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A Computational Method in Ship Routing Using the Concept of Limited Manoeuvrability

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Under some circumstances, dependent on a ship's velocity, the wave period and wave direction, certain courses induce heavy rolling and must be avoided. This paper proposes a computational method for the solution of optimal control problems in ship routing for ships with such limited manoeuvrability. Known results for the control problem of Bolza with additional constraints are interpreted in terms of this new problem. This approach is equivalent to the application of Pontryagin's maximum principle. The method is an extension of an earlier method dealing with the meteorological navigation of ships with unrestricted manoeuvrability and gives a more realistic picture of what really could happen in practice.

KEY WORDS

1. Ship routing. 2. Limited manoeuvrability.

1. INTRODUCTION. Ship routing involves the solution of optimal control problems resulting from the meteorological effects on the navigation of ships. For instance in ship routing, the course of a ship between two given points may be determined so that the sailing time is minimized, given the maximum speed that the ship can maintain during 6 or 12 hours in different directions and subject to the disturbing forces of the ocean surface. Alternatively, other cost functions such as fuel consumption could be minimized leading to a more complex optimization problem considering the course and the speed of the ship as control variables. To compute an optimal path all the disturbing forces that could influence the ship on its way must be known. Since one of the most important forces is formed by the disturbed ocean surface it is assumed here that the state of the ocean surface is fully known beforehand for the complete passage.

After the appearance of the innovative paper by Hanssen and James (1960), several methods were proposed to solve the relatively simple problem of minimizing sailing time. In a series of papers we showed how these methods could be generalized to include minimization of the fuel consumption. For instance in Bijlsma (2001) application of the calculus of variations was extended to the minimization of fuel consumption, in Bijlsma (2002) the relation between optimal control theory and dynamic programming was elucidated for the case of minimal fuel routing, and in Bijlsma (2004) this was done for the relation between time front and energy front methods. In all these methods unrestricted manoeuvrability was assumed, ignoring the fact that some courses could be forbidden due to heavy rolling which occurs if the course of the ship is at a certain angle, depending on the ship's speed and wave period, with respect to the wave direction. As a more realistic approach, in this paper a computational method will be discussed which takes into account this conditionally restricted manoeuvrability. For the sake of simplicity it is assumed that the occurrence of forbidden courses merely depends on the significant wave height (which in fact determines the optimal ship's speed with respect to the minimization of sailing time or fuel consumption, and in a sense the presence of critical wave periods) and wave direction. We will return to this subject below. Relevant here is a paper by France et al (2002) on the largest container casualty in history. It concerns, we quote, a laden, post-Panamax, C11 class containership, eastbound from Kaohsiung to Seattle that was overtaken by a violent storm in the North Pacific Ocean in late October 1998. The encounter with the storm continued for some 12 hours, mostly at night, during which the master reduced speed and attempted to steer into increasingly higher seas off the vessel's starboard bow. Ultimately, the seas became completely confused and violent. The master later described the ship as absolutely out of control during the worst storm conditions. Investigation of the motion of the vessel during this storm event through a series of model tests and numerical analyses confirmed the vessel's parametric rolling response in the head seas prevailing at the time of the casualty. The responses can occur in extreme head, or near head, seas when unfavorable tuning is combined with low roll damping (reduced speed) and large stability variations (governed by wave length, wave height, general hull form, bow flare and stern shapes). One of the recommendations made by the authors is that safe and unsafe combinations of heading and speed for various sea state/loading combinations (in this case focused on head-sea parametric rolling) should be identified and presented to the master in the form of polar plots or other diagrams, or included in the ship's routing computer software. For more information on the response of a ship to waves the reader is referred for instance to publications of the Maritime Research Institute Netherlands (MARIN) at Wageningen, The Netherlands.

In this paper a simple model is used for the ship's performance characteristics. This is obviously in accordance with the wishes of those who make use of an on board routing system (see Spaans and Stoter (2000)). In the case of minimal-time routing the ship's performance is represented by a polar velocity diagram of elliptic form giving the ship's velocity as a function of the angle between the ship's heading and the wave direction for specific values of the significant wave height. It is constructed with the aid of the actual maximum values of the ship's speed in the case of following, beam and head waves obtained from empirical data. In the case of minimum fuel routing, it is not the maximum speed but the speed that maximizes the quotient of speed and fuel consumption per unit of time that plays an important role, as we shall see below. Sectors of forbidden courses are included in these polar velocity diagrams.

In Bijlsma (2001) a method was presented for the computation of a minimum-fuel route for ships with unrestricted manoeuvrability based on known results for the control problem of Bolza (1909) from the classical calculus of variations. The method presented here is an extension of that method by imposing additional constraints on the course of the ship, so allowing for restricted manoeuvrability. This approach is equivalent to application of the maximum principle from optimal control theory (Pontryagin et al (1964), see also Hestenes (1966)). No conditions are imposed on the spatial variables. For instance, it is not assumed here that a ship will follow a route partly or wholly belonging to the fixed boundary that may be formed by land or ice (see Bijlsma (1975)).

In this paper we give the mathematical tools that are needed to introduce forbidden courses in the form of modified polar velocity diagrams and to solve the corresponding optimal control problem in ship routing. Those familiar with shipping should be consulted for providing practical information on the behaviour of a ship in a seaway. The paper is organized as follows. In the next section the problem of Bolza is discussed for ships with unlimited manoeuvrability. Section 3 then pays attention to modifications that should be introduced in the case of forbidden courses; the practical aspects of this approach are in section 4. Our conclusions are presented in section 5.

2. THE PROBLEM OF BOLZA FOR UNLIMITED MANOEUVRABILITY.

Let us consider the minimization of the costs of a ship during its transit over the ocean between two given points. It is assumed here that the costs of a ship during an ocean crossing are due mainly to fuel consumption. For the sake of completeness it is noted that the resulting equations also apply in the case of other penalty or cost functions such as damage to cargo or passenger discomfort. We assume that the state of the ocean surface for the time of the complete passage is fully known beforehand. The navigation area is mapped conformally onto a plane, for instance by means of stereographic projection (see Bijlsma, 1975, p. 14). Introducing a Cartesian coordinate system with coordinates x_1 and x_2, it is assumed that the rate of decrease of fuel can be described by the equation

$$\dot{x}_0 = f_0(t, x_1, x_2, V, p)$$

where the variable x_0 denotes the fuel consumption and where the dot denotes differentiation to the time t. The speed V and the course p are control variables. It is assumed that these control variables belong to open sets. We now seek to find continuous control functions $V(t)$ and $p(t)$ and a corresponding trajectory $x(t) = (x_1(t), x_2(t))$ $(0 \leqslant t \leqslant t_1)$ satisfying the equations:

$$\dot{x}_1 = V \cos p + S_1(t, x_1, x_2) \tag{1}$$

$$\dot{x}_2 = V \sin p + S_2(t, x_1, x_2) \tag{2}$$

with

$$x_i(0) = x_{i0}, \quad x_i(t_1) = x_{i1} \quad (i = 1, 2) \tag{3}$$

which minimize

$$\int_0^{t_1} f_0(t, x_1(t), x_2(t), V(t), p(t))dt \tag{4}$$

This problem is called the control problem of Bolza (1909). The functions $S_1(t, x_1, x_2)$ and $S_2(t, x_1, x_2)$ are the components of the ocean current and will be omitted for notational simplicity. Inclusion is possible and will result in additional terms in equations (6), (7) and (10) or (13) (see also Bijlsma (2001), p. 147). The problem of Bolza has been studied extensively in the literature. A comprehensive treatment can be found in Hestenes (1966). The functions $V(t)$ and $p(t)$ satisfying the control problem of Bolza are optimal control functions and the arc $x(t)$ is an optimal trajectory. It is supposed here that the arc $x(t)$ is normal which implies the existence of a one-parameter family of arcs satisfying equations (1) and (2), with initial and end conditions given by equation (3), and containing $x(t)$ for a specific parameter value. The necessary condition for the control functions $V(t)$ and $p(t)$ and the trajectory $x(t)$ to be optimal is that there exist continuously differentiable multipliers $\lambda(t) = (\lambda_0(t), \lambda_1(t), \lambda_2(t))$, $\lambda_0(t) = \text{constant} \leqslant 0$ and a function

$$H(t, x, V, p, \lambda) = \lambda_0 f_0 + \lambda_1 V \cos p + \lambda_2 V \sin p$$

so that the following conditions hold on $x(t)$:

(a) The first necessary condition. On $x(t)$ the Euler-Lagrange equations

$$\dot{x}_i = H_{\lambda_i},\ \dot{\lambda}_i = -H_{x_i},\ H_V = 0,\ H_p = 0 \quad (i = 1, 2)$$

hold. Variables as subscripts denote partial differentiation.

(b) The necessary condition of Weierstrass. Along $x(t)$ the inequality

$$H(t, x(t), V, p, \lambda(t)) \leqslant H(t, x(t), V(t), p(t), \lambda(t))$$

must hold for any t, $0 \leqslant t \leqslant t_1$. In addition

$$H(t_1, x(t_1), V(t_1), p(t_1), \lambda(t_1)) = 0.$$

As a consequence of normality the equality sign for the multiplier λ_0 is excluded. The generalization of the conditions (a) and (b) in the case that $V(t)$ or $p(t)$ will assume boundary values is an extension of the classical calculus of variations and is known as Pontryagin's maximum principle. This situation may occur for instance if the course p is restricted to a closed set so that certain courses are forbidden. This case is dealt with in the next section. The speed V is restricted to the open range of values given by:

$$V_{\text{min}}(t, x_1, x_2, p) < V < V_{\text{max}}(t, x_1, x_2, p)$$

where $V_{\text{min}}(t, x_1, x_2, p)$ denotes an acceptable minimum speed depending on wave height and wave direction, and $V_{\text{max}}(t, x_1, x_2, p)$ denotes a maximum speed.

In the case of a movable end point the differentials of x_{i1} $(i = 1, 2)$ are connected by the transversality condition:

$$\sum_{i=1}^{2} \lambda_i(t_1) dx_{i1} = 0$$

which is an additional necessary condition for $x(t)$ to yield the solution of the optimal problem with movable ends. It should be noted that the navigation area is assumed to be an open region, which means that no constraints are imposed on the spatial variables so that ships are not assumed to follow a route partly, or wholly belonging to the boundary of the navigation area. Solutions of the Euler-Lagrange equations with continuous control functions are called extremals. An optimal trajectory is obviously an extremal. We observe that every part of an optimal trajectory is an optimal trajectory itself, which is a direct consequence of the principle of optimality (Bellman, 1957). A proof by contradiction is immediate. This means that every point of an extremal can be considered as an end point. Therefore equation $H = 0$ is supposed to be satisfied for all values of the time t with $0 \leqslant t \leqslant t_1$. Combination of the equations $H=0$ and $H_V=0$ shows that the speed V along an extremal must be chosen so that it maximizes the quotient:

$$\frac{V}{f_0(t, x_1, x_2, V, p)} \tag{5}$$

As a result we could describe in every point of the (x_1, x_2) plane an optimal speed $V(t, x_1, x_2, p)$ maximizing relation (5) leading to a more direct approach to the problem analogous to the time-optimal case. This approach has been treated extensively in Bijlsma (2001) and will be summarized here. Note that it is assumed that wave height and direction are known for the complete passage in every point of the (x_1, x_2) plane. The new problem under consideration is to determine the course $p(t)$ and corresponding trajectory $x(t)$, $0 \leqslant t \leqslant t_1$, satisfying the equations:

$$\dot{x}_1 = V(t, x_1, x_2, p)\cos p \tag{6}$$

$$\dot{x}_2 = V(t, x_1, x_2, p)\sin p \tag{7}$$

with initial and end conditions given by equation (3) that minimize the integral:

$$\int_0^{t_1} f_0(t, x_1(t), x_2(t), p(t))dt \tag{8}$$

The fuel consumption (8) is obtained by substituting $V=V(t, x_1, x_2, p)$ in integral (4). However the same notation is used for the resulting fuel consumption per unit of time f_0. Again we can apply the necessary conditions (a) and (b) for the problem of Bolza observing that the speed is now a known function of t, x_1, x_2 and p so that

$$H(t, x, p, \lambda) = \lambda_0 f_0(t, x, p) + \lambda_1 V(t, x, p)\cos p + \lambda_2 V(t, x, p)\sin p$$

The Euler-Lagrange equations now read

$$\dot{x}_i = H_{\lambda_i}, \ \dot{\lambda}_i = -H_{x_i}, \ H_p = 0 \quad (i=1, 2)$$

and the Weierstrass condition

$$H(t, x(t), p, \lambda(t)) \leqslant H(t, x(t), p(t), \lambda(t))$$

with

$$H(t_1, x(t_1), p(t_1), \lambda(t_1)) = 0$$

Combination of the equations $H=0$ and $H_p=0$ gives

$$\sum_{i=1}^{2} \lambda_i V_{ip} = 0 \tag{9}$$

where

$$V_1 = \frac{V}{f_0}\cos p, \quad V_2 = \frac{V}{f_0}\sin p$$

Equation (9) has a simple geometrical meaning. This point is illustrated in Figure 1, where we introduced a polar 'velocity' diagram giving the ship's 'velocity' (V_1, V_2) as a function of the angle between the ship's heading and the wave direction for fixed

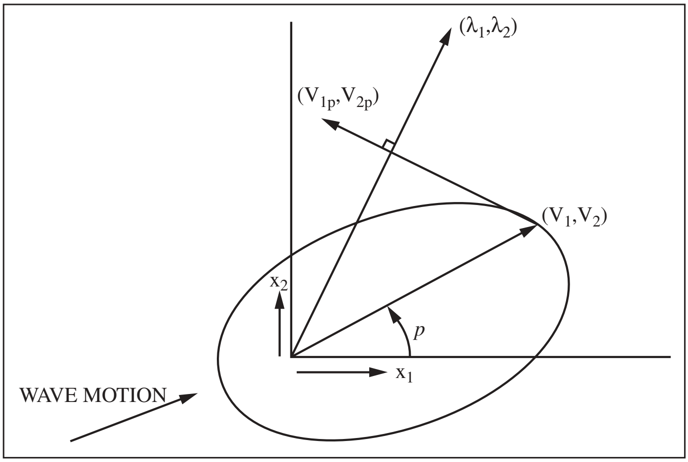

Figure 1. In view of the Weierstrass or Legendre condition there is a unique choice for p. The course p is measured as indicated.

values of t, x_1 and x_2. We note that the solution of the Euler-Lagrange equations

$$\dot{\lambda}_i = -\lambda_0 f_{0x_i} - \lambda_1 V_{x_i} \cos p - \lambda_2 V_{x_i} \sin p \quad (i=1,\ 2) \tag{10}$$

does not change if the Lagrange multipliers λ_1 and λ_2 are multiplied by an arbitrary constant. This is the case because the multiplier λ_0 can be chosen arbitrarily and the multipliers λ_0, λ_1 and λ_2 are defined up to a common factor of proportionality. Therefore we may write initial conditions as $\lambda_1(0) = \cos a$ and $\lambda_2(0) = \sin a$ for every choice of λ_0. For instance we can choose $\lambda_0 = -1$. All extremals emanating from the starting point are found by varying the parameter a. In the following we are interested in solutions of equations (6), (7) and (10), which are continuous in their dependence on the parameter a. Therefore we make use of the two following theorems.

- From a theorem on the initial value problem for systems of differential equations (Walter (1972), p. 93) we learn that the right-hand sides of equations (6), (7) and (10), where p is implicitly defined by relation (9), must be continuous and must satisfy a Lipschitz condition with respect to x_1, x_2, λ_1 and λ_2 so that the solutions depend continuously on the parameter a.
- Application of a theorem on implicit functions (Hestenes (1966), p.22) to relation (9) shows that V_{ip} and V_{ipp} $(i=1, 2)$ must be continuous functions with respect to t, x_1, x_2 and p and that the Legendre condition

$$\sum_{i=1}^{2} \lambda_i V_{ipp} < 0 \tag{11}$$

must hold so that p is a continuous function of x_1, x_2, λ_1, λ_2 and t.

The Legendre condition, which is a direct consequence of the Weierstrass condition, implies that the polar 'velocity' diagram must be convex. Summarizing we have the following result: Let the functions V, V_{x_1}, V_{x_2}, f_{0x_1}, f_{0x_2}, V_{ip} and $V_{ipp}(i=1,2)$ be continuous with respect to their arguments and let relation (11) be valid for $0 \leqslant t \leqslant t_1$.

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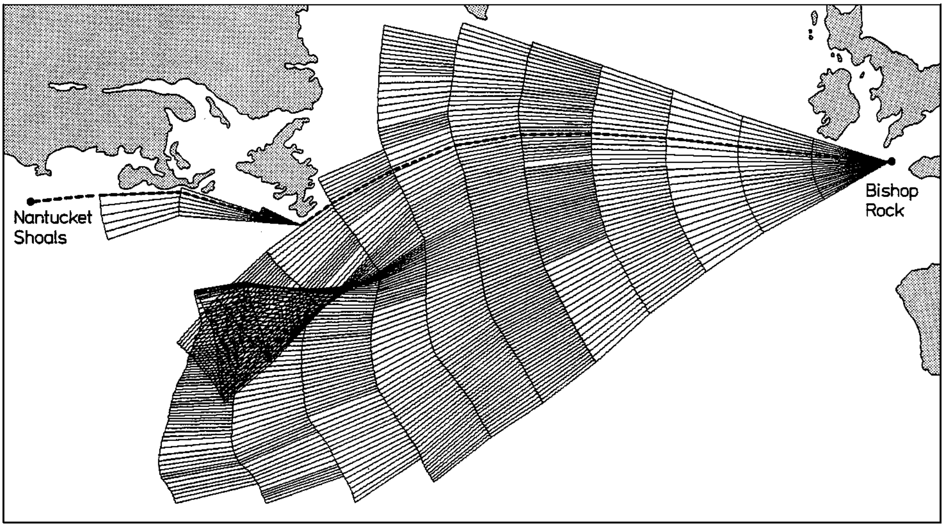

Figure 2. Computer produced by means of an incremental plotter using wave information over the period 17 January–23 January 1970, fictitious ship's data and a 12 hour time step. The least-time track is indicated by the dashed line.

Further let the right-hand sides of equations (6), (7) and (10), where p is given by equation (9), satisfy a Lipschitz condition with respect to x_1, x_2, λ_1 and λ_2. Then $x_i(t, a)$ and $\lambda_i(t, a)$ $(i=1,2;\ 0\leqslant t\leqslant t_1)$ as solutions of equations (6), (7) and (10) with $x_i(0)=x_{i0}$ $(i=1,2)$, where p is given by (9), are continuously differentiable with respect to t and continuous in their dependence on the parameter a defined by $\lambda_1(0)=\cos a$ and $\lambda_2(0)=\sin a$.

This result enables us to introduce a numerical method for the solution of optimal control problems in ship routing. An optimal route is obtained by integrating the system of equations, varying the parameter a and selecting that extremal which ends closest to the destination. The method appears to be very suitable in practical cases because the above requirements can be satisfied rather simply. The essence of the method is demonstrated in Figure 2, which has been taken from Bijlsma (1975). As a consequence of the transversality condition (cf. Bijlsma (2001), p. 149) the vector $(\lambda_1(t, a), \lambda_2(t, a))$ is orthogonal to the manifold consisting of points that can be reached along extremals in a given time t, in parametric representation given by

$$x_1=x_1(t,\ a),\ \ x_2=x_2(t,\ a)$$

so that

$$\sum_{i=1}^{2}\lambda_i(t,\ a)x_{ia}(t,\ a)=0$$

where $(x_{1a}(t, a), x_{2a}(t, a))$ is the tangent vector to the manifold.

3. MODIFICATIONS IN THE CASE OF LIMITED MANOEUVRABILITY.

So far we ignored the fact that some courses could be forbidden. This point is considered here. We have already noted that the prohibition of

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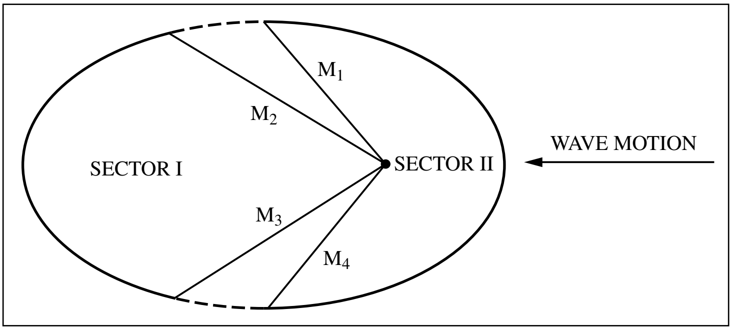

Figure 3. Polar velocity diagram of elliptic form showing the sectors of forbidden courses, indicated by the dashed lines. The non-coinciding boundary courses $M_l(t, x_1, x_2)$, $(l = 1, 2, 3, 4)$ are measured in the same way as the course p.

a course is due to heavy rolling that occurs if the course is at a certain angle, depending on the ship's speed and wave period, with respect to the wave direction. To be more specific, consider a ship with zero velocity in a wave field composed of waves of a single period. If the direction of the waves lies in a suitable sector and if their period corresponds to the resonance period of the ship, then the ship will undergo violent movements. As a consequence, one has to change the position of the ship with respect to the wave direction in order to avoid heavy rolling. If we now consider the more realistic situation of a moving ship in a seaway, some modifications will appear due to the fact that the wave period that causes resonance and the corresponding sector of relative wave direction are dependent on the velocity of the ship and that instead of waves of a single period, a spectrum of wave periods is present. As a representative period, one could take in this case the period that is related to the maximum energy in the spectrum. The corresponding polar velocity diagram, which for simplicity is assumed to be of elliptic form, is then changed as indicated in Figure 3. The course p is here restricted to sector I or II determined by the angles $M_l(t,\ x_1,\ x_2)$ $(l = 1,\ 2,\ 3,\ 4)$.

We are now in a position to extend the results of the previous section for the case in which the variables x_1 and x_2, and course p not only satisfy equations (6) and (7) with initial and end conditions given by (3) but also a set of additional constraints of the form:

$$\phi_l(t, x_1, x_2, p) \leqslant 0 \quad (l = 1, 2, 3, 4) \tag{12}$$

which involve the control variable p explicitly. The functions $\phi_l(t,\ x_1,\ x_2,\ p)$ are assumed to be continuously differentiable with respect to x_1, x_2 and p. These constraints can be constructed, for instance, by writing

$$\phi_j(t, x_1, x_2, p) = p - M_j(t, x_1, x_2) \quad \text{for} \quad j = 1, 3$$

$$\phi_j(t, x_1, x_2, p) = M_j(t, x_1, x_2) - p \quad \text{for} \quad j = 2, 4$$

Let $p(t)$ be an optimal course and $x(t)$ the corresponding optimal trajectory, which minimize (8) subject to the equations (6) and (7) with (3), and condition (12).

Then there exist multipliers $\lambda(t)=(\lambda_0(t), \lambda_1(t), \lambda_2(t))$, $\lambda_0(t)=\text{constant}\leqslant 0$ and $\mu(t)=(\mu_1(t), \mu_2(t), \mu_3(t), \mu_4(t))$, respectively continuously differentiable and continuous, and a function:

$$H(t, x, p, \lambda, \mu)=\lambda_0 f_0+\lambda_1 V\cos p+\lambda_2 V\sin p-\sum_{l=1}^{4}\mu_l\phi_l(t, x, p)$$

so that conditions (a) and (b) described below hold along $x(t)$.

(a) The first necessary condition. On $x(t)$ the equations

$$\dot{x}_i=H_{\lambda_i},\ \dot{\lambda}_i=-H_{x_i},\ H_p=0,\ \phi_l\leqslant 0 \quad (i=1,\ 2;\ l=1,\ 2,\ 3,\ 4)$$

hold. Moreover $\mu_l(t)\geqslant 0$ $(l=1, 2, 3, 4)$ where $\mu_l(t)=0$ at each point of $x(t)$ at which $\phi_l<0$.

(b) The second necessary condition. Along $x(t)$ the inequality

$$H(t,\ x(t),\ p,\ \lambda(t),\ 0)\leqslant H(t,\ x(t),\ p(t),\ \lambda(t),\ 0)$$

must hold for any t, $0\leqslant t\leqslant t_1$ and every admissible p for which $\phi_l(t, x(t), p)\leqslant 0$. In addition:

$$H(t_1,\ x(t_1),\ p(t_1),\ \lambda(t_1),\ 0)=0.$$

The necessary condition for optimality, contained in conditions (a) and (b), is an appropriate formulation of Pontryagin's maximum principle for the present problem. For courses p that are interior courses of sectors I or II, the results from the foregoing section remain valid. If p would exceed one of the boundary courses, say M_j, it is equalized to this value. In the following we choose $j=1$ or $j=3$ so that the formulae given below apply. The cases $j=2$ or $j=4$ can be treated accordingly. Application of the maximum principle gives the modified equations (cf. equation (10))

$$\dot{\lambda}_i=-\lambda_0 f_{0x_i}-\lambda_1 V_{x_i}\cos p-\lambda_2 V_{x_i}\sin p-\mu_j M_{jx_i} \quad (i=1,\ 2;\ j=1,\ 3) \tag{13}$$

Let us consider the variables p and μ_j in their dependence on x_1, x_2, λ_1, λ_2, and t. If the course p is an interior course of sector I or II denoted by $p=p(t, x_1, x_2, \lambda_1, \lambda_2)$ it depends continuously on $\theta=\arctan(\lambda_2/\lambda_1)$ (in the first instance x_1, x_2, and t are assumed to be fixed) according to equation (9) of the previous section. As soon as this angle equals or exceeds a boundary value, say $\theta(M_j)=\arctan(\lambda_2(M_j)/\lambda_1(M_j))$, p obtains the value $M_j\ (t, x_1, x_2)$. So at the boundary the following relation holds:

$$p=\begin{cases} M_j(t,\ x_1,\ x_2) & \theta\geqslant\theta(M_j) \\ p(t,\ x_1,\ x_2,\ \lambda_1,\ \lambda_2) & \theta<\theta(M_j). \end{cases} \tag{14}$$

Of course the inequality sign is changed in the opposite sense if $j=2$ or $j=4$. If the functions $M_l(t, x_1, x_2)(l=1, 2, 3, 4)$ are required to be continuous with respect to x_1, x_2, and t and if p is restricted to lie in one of the sectors I or II, then p is continuous in its dependence on x_1, x_2, λ_1, λ_2 and t. With respect to the

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parameter μ_j we have the following relations, which follow directly from application of equation $H_p=0$.

$$\mu_j=\begin{cases}\lambda_0 f_{0p}+\lambda_1(V\cos p)_p+\lambda_2(V\sin p)_p \text{ with } p=M_j & \theta\geqslant\theta(M_j)\\ 0 & \theta<\theta(M_j)\end{cases} \tag{15}$$

If $j=2$ or $j=4$ the result for μ_j in the first equation of (15) must be provided with a minus sign. Analogous to the previous section we have the following result: Let $M_j(t, x_1, x_2)$ and M_{jx_i} $(j=1, 2, 3, 4; i=1, 2)$ be continuous with respect to x_1, x_2 and t. Further let the conditions of the main result of the previous section be fulfilled for courses in sectors I and II including the boundary courses for $0\leqslant t\leqslant t_1$. Then $x_i(t, a)$ and $\lambda_i(t, a)(i=1, 2;\ 0\leqslant t\leqslant t_1)$ as solutions of equations (6), (7) and (13) with $x_i(0)=x_{i0}(i=1, 2)$, where p and $\mu_j(j=1, 2, 3, 4)$ are given by equations (14) and (15), are continuously differentiable with respect to t and continuous in their dependence on the parameter a defined by $\lambda_1(0)=\cos a$ and $\lambda_2(0)=\sin a$.

Regions of limited and unlimited manoeuvrability are separated from each other by closed curves in the (x_1, x_2) plane changing continuously with time and containing the limiting points where M_1 and M_2 as well as M_3 and M_4 coincide. So far it has been shown that $x_i(t, a)$ and $\lambda_i(t, a)$ $(i=1, 2)$ are continuous in their dependence on the parameter a for ships with unrestricted manoeuvrability and for ships with restricted manoeuvrability provided that the courses along the neighbouring extremals fall within the same sector I or II at a specific time. If not, a discontinuity in the dependence on the parameter a arises since the ship cannot bridge the gap between two boundary courses if it has to change from sector I to sector II or vice versa, owing to the danger of heavy rolling and the possible damage attendant upon it. This may be the case if the courses on two neighbouring extremals fall within different sectors. In that case we may distinguish between separate groups of extremals, each group depending continuously on the parameter a. For each of these separate groups the main result of this section applies.

4. PRACTICAL ASPECTS.

In order to simplify the discussion the discretized equations are considered on an orthogonal grid with coordinates x_1 and x_2 and a given mesh distance. The maximum time step in the numerical solution of equations (6), (7) and (10) for unlimited manoeuvrability, or equations (6), (7) and (13) for limited manoeuvrability, is determined by the fact that wave charts are usually available every 12 hours. Depending on the ship's speed one can introduce 6-hour or 12-hour time steps interpolating between two successive wave charts so that the distance which can be covered by the ship is of the order of magnitude of the mesh distance. Significant wave height and wave direction are assumed to be known at grid points for the complete passage. It is assumed that wave height and wave direction between grid points can be obtained by bilinear interpolation. The same interpolation procedure could be applied to the approximation of the spatial derivatives in the equations that are computed at grid points by means of central differences. For instance, for computational simplicity we could write equation (13) at the boundary $p=M_j(t, x)$ as

$$\dot{\lambda}_i=-H_{x_i}(t, x, M_j(t, x), \lambda, 0)$$

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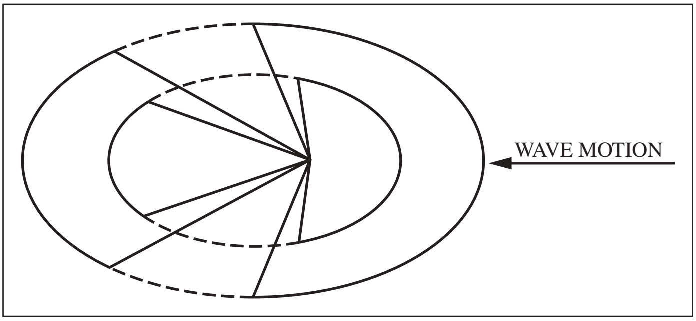

Figure 4. Polar velocity diagrams of elliptic form are shown for two different values of the significant wave height. Note that the sectors of forbidden courses, indicated by the dashed lines, are increasing with increasing wave height.

and apply foregoing interpolation procedures. The validity of this simple interpolation rule has been demonstrated for instance in the experiments shown in Bijlsma (2002). For more details concerning practical problems the flow diagram and corresponding computer program in Bijlsma (1975) might be helpful, minor programming errors reserved.

Just like the polar velocity diagram for the optimal ship's velocity with respect to the minimization of sailing time or fuel consumption also the polar 'velocity' diagram used in equation (9), showing the relation between the ship's 'velocity' and wave direction (and wave height), is assumed to be of elliptic form. It is constructed with the aid of the values of the ship's 'speed' (which is equal to the value of the quotient of the optimal ship's speed and the corresponding fuel consumption per unit of time) in the case of following, beam and head waves. These values are assumed to be known for a range of values of significant wave height, just like the boundary courses which define the forbidden sectors as shown in Figure 4. Intervening values can be obtained by linear interpolation. In fact the ship's 'speed' corresponds with the distance that can be covered ultimately per unit of fuel consumption. If one of the directions defining following, beam or head waves falls into a forbidden sector its contribution might be obtained by extrapolation.

Let us now consider the continuous dependence of the solutions of the system of equations on the initial values of the Lagrange multipliers, discussed in the preceding sections, from a practical point of view. Therefore we regard a one-parameter family of extremals emanating from the point of departure. We may start a new extremal between two neighbouring extremals at a specific time if the distance between these neighbours becomes too large at that time. If this distance is chosen sufficiently small the starting values of the new extremal can be obtained by linear interpolation between the corresponding values of its neighbours (see Figure 2). An optimal route is obtained by selecting that extremal which ends closest to the destination. This procedure can also be applied in the case of limited manoeuvrability if the courses associated with the two neighbouring extremals at a certain time belong to sector I or II (see Figure 3). The computation of an extremal is stopped if it intersects the boundary of the navigation area (see again Figure 2).

A final remark concerns the introduction of forbidden courses. Owing to the danger of heavy rolling, a ship will in general not follow a strategy as mentioned by de Wit (1968), where points in the forbidden sectors can be reached by steering a combination of boundary courses. Therefore it may occur that it is even impossible to reach the destination because the ship cannot bridge the gap formed by the boundary courses. In that case it is up to the master to take measures to change course. For that reason the master should have the possibility of an interactive communication with the ship's routing computer program in order to anticipate and avoid undesirable situations.

5. CONCLUSIONS.

In this paper a ship routing method is presented for ships with limited manoeuvrability, which means that certain courses of the ship are forbidden due to the heavy rolling that occurs if the course of the ship is at a certain angle depending on the ship's speed and wave period with respect to the wave direction. It is assumed that sectors of forbidden courses merely depend on the significant wave height, which in fact determines the optimal ship's speed with respect to minimization of sailing time or fuel consumption and in a sense the presence of critical wave periods, and the wave direction. Then inclusion of these sectors in the polar velocity diagram, giving the ship's speed as a function of the angle between the ship's heading and the wave direction for a specific value of the significant wave height, is very simple. These modified polar velocity diagrams can easily be implemented in the ship's routing computer software for a range of values of significant wave height. It should be noted that there are ships, which are extremely sensitive to rolling within certain sectors of incoming waves and which are liable to heavy rolling even in wave fields of moderate significant wave heights, demonstrating that inclusion of the modified velocity diagrams in the ship's routing software is not merely a matter of theoretical interest. Because of its practical importance the master should have the opportunity to control these developments for instance by interactive communication with the ship's computer routing program to avoid undesirable situations such as missing the destination because it is located within a sector of forbidden courses.

To sum up, it may be said given the paper of France et al. (2002) that the above discussion is obviously in accordance with what really could happen to a ship in certain circumstances. Of course the implementation of sectors of forbidden courses will create additional complications in programming. Unfortunately we were not in a position to do experiments. Nevertheless we are of the opinion that the inclusion of limited manoeuvrability in the ship's routing computer software deserves further consideration in view of the realistic approach of the meteorological navigation of ships.

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