$ such that $L = \langle \partial_{y_1}F, \partial_{y_2}F, \partial_{y_3}F \rangle$. Hence, there is a non-singular matrix C with entries in $\boldsymbol{k}$ such that

$$\begin{pmatrix} y_1^2 \\ y_2y_3 \\ y_1y_3 \end{pmatrix} = C \begin{pmatrix} \partial_{y_1}F \\ \partial_{y_2}F \\ \partial_{y_3}F \end{pmatrix}$$

so

$$\begin{pmatrix} \partial_{y_1}F \\ \partial_{y_2}F \\ \partial_{y_3}F \end{pmatrix} = C^{-1} \begin{pmatrix} y_1^2 \\ y_2y_3 \\ y_1y_3 \end{pmatrix}.$$

Considering Schwartz's conditions $\partial_{y_iy_j}F = \partial_{y_jy_i}F$, $1 \leqslant i < j \leqslant 3$, we get that C^{-1} is singular. Hence, A is a non-Teter ring.

5. Case studies: stretched, almost stretched and low socle degree rings

The aim of this section is to study stretched, almost stretched and short Teter rings. Our strategy is to use the known results on the minimal system of generators of such rings.

Following Sally (see [19]), we say that an Artin local ring $A = R/I$ of socle degree s is stretched if $\mathrm{HF}_A(2) = 1$. We call an Artin local ring A almost stretched if $\mathrm{HF}_A(2) = 2$. By Macaulay's characterization of Hilbert functions, the Hilbert function of A is given by

i	0	1	2	$\cdots$	s	$s+1$
$\mathrm{HF}_A(i)$	1	h	1	$\cdots$	1	0

with $(s \geqslant 2)$ if A is stretched, or by

i	0	1	2	$\cdots$	t	$t+1$	$\cdots$	s	$s+1$
$\mathrm{HF}_A(i)$	1	h	2	$\cdots$	2	1	$\cdots$	1	0

with $s \geqslant t \geqslant 2$ if A is almost stretched. If $t = s$, then we assume that $\mathrm{HF}_A(t) = \mathrm{HF}_A(s) = 2$. We say that A is almost stretched of type (s, t).

The case of the stretched Artin Gorenstein local ring was studied in [19] by Sally, who found a structure theorem for the corresponding ideals. Elias and Valla [11] extended this result to the case of stretched Artin local rings of any Cohen–Macaulay type. Moreover, a structure theorem for the minimal systems of generators of the ideals defining almost stretched Gorenstein ideals is given in [11]. In [12] Elias and Valla presented an analytic classification of almost stretched algebras under the assumption $s \geqslant 2t$, and Elias and Homs presented a classification for all pairs (s, t) in [9].

From the definition of stretched and almost stretched rings it is easy to prove the following.

Proposition 5.1. *Let A be a Teter Artin ring and let G be a Teter cover of A. If A is stretched (respectively, almost stretched) of socle degree $s-1$, then G is an Artin Gorenstein stretched ring (respectively, an Artin Gorenstein almost stretched ring) of socle degree s.*

In the next proposition we collect some results of [11].

Proposition 5.2 (Elias and Valla [11]). *Let $A = R/I$ be an Artin ring of socle degree s and Cohen–Macaulay type t.*

(i) *If A is stretched, then $1 \leqslant t \leqslant n$.*

(ii) *If A is stretched and $t < n$, then, after a $\boldsymbol{k}$-algebra isomorphism of R, I is minimally generated by the elements $\{x_i x_j\}_{1\leqslant i<j\leqslant n}$, $\{x_j^2\}_{2\leqslant j\leqslant t}$, $\{x_i^2 - u_i x_1^s\}_{t+1\leqslant i\leqslant n}$, where the u_i are units in R.*

(iii) *If A is stretched and $t = n$, then, after a $\boldsymbol{k}$-algebra isomorphism of R, I is minimally generated by the elements $\{x_1 x_j\}_{2\leqslant j\leqslant n}$, $\{x_i x_j\}_{2\leqslant i\leqslant j\leqslant n}$ and x_1^{s+1}.*

If A is Gorenstein, i.e. $t = 1$, the coset of x_1^s in A is a generator of $\operatorname{soc}(A)$.

From this structure theorem we deduce a characterization of stretched Teter rings, which we prove are unique up to isomorphisms.

Proposition 5.3. *Let $A = R/I$ be an Artin stretched Teter ring of socle degree $s-1$. Then, after a $\boldsymbol{k}$-algebra isomorphism of R, I is minimally generated by the elements $\{x_1 x_j\}_{2\leqslant j\leqslant n}$, $\{x_i x_j\}_{2\leqslant i\leqslant j\leqslant n}$ and x_1^s. In particular, I is unique up to a $\boldsymbol{k}$-algebra isomorphism of R.*

Proof. Let G be a Teter cover of A. Since $G = R/J$ is stretched and Gorenstein of socle degree s, we may assume that J is minimally generated by $\{x_i x_j\}_{1\leqslant i<j\leqslant n}$, $\{x_i^2 - u_i x_1^s\}_{2\leqslant i\leqslant n}$, where the u_i are units in R. Then $I = J + (x_1^s)$ is generated by $\{x_i x_j\}_{1\leqslant i<j\leqslant n}$, $\{x_i^2\}_{2\leqslant i\leqslant n}$, x_1^s (proposition 5.2(iii)). □

Remark 5.4. For the class of almost stretched Teter rings, we can prove a similar result by using the main result of [9]. We do not present it here due to the many cases that should be considered.

Proposition 5.5. *Let $A = R/I$ be an Artin ring.*

(i) *If the Hilbert function of A is $\{1, n, m\}$ and A is Teter then $m \leqslant n$.*

(ii) *For all $m \leqslant n$ there is a Teter ring A with Hilbert function $\{1, n, m\}$.*

Proof.
(i) Let G be a Teter cover of A. Then G is a short Gorenstein ring, i.e. a ring with Hilbert function $\{1, n, m, 1\}$. From Iarrobino's shell formula we get that $m \leqslant n$ (see [10, 17]).

(ii) Given an integer $m \leqslant n$, consider the polynomial

$$F = y_1^3 + \cdots + y_m^3 + y_{m+1}^2 + \cdots + y_n^2$$

and the Gorenstein ring G with Macaulay inverse system F. A simple computation shows that $\mathrm{HF}_G = \{1, n, m, 1\}$. Hence, $A = G/\operatorname{soc}(G)$ is a Teter ring with $\{1, n, m\}$ as its Hilbert function. □

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Remark 5.6. Note that from Macaulay's characterization of Hilbert functions we get that $\{1, n, m\}$ is the Hilbert function of an Artin algebra if and only if $0 \leqslant m \leqslant \binom{n+1}{2}$ [4, 20]. For Teter rings we have a stronger condition: $m \leqslant n$.

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References

1 H. Ananthnarayan. The Gorenstein colength of an Artinian local ring. *J. Alg.* **320** (2008), 3438–3446.

2 H. Ananthnarayan. Computing Gorenstein colength. *J. Commut. Alg.* **1** (2009), 343–359.

3 H. Ananthnarayan, L. L. Avramov and W. F. Moore. Connected sums of Gorenstein local rings. *J. Reine Angew. Math.* **667** (2012), 149–176.

4 W. Bruns and J. Herzog. *Cohen–Macaulay rings*, revised edn, Cambridge Studies in Advanced Mathematics, vol. 39 (Cambridge University Press, 1997).

5 A. De Stefani. Artinian level algebras of low socle degree. *Commun. Alg.* **42** (2014), 729–754.

6 W. Decker, G.-M. Greuel, G. Pfister and H. Schönemann. Singular 4-0-1: a computer algebra system for polynomial computations (available at www.singular.uni-kl.de, accessed 2014).

7 J. Elias. Inverse systems of local rings. Notes of a course given at the Vietnam Institute for Advanced Study in Mathematics, Hanoi. Preprint, 2014.

8 J. Elias. Inverse-syst.lib: Singular library for computing Macaulay's inverse systems. Preprint arXiv:1501.01786, 2015.

9 J. Elias and R. Homs. On the analytic type of Artinian algebras. *Commun. Alg.* **44** 2277–2304.

10 J. Elias and M. E. Rossi. Isomorphism classes of short Gorenstein local rings via Macaulay's inverse system. *Trans. Am. Math. Soc.* **364** (2012), 4589–4604.

11 J. Elias and G. Valla. Structure theorems for certain Gorenstein ideals. *Michigan Math. J.* **57** (2008), 269–292.

12 J. Elias and G. Valla. Isomorphism classes of certain Gorenstein ideals. *Alg. Representat. Theory* **14** (2011), 429–448.

13 J. Emsalem. Géométrie des points épais. *Bull. Soc. Math. France* **106** (1978), 399–416.

14 P. Gabriel. Objects injectifs dans les catégories abéliennes. *Sém. Dubriel Alg. Théor. Nombr.* **12**(2) (1959), 17–32.

15 C. Huneke and A. Vraciu. Rings that are almost Gorenstein. *Pac. J. Math.* **225** (2006), 85–102.

16 A. Iarrobino. Compressed algebras: Artin algebras having given socle degrees and maximal length. *Trans. Am. Math. Soc.* **285** (1984), 337–378.

17 A. Iarrobino. *Associated graded algebra of a Gorenstein Artin algebra*. Memoirs of the American Mathematical Society, vol. 107 (Providence, RI: American Mathematical Society, 1994).

18 F. S. Macaulay. *The algebraic theory of modular systems* (Cambridge University Press, 1916).

19 J. D. Sally. Stretched Gorenstein rings. *J. Lond. Math. Soc.* **20** (1979), 19–26.

20 R. P. Stanley. Hilbert functions of graded algebras. *Adv. Math.* **28** (1978), 57–83.

21 J. Striuli and A. Vraciu. Some homological properties of almost Gorenstein rings. In *Commutative algebra and its connections to geometry*. Contemporary Mathematics, vol. 555, pp. 201–215 (Providence, RI: American Mathematical Society, 2011).

22 W. Teter. Rings which are a factor of a Gorenstein ring by its socle. *Invent. Math.* **23** (1974), 153–162.