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The interface dynamics of a surfactant drop on a thin viscous film†

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We study a system of two coupled parabolic equations that models the spreading of a drop of an insoluble surfactant on a thin liquid film. Unlike the previously known results, the surface diffusion coefficient is not assumed constant and depends on the surfactant concentration. We obtain sufficient conditions for finite speed of support propagation and for waiting-time phenomenon by application of an extension of Stampacchia's lemma for a system of functional equations.

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1 Introduction

We study the interface dynamics of thin liquid films influenced by an insoluble surfactant (a surface-tension-reducing agent) on a horizontal plane in the presence of gravity. The motion of the film is modelled in the lubrication approximation by a coupled system of two non-linear parabolic equations.

The investigation of surfactant spreading on a thin liquid film goes back several decades, and includes experiments, asymptotic analysis and numerical simulations. For example, in [18], the first quantitative, spatio-temporally resolved measurements were performed for the spreading of an insoluble surfactant on a thin fluid layer. They directly observed

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the radial height profile of the spreading droplet and the spatial distribution of the fluorescently tagged surfactant. It was discovered that the leading edge of a spreading circular layer of surfactant forms a Marangoni ridge on the underlying fluid. A peak in the surfactant concentration was observed that trailled the leading edge.

The authors of [32] implemented a Godunov scheme for the lubrication approximation to the Stokes system to study numerically the development of the leading order insoluble surfactant for two different equations of state (dependence of the surface tension σ on the surfactant concentration Γ): linear $\sigma(\Gamma) = 1 - \Gamma$ and non-linear $\sigma(\Gamma) = (1+\theta\Gamma)^{-3}$, where θ is an empirical material parameter. They showed that the leading edge of the surfactant travels with speed which is equal to the surface fluid velocity at that point. It was shown numerically and confirmed experimentally in [34] that the spreading dynamics of an insoluble surfactant on a thin liquid film confined by chemical surface patterns can be represented locally in time by a power law. They determined the time evolution of the liquid film thickness and the corresponding spreading exponents both from experiments using interference microscopy and by numerical finite element simulations. It was also shown in [29] that, with the exception of monolayers with strong cohesive interactions (strong intermolecular bounds) which tend to retard the spreading process, an insoluble monolayer increases the rate of drop spreading. Simulation results (with surface equation of state based on the Frumkin adsorption framework) for small Bond numbers indicated the existence of a power-law region for the time-dependence of the basal radius of the drop, consistent with experimental measurements.

Interesting results from experimental study of the spreading of an insoluble surfactant over a thin liquid layer were reported in [36]. Initial concentrations of surfactant above and below critical micelle concentration were considered. If the concentration was above critical, two distinct stages of spreading were found: The first stage was connected with micelle dissolution; the second one was the transfer stage and this stage was much slower than the first one. When the surfactant concentration was in the second stage, the formation of a dry spot in the middle of the film was observed.

Recent modelling work has incorporated aspects of micelle formation into the governing equations and resulted in trends that are qualitatively consistent with experimental observations. The following model of a thin film of viscous, incompressible, Newtonian fluid lying on a horizontal plane, with a monolayer of an insoluble surfactant on its surface, was derived by Jensen and Grotberg (see [21]):

$$(S)\begin{cases} h_t + \frac{1}{3}(h^3(\mathcal{S}h_{xxx} - \mathcal{G}h_x + 3\mathcal{A}h^{-4}h_x))_x + \frac{1}{2}(h^2\sigma_x)_x = 0, & (1.1)\\ \Gamma_t + \frac{1}{2}(\Gamma h^2(\mathcal{S}h_{xxx} - \mathcal{G}h_x + 3\mathcal{A}h^{-4}h_x))_x + (\Gamma h\sigma_x)_x = (\mathcal{D}(\Gamma)\Gamma_x)_x, & (1.2)\end{cases}$$

where h is the film height; Γ is the surfactant concentration in the monolayer; $\sigma(\Gamma)$ is the surface tension; $\mathcal{G}$ is a parameter representing a gravitational force directed vertically downwards; $\mathcal{A}$ is related to the Hamaker constant, being connected with intermolecular van der Waals forces; $\mathcal{S}$ is connected with capillarity forces; $\mathcal{D}$ is related to the surface diffusion and is assumed constant in [21].

Neglecting temperature effects, a fundamental equation of chemical thermodynamics (see [44]) relates the concentration-dependent surface tension σ to the free energy, Φ, and the chemical potential, Φ', where both functions depend on the surfactant concentration Γ :

$$\sigma(\Gamma) = \Phi(\Gamma) - \Gamma \Phi'(\Gamma). \tag{1.3}$$

By convexity of the free energy, this relation implies a monotone decrease of surface tension for non-negative concentration. It seems more realistic to assume that the surface diffusivity of surfactant is not a constant [16, 17, 22], and given by a non-linear function of the surfactant concentration Γ (see, e. g. [8, (6.1) and (6.2), pp. 158–159]), namely in the dimensionless form

$$\sigma(\Gamma) = (1 + \theta\Gamma)^{-3}, \ \mathcal{D}(\Gamma) = (1 + \tau\Gamma)^{-k}, \tag{1.4}$$

where θ, τ and k are positive empirical parameters. It was shown that the parameter θ depends on the material properties of the monolayer (cf. [8] for details). The empirical relation (1.4) is based on experimental data obtained for the subinterval $0 < \sigma \leqslant 1$. For example, if $\theta = 0.15$, then (1.4) well describes an oil layer on water [8, Figure 2, p. 159].

The main goals of this paper are to study waiting-time and finite-speed phenomena for surfactant-driven flows. Our approach is based on now well established non-linear PDE analysis for degenerate higher order parabolic equations [1, 4, 9, 12, 27, 33, 39]. We find power-law behaviour for finite-speed propagation and define sufficient conditions for waiting-time phenomena.

The sufficient conditions: $h_0(x) \leqslant A|x|^{\frac{4}{n}}$ for $0 < n < 2$, $|h_{0x}(x)| \leqslant B|x|^{\frac{4}{n}-1}$ for $2 \leqslant n < 3$ (where A and B are some positive constants) on non-negative initial data, h_0, for the occurrence of waiting-time phenomenon were derived by Dal Passo, Giacomelli and Grün [13] for the classic multi-dimensional thin-film equation:

$$h_t + \nabla \cdot (|h|^n \nabla \triangle h) = 0. \tag{1.5}$$

For the multi-dimensional case and $2 \leqslant n < 3$, see [26].

In [7], the waiting-time phenomenon in the classic one-dimensional thin-film equation (1.5) was identified for $h_0(x) \sim |x|^\alpha$ for $2 < \alpha < \frac{4}{n}$. The result was obtained by means of matching asymptotic methods and was supported by numerous numerical simulations. The upper bounds on the waiting time of solutions to the thin-film equation in the regime of weak slippage was obtained by Fischer in [20]. Fischer also derived lower bounds on asymptotic support propagation rates for strong solutions to the thin-film equation in [19].

For more general non-linear degenerate parabolic equations with non-linear lower order terms, the waiting-time phenomenon was analysed in [24, 31, 33, 39]. In the present paper, we find sufficient conditions on initial data (h_0, Γ_0) such that interfaces of the liquid film and of the surfactant do not move. These conditions were not known before and would be interesting to investigate experimentally.

The first finite-speed results for non-negative generalised solutions of the classic thin-film equation (1.5) were obtained in [2, 3] for the cases $0 < n < 2$ and $2 \leqslant n < 3$, respectively. For more general types of thin-film equations, the finite speed of support propagation phenomenon was studied in [5,6,25,38,40,41] (see also references therein). In this paper, we analyse the speed of interface propagation, $\gamma(t)$, for a model (P), defined below. This model is a generalisation of the original system (S) for partial-slip regimes. The original system (S) is a special case (no-slip regime) of the model (P) which corresponds to $n = 3$ and does not possess a finite-speed propagation property. For the subcritical case $\frac{n}{2} < m < n+2$ in the partial wetting regime $2 \leqslant n \leqslant \frac{5}{2}$, and $0 < q < 4n + 7 + 3\min\{0, 6m - 5n + 2\}$, we prove that the speed is finite (q is a parameter related to the type of diffusion). Note that our threshold $5/2$ is connected with existence of strong solutions (C^1 smooth ones) as we use this regularity to prove finite speed. Therefore, we leave the question open for the range $5/2 < n < 3$. For a subset $n - 1 < m < n + 2$, we obtain an upper bound on it:

$$\gamma(t) \leqslant K(t^{\frac{1}{n+7}} + t^{\frac{1}{3(q+2)}}),$$

where K is a positive constant depending only on the parameters in the problem and on the initial data. The first term $t^{\frac{1}{n+7}}$ to the best of the authors' knowledge is new, and, apparently, cannot be obtained from standard self-similar type solutions as they have the asymptotic behaviour $t^{\frac{1}{n+4}}$, see for example [2,3]. The new asymptotic $t^{\frac{1}{n+7}}$ for $n = 3$ has better agreement with the spreading rate $t^{\frac{1}{10}}$ that was experimentally observed for the leading edge of the surfactant in [37]. The last term $t^{\frac{1}{3(q+2)}}$ coincides with $t^{\frac{1}{6}}$ as $q \to 0$. This asymptotic behaviour was discovered in [21] for self-similar solutions of (S) with constant diffusion.

We also prove finite-speed propagation of the support in critical and supercritical cases for $2 \leqslant n \leqslant \frac{5}{2}$, $n+2 \leqslant m < n+2+3\min\{n, 2q\}$ and $0 < q < 4n+7+3\min\{0, 6m-5n+2\}$. Under appropriate flatness conditions on the initial data and $2 \leqslant n < 3$, we prove existence of a waiting-time phenomenon in the subcritical case for $\frac{2n}{3} < m < n+2$ and $0 < q < 4n + 7 + 3\min\{0, 6m - 5n + 2\}$ and in critical and supercritical cases for $n + 2 \leqslant m < n + 2 + 2\min\{n, 3q\}$ and $0 < q < 4n + 7 + 3\min\{0, 6m - 5n + 2\}$.

An outline of the paper is as follows. Section 2 is devoted to a generalised version of the original system and to the description of our main results. In Section 3, we prove the finite-speed interface propagation. In Section 4, we find sufficient conditions for waiting-time phenomena. In the Appendixes A and B, we collect technical proofs of the paper's statements.

2 Main results

We will consider the generalisation of (S) for the case of partial-slip conditions in a dimensionless form introduced in [10], namely, the following problem:

$$(P)\begin{cases} h_t + (f_n(h)(h_{xxx} - h_x + F''_{n,m}(h)h_x))_x + (f_{n-1}(h)\sigma_x)_x = 0, & (2.1)\\ \Gamma_t + (\Gamma f_{n-1}(h)(h_{xxx} - h_x + F''_{n,m}(h)h_x))_x + \\ \qquad (\Gamma f_{n-2}(h)\sigma_x)_x = (D(\Gamma)\Gamma_x)_x, & (2.2)\\ h_x(\pm a, t) = h_{xxx}(\pm a, t) = \Gamma_x(\pm a, t) = 0 \text{ for } t > 0, & (2.3)\\ \quad h(x, 0) = h_0(x),\ \Gamma(x, 0) = \Gamma_0(x), & (2.4)\end{cases}$$

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in $Q_T = (0,T)\times\Omega$, where $\Omega = (-a,a)$, $n \geqslant 2$, $f_n(z) = \frac{|z|^n}{n}$, $f_0(z) = 1$, and

$$F_{n,m}(z) = \begin{cases} \dfrac{n\,|z|^{m-n+2}}{(m-n+1)(m-n+2)} - \dfrac{n\,z}{m-n+1} + \dfrac{n}{m-n+2} \text{ if } m-n < -1, \\ n(z - \ln|z| - 1) \text{ if } m-n = -2, \\ n(z\ln|z| - z + 1) \text{ if } m-n = -1, \\ \dfrac{n\,|z|^{m-n+2}}{(m-n+1)(m-n+2)} \text{ if } m-n > -1, \end{cases} \tag{2.5}$$

$F''_{n,m}(z) = \frac{|z|^m}{f_n(z)} \geqslant 0$, $F_{n,m}(z) \in C^2_{loc}(\mathbb{R}^1)$ when $m \geqslant n$ and $F_{n,m}(z) \in C^2_{loc}(\mathbb{R}^1_- \cup \mathbb{R}^1_+)$ when $m < n$. The well-known physical situation corresponds to $n = 3,\ m = -1$ (see (S)), hence, $F_{3,-1}(z) = \frac{1}{2}z^{-2} + z - \frac{3}{2}$. Moreover,

$$\int_\Omega h(x,t)\,dx = \int_\Omega h_0(x)\,dx = M, \int_\Omega \Gamma(x,t)\,dx = \int_\Omega \Gamma_0(x)\,dx = \mathfrak{S}. \tag{2.6}$$

We assume that the initial data satisfy the conditions:

$$\begin{gathered} 0 \leqslant h_0 \in H^1(\Omega),\ F_{n,m}(h_0) \in L^1(\Omega), \\ \Gamma_0 \in L^2(\Omega),\ 0 \leqslant \Gamma_0 \leqslant 1,\ \Phi(\Gamma_0) \in L^1(\Omega), \end{gathered} \tag{2.7}$$

where Φ is from (1.3).

Before we state our main result, we need to recall certain restrictions on the regularity of the coefficients that we introduced in [10]. Namely, it was assumed that

(A1) the function $\Phi : [0,1] \to \mathbb{R}^+_0$ is convex, and

$$\lim_{z\to 0^+} z\,\Phi''(z) = C_0 \quad (\Rightarrow \lim_{z\to 0+} z\,\Phi'(z) = 0), \tag{2.8}$$

where $0 < C_0 < +\infty$;
(A2) the function $D : [0,1] \to \mathbb{R}^+_0$ is non-increasing and

$$\lim_{z\to 1^-} D(z)\Phi''(z) = C_1, \tag{2.9}$$

where $0 < C_1 \leqslant \infty$ if $D(1) = 0$.

Note that the conditions above are in good agreement with many equations of state given in the fluid mechanics literature, for example, see the Frumkin equation, the Langmuir equation and the Borgas–Grotberg equation of state (for more examples, see [30]).

Definition 2.1 Let $n \geqslant 2$, $m \in \mathbb{R}^1$. A generalised weak solution of the problem (2.1)–(2.4) with initial data $(h_0, \Gamma_0) \in H^1(\Omega) \times L^2(\Omega)$, $h_0 \geqslant 0$, $0 \leqslant \Gamma_0 \leqslant 1$ is a pair (h, Γ) satisfying

$$h \geqslant 0,\ 0 \leqslant \Gamma \leqslant 1 \text{ a.e. in } Q_T, \tag{2.10}$$

$$h \in C^{1/2,1/8}_{x,t}(\overline{Q}_T) \cap L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))^*), \tag{2.11}$$

$$\Gamma \in L^6(Q_T) \cap L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;(H^1(\Omega))^*), \tag{2.12}$$

$$\sqrt{f_n(h)}(h_{xxx} - h_x + F''_{n,m}(h)h_x) \in L^2(\mathcal{P}_T), \tag{2.13}$$

$$f_{n-1}(h)\sigma_x,\ \Gamma f_{n-2}(h)\sigma_x \in L^2(\mathcal{P}_T),\ D(\Gamma)\Gamma_x \in L^2(Q_T), \tag{2.14}$$

$$\Gamma f_{n-1}(h)(h_{xxx} - h_x + F''_{n,m}(h)h_x) \in L^2(\mathcal{P}_T), \tag{2.15}$$

where $\mathcal{P}_T := \{(x,t) \in \bar{Q}_T : h > 0\}$ and (h, Γ) satisfies (2.1), (2.2) in the following sense:

$$\int_0^T \langle h_t(\cdot,t), \phi\rangle\, dt - \iint_{\mathcal{P}_T} f_{n-1}(h)\sigma_x\phi_x\, dxdt - \iint_{\mathcal{P}_T} f_n(h)(h_{xxx} - h_x + F''_{n,m}h_x)\phi_x\, dxdt = 0, \tag{2.16}$$

$$\int_0^T \langle \Gamma_t(\cdot,t), \phi\rangle\, dt - \iint_{\mathcal{P}_T} \Gamma f_{n-2}(h)\sigma_x\phi_x dxdt + \iint_{Q_T} D(\Gamma)\Gamma_x\phi_x dxdt - \iint_{\mathcal{P}_T} \Gamma f_{n-1}(h)(h_{xxx} - h_x + F''_{n,m}h_x)\phi_x\, dxdt = 0, \tag{2.17}$$

for all $\phi \in L^4(0,T;H^2_N(\Omega))$, where $H^2_N(\Omega) := \{v \in H^2(\Omega) : v_x = 0 \text{ on } \partial\Omega\}$;

$$h(\cdot,t) \to h(\cdot,0) = h_0 \text{ pointwise and strongly in } L^2(\Omega) \text{ as } t \to 0, \tag{2.18}$$

$$\Gamma(\cdot,t) \to \Gamma(\cdot,0) = \Gamma_0 \text{ strongly in } (H^1(\Omega))^* \text{ as } t \to 0; \tag{2.19}$$

and $h_x(\pm a,t) = h_{xxx}(\pm a,t) = 0$ and $\Gamma_x(\pm a,t) = 0$ at all points of the lateral boundary, where $\{h \neq 0\}$ and $\{\Gamma \neq 1\}$ correspondingly.

In two cases – $m > n-2$ for $n \in [2,4)$ and $m \geqslant \frac{n}{2}$ for $n \in [4,\infty)$ – local in time existence of a weak solution (h,Γ) of the problem (P) in the sense of the Definition 2.1 was proved in [10, Theorem 1] when the assumptions (A1), (A2) hold. In addition, the global in time existence was shown if $m \leqslant n+2$ (and $M < M_c$ for $m = n+2$, where M_c is some critical mass). Unfortunately, we cannot show finite speed of the support propagation for all possible forms of surface tension dependence on surfactant concentration and for all values of parameters n, m where existence of non-negative weak solutions was obtained. This is primarily due to the additional smoothness requirements for the weak solutions in a neighbourhood of a touchdown point that will lead to finite-speed result. Our main goal is to find the condition on the surface tension behaviour which will guarantee this additional smoothness property.

Theorem 1 *Let functions $D(z)$ and $\Phi(z)$ satisfy*

$$D(z)\Phi''(z) \geqslant C_2\Phi^{q-1}(z)(\Phi'(z))^2, \text{ where } 0 < C_2 < +\infty, \tag{2.20}$$

$$\left|\int_1^z D(s)\Phi'(s)ds\right| \leqslant C_3\Phi^{q+1}(z), \quad |z\,\Phi'(z)| \leqslant C_4\Phi^{\nu}(z), \tag{2.21}$$

where $\nu \in \left(\frac{2}{n+2}, \min\left\{1, \frac{2(q+1)}{n+2}\right\}\right)$ *and* $\Phi(z) \in C[0,1] \cap C^1(0,1] : \Phi(1) = \Phi'(1^-) = 0$. *Assume also that*

$$\begin{gathered} 2 \leqslant n \leqslant \frac{5}{2}, \ \frac{n}{2} < m < n+2+3\,\min\{n, 2q\}, \\ 0 < q < 4n+7+3\min\{0, 6m-5n+2\} \end{gathered} \tag{2.22}$$

and (h_0, Γ_0) *satisfies to (2.7),* $\operatorname{supp} h_0 \subset (-r_0, r_0) \Subset \Omega$, $\operatorname{supp}\Phi(\Gamma_0) \subset (-r_0, r_0) \Subset \Omega$, *and the conditions (A1), (A2) hold. Then, there exists a time* $T_0 > 0$ *such that a weak solution* (h, Γ) *of the Definition 2.1 has finite speed of propagation for all* $t \leqslant T_0 := \gamma^{-1}(a - r_0)$, *i.e. there exists a continuous function* $\gamma(t) \in C[0, T_0]$, $\gamma(0) = 0$ *such that* $\operatorname{supp} h(t,.), \operatorname{supp}\Phi(\Gamma(t,.)) \subset (-r_0 - \gamma(t), r_0 + \gamma(t)) \Subset \Omega$. *Moreover, if* $\Omega = \mathbb{R}^1$, $n-1 < m < n+2$, *then there exist a solution of the Cauchy problem* (h, Γ) *and small enough time* $T_0 > 0$ *such that following upper estimate*

$$\gamma(t) \leqslant K\left(t^{\frac{1}{n+7}} + t^{\frac{1}{3(q+2)}}\right) \tag{2.23}$$

holds for $t \leqslant T_0$. *Here, the constant* K *depends only on the parameters of the problem and on the initial data.*

The theorem above provides control over rate of expansion of domain where $\{h > 0, \Gamma < 1\}$ by analysing the time evolution of the free boundary, where $\{h = 0, \Gamma = 1\}$. This model assumes that the concentration of the surfactant is equal to $\Gamma = 1$ if it covers the dry area of the thin film. After the thin film spreading towards this dry area with the highest surfactant concentration, the concentration of the surfactant starts decreasing $\Gamma < 1$.

Remark 2.1 *Note that our admissible* σ *and* D *satisfying (2.20) and (2.21) have at least such asymptotic* $\sigma(z) \sim 1 - C_0 z$ *as* $z \to 0^+$, $\sigma(z) \sim (1-z)^r$ *as* $z \to 1^-$, $D(z) \sim (1-z)^k$, *with* $\frac{2}{n} < r < 1$, $0 < k \leqslant 1 - r$, $q = \frac{k}{r+1}$, $\frac{2}{n+2} < \nu \leqslant \min\left\{\frac{r}{r+1}, \frac{2(q+1)}{n+2}\right\}$. *Such behaviour was observed experimentally in [42] for the case when the surface tension depends on concentration of surfactant in the presence of a polymer. Also, we note that our restrictions on physical parameters are inconsistent with the Frumkin equation of state.*

Remark 2.2 *Note, the assumption* $\operatorname{supp}\Phi(\Gamma_0) \subset (-r_0, r_0)$ *means that* $\Gamma_0 = 1$ *for all* $x \in \bar{\Omega} \setminus (-r_0, r_0)$.

Remark 2.3 *Note, the assumption* $\operatorname{supp}\Phi(\Gamma_0) \subset (-r_0, r_0)$ *means that* $\Gamma_0 = 1$ *for all* $x \in \bar{\Omega} \setminus (-r_0, r_0)$.

Let $\Omega(s) = \{x : x \geqslant s\}$, $Q_T(s) = (0, T) \times \Omega(s)$ for all $s \in \mathbb{R}^1$, and

$$\mathbf{k}_0(s) := \int\limits_{\Omega(s)} \{\frac{1}{2}h_{0x}^2 + \Phi(\Gamma_0)\}dx, \quad \mathbf{k}_0(s) = 0 \ \ \forall s \geqslant r_0. \tag{2.24}$$

Let us assume that the function $\mathbf{k}_0(s)$ satisfies a *flatness* condition. Namely, for every $s : 0 < s < r_0$, the estimate

$$\mathbf{k}_0(s) \leqslant \chi(r_0 - s)_+^{\gamma - 1} \tag{2.25}$$

is valid, where $\chi > 0$ and

$$\gamma = \begin{cases} \dfrac{2(n+6)}{n} \text{ for } m \leqslant n, \\ \dfrac{3m - n + 12}{3m - 2n} \text{ for } m > n. \end{cases} \tag{2.26}$$

Theorem 2 *Let*

$$\begin{gathered} 2 \leqslant n \leqslant \frac{5}{2}, \ \frac{2n}{3} < m < n + 2 + 2\,\min\{1, 3q\}, \\ 0 < q < 4n + 7 + 3\min\{0, 6m - 5n + 2\}. \end{gathered} \tag{2.27}$$

Assume that (h_0, Γ_0) satisfies to (2.7) with $\Omega = \mathbb{R}^1$, and meas$\{\Omega(s) \cap \operatorname{supp} h_0\} = \emptyset$, meas$\{\Omega(s) \cap \operatorname{supp} \Phi(\Gamma_0)\} = \emptyset$ for all $s \geqslant r_0$, i.e. condition (2.24) is valid, and the flatness condition (2.25) holds.

Then, for a weak solution (h, Γ) of the Definition 2.1 (with $\Omega = \mathbb{R}^1$), there exists sometime $T^* = T^*(\chi) > 0$ depending on the known parameters only such that

$$\operatorname{supp} h(t, \cdot), \operatorname{supp} \Phi(\Gamma(t, \cdot)) \cap \Omega(r_0) = \emptyset \ \forall\, 0 \leqslant t \leqslant T^*, \tag{2.28}$$

where χ is the constant from the flatness condition. Note that $T^* \to +\infty$ as $\chi \to 0$.

Note that the condition (2.24) is the interplay between the thin-film flatness and the surfactant concentration. The higher level of the concentration the less flatness of the thin film is required for the waiting-time phenomenon to occur.

3 Finite speed of propagation

To prove the Theorem 1, we use the following energy inequality, Appendix A contains the proof of it.

3.1 Proof of finite speed

Lemma 3.1 *Let $2 \leqslant n \leqslant \frac{5}{2}$, $m \geqslant \frac{2(n-1)}{3}$, $q \geqslant 0$ and $\beta > \frac{1-n}{3}$. Let $\zeta \in C^{2,1}_{x,t}(\overline{Q}_T)$ be an arbitrary non-negative function ($T \leqslant T_{loc}$ if $m > n + 2$) such that $\zeta_x = 0$ on $\partial\Omega$. Then, there exists a solution (h, Γ) in the sense of the Definition 2.1, constants C_1, C_2 dependent*

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on n, m and q but independent of Ω, such that for all $t \leqslant T$

$$\frac{1}{2}\int_{\Omega}(h_x^2+2\Phi(\Gamma))\zeta^6\,dx+\int_{\Omega}\zeta^4 h^{\beta+1}\,dx$$
$$-\frac{1}{2}\iint_{Q_t}(h_x^2+2\Phi(\Gamma))(\zeta^6)_t\,dx-\iint_{Q_t}h^{\beta+1}(\zeta^4)_t\,dx$$
$$+C_1\iint_{Q_t}\big((h^{\frac{n+2}{2}})^2_{xxx}+(\Phi^{\frac{q+1}{2}}(\Gamma))^2_x+f_{n-2}(h)\sigma_x^2\big)\zeta^6\,dxdt$$
$$\leqslant\frac{1}{2}\int_{\Omega}(h_{0x}^2+2\Phi(\Gamma_0))\zeta^6\,dx+\int_{\Omega}\zeta^4 h_0^{\beta+1}\,dx$$
$$+C_2\iint_{Q_t}\Phi^{\frac{\nu(n+2)}{2}}(\Gamma)\zeta^4\zeta_x^2\,dxdt+C_3\iint_{Q_t}\Phi^{q+1}(\Gamma)\zeta^4(\zeta_x^2+\zeta|\zeta_{xx}|)\,dxdt$$
$$+C_4\iint_{Q_t}(h^{n+2}(\zeta^6+\zeta_x^6+\zeta^4\zeta_x^2+\zeta^{\frac{9}{2}}|\zeta_{xx}|^{\frac{3}{2}}+\zeta^3|\zeta_{xx}|^3)\,dxdt$$
$$+C_5\iint_{Q_t}\{\chi_{\{\zeta>0\}}h^{n+3\beta-1}+h^{3m-2n+2}\zeta^6\}\,dxdt. \tag{3.1}$$

The proof of Lemma 3.1 is in Appendix A.

For an arbitrary $s\in(0,a-r_0)$ and $\delta>0$, we consider the families of sets

$$\widetilde{\Omega}(s)=\Omega\setminus(-r_0-s,r_0+s),\ \ Q_T(s)=(0,T)\times\widetilde{\Omega}(s). \tag{3.2}$$

We also define

$$K(s,\delta)=\{x\in\bar{\Omega}\,:\ r_0+s\leqslant|x|<r_0+s+\delta\},\quad K_T(s,\delta)=(0,T)\times K(s,\delta).$$

We introduce a non-negative cut-off function $\eta(\tau)$ from the space $C^2(\mathbb{R}^1)$ with the following properties:

$$\eta(\tau)=\begin{cases}0 & \tau\leqslant 0,\\ \tau^3(6\tau^2-15\tau+10) & 0<\tau<1,\\ 1 & \tau\geqslant 1.\end{cases} \tag{3.3}$$

Next, we introduce our main cut-off functions $\eta_{s,\delta}(x)\in C^2(\bar{\Omega})$ such that $0\leqslant\eta_{s,\delta}(x)\leqslant 1\ \forall\,x\in\bar{\Omega}$ that possess the following properties:

$$\eta_{s,\delta}(x)=\eta\Big(\frac{|x|-(r_0+s)}{\delta}\Big)=\begin{cases}1\,, x\in\widetilde{\Omega}(s+\delta),\\ 0\,, x\in\Omega\setminus\widetilde{\Omega}(s),\end{cases}\quad|(\eta_{s,\delta})_x|\leqslant\frac{15}{8\delta},\ \ |(\eta_{s,\delta})_{xx}|\leqslant\frac{5(\sqrt{3}-1)}{\delta^2} \tag{3.4}$$

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for all $s \in [0, a - r_0)$, $\delta > 0 : r_0 + s + \delta < a$. Setting $\zeta(x) = \eta_{s,\delta}(x)$ into (3.1), after simple transformations, we obtain

$$
\begin{aligned}
LHS :=& \int\limits_{\widetilde{\Omega}(s+\delta)} \{h_x^2(x,T) + 2\Phi(\Gamma(T)) + h^{\beta+1}(x,T)\}\,dx \\
&+ C \iint\limits_{Q_T(s+\delta)} \{(h^{\frac{n+2}{2}})^2_{xxx} + (\Phi^{\frac{q+1}{2}}(\Gamma))^2_x\}\,dxdt \leqslant \frac{C}{\delta^6} \iint\limits_{K_T(s,\delta)} h^{n+2}\,dxdt \\
&+ C \iint\limits_{K_T(s,\delta)} \{h^{n+3\beta-1} + h^{3m-2n+2}\}\,dxdt + \frac{C}{\delta^2} \iint\limits_{K_T(s,\delta)} (\Phi^{\frac{v(n+2)}{2}}(\Gamma) + \Phi^{q+1}(\Gamma))\,dxdt \\
&= C \sum_{i=1}^{3} \delta^{-\alpha_i} \iint\limits_{K_T(s,\delta)} h^{\xi_i} + C\,\delta^{-2} \iint\limits_{K_T(s,\delta)} (\Phi^{\frac{v(n+2)}{2}}(\Gamma) + \Phi^{q+1}(\Gamma))\,dxdt
\end{aligned}
\tag{3.5}
$$

for all $s \in [0, a - r_0)$, $\delta > 0 : r_0 + s + \delta < a$, where we use that

$$
\int\limits_{\widetilde{\Omega}(s)} \{h_{0x}^2 + 2\Phi(\Gamma_0) + h_0^{\beta+1}\}dx = 0 \;\forall\, s \geqslant 0,
$$

and $\alpha_1 = 6$, $\alpha_2 = \alpha_3 = 0$.

Next, we apply the Nirenberg–Gagliardo interpolation inequality (see Lemma B.3) in the region $K(s,\delta)$ to a function $v := h^{\frac{n+2}{2}}$ with $a = \frac{2\xi_i}{n+2}$, $b = \frac{2(\beta+1)}{n+2}$, $d = 2$, $k = 0$, $j = 3$, $d_2 = c^*\,\delta^{-\frac{(n+2)(\xi_i-\beta-1)}{2\xi_i(\beta+1)}}$ and $\theta_i = \frac{(n+2)(\xi_i-\beta-1)}{\xi_i(n+5\beta+7)}$ under the conditions:

$$
\beta < \xi_i - 1 \text{ for } i = \overline{1,3}. \tag{3.6}
$$

Moreover, we apply Lemma B.3 in the region $K(s,\delta)$ to a function $v := \Phi^{\frac{q+1}{2}}(\Gamma)$ with $a = \frac{v(n+2)}{q+1}$ (or $a = 2$), $b = \frac{2}{q+1}$, $d = 2$, $k = 0$, $j = 1$, $d_2 = c\,\delta^{-\frac{(q+1)(v(n+2)-2)}{2v(n+2)}}$ (or $d_2 = c\,\delta^{-\frac{q}{2}}$) and $\theta_4 = \frac{(q+1)(v(n+2)-2)}{v(q+2)(n+2)}$, $v > \frac{2}{n+2}$ (or $\theta_5 = \frac{q}{q+2}$). Integrating the resulted inequalities with respect to time, applying the Young inequality and taking into account (3.5), we arrive at the following relations:

$$
\begin{aligned}
\delta^{-\alpha_i} \iint\limits_{K_T(s,\delta)} h^{\xi_i} \leqslant& \epsilon_i C \iint\limits_{K_T(s,\delta)} \left(h^{\frac{n+2}{2}}\right)^2_{xxx} \\
&+ C\delta^{-\alpha_i\left(1-\frac{\xi_i\theta_i}{n+2}\right)^{-1}} \int_0^T \left(\int\limits_{K(s,\delta)} h^{\beta+1}\right)^{\frac{\xi_i(1-\theta_i)}{\beta+1}\left(1-\frac{\xi_i\theta_i}{n+2}\right)^{-1}} \\
&+ C\,\delta^{-\alpha_i-\frac{\xi_i-\beta-1}{n+2}} \int_0^T \left(\int\limits_{K(s,\delta)} h^{\beta+1}\right)^{\frac{\xi_i}{\beta+1}},
\end{aligned}
\tag{3.7}
$$

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$$\delta^{-2}\iint\limits_{K_T(s,\delta)}\Phi^{\frac{v(n+2)}{2}}(\Gamma)\leqslant\epsilon_4 C\iint\limits_{K_T(s,\delta)}\left(\Phi^{\frac{q+1}{2}}(\Gamma)\right)_x^2$$
$$+C\delta^{-2\left(1-\frac{v(n+2)\theta_4}{2(q+1)}\right)^{-1}}\int_0^T\left(\int\limits_{K(s,\delta)}\Phi(\Gamma)\right)^{\frac{v(n+2)(1-\theta_4)}{2}\left(1-\frac{v(n+2)\theta_4}{2(q+1)}\right)^{-1}}$$
$$+C\delta^{-2-\frac{v(n+2)-2}{2}}\int_0^T\left(\int\limits_{K(s,\delta)}\Phi(\Gamma)\right)^{\frac{v(n+2)}{2}}, \tag{3.8}$$
$$\delta^{-2}\iint\limits_{K_T(s,\delta)}\Phi^{q+1}(\Gamma)\leqslant\epsilon_5 C\iint\limits_{K_T(s,\delta)}\left(\Phi^{\frac{q+1}{2}}(\Gamma)\right)_x^2$$
$$+C\delta^{-2-q}\int_0^T\left(\int\limits_{K(s,\delta)}\Phi(\Gamma)\right)^{q+1}, \tag{3.9}$$

where $i=1,2,3$, $v_i=\frac{6(\xi_i-\beta-1)}{n+5\beta+7}$, $\frac{2}{n+2}<v<1<\frac{2(q+3)}{n+2}$. Substituting these estimates into (3.5) and making the standard iterative procedure similar to [19, Lemma 4.2] for small enough $0<\epsilon_i<1$, we arrive at the inequality

$$LHS\leqslant C\sum_{i=1}^{3}\Bigg[\delta^{-\alpha_i\left(1-\frac{\xi_i\theta_i}{n+2}\right)^{-1}}\int_0^T\left(\int\limits_{\widetilde{\Omega}(s)}h^{\beta+1}\right)^{\frac{\xi_i(1-\theta_i)}{\beta+1}\left(1-\frac{\xi_i\theta_i}{n+2}\right)^{-1}}$$
$$+C\delta^{-\alpha_i-\frac{\xi_i-\beta-1}{n+2}}\int_0^T\left(\int\limits_{\widetilde{\Omega}(s)}h^{\beta+1}\right)^{\frac{\xi_i}{\beta+1}}\Bigg]$$
$$+C\delta^{-2\left(1-\frac{v(n+2)\theta_4}{2(q+1)}\right)^{-1}}\int_0^T\left(\int\limits_{\widetilde{\Omega}(s)}\Phi(\Gamma)\right)^{\frac{v(n+2)(1-\theta_4)}{2}\left(1-\frac{v(n+2)\theta_4}{2(q+1)}\right)^{-1}}$$
$$+C\delta^{-2-\frac{v(n+2)-2}{2}}\int_0^T\left(\int\limits_{\widetilde{\Omega}(s)}\Phi(\Gamma)\right)^{\frac{v(n+2)}{2}}$$
$$+C\delta^{-2-q}\int_0^T\left(\int\limits_{K(s,\delta)}\Phi(\Gamma)\right)^{q+1}=:RHS, \tag{3.10}$$

where

$$\frac{\xi_i(1-\theta_i)}{\beta+1}(1-\frac{\xi_i\theta_i}{n+2})^{-1}=1+\frac{6(\xi_i-\beta-1)}{n+6\beta+8-\xi_i}>1,$$
$$\frac{v(n+2)(1-\theta_4)}{2}(1-\frac{v(n+2)\theta_4}{2(q+1)})^{-1}=1+\frac{2[v(n+2)-2]}{2(q+3)-v(n+2)}>1. \tag{3.11}$$

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Let us denote by $G(s) := \int_{\widetilde{\Omega}(s)} (h^{\beta+1} + 2\Phi(\Gamma))dx$. Then, from (3.10), we deduce that

$$G(s+\delta) \leqslant C(T)\delta^{-\alpha} G^k(s),$$

where

$$\alpha := \max\left\{\alpha_i\left(1-\frac{\xi_i\theta_i}{n+2}\right)^{-1},\ \alpha_i+\frac{\xi_i-\beta-1}{n+2},\right.$$
$$\left. 2\left(1-\frac{\nu(n+2)\theta_4}{2(q+1)}\right)^{-1},\ 2+\frac{\nu(n+2)-2}{2},\ 2+q\right\},$$
$$1<k:=\min\left\{\frac{\xi_i(1-\theta_i)}{\beta+1}\left(1-\frac{\xi_i\theta_i}{n+2}\right)^{-1},\ \frac{\xi_i}{\beta+1},\right.$$
$$\left.\frac{\nu(n+2)(1-\theta_4)}{2}\left(1-\frac{\nu(n+2)\theta_4}{2(q+1)}\right)^{-1},\ \frac{\nu(n+2)}{2},\ q+1\right\}. \quad (3.12)$$

These inequalities hold provided that

$$\frac{\theta_i\xi_i}{n+2} < 1 \Leftrightarrow \beta > \frac{\xi_i-n-8}{6} \text{ for } i=\overline{1,3}. \quad (3.13)$$

Simple calculations show that inequalities (3.6) and (3.13) are valid with some β such that

$$\max\{\frac{1-n}{3}, \frac{2-n}{2}, \frac{m-n-2}{6}, \frac{q-n-7}{6}\} < \beta < \min\{\frac{n}{2}, q, 3m-2n+1\},$$

then the inequalities (3.6) and (3.13) hold if and only if the restrictions (2.22) are true.

Note that for small enough T, taking into account the boundedness of RHS by a constant depending on the initial data (namely, $\|h_{0,x}\|_2$ and $\|\Phi(\Gamma_0)\|_1$) and the known parameters, we can estimate the second summands in (3.7)–(3.9) using the first ones. Moreover, as $\Phi^{\frac{q+1}{2}}(\Gamma)$ is bounded in $L^2(0,T;H^1(\Omega))$, then $\Phi(\Gamma)$ is bounded in $L^{\frac{\nu(n+2)}{2}}(Q_T)$ provided $\nu \leqslant \frac{2(q+1)}{n+2}$. Without loss of generality, we can take $G(s)=0$ for all $s \geqslant a-r_0$, therefore $G(s)$ is defined for all $s \geqslant 0$. So, finite speed of propagations follows from (3.7)–(3.9) by Lemma B.2 with $s_1=0$ and sufficiently small T. Hence,

$$\operatorname{supp} h(T,.),\ \operatorname{supp} \Phi(\Gamma(T,.)) \subset (-r_0-\gamma(T), r_0+\gamma(T)) \Subset \Omega \quad (3.14)$$

for all $T \in [0, T_0]$, where $T_0 := \gamma^{-1}(a-r_0)$.

Using pseudo-spectral Galerkin method, we computed weak solutions of the system (S) numerically (see Figure 1). As initial data, we took $\Gamma_0(x) = 0.5-0.5\cos(\pi x/0.7)$ smoothly continued by 1 for $|x|>0.7$ and $h_0(x) = 0.16+0.12\cos(\pi x)$. All coefficients $\mathcal{S}, \mathcal{G}, \mathcal{A}$ are taken equal to 1. Non-linearity powers are $n=2.3$ and $m=3$ (partial slip conditions). The diffusion coefficient $\mathcal{D}(\Gamma) = (1-\Gamma)_+^{1/23}$, $q=1/44$ and $\sigma(\Gamma) = (1-\Gamma)_+^{21/23}$. This numerical result illustrates decreasing of thin-film thickness in the middle point where the concentration of the surfactant is growing.

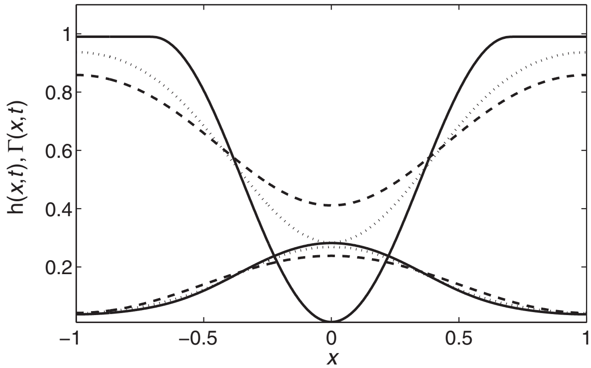

FIGURE 1. Numerical evaluation of the surfactant concentration $\Gamma(x,t)$ and of the thin-film drop height $h(x,t)$ from (S). Snap shorts of the solutions are given at times $t = 0, 0.02, 0.08$ by solid, dotted and dashed lines, respectively.

3.2 Upper bound for the speed of the interface propagation

Now, we establish an exact upper estimate for $\gamma(t)$ for a solution of the corresponding Cauchy problem with a compactly supported non-negative initial data such that $h_0 \in H^1(\mathbb{R}^1)$ and $\Phi(\Gamma_0) \in L^1(\mathbb{R}^1)$. Using the uniform bound of $\|h\|_{H^1}$ in Ω for $n-1 < m \leqslant n+2$ (see [10, Lemma 3.2, p. 112]), we can show that the upper bound of $\gamma(T)$ is independent of Ω; therefore, the solution can be extended for h by zero and for Γ by 1 on $|x| > r_0 + \gamma(T)$ and thus is a solution on the line for all $T \leqslant T_0$. Performing a similar procedure in $[T_0, 2T_0], \ldots, [mT_0, (m+1)T_0], \ldots$, we obtain a compactly supported solution of the Cauchy problem for all $T \geqslant 0$.

Suppose that $\Omega(s) = \mathbb{R}^1 \setminus \{x : |x| < s\}$, $Q_T(s) = (0,T) \times \Omega(s)$ for all $s > r_0$, $\operatorname{supp} h_0 \subseteq (-r_0, r_0)$, and $\gamma(T) = r(T) - r_0$. Since the time interval is small, we can assume that $r(T) < 2r_0$. Hence, for all $s \in (r_0, 2r_0)$, we can take (up to a smooth C^2 approximation) $\zeta(x) = (|x| - s)_+$ in (A.28). As a result, we obtain

$$\frac{1}{2}\int\limits_{\Omega(s)} (|x|-s)_+^6 h_x^2\,dx + \delta^6 \int\limits_{\Omega(s+\delta)} \Phi(\Gamma)\,dx + \delta^6 C_1 \iint\limits_{Q_T(s+\delta)} (h^{\frac{n+2}{2}})_{xxx}^2\,dxdt$$
$$+ \delta^6 C_1 \iint\limits_{Q_T(s+\delta)} (\Phi^{\frac{q+1}{2}}(\Gamma))_x^2\,dxdt \leqslant C_3 \iint\limits_{Q_T(s)} (r(T)-s)_+^6 h^{3m-2n+2}\,dxdt$$
$$+ C_3 \iint\limits_{Q_T(s)} \{h^{n+2} + \Phi^{\frac{\nu(n+2)}{2}}(\Gamma) + \Phi^{q+1}(\Gamma)\}\,dxdt \qquad (3.15)$$

for all $T \leqslant T_0$, $s \in (r_0, 2r_0)$. Using the Hardy type inequality

$$\int\limits_{\Omega(s)} (|x|-s)_+^{\alpha} f^2\,dx \leqslant C_0 \int\limits_{\Omega(s)} (|x|-s)_+^{\alpha+2} f_x^2\,dx, \qquad (3.16)$$

where f_x is an integrable function such that $\int_{\Omega(s)} (|x|-s)_+^{\alpha+2} f_x^2 dx < \infty$, $C_0 = \frac{4}{(\alpha+1)^2}$ (this constant is sharp) and $\alpha \neq -1$, we deduce that

$$\int\limits_{\Omega(s+\delta)} h\,dx \leqslant \Big(\int\limits_{\Omega(s+\delta)} (|x|-s)_+^4 h^2\,dx \Big)^{\frac{1}{2}} \Big(\int\limits_{\Omega(s+\delta)} (|x|-s)_+^{-4}\,dx \Big)^{\frac{1}{2}}$$
$$\leqslant (\frac{C_0}{3\delta^3})^{1/2} \Big(\int\limits_{\Omega(s)} (|x|-s)_+^6 h_x^2\,dx \Big)^{\frac{1}{2}},$$

whence

$$\Big(\int\limits_{\Omega(s+\delta)} h\,dx \Big)^2 \leqslant \frac{C_0}{3} \delta^{-3} \int\limits_{\Omega(s)} (|x|-s)_+^6 h_x^2\,dx \tag{3.17}$$

for all $\delta > 0$, $s \in (r_0, 2r_0)$. Substituting (3.17) in (3.15), we get

$$\frac{3}{2C_0} \sup_t \Big(\int\limits_{\Omega(s+\delta)} h\,dx \Big)^2 + \delta^3 \int\limits_{\Omega(s+\delta)} \Phi(\Gamma)\,dx + C_1\delta^3 \iint\limits_{Q_T(s+\delta)} (h^{\frac{n+2}{2}})_{xxx}^2\,dxdt$$
$$+ C_1\delta^3 \iint\limits_{Q_T(s+\delta)} (\Phi^{\frac{q+1}{2}}(\Gamma))_x^2\,dxdt \leqslant \frac{C_3}{\delta^3} \iint\limits_{Q_T(s)} \{h^{n+2} + \Phi^{\frac{\nu(n+2)}{2}}(\Gamma) + \Phi^{q+1}(\Gamma)\}\,dxdt$$
$$+ \frac{C_3}{\delta^3}\gamma^6(T) \iint\limits_{Q_T(s)} h^{3m-2n+2}\,dxdt \tag{3.18}$$

for all $T \leqslant T_{loc}$, $s \in (r_0, 2r_0)$. By the Nirenberg–Gagliardo, Hölder and Young inequalities, after simple transformations, for $\epsilon_i > 0$ and $\frac{2n-1}{3} < m < n+2$, we have

$$\frac{C_6}{\delta^3} \iint\limits_{Q_T(s)} h^{n+2}\,dxdt \leqslant \epsilon_1\delta^3 \iint\limits_{Q_T(s)} (h^{\frac{n+2}{2}})_{xxx}^2\,dxdt + \frac{C(\epsilon_1)}{\delta^{n+4}} \int\limits_0^T \Big(\int\limits_{\Omega(s)} h\,dx \Big)^{n+2} dt, \tag{3.19}$$

$$\frac{C_3\gamma^6(T)}{\delta^3} \iint\limits_{Q_T(s)} h^{3m-2n+2}\,dxdt \leqslant \epsilon_2\delta^3 \iint\limits_{Q_T(s)} (h^{\frac{n+2}{2}})_{xxx}^2\,dxdt$$
$$+ C(\epsilon_2) \frac{\gamma(T)^{\frac{2(n+7)}{n-m+2}}}{\delta^{\frac{3m-n+8}{n-m+2}}} \int\limits_0^T \Big(\int\limits_{\Omega(s)} h\,dx \Big)^{\frac{5m-3n+4}{n-m+2}} dt. \tag{3.20}$$

$$\frac{C_3}{\delta^3} \iint\limits_{Q_T(s)} \Phi^{\frac{\nu(n+2)}{2}}(\Gamma)\,dxdt \leqslant \epsilon_3\delta^3 \iint\limits_{Q_T(s)} (\Phi^{\frac{q+1}{2}}(\Gamma))_x^2\,dxdt$$
$$+ \frac{C(\epsilon_3)}{\delta^{\frac{3[2(q+1)+\nu(n+2)]}{2(q+3)-\nu(n+2)}}} \int\limits_0^T \Big(\int\limits_{\Omega(s)} \Phi(\Gamma)\,dx \Big)^{\frac{2(q+1)+\nu(n+2)}{2(q+3)-\nu(n+2)}} dt, \tag{3.21}$$

$$\frac{C_3}{\delta^3} \iint\limits_{Q_T(s)} \Phi^{q+1}(\Gamma)\,dxdt \leqslant \epsilon_4 \delta^3 \iint\limits_{Q_T(s)} (\Phi^{\frac{q+1}{2}}(\Gamma))_x^2\,dxdt$$

$$+ \frac{C(\epsilon_4)}{\delta^{3(q+1)}} \int\limits_0^T \Big(\int\limits_{\Omega(s)} \Phi(\Gamma)\,dx \Big)^{q+1} dt. \tag{3.22}$$

Substituting the estimates (3.19)–(3.22) to (3.18) and making the standard iterative procedure similar to [19, Lemma 4.2] for small enough $0 < \epsilon_i < 1$, we arrive at the inequality

$$\frac{3}{2C_0} \sup_t \Big(\int\limits_{\Omega(s+\delta)} h\,dx \Big)^2 + \delta^3 \sup_t \int\limits_{\Omega(s+\delta)} \Phi(\Gamma)\,dx$$

$$+ C_4 \delta^3 \iint\limits_{Q_T(s+\delta)} \{ (h^{\frac{n+2}{2}})_{xxx}^2 + (\Phi^{\frac{q+1}{2}}(\Gamma))_x^2 \}\,dxdt \leqslant C_5 \sum_{i=1}^4 \frac{G_i(s)}{\delta^{\alpha_i}}, \tag{3.23}$$

where

$$G_1(s) := \int\limits_0^T \Big(\int\limits_{\Omega(s)} h\,dx \Big)^{n+2} dt,\ \alpha_1 = n+4,$$

$$G_2(s) := \gamma^{\frac{2(n+7)}{n-m+2}}(T) \int\limits_0^T \Big(\int\limits_{\Omega(s)} h\,dx \Big)^{\frac{5m-3n+4}{n-m+2}} dt,\ \alpha_2 = \frac{3m-n+8}{n-m+2},$$

$$G_3(s) := \int\limits_0^T \Big(\int\limits_{\Omega(s)} \Phi(\Gamma)\,dx \Big)^{\frac{2(q+1)+\nu(n+2)}{2(q+3)-\nu(n+2)}} dt,\ \alpha_3 = \frac{3[2(q+1)+\nu(n+2)]}{2(q+3)-\nu(n+2)},$$

$$G_4(s) := \int\limits_0^T \Big(\int\limits_{\Omega(s)} \Phi(\Gamma)\,dx \Big)^{q+1} dt,\ \alpha_4 = 3(q+1).$$

Thus, taking into account $G_i(s) = 0$ for all $s \geqslant 2r_0$, (3.23) yields

$$G_i(s+\delta) \leqslant C_6 T\, \gamma^{\mu_i}(T) \Big(\sum_{k=1}^4 \frac{G_k(s)}{\delta^{\alpha_k+3}} \Big)^{\beta_i} \tag{3.24}$$

for all $s > r_0$ and $\delta > 0$, where $\mu_1 = \mu_3 = \mu_4 = 0$, $\mu_2 = \frac{2(n+7)}{n-m+2}$, $\beta_1 = \frac{n+2}{2}$, $\beta_2 = \frac{5m-3n+4}{2(n-m+2)}$, $\beta_3 = \frac{2(q+1)+\nu(n+2)}{2(q+3)-\nu(n+2)}$, $\beta_4 = q+1$. By Lemma B.2 with $s_1 = 0$, we find that $G_i(s_0) = 0$, where

$$\gamma(T) \leqslant s_0(T) = C_7 \Big(T^{\frac{1}{\alpha_1+3}} + T^{\frac{1}{\alpha_2+3}} \gamma^{\frac{\mu_2}{\alpha_2+3}}(T) + T^{\frac{1}{6}+\frac{2-\nu(n+2)}{12(q+2)}} + T^{\frac{1}{\alpha_4+3}} \Big).$$

As $\frac{\mu_2}{\alpha_2+3} = 1$, then we obtain (2.23) for enough small $T_0 > 0$ and for any $0 < T \leqslant T_0$.

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4 Waiting-time phenomenon

Similarly to (3.15) using the Hardy inequality (3.16), we find that

$$\frac{C_0}{2}\int\limits_{\Omega(s+\delta)} h^2\,dx+\delta^2\int\limits_{\Omega(s+\delta)}\Phi(\Gamma)\,dx+\delta^2C_1\iint\limits_{Q_T(s+\delta)}(h^{\frac{n+2}{2}})^2_{xxx}\,dxdt$$
$$+\delta^2C_1\iint\limits_{Q_T(s+\delta)}(\Phi^{\frac{q+1}{2}}(\Gamma))^2_x\,dxdt\leqslant\delta^{-4}H_0(s)$$
$$+\frac{C_3}{\delta^4}\iint\limits_{Q_T(s)}\{h^{n+2}+\Phi^{\frac{v(n+2)}{2}}(\Gamma)+\Phi^{q+1}(\Gamma)+h^{3m-2n+2}\}\,dxdt \tag{4.1}$$

for all $T\leqslant T_0$, $s\in\mathbb{R}^1$ and $\delta>0$, where, due to (2.24),

$$H_0(s):=\frac{1}{2}\int\limits_{\Omega(s)}(x-s)^6_+\{h^2_{0x}+2\Phi(\Gamma_0)\}dx,\ \ H_0(s)=0\ \forall\,s\geqslant r_0. \tag{4.2}$$

By the Nirenberg–Gagliardo, Hölder and Young inequalities, after simple transformations, for $\epsilon_i>0$, we have

$$\frac{C_3}{\delta^4}\iint\limits_{Q_T(s)}h^{n+2}\,dxdt\leqslant\epsilon_1\delta^2\iint\limits_{Q_T(s)}(h^{\frac{n+2}{2}})^2_{xxx}\,dxdt+\frac{C(\epsilon_1)}{\delta^{\frac{n+8}{2}}}\int\limits_0^T\Big(\int\limits_{\Omega(s)}h^2\,dx\Big)^{\frac{n+2}{2}}dt, \tag{4.3}$$

$$\frac{C_3}{\delta^4}\iint\limits_{Q_T(s)}h^{3m-2n+2}\,dxdt\leqslant\epsilon_2\delta^2\iint\limits_{Q_T(s)}(h^{\frac{n+2}{2}})^2_{xxx}\,dxdt$$
$$+\frac{C(\epsilon_2)}{\delta^{\frac{2(m+8)}{n-m+4}}}\int\limits_0^T\Big(\int\limits_{\Omega(s)}h^2\,dx\Big)^{\frac{2(q+1)+v(n+2)}{2(q+3)-v(n+2)}}dt \tag{4.4}$$

for $\frac{2n}{3}<m<n+4$,

$$\frac{C_3}{\delta^4}\iint\limits_{Q_T(s)}\Phi^{\frac{v(n+2)}{2}}(\Gamma)\,dxdt\leqslant\epsilon_3\delta^2\iint\limits_{Q_T(s)}(\Phi^{\frac{q+1}{2}}(\Gamma))^2_x\,dxdt$$
$$+\frac{C(\epsilon_3)}{\delta^{\frac{2[4(q+2)+v(n+2)-2]}{2(q+3)-v(n+2)}}}\int\limits_0^T\Big(\int\limits_{\Omega(s)}\Phi(\Gamma)\,dx\Big)^{\frac{2(q+1)+v(n+2)}{2(q+3)-v(n+2)}}dt, \tag{4.5}$$

$$\frac{C_3}{\delta^4}\iint\limits_{Q_T(s)}\Phi^{q+1}(\Gamma)\,dxdt\leqslant\epsilon_4\delta^2\iint\limits_{Q_T(s)}(\Phi^{\frac{q+1}{2}}(\Gamma))^2_x\,dxdt$$
$$+\frac{C(\epsilon_4)}{\delta^{3q+4}}\int\limits_0^T\Big(\int\limits_{\Omega(s)}\Phi(\Gamma)\,dx\Big)^{q+1}dt. \tag{4.6}$$

Substituting the estimates (4.3)–(4.6) to (4.1) and again making the standard iterative procedure similar to [19, Lemma 4.2] for small enough $0<\epsilon_i<1$, we arrive at the

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inequality

$$\frac{C_0}{2}\int\limits_{\Omega(s+\delta)} h^2\,dx + \delta^2 \int\limits_{\Omega(s+\delta)} \Phi(\Gamma)\,dx + C_8\delta^2 \iint\limits_{Q_T(s+\delta)} (h^{\frac{n+2}{2}})^2_{xxx}\,dxdt$$
$$+ C_8\delta^2 \iint\limits_{Q_T(s+\delta)} (\Phi^{\frac{q+1}{2}}(\Gamma))^2_x\,dxdt \leqslant \delta^{-4}H_0(s) + C_9\sum_{i=1}^{4}\frac{G_T^{(i)}(s)}{\delta^{\alpha_i}}, \tag{4.7}$$

where

$$G_T^{(1)}(s) := \int_0^T \Big(\int\limits_{\Omega(s)} h^2\,dx\Big)^{\frac{n+2}{2}} dt,\ \alpha_1 = \frac{n+8}{2},$$

$$G_T^{(2)}(s) := \int_0^T \Big(\int\limits_{\Omega(s)} h^2\,dx\Big)^{\frac{5m-3n+4}{n-m+4}} dt,\ \alpha_2 = \frac{2(m+8)}{n-m+4},$$

$$G_T^{(3)}(s) := \int_0^T \Big(\int\limits_{\Omega(s)} \Phi(\Gamma)\,dx\Big)^{\frac{2(q+1)+\nu(n+2)}{2(q+3)-\nu(n+2)}} dt,\ \alpha_3 = \frac{2[4(q+2)+\nu(n+2)-2]}{2(q+3)-\nu(n+2)},$$

$$G_T^{(4)}(s) := \int_0^T \Big(\int\limits_{\Omega(s)} \Phi(\Gamma)\,dx\Big)^{q+1} dt,\ \alpha_4 = 3q+4.$$

Hence, from (4.7), we deduce

$$G_T^{(i)}(s+\delta) \leqslant C_{10}T\Big(\sum_{k=1}^{4}\frac{G_T^{(k)}(s)}{\delta^{\alpha_k+2}} + \delta^{-6}H_0(s)\Big)^{\beta_i}$$
$$= C_{10}T\Big(\sum_{k=1}^{4}\delta^{-(\alpha_k+2)}\big(G_T^{(k)}(s) + \delta^{\alpha_k-4}H_0(s)\big)\Big)^{\beta_i} \tag{4.8}$$

for all $s \in \mathbb{R}^1$ and $\delta > 0$, where $\beta_1 = \frac{n+2}{2}$, $\beta_2 = \frac{5m-3n+4}{n-m+4}$, $\beta_3 = \frac{2(q+1)+\nu(n+2)}{2(q+3)-\nu(n+2)}$, $\beta_4 = q+1$. From (4.2) and (2.25), we find

$$H_0(s) \leqslant (r_0 - s)_+^6\,\mathbf{k}_0(s) \leqslant \chi(r_0 - s)_+^{\gamma+5}. \tag{4.9}$$

Assume that $s+\delta \leqslant r_0$. As $\alpha_k > 4$, then from (4.8), due to (4.9), we arrive at

$$G_T^{(i)}(s+\delta) \leqslant C_{10}T\Big(\sum_{k=1}^{4}\delta^{-(\alpha_k+2)}\big(G_T^{(k)}(s) + \chi(r_0 - s)_+^{\alpha_k+1+\gamma}\big)\Big)^{\beta_i} \tag{4.10}$$

for all $s+\delta \leqslant r_0,\ \delta > 0$. Let us denote by

$$\bar{G}_T^{(i)}(s) := G_T^{(i)}(s) + \chi(r_0 - s)_+^{\alpha_k+1+\gamma} \geqslant 0.$$

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Then, (4.10) can be rewritten in the form

$$\bar{G}_T^{(i)}(s+\delta) \leqslant C_{10}T\Big(\sum_{k=1}^{4}\delta^{-(\alpha_k+2)}\bar{G}_T^{(k)}(s)\Big)^{\beta_i} + \chi(r_0-s)_+^{\alpha_i+1+\gamma}$$
$$\leqslant C_{10}T\Big(\sum_{k=1}^{4}\delta^{-(\alpha_k+2)}\bar{G}_T^{(k)}(s) + \big(\frac{\chi}{C_{10}T}\big)^{\frac{1}{\beta_i}}(r_0-s)_+^{\frac{\alpha_i+1+\gamma}{\beta_i}}\Big)^{\beta_i}, \qquad (4.11)$$

where we use the simple inequality $a^m + b^m \leqslant (a+b)^m$ for positive a, b and $m > 1$. Let us now check that all conditions of Lemma B.4 are satisfied. We denote

$$G_{\max}(s) := \max_{i=\overline{1,4}}\{c_0 2^{\beta+1}(\sum_{k=1}^{4}\Big(\bar{G}_k(s)\Big)^{\overline{\beta}_k})^{\beta_i-1}(s)\}^{\frac{1}{(\alpha_i+2)\beta}},$$

$$g_{\max}(s) := \max_{i=\overline{1,4}}\{2^{\frac{\beta+1}{(\alpha_i+2)\beta}}\Big(2^{\beta-1}\sum_{k=1}^{4}(C_{10}T)^{\overline{\beta}_k}\Big)^{\frac{\overline{\beta}_i}{(\alpha_i+2)}}\big(\big(\frac{\chi}{C_{10}T}\big)^{\frac{1}{\beta_i}}(r_0-s)_+^{\frac{\alpha_i+3+\gamma}{\beta_i}}\big)^{\frac{\beta_i-1}{\alpha_i+2}}\},$$

$$c_0 = 2^{\beta-1}\sum_{k=1}^{4}(C_{10}T)^{\overline{\beta}_k},\ \beta = \beta_1\beta_2\beta_3\beta_4.$$

Taking $s = -2\delta$ in (4.11) and passing to the limit $\delta \to \infty$, due to the boundedness of functions $\bar{G}_k(s)$, we deduce

$$\bar{G}_k(-\infty) \leqslant \chi r_0^{\alpha_k+1+\gamma}. \qquad (4.12)$$

This implies that the condition (i) of Lemma B.4 is fulfilled for $s_0 = r_0$. Because of the assumption (2.25) on the function $\mathbf{k}_0(s)$ and

$$\gamma \geqslant \min_{i,k=\overline{1,4}}\Big\{\frac{\beta_k(\alpha_i+2)}{\beta_i-1} - \alpha_k - 1\Big\} = \min\Big\{\frac{2(n+6)}{n}, \frac{3m-n+12}{3m-2n}, 1+\frac{6(q+2)}{v(n+2)-2}, 4+\frac{6}{q}\Big\}$$
$$= \min\Big\{\frac{2(n+6)}{n}, \frac{3m-n+12}{3m-2n}, 1+\frac{6(q+2)}{v(n+2)-2}\Big\}, \qquad (4.13)$$

we can find T^* such that the condition (ii) of Lemma B.4 is valid for all $T \in [0, T^*]$. Here, $T^* = T^*(\chi)$ goes to infinity as $\chi \to 0$. Hence, the application of Lemma B.4 ends the proof.

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Appendix A Proof of lemma 3.1

Given $\delta, \varepsilon > 0$, a regularised parabolic problem, similar to that of [10], is considered:

$$\begin{cases} h_t + (f_n^{\delta\varepsilon}(h_{xxx} - h_x + (F_{n,m}^{\varepsilon})'' h_x))_x + (f_{n-1}^{\varepsilon}\sigma_x^{\varepsilon})_x = 0, & \text{(A.1)} \\ \Gamma_t + (\Gamma f_{n-1}^{\varepsilon}(h_{xxx} - h_x + (F_{n,m}^{\varepsilon})'' h_x))_x + (\Gamma f_{n-2}^{\varepsilon}\sigma_x^{\varepsilon})_x & \\ \qquad + \delta(\frac{1}{\Phi_\varepsilon''(\Gamma)}\Gamma_{xxx})_x = (D^\varepsilon(\Gamma)\Gamma_x)_x, & \text{(A.2)} \\ h_x(\pm a, t) = h_{xxx}(\pm a, t) = \Gamma_x(\pm a, t) = \Gamma_{xxx}(\pm a, t) = 0 \text{ for } t > 0, & \text{(A.3)} \\ \quad h(x, 0) = h_{0,\varepsilon}(x),\ \Gamma(x, 0) = \Gamma_{0,\varepsilon}(x), & \text{(A.4)} \end{cases}$$

where

$$f_k^{\delta\varepsilon}(z) := f_k^{\varepsilon}(z) + \delta = \frac{|z|^s f_k(z)}{|z|^s + \varepsilon f_n(z)} + \delta,\ (F_{n,m}^{\varepsilon})''(z) := \frac{|z|^m}{f_n(z) + \varepsilon |z|^m}, \tag{A.5}$$

$$D^\varepsilon(z) \in C^{1+\gamma}(\mathbb{R}^1) : D^\varepsilon(z) \geqslant \varepsilon, \tag{A.6}$$

$$\Phi_\varepsilon''(z) \geqslant \begin{cases} \Phi''(\varepsilon) \text{ for } z < \varepsilon, \\ \Phi''(z) \text{ for } \varepsilon \leqslant z \leqslant 1 - \varepsilon, \\ \Phi''(1 - \varepsilon) \text{ for } z > 1 - \varepsilon, \end{cases} \quad \frac{1}{\Phi_\varepsilon''(z)} \in C^{1+\gamma}(\mathbb{R}^1), \quad (\sigma^\varepsilon)'(z) = -z\Phi_\varepsilon''(z) \tag{A.7}$$

$\forall\, z \in \mathbb{R}^1,\ \delta > 0,\ \varepsilon > 0,\ 2 \leqslant k \leqslant n,\ s \geqslant \max\{n, 8\},\ \gamma \in (0, 1)$. Note that the parameter n is fixed in the definition of $f_k^{\delta\varepsilon}(z)$.

We construct $\Phi_\varepsilon(z)$ that satisfy (A.7) and also satisfy $\Phi_\varepsilon(z) = \Phi(z)$ if $\varepsilon \leqslant z \leqslant 1 - \varepsilon$. The $\delta > 0$ in (A.5) and (A.6) makes the system (A.1)–(A.2) regular (i. e. uniformly parabolic). The parameter ε is an approximating parameter which has the effect of increasing the degeneracy from $f_k(h) \sim |h|^k$ to $f_k^\varepsilon(h) \sim \varepsilon^{-1}|h|^{s+k-n}$. This will allow us to construct positive approximations $h_\varepsilon > 0$ when $\delta \to 0$. The initial data, (h_0, Γ_0), are approximated via

$$\begin{gathered} h_0 + \varepsilon^\theta \leqslant h_{0,\varepsilon} \in C^{4+\gamma}(\Omega) \text{ for some } 0 < \theta < 2/(s-4) \\ h_{0,\varepsilon} \underset{\varepsilon \to 0}{\to} h_0 \text{ strongly in } H^1(\Omega), \\ \Gamma_{0,\varepsilon} \in C^{4+\gamma}(\Omega) : \Gamma_{0,\varepsilon} \in [\varepsilon, 1 - \varepsilon],\ \Gamma_{0,\varepsilon} \underset{\varepsilon \to 0}{\to} \Gamma_0 \text{ strongly in } L^2(\Omega). \end{gathered} \tag{A.8}$$

The ε term in (A.8) “lifts” the initial data so that they are smoothing from $H^1(\Omega) \times L^2(\Omega)$ to $C^{4+\gamma}(\Omega) \times C^{4+\gamma}(\Omega)$. Note that the system (A.1)–(A.2) without the term $\delta(\frac{1}{\Phi_\varepsilon''(\Gamma)}\Gamma_{xxx})_x$ is uniformly parabolic in the sense of Douglis–Nirenberg because $f_n^{\delta\varepsilon}(h) \geqslant \delta > 0$ and

$$\begin{aligned} &f_n^{\delta\varepsilon}(h)\big(D^\varepsilon(\Gamma) - \Gamma f_{n-2}^\varepsilon(h)(\sigma^\varepsilon)'(\Gamma)\big) + \Gamma (f_{n-1}^\varepsilon(h))^2 (\sigma^\varepsilon)'(\Gamma) \\ &\qquad = f_n^{\delta\varepsilon}(h) D^\varepsilon(\Gamma) + \Gamma^2 \Phi_\varepsilon''(\Gamma)\big(f_n^{\delta\varepsilon}(h) f_{n-2}^\varepsilon(h) - (f_{n-1}^\varepsilon(h))^2\big) \geqslant \delta\varepsilon > 0. \end{aligned}$$

The term $\delta(\frac{1}{\Phi_\varepsilon''(\Gamma)}\Gamma_{xxx})_x$ makes this system uniformly parabolic in the sense of Petrovskii. Moreover, the boundary conditions (A.3) are of Lopatinskii–Shapiro type. This implies existence of a proper continuation on the whole line (cf. [35] for example). Using the parabolic Schauder estimates from [35], one can generalise [15, Theorem 6.3, p. 302] and

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show that the regularised problem has a unique classical solution $(h_{\delta\varepsilon}, \Gamma_{\delta\varepsilon}) \in C_{x,t}^{4+\gamma,1+\gamma/4}(\Omega \times [0,\tau_{\delta\varepsilon}]) \times C_{x,t}^{4+\gamma,1+\gamma/4}(\Omega \times [0,\tau_{\delta\varepsilon}])$ for sometime $\tau_{\delta\varepsilon} > 0$.

Now, we assume that $\tau_{\delta\varepsilon}$ is the local existence time from [15, Theorem 6.3, p. 302]. Following Kamin and Vazquez, we introduce change of variables (see [28, p. 41])

$$v_{\delta\varepsilon}(x,t) := \int_{-a}^{x} (h_{\delta\varepsilon}(y,t) - \bar{h}_\varepsilon)\, dy, \;\; u_{\delta\varepsilon}(x,t) := \int_{-a}^{x} (\Gamma_{\delta\varepsilon}(y,t) - \bar{\Gamma}_\varepsilon)\, dy,$$

where $\bar{h}_\varepsilon := \frac{1}{|\Omega|}\int_\Omega h_{0\varepsilon}(x)\,dx$, $\bar{\Gamma}_\varepsilon := \frac{1}{|\Omega|}\int_\Omega \Gamma_{0\varepsilon}(x)\,dx$. Then, $(v_{\delta\varepsilon}, u_{\delta\varepsilon})$ is a solution of the following problem:

$$\begin{cases} v_{\delta\varepsilon,t} + f_n^{\delta\varepsilon}(v_{\delta\varepsilon,x} + \bar{h}_\varepsilon) J_{\delta\varepsilon}(v_{\delta\varepsilon}) \\ \qquad\qquad + f_{n-1}^{\varepsilon}(v_{\delta\varepsilon,x} + \bar{h}_\varepsilon)(\sigma^\varepsilon(u_{\delta\varepsilon,x} + \bar{\Gamma}_\varepsilon))' u_{\delta\varepsilon,xx} = 0, & \text{(A.9)} \\ u_{\delta\varepsilon,t} + (u_{\delta\varepsilon,x} + \bar{\Gamma}_\varepsilon) f_{n-1}^{\varepsilon} J_{\delta\varepsilon}(v_{\delta\varepsilon}) \\ \quad + ((u_{\delta\varepsilon,x} + \bar{\Gamma}_\varepsilon) f_{n-2}^{\varepsilon}(v_{\delta\varepsilon,x} + \bar{h}_\varepsilon)(\sigma^\varepsilon(u_{\delta\varepsilon,x} + \bar{\Gamma}_\varepsilon))' \\ \qquad\qquad - D^\varepsilon(u_{\delta\varepsilon,x} + \bar{\Gamma}_\varepsilon)) u_{\delta\varepsilon,xx} + \dfrac{\delta}{\Phi_\varepsilon''(u_{\delta\varepsilon,x} + \bar{\Gamma}_\varepsilon)} u_{\delta\varepsilon,xxxx} = 0, & \text{(A.10)} \\ v_{\delta\varepsilon}(\pm a,t) = v_{\delta\varepsilon,xx}(\pm a,t) = u_{\delta\varepsilon}(\pm a,t) = u_{\delta\varepsilon,xx}(\pm a,t) = 0, & \text{(A.11)} \end{cases}$$

where $J_{\delta\varepsilon}(v_{\delta\varepsilon}) := v_{\delta\varepsilon,xxxx} - v_{\delta\varepsilon,xx} + (F_{n,m}^\varepsilon(v_{\delta\varepsilon,x} + \bar{h}_\varepsilon))'' v_{\delta\varepsilon,xx}$.

The problem (A.9)–(A.11) has a unique classic solution $(v_{\delta\varepsilon}, u_{\delta\varepsilon}) \in C_{x,t}^{4+\gamma,1+\gamma/4}(\Omega \times [0,\tau_{\delta\varepsilon}]) \times C_{x,t}^{4+\gamma,1+\gamma/4}(\Omega \times [0,\tau_{\delta\varepsilon}])$. It follows that $\|v_{\delta\varepsilon}\|_{C^{\frac{3}{2},\frac{3}{8}}(\bar{Q}_T)}$ and $\|u_{\delta\varepsilon}\|_{C^{\frac{1}{2},\frac{1}{18}}(\bar{Q}_T)}$ are uniformly bounded with respect to δ, ε and $\tau_{\delta\varepsilon}$. Note that these uniform bounds follow from

$$(v_{\delta\varepsilon}, u_{\delta\varepsilon}) \in L^\infty(0,T;H^2(\Omega)) \times L^\infty(0,T;H^1(\Omega))$$

and

$$(v_{\delta\varepsilon,t}, u_{\delta\varepsilon,t}) \in L^2(Q_T) \times L^{\frac{4}{3}}(0,T;(H^1(\Omega))^*)$$

(cf. for details [43, Lemma 7.19, p.175] to $v_{\delta\varepsilon}$ and [1, Lemma 2.1, p.183] to $u_{\delta\varepsilon}$). Therefore, for any fixed values of δ, ε and $T_{loc} > \tau_{\delta\varepsilon}$, by Eĭdelman [15, Theorem 9.3, p.316], we can continue our local in time solution $(v_{\delta\varepsilon}, u_{\varepsilon\delta})$, and as a consequence $(h_{\delta\varepsilon}, \Gamma_{\varepsilon\delta})$ from $[0,\tau_{\delta\varepsilon}]$ to $[\tau_{\delta\varepsilon}, T_{loc}]$. Indeed, since $(v_{\delta\varepsilon}, u_{\delta\varepsilon})$ is continuous with $(v_{\delta\varepsilon,x}, u_{\delta\varepsilon,x})$ continuous, then $(v_{\delta\varepsilon,x} + \bar{h}_\varepsilon, u_{\delta\varepsilon,x} + \bar{\Gamma}_\varepsilon)$ is a solution of (A.1)–(A.4) and $(h_{\delta\varepsilon}, \Gamma_{\varepsilon\delta}) = (v_{\delta\varepsilon,x} + \bar{h}_\varepsilon, u_{\delta\varepsilon,x} + \bar{\Gamma}_\varepsilon)$. Hence, we can continue a solution $(h_{\delta\varepsilon}, \Gamma_{\varepsilon\delta})$ of the original problem (A.1)–(A.4) until time T_{loc} that does not depend on δ and ε, see [10] for details. We also omit the details of the limit process, positivity of h_ε and upper and lower bounds $0 \leqslant \Gamma \leqslant 1$.

Lemma A.1 (*entropy estimate*) *Let $(h_\varepsilon, \Gamma_\varepsilon)$ be a solution of the problem (A.1)–(A.4), where $\delta = 0$, $2 \leqslant n \leqslant \frac{5}{2}$, $m \geqslant \frac{2n-3}{2}$. Then, there exist independent of ε, positive constants c_1, c_2 and*

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c_3 such that the following entropy estimate

$$\int\limits_{\Omega} G_\varepsilon(h)\,dx + c_1 \iint\limits_{Q_T} h^\alpha h_{xx}^2\,dxdt + c_2 \iint\limits_{Q_T} h^{\alpha-2} h_x^4\,dxdt \leqslant \int\limits_{\Omega} G_\varepsilon(h_0)\,dx + c_3 \tag{A.12}$$

holds for any $\alpha \in [\max\{-2(m-n+1), \frac{2}{3}(n-1)\}, 1]$, where $c_2 = 0$ if $\alpha = 1$.

Proof of Lemma A.1. First, we obtain the entropy estimate for the solution $(h_\varepsilon, \Gamma_\varepsilon)$, where $h_\varepsilon > 0$ (see for a proof of positivity [10, Section A.4, p.122]. Multiplying (A.1) with $\delta = 0$ by $G'_\varepsilon(h)$, where $G''_\varepsilon(z) = \frac{z^\alpha}{f_n^\varepsilon(z)}$, and integrating over Ω, we deduce that

$$\frac{d}{dt}\int\limits_{\Omega} G_\varepsilon(h)\,dx + \int\limits_{\Omega} (h^\alpha h_{xx}^2 + h^\alpha h_x^2)\,dx + \frac{\alpha(1-\alpha)}{3}\int\limits_{\Omega} h^{\alpha-2} h_x^4\,dx$$
$$= \int\limits_{\Omega} h^\alpha (F_{n,m}^\varepsilon)''(h) h_x^2\,dx + \int\limits_{\Omega} \frac{h^\alpha f_{n-1}^\varepsilon(h)}{f_n^\varepsilon(h)} \sigma_x^\varepsilon h_x\,dx. \tag{A.13}$$

Due to the Cauchy inequality, we arrive at the estimates

$$\int\limits_{\Omega} h^\alpha (F_{n,m}^\varepsilon)''(h) h_x^2\,dx \leqslant n \int\limits_{\Omega} h^{m-n+\alpha} h_x^2\,dx = -\frac{n}{m-n+\alpha+1}\int\limits_{\Omega} h^{m-n+\alpha+1} h_{xx}\,dx$$
$$\leqslant \epsilon_1 \int\limits_{\Omega} h^\alpha h_{xx}^2\,dx + \frac{n^2}{4\epsilon_1(m-n+\alpha+1)^2}\int\limits_{\Omega} h^{2(m-n+1)+\alpha}\,dx,$$
$$\int\limits_{\Omega} \frac{h^\alpha f_{n-1}^\varepsilon(h)}{f_n^\varepsilon(h)} \sigma_x^\varepsilon h_x\,dx \leqslant \epsilon_2 \int\limits_{\Omega} h^{2\alpha-n} h_x^2\,dx + \frac{n^2(n-2)}{4\epsilon_2(n-1)^2}\int\limits_{\Omega} f_{n-2}^\varepsilon(h)(\sigma_x^\varepsilon)^2\,dx$$
$$= -\frac{\epsilon_2}{2\alpha-n+1}\int\limits_{\Omega} h^{2\alpha-n+1} h_{xx}\,dx + \frac{n^2(n-2)}{4\epsilon_2(n-1)^2}\int\limits_{\Omega} f_{n-2}^\varepsilon(h)(\sigma_x^\varepsilon)^2\,dx$$
$$\leqslant \epsilon_3 \int\limits_{\Omega} h^\alpha h_{xx}^2\,dx + \frac{\epsilon_2^2}{4\epsilon_3(2\alpha-n+1)^2}\int\limits_{\Omega} h^{3\alpha-2n+2}\,dx$$
$$+ \frac{n^2(n-2)}{4\epsilon_2(n-1)^2}\int\limits_{\Omega} f_{n-2}^\varepsilon(h)(\sigma_x^\varepsilon)^2\,dx$$

for arbitrary $\epsilon_i > 0$. Using these estimates in (A.13), we find that

$$\frac{d}{dt}\int\limits_{\Omega} G_\varepsilon(h)\,dx + (1-\epsilon_1-\epsilon_3)\int\limits_{\Omega} h^\alpha h_{xx}^2\,dx + \frac{\alpha(1-\alpha)}{3}\int\limits_{\Omega} h^{\alpha-2} h_x^4\,dx$$
$$\leqslant \frac{n^2}{4\epsilon_1(m-n+\alpha+1)^2}\int\limits_{\Omega} h^{2(m-n+1)+\alpha}\,dx + \frac{\epsilon_2^2}{4\epsilon_3(2\alpha-n+1)^2}\int\limits_{\Omega} h^{3\alpha-2n+2}\,dx$$
$$+ \frac{n^2(n-2)}{4\epsilon_2(n-1)^2}\int\limits_{\Omega} f_{n-2}^\varepsilon(h)(\sigma_x^\varepsilon)^2\,dx. \tag{A.14}$$

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Choosing $\epsilon_i > 0$ such that $\epsilon_1 + \epsilon_3 < 1$, integrating (A.14) in time, and taking into account uniform boundedness of $\|h_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}$ and $\|(f^\varepsilon_{n-2}(h_\varepsilon))^{1/2}\sigma^\varepsilon_x\|_{L^2(Q_T)}$ in $\varepsilon > 0$ (see [10]), from (A.14), we deduce the entropy estimate. □

By (A.12), we have

$$\{h_\varepsilon^{\frac{\alpha}{2}} h_{\varepsilon,xx}\}_{\varepsilon>0} \text{ is uniformly bounded in } L^2(Q_T), \tag{A.15}$$

$$\{h_\varepsilon^{\frac{\alpha-2}{4}} h_{\varepsilon,x}\}_{\varepsilon>0} \text{ is uniformly bounded in } L^4(Q_T). \tag{A.16}$$

In particular, using (A.15) and (A.16) with $\varepsilon \to 0$, we can show that our solution $h \in C^1(\bar{\Omega})$ for a. e. $t > 0$ (proof is similar to [1,Theorem 3.1, p.190]). Note that this regularity allows us to take limit in local energy estimate when $\varepsilon \to 0$ on the set $\{h \leqslant \mu\}$.

Next, we obtain the local energy estimate for the solution $(h_\varepsilon, \Gamma_\varepsilon)$, where $h_\varepsilon > 0$. Multiplying (A.1) with $\delta = 0$ by $-(\phi h_x)_x$ and integrating in Q_T, we find

$$\begin{aligned}
&\frac{1}{2}\int\limits_\Omega h_x^2\phi\,dx - \frac{1}{2}\iint\limits_{Q_T} h_x^2\phi_t\,dxdt + \iint\limits_{Q_T} f_n^\varepsilon(h)h_{xxx}^2\phi\,dxdt \\
&\quad = \frac{1}{2}\int\limits_\Omega h_{0,x}^2\phi\,dx + \iint\limits_{Q_T} f_n^\varepsilon(h)(1-(F_{n,m}^\varepsilon)''(h))h_x h_{xxx}\phi\,dxdt \\
&\quad - \iint\limits_{Q_T} f_n^\varepsilon(h)(h_{xxx} - h_x + (F_{n,m}^\varepsilon)''(h)h_x)(2h_{xx}\phi_x + h_x\phi_{xx})\,dxdt \\
&\quad - \iint\limits_{Q_T} f_{n-1}^\varepsilon(h)\sigma_x^\varepsilon(h_{xxx}\phi + 2h_{xx}\phi_x + h_x\phi_{xx})\,dxdt.
\end{aligned} \tag{A.17}$$

It is easy to check, using *a priori* estimates obtained in [10], that the integrals on the right-hand side of (A.17) are uniformly bounded with respect to $\varepsilon > 0$. For arbitrary $\mu > 0$, $h_\varepsilon \to h$ strongly in the space $C^{4,1}_{x,t}(\{h > \mu\})$. Therefore, passage to the limit $\varepsilon \to 0$ in all of the integrals in (A.17) over the domain $\{h > \mu\}$ is straightforward. As to integrals over the domain $\{h \leqslant \mu\}$, we have, for example, by virtue of (A.15) and uniform boundedness of $\|(f^\varepsilon_{n-2}(h_\varepsilon))^{1/2}\sigma^\varepsilon_x\|_{L^2(Q_T)}$ in $\varepsilon > 0$,

$$\begin{aligned}
&\iint\limits_{\{h\leqslant\mu\}} f_{n-1}^\varepsilon(h)\sigma_x^\varepsilon h_{xx}\phi_x\,dxdt \\
&\quad \leqslant \Big(\iint\limits_{Q_T} f_{n-2}^\varepsilon(h)(\sigma_x^\varepsilon)^2\phi_x^2\,dxdt\Big)^{\frac{1}{2}}\Big(\iint\limits_{\{h\leqslant\mu\}} \frac{(f_{n-1}^\varepsilon(h))^2}{f_{n-2}^\varepsilon(h)} h_{xx}^2\,dxdt\Big)^{\frac{1}{2}} \\
&\quad \leqslant C\mu^{\frac{n-\alpha}{2}}\Big(\iint\limits_{Q_T} h^\alpha h_{xx}^2\,dxdt\Big)^{\frac{1}{2}} \leqslant C\mu^{\frac{n-\alpha}{2}} \to 0 \text{ as } \mu \to 0.
\end{aligned}$$

Analogously, it is easy to check that all of the other integrals in (A.17) over $\{h \leqslant \mu\}$ are bounded from above by some continuous function, $k(\mu)$, such that $k(\mu) \to 0$ as $\mu \to 0$.

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Therefore, first passing to the limit $\varepsilon \to 0$, and afterwards letting $\mu \to 0$, we easily obtain

$$\begin{aligned}
&\frac{1}{2}\int\limits_{\Omega} h_x^2\phi\,dx - \frac{1}{2}\iint\limits_{Q_T} h_x^2\phi_t\,dxdt + \iint\limits_{\{h>0\}} f_n(h)h_{xxx}^2\phi\,dxdt \\
&\quad\leqslant \frac{1}{2}\int\limits_{\Omega} h_{0,x}^2\phi\,dx + \iint\limits_{\{h>0\}} f_n(h)(1-F''_{n,m}(h))h_x h_{xxx}\phi\,dxdt \\
&\quad - \iint\limits_{\{h>0\}} f_n(h)(h_{xxx} - h_x + F''_{n,m}(h)h_x)(2h_{xx}\phi_x + h_x\phi_{xx})\,dxdt \\
&\quad - \iint\limits_{\{h>0\}} f_{n-1}(h)\sigma_x(h_{xxx}\phi + 2h_{xx}\phi_x + h_x\phi_{xx})\,dxdt \\
&\quad = \frac{1}{2}\int\limits_{\Omega} h_{0,x}^2\phi\,dx + I_1 + I_2 + I_3. \qquad (A.18)
\end{aligned}$$

Now, we bound the terms I_i. First,

$$\begin{aligned}
I_1 &= -\iint\limits_{\{h>0\}} f_n(h)h_{xx}^2\phi\,dxdt + \frac{1}{3}\iint\limits_{\{h>0\}} f''_n(h)h_x^4\phi\,dxdt + \frac{5}{6}\iint\limits_{\{h>0\}} f'_n(h)h_x^3\phi_x\,dxdt \\
&+\frac{1}{2}\iint\limits_{\{h>0\}} f_n(h)h_x^2\phi_{xx}\,dxdt - \iint\limits_{\{h>0\}} f_n(h)F''_{n,m}(h)h_{xxx}h_x\phi\,dxdt \\
&\leqslant -\iint\limits_{\{h>0\}} f_n(h)h_{xx}^2\phi\,dxdt + \epsilon_1\iint\limits_{\{h>0\}} (f_n(h)h_{xxx}^2 + h^{n-4}h_x^6)\phi\,dxdt \\
&+ C(\epsilon_1)\iint\limits_{Q_T} h^{n+2}\Big(\phi + \frac{\phi_x^2}{\phi} + \frac{|\phi_{xx}|^{\frac{3}{2}}}{\phi^{\frac{1}{2}}}\Big)\,dxdt + C(\epsilon_1)\iint\limits_{Q_T} h^{3m-2n+2}\phi\,dxdt, \qquad (A.19)
\end{aligned}$$

$$\begin{aligned}
I_2 &= 2\iint\limits_{\{h>0\}} f_n(h)h_{xxx}h_{xx}\phi_x\,dxdt - 2\iint\limits_{\{h>0\}} f_n(h)h_{xx}h_x\phi_x\,dxdt \\
&+2\iint\limits_{\{h>0\}} f_n(h)F''_{n,m}(h)h_{xx}h_x\phi_x\,dxdt + \iint\limits_{\{h>0\}} f_n(h)h_{xxx}h_x\phi_{xx}\,dxdt \\
&-\iint\limits_{\{h>0\}} f_n(h)h_x^2\phi_{xx}\,dxdt + \iint\limits_{\{h>0\}} f_n(h)F''_{n,m}(h)h_x^2\phi_{xx}\,dxdt \\
&\leqslant \epsilon_2\iint\limits_{\{h>0\}} (f_n(h)h_{xxx}^2 + h^{n-1}|h_{xx}|^3 + h^{n-2}h_x^2h_{xx}^2 + h^{n-4}h_x^6)\phi\,dxdt \\
&+C(\epsilon_2)\iint\limits_{Q_T} h^{n+2}\Big(\frac{\phi_x^6}{\phi^5} + \frac{\phi_x^2}{\phi} + \frac{|\phi_{xx}|^{\frac{3}{2}}}{\phi^{\frac{1}{2}}} + \frac{|\phi_{xx}|^3}{\phi^2}\Big)\,dxdt + C(\epsilon_2)\iint\limits_{Q_T} h^{3m-2n+2}\phi\,dxdt,
\end{aligned}$$

(A.20)

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$$I_3 = -\iint\limits_{\{h>0\}} f_{n-1}(h)\sigma_x h_{xxx}\phi\,dxdt - 2\iint\limits_{\{h>0\}} f_{n-1}(h)\sigma_x h_{xx}\phi_x\,dxdt$$
$$-\iint\limits_{\{h>0\}} f_{n-1}(h)\sigma_x h_x\phi_{xx}\,dxdt \leqslant \epsilon_3\iint\limits_{\{h>0\}} f_n(h)h_{xxx}^2\phi\,dxdt$$
$$+\epsilon_3\iint\limits_{\{h>0\}} f_n(h)h_{xx}^2\frac{\phi_x^2}{\phi}\,dxdt + \epsilon_3\iint\limits_{\{h>0\}} f_n(h)h_x^2\frac{\phi_{xx}^2}{\phi}\,dxdt + \frac{3n(n-2)}{2\epsilon_3(n-1)^2}\iint\limits_{Q_T} f_{n-2}(h)\sigma_x^2\phi\,dxdt$$
$$\leqslant \epsilon_3\iint\limits_{\{h>0\}}(f_n(h)h_{xxx}^2 + h^{n-1}|h_{xx}|^3 + h^{n-4}h_x^6)\phi\,dxdt$$
$$+C(\epsilon_3)\iint\limits_{Q_T} h^{n+2}\left(\frac{\phi_x^6}{\phi^5} + \frac{|\phi_{xx}|^3}{\phi^2}\right)dxdt + \frac{3n(n-2)}{2\epsilon_3(n-1)^2}\iint\limits_{Q_T} f_{n-2}(h)\sigma_x^2\phi\,dxdt, \qquad \text{(A.21)}$$

where $f_{n-1}(z) = \left(\frac{n(n-2)}{(n-1)^2}f_n(z)f_{n-2}(z)\right)^{\frac{1}{2}}$, $0 < \frac{n(n-2)}{(n-1)^2} < 1$.

Multiplying (A.2) with $\delta = 0$ by $\Phi'_\varepsilon(\Gamma)\phi$ and integrating in Q_T, we deduce

$$\int\limits_{\Omega}\Phi_\varepsilon(\Gamma)\phi\,dx - \iint\limits_{Q_T}\Phi_\varepsilon(\Gamma)\phi_t\,dxdt$$
$$+\iint\limits_{Q_T} D^\varepsilon(\Gamma)\Phi''_\varepsilon(\Gamma)\Gamma_x^2\phi\,dxdt + \iint\limits_{Q_T} f_{n-2}^\varepsilon(h)(\sigma_x^\varepsilon)^2\phi\,dxdt = \int\limits_{\Omega}\Phi_\varepsilon(\Gamma_0)\phi\,dx$$
$$+\iint\limits_{Q_T}\tilde{D}_\varepsilon(\Gamma)\phi_{xx}dxdt + \iint\limits_{Q_T} f_{n-2}^\varepsilon(h)\Gamma\,\Phi'_\varepsilon(\Gamma)\sigma_x^\varepsilon\phi_x\,dxdt$$
$$-\iint\limits_{Q_T} f_{n-1}^\varepsilon(h)\sigma_x^\varepsilon(h_{xxx} - h_x + (F_{n,m}^\varepsilon)''(h)h_x)\phi\,dxdt$$
$$+\iint\limits_{Q_T}\Gamma\,\Phi'_\varepsilon(\Gamma)f_{n-1}^\varepsilon(h)(h_{xxx} - h_x + (F_{n,m}^\varepsilon)''(h)h_x)\phi_x\,dxdt, \qquad \text{(A.22)}$$

where $\tilde{D}_\varepsilon(z) := \int_1^z D^\varepsilon(s)\Phi'_\varepsilon(s)ds$. Without loss of generality, we will assume that estimates (2.20), (2.21) are valid for approximations $D^\varepsilon(z)$ and $\Phi_\varepsilon(z)$ for all $z \in \mathbb{R}^1$. Then from (A.22) with $\phi = 1$, we arrive at the uniform boundedness in $\varepsilon > 0$ of $\|\Phi_\varepsilon^{\frac{q+1}{2}}(\Gamma)\|_{L^2(0,T;H^1(\Omega))}$, $\|\Phi_\varepsilon(\Gamma)\|_{L^{q+1}(Q_T)}$ and, in particular, $\|\Gamma\,\Phi'_\varepsilon(\Gamma)\|_{L^{\frac{q+1}{\nu}}(Q_T)}$ for $\nu \in (0,1)$. Using this information, it is easy to check that the integrals on the right-hand side of (A.22) are uniformly bounded with respect to ε. For arbitrary $\mu > 0$, $h_\varepsilon \to h$ strongly in the space $C_{x,t}^{4,1}(\{h > \mu\})$. Therefore, passage to the limit $\varepsilon \to 0$ in last two integrals from the right-hand side of (A.22) over the domain $\{h > \mu\}$ is straightforward. As to integrals over the domain $\{h \leqslant \mu\}$, we have, for example, by virtue of uniform boundedness of $\|(f_n^\varepsilon(h_\varepsilon))^{1/2}(h_{xxx} - h_x + (F_{n,m}^\varepsilon)''(h)h_x)\|_{L^2(Q_T)}$

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and $\|\Gamma\,\Phi'_\varepsilon(\Gamma)\|_{L^{\frac{q+1}{\nu}}(Q_T)}$ in $\varepsilon > 0$,

$$\iint\limits_{\{h\leqslant\mu\}} \Gamma\,\Phi'_\varepsilon(\Gamma) f^\varepsilon_{n-1}(h)(h_{xxx} - h_x + (F^\varepsilon_{n,m})''(h)h_x)\phi_x\,dxdt$$
$$\leqslant \Big(\iint\limits_{Q_T} f^\varepsilon_n(h)(h_{xxx} - h_x + (F^\varepsilon_{n,m})''(h)h_x)^2\phi_x^2\,dxdt\Big)^{\frac{1}{2}} \Big(\iint\limits_{\{h\leqslant\mu\}} \frac{(f^\varepsilon_{n-1}(h))^2}{f^\varepsilon_n(h)}(\Gamma\,\Phi'_\varepsilon(\Gamma))^2\,dxdt\Big)^{\frac{1}{2}}$$
$$\leqslant C\mu^{\frac{n-\alpha}{2}}\Big(\iint\limits_{Q_T}(\Gamma\,\Phi'_\varepsilon(\Gamma))^2\,dxdt\Big)^{\frac{1}{2}} \leqslant C\mu^{\frac{n-\alpha}{2}} \to 0 \text{ as } \mu \to 0$$

for $\nu \in (0, \frac{q+1}{2})$. Analogously, it is easy to check that the another integral in (A.22) over $\{h \leqslant \mu\}$ is bounded from above by some continuous function, $k(\mu)$, such that $k(\mu) \to 0$ as $\mu \to 0$. Therefore, first passing to the limit $\varepsilon \to 0$, and afterwards letting $\mu \to 0$, taking into account (2.20), (2.21) and $0 \leqslant \Gamma \leqslant 1$ a.e. in Q_T, we easily derive

$$\int\limits_\Omega \Phi(\Gamma)\phi\,dx - \iint\limits_{Q_T}\Phi(\Gamma)\phi_t\,dxdt$$
$$+ \frac{4C_2}{(q+1)^2}\iint\limits_{Q_T}(\Phi^{\frac{q+1}{2}}(\Gamma))_x^2\phi\,dxdt + \iint\limits_{Q_T} f_{n-2}(h)\sigma_x^2\phi\,dxdt \leqslant \int\limits_\Omega \Phi(\Gamma_0)\phi\,dx$$
$$+ C_3\iint\limits_{Q_T}\Phi^{q+1}(\Gamma)|\phi_{xx}|dxdt + \iint\limits_{Q_T} f_{n-2}(h)\Gamma\,\Phi'(\Gamma)\sigma_x\phi_x\,dxdt$$
$$- \iint\limits_{\{h>0\}} f_{n-1}(h)\sigma_x(h_{xxx} - h_x + F''_{n,m}(h)h_x)\phi\,dxdt$$
$$+ \iint\limits_{\{h>0\}} \Gamma\,\Phi'(\Gamma) f_{n-1}(h)(h_{xxx} - h_x + F''_{n,m}(h)h_x)\phi_x\,dxdt$$
$$=: \int\limits_\Omega \Phi(\Gamma_0)\phi\,dx + C_3\iint\limits_{Q_T}\Phi^{q+1}(\Gamma)|\phi_{xx}|dxdt + J_1 + J_2 + J_3. \tag{A.23}$$

Now, we bound the terms J_i.

$$J_1 \leqslant \delta_1\iint\limits_{Q_T} f_{n-2}(h)\sigma_x^2\phi\,dxdt + C(\delta_1)\iint\limits_{Q_T} f_{n-2}(h)(\Gamma\,\Phi'(\Gamma))^2\frac{\phi_x^2}{\phi}\,dx$$
$$\leqslant \delta_1\iint\limits_{Q_T} f_{n-2}(h)\sigma_x^2\phi\,dxdt + C(\delta_1)\iint\limits_{Q_T} h^{n+2}\frac{\phi_x^2}{\phi}\,dxdt + C(\delta_1)\iint\limits_{Q_T}|\Gamma\,\Phi'(\Gamma)|^{\frac{n+2}{2}}\frac{\phi_x^2}{\phi}\,dxdt, \tag{A.24}$$

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$$J_2 = -\iint\limits_{\{h>0\}} f_{n-1}(h)\sigma_x h_{xxx}\phi\, dxdt + \iint\limits_{\{h>0\}} f_{n-1}(h)\sigma_x h_x \phi\, dxdt$$
$$- \iint\limits_{\{h>0\}} f_{n-1}(h)\sigma_x F''_{n,m}(h) h_x \phi\, dxdt \leqslant \delta_2 \iint\limits_{\{h>0\}} f_n(h) h_{xxx}^2 \phi\, dxdt + \delta_2 \iint\limits_{\{h>0\}} f_n(h) h_x^2 \phi\, dxdt$$
$$+\delta_2 \iint\limits_{\{h>0\}} f_n(h)((F''_{n,m}(h))^2 h_x^2 \phi\, dxdt + \frac{3n(n-2)}{4\delta_2(n-1)^2} \iint\limits_{Q_T} f_{n-2}(h)\sigma_x^2 \phi\, dxdt$$
$$\leqslant \delta_2 \iint\limits_{\{h>0\}} (f_n(h) h_{xxx}^2 + h^{n-4} h_x^6)\phi\, dxdt + C(\delta_2) \iint\limits_{Q_T} h^{n+2}\phi\, dxdt$$
$$+C(\delta_2) \iint\limits_{Q_T} h^{3m-2n+2}\phi\, dxdt + \frac{3n(n-2)}{4\delta_2(n-1)^2} \iint\limits_{Q_T} f_{n-2}(h)\sigma_x^2 \phi\, dxdt, \tag{A.25}$$

$$J_3 = \iint\limits_{\{h>0\}} \Gamma\, \Phi'(\Gamma) f_{n-1}(h) h_{xxx}\phi_x\, dxdt - \iint\limits_{\{h>0\}} \Gamma\, \Phi'(\Gamma) f_{n-1}(h) h_x \phi_x\, dxdt$$
$$+ \iint\limits_{\{h>0\}} \Gamma\, \Phi'(\Gamma) f_{n-1}(h) F''_{n,m}(h) h_x \phi_x\, dxdt \leqslant \delta_3 \iint\limits_{\{h>0\}} f_n(h) h_{xxx}^2 \phi\, dxdt$$
$$+\delta_3 \iint\limits_{\{h>0\}} f_n(h) h_x^2 \phi\, dxdt + \delta_3 \iint\limits_{\{h>0\}} f_n(h)(F''_{n,m}(h))^2 h_x^2 \phi\, dxdt$$
$$+C(\delta_3) \iint\limits_{Q_T} f_{n-2}(h)(\Gamma\, \Phi'(\Gamma))^2 \frac{\phi_x^2}{\phi}\, dxdt \leqslant \delta_3 \iint\limits_{\{h>0\}} (f_n(h) h_{xxx}^2 + h^{n-4} h_x^6)\phi\, dxdt$$
$$+C(\delta_3) \iint\limits_{Q_T} h^{n+2}(\phi + \frac{\phi_x^2}{\phi})\, dxdt + C(\delta_3) \iint\limits_{Q_T} h^{3m-2n+2}\phi\, dxdt$$
$$+C(\delta_3) \iint\limits_{Q_T} |\Gamma\, \Phi'(\Gamma)|^{\frac{n+2}{2}} \frac{\phi_x^2}{\phi}\, dxdt. \tag{A.26}$$

Summing (A.18) and (A.23), in view of estimates (A.19)–(A.21), (A.24)–(A.26) and (2.20), (2.21), using Lemma B.1, we arrive at

$$\frac{1}{2}\int\limits_{\Omega} (h_x^2 + 2\Phi(\Gamma))\phi\, dx - \frac{1}{2}\iint\limits_{Q_T} (h_x^2 + 2\Phi(\Gamma))\phi_t\, dxdt$$
$$+ C_1 \iint\limits_{Q_T} (h^{\frac{n+2}{2}})_{xxx}^2 \phi\, dxdt + C_0 \iint\limits_{Q_t} (\Phi^{\frac{q+1}{2}}(\Gamma))_x^2 \phi\, dxdt$$
$$+ C_2 \iint\limits_{Q_T} f_{n-2}(h)(\sigma_x)^2 \phi\, dxdt \leqslant \frac{1}{2}\int\limits_{\Omega} (h_{0x}^2 + 2\Phi(\Gamma_0))\phi\, dx$$

$$+ C_3 \iint\limits_{Q_T} \Phi^{\frac{\nu(n+2)}{2}}(\Gamma)\frac{\phi_x^2}{\phi}\,dxdt + C_4 \iint\limits_{Q_T} \Phi^{q+1}(\Gamma)\phi_{xx}\,dxdt$$
$$+ C_5 \iint\limits_{Q_T} h^{n+2}(\phi + \frac{\phi_x^6}{\phi^5} + \frac{\phi_x^2}{\phi} + \frac{|\phi_{xx}|^{\frac{3}{2}}}{\phi^{\frac{1}{2}}} + \frac{|\phi_{xx}|^3}{\phi^2})\,dxdt$$
$$+ C_6 \iint\limits_{Q_T} h^{3m-2n+2}\phi\,dxdt. \tag{A.27}$$

Taking $\phi = \zeta^6$ in (A.27), we deduce that

$$\frac{1}{2}\int\limits_{\Omega} (h_x^2 + 2\Phi(\Gamma))\zeta^6\,dx - \frac{1}{2}\iint\limits_{Q_T} (h_x^2 + 2\Phi(\Gamma))(\zeta^6)_t\,dx$$
$$+ C_1 \iint\limits_{Q_T} \{(h^{\frac{n+2}{2}})_{xxx}^2 + (\Phi^{\frac{q+1}{2}}(\Gamma))_x^2 + f_{n-2}(h)(\sigma_x)^2\}\zeta^6\,dxdt$$
$$\leqslant \frac{1}{2}\int\limits_{\Omega} (h_{0x}^2 + 2\Phi(\Gamma_0))\zeta^6(x,0)\,dx + C_3 \iint\limits_{Q_T} \Phi^{\frac{\nu(n+2)}{2}}(\Gamma)\zeta^4\zeta_x^2\,dxdt$$
$$+ C_4 \iint\limits_{Q_T} \Phi^{q+1}(\Gamma)\zeta^4(\zeta_x^2 + \zeta|\zeta_{xx}|)\,dxdt + C_6 \iint\limits_{Q_T} h^{3m-2n+2}\zeta^6\,dxdt$$
$$+ C_5 \iint\limits_{Q_T} (h^{n+2}(\zeta^6 + \zeta_x^6 + \zeta^4\zeta_x^2 + \zeta^{\frac{9}{2}}|\zeta_{xx}|^{\frac{3}{2}} + \zeta^3|\zeta_{xx}|^3))\,dxdt. \tag{A.28}$$

Multiplying (A.1) with $\delta = 0$ by $\zeta^4(h+\tilde{\gamma})^{\beta}$, $\beta > \frac{1-n}{3}$, $\tilde{\gamma} > 0$ and integrating on Q_T, using Young's inequality, and letting $\tilde{\gamma} \to 0$ and $\varepsilon \to 0$, we obtain the following estimate:

$$\int\limits_{\Omega} \zeta^4 h^{\beta+1}(T)\,dx - \iint\limits_{Q_T} h^{\beta+1}(\zeta^4)_t \leqslant \int\limits_{\Omega} \zeta^4 h_0^{\beta+1}\,dx$$
$$+ \epsilon_4 \iint\limits_{\{h>0\}} \zeta^6\{f_n(h)h_{xxx}^2 + h^{n-4}h_x^6\}\,dxdt + \epsilon_4 \iint\limits_{Q_T} \zeta^6 f_{n-2}(h)(\sigma_x)^2\,dxdt$$
$$+ C(\epsilon_4) \iint\limits_{Q_T} \{\chi_{\{\zeta>0\}}h^{n+3\beta-1} + h^{n+2}(\zeta^6 + \zeta_x^6) + h^{3m-2n+2}\zeta^6\}\,dxdt, \tag{A.29}$$

where $\beta > \frac{1-n}{3}$. Summing (A.29) and (A.28), we obtain (3.1).

Appendix B

Lemma B.1 ([3,23,25]) *Let $\Omega \subset \mathbb{R}^1$ be a bounded domain, and let $\frac{1}{2} < n < 3$. Then, the following estimates hold for any function $v \in C^1(\bar{\Omega}) \cap H^3_{loc}(\{v > 0\})$ such that $v \geqslant 0$, $v_x = 0$*

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on $\partial\Omega$ *and* $\int_{\{v>0\}} v^n v_{xxx}^2 dx < \infty$:

$$\int\limits_{\Omega} \varphi^6 \{v^{n-4} v_x^6 + v^{n-2} v_x^2 v_{xx}^2 + v^{n-1} |v_{xx}|^3\} dx \leqslant c \Big\{ \int\limits_{\{v>0\}} \varphi^6 v^n v_{xxx}^2 dx + \int\limits_{\{\varphi>0\}} v^{n+2} \varphi_x^6 dx \Big\},$$

$$\int\limits_{\Omega} \varphi^6 (v^{\frac{n+2}{2}})_{xxx}^2 dx \leqslant c \Big\{ \int\limits_{\{v>0\}} \varphi^6 v^n v_{xxx}^2 dx + \int\limits_{\{\varphi>0\}} v^{n+2} \{\varphi_x^6 + \varphi^2 \varphi_x^2 \varphi_{xx}^2 + \varphi^3 |\varphi_{xx}|^3\} dx \Big\},$$

where $\varphi \in C^2(\Omega)$ *is an arbitrary non-negative function such that* $\varphi_x = 0$ *on* $\partial\Omega$, *and the constants* $c > 0$ *are independent of* v.

Lemma B.2 (*Stampacchia's Lemma for Systems* [23]) *Let* $(\beta_1, \ldots, \beta_m) \in \mathbb{R}^m, m \geqslant 1$, *and let* $\beta = \prod_{j=1}^m \beta_j$, $\overline{\beta_i} = \frac{\beta}{\beta_i} = \prod_{j=1, j\neq i}^m \beta_j$. *Assume that* $G_i(s)$ *are non-negative non-increasing functions satisfying the inequalities*

$$G_i(s+\delta) \leqslant c_i \Big(\sum_{k=1}^m \frac{G_k(s)}{\delta^{\alpha_k}} \Big)^{\beta_i} \ \forall\, s > 0, \ \delta > 0, \ i = \overline{1, m}$$

with real constants $c_i > 0$, $\beta_i > 1$, *and* $\alpha_i \geqslant 0$ *for* $i = \overline{1, m}$, *and* $\alpha_i > 0$ *for* $i = \overline{1, \ell}$ *for some* $1 \leqslant \ell \leqslant m$. *Let* $G(s) = \sum_{k=1}^m \overline{(c_k^{\overline{\beta}_k})} \Big(G_k(s) \Big)^{\overline{\beta}_k}$, *and let the function* $H(s) = m^{\beta} \sum_{k=\ell+1}^m c_k^{\overline{\beta}_k} \overline{(c_k^{\overline{\beta}_k})}^{1-\beta_k} \Big(G_k(s) \Big)^{\beta_k - 1}$ *be such that* $H(s_1) < 1$ *at a some* $s_1 \geqslant 0$. *Then, there exists a positive constant* $c > 1$ *depending on* m, α_i, β_i, ℓ *and* $H(s_1)$ *such that* $G_i(s_0) \equiv 0$ *for all* $i = \overline{1, \ell}$, *where* $s_0 = s_1 + c \sum_{k=1}^{\ell} \Big(c_k^{\overline{\beta}_k} \overline{(c_k^{\overline{\beta}_k})}^{1-\beta_k} \Big(G(s_1) \Big)^{\beta_k - 1} \Big)^{\frac{1}{\alpha_k \beta}}$. *Note, if* $\ell = m$, *then* $s_1 = 0$.

Lemma B.3 ([14]) *If* $\Omega \subset \mathbb{R}^N$ *is a bounded domain with piecewise-smooth boundary,* $a > 1$, $b \in (0, a)$, $d > 1$, *and* $0 \leqslant k < j$, $k, j \in \mathbb{N}$, *then there exist positive constants* d_1 *and* d_2 ($d_2 = 0$ *if* Ω *is unbounded*) *depending only on* Ω, d, j, b *and* N *such that the following inequality is valid for every* $v(x) \in W^{j,d}(\Omega) \cap L^b(\Omega)$:

$$\left\| D^k v \right\|_{L^a(\Omega)} \leqslant d_1 \left\| D^j v \right\|_{L^d(\Omega)}^{\theta} \left\| v \right\|_{L^b(\Omega)}^{1-\theta} + d_2 \left\| v \right\|_{L^b(\Omega)}, \ \theta = \frac{\frac{1}{b} + \frac{k}{N} - \frac{1}{a}}{\frac{1}{b} + \frac{j}{N} - \frac{1}{d}} \in \Big[\frac{k}{j}, 1 \Big).$$

Note that if $\Omega = B(0, R) \setminus B(0, r)$ *and* $k = 0$, *where* $B(0, x)$ *is ball with the radius* x *and the origin at* 0, *then* $d_2 = c(R - r)^{-\frac{(a-b)N}{ab}}$.

Lemma B.4 *(see [11]) Let* $(\beta_1, \ldots, \beta_m) \in \mathbb{R}^m, m \geqslant 1$, *and let* $\beta = \prod_{j=1}^m \beta_j$, $\overline{\beta_i} = \frac{\beta}{\beta_i} = \prod_{j=1, j\neq i}^m \beta_j$. *Assume that* $G_i(s)$, $g(s)$ *are non-negative non-increasing functions satisfying the inequalities*

$$G_i(s+\delta) \leqslant c_i \Big(\sum_{k=1}^m \frac{G_k(s)}{\delta^{\alpha_k}} + g(s) \Big)^{\beta_i} \ \forall\, s \in \mathbb{R}^1, \ \delta > 0, \ i = \overline{1, m} \qquad \text{(B.1)}$$

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with real constants $c_i > 0,\ \beta_i > 1$ *and* $\alpha_i > 0$. *Let the functions*

$$G_{\max}(s) := \max_{i=\overline{1,m}}\{mc_0 2^{\beta}(\sum_{k=1}^{m}\left(G_k(s)\right)^{\overline{\beta}_k})^{\beta_i-1}\}^{\frac{1}{\alpha_i\beta}},\ c_0 = 2^{\beta-1}\sum_{k=1}^{m}(c_k)^{\overline{\beta}_k},$$

and $g_{\max}(s) := \max_{i=\overline{1,m}}\left(m2^{\beta}\right)^{\frac{1}{\alpha_i\beta}}\left(2^{\beta-1}\sum_{k=1}^{m}(c_k)^{\overline{\beta}_k}\right)^{\frac{\overline{\beta}_i}{\alpha_i}}(g(s))^{\frac{\beta_i-1}{\alpha_i}}$ *be such that*

(*i*) *for some* $s_1 \in (-\infty, s_0)$, *the inequality* $G_{\max}(s) \leqslant k_1 g_{\max}(s)$ *holds for all* $s < s_1$,

(*ii*) $g_{\max}(s) \leqslant k_2(s_0 - s)$ *for all* $s \leqslant s_0$,

where $k_1 > \left(1 - \max_{i=\overline{1,m}}\{2^{-\frac{\beta_i-1}{\alpha_i\beta}}\}\right)^{-1}$ *and* $0 < k_2 < k_1^{-1}(1 - k_1^{-1} - \max_{i=\overline{1,m}}\{2^{-\frac{\beta_i-1}{\alpha_i\beta}}\})$. *Then,* $G_i(s) \equiv 0$ *for all* $s \geqslant s_0$.