



Retract Rationality and Algebraic Tori

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Abstract. For any prime number p and field k , we characterize the p -retract rationality of an algebraic k -torus in terms of its character lattice. We show that a k -torus is retract rational if and only if it is p -retract rational for every prime p , and that the Noether problem for retract rationality for a group of multiplicative type G has an affirmative answer for G if and only if the Noether problem for p -retract rationality for G has a positive answer for all p . For every finite set of primes S we give examples of tori that are p -retract rational if and only if $p \notin S$.

1 Introduction

A classical question in algebraic geometry is to decide whether a given variety is rational, that is, birational to some affine space. Questions of this type are called *rationality problems*. For a given variety, it is often easier to establish weaker properties, such as unirationality, stable rationality, or retract rationality. However, deciding under what circumstances these weaker properties imply rationality leads to difficult questions. In particular, the Lüroth problem asks whether every unirational variety is rational and the Zariski problem asks whether every stably rational variety is rational. The Noether problem asks whether, for a given algebraic group G and generically free finite-dimensional linear G -representation V , the rational quotient variety V/G is rational.

The first constructions of varieties that are unirational but not rational (over \mathbb{C}) were given by Clemens and Griffiths [4] and Manin and Iskovskikh [12], thus answering the Lüroth problem in the negative. Unramified cohomology, defined by Artin and Mumford [1] and Saltman [24], was used to construct further examples of unirational varieties that are not rational. In particular, Saltman showed that the Noether problem over \mathbb{C} has a negative answer for some finite groups G . The first counterexamples to the Zariski problem were constructed by Beauville, Colliot-Thélène, Sansuc, and Swinnerton-Dyer [2]. There has been much recent work on the rationality problem for projective hypersurfaces using the degeneration method due to Voisin [27] and subsequently generalized by Colliot-Thélène and Pirutka [5] and others. For further developments, see [25, 26].

Recall that a variety X is defined to be retract rational if the identity map of X factors rationally through some affine space. If p is a prime, X is defined to be p -retract

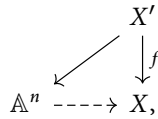
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rational if there exists a diagram



where f is a finite dominant morphism of degree not divisible by p . The definition of retract rationality is due to Saltman [23]. The notion of p -retract rationality, recently introduced by Merkurjev [18], naturally leads to the following question: if X is a p -retract rational variety for every prime p , is it retract rational? In other words, is retract rationality a p -local property? As Merkurjev showed [18, Corollary 7.5], unramified cohomology cannot tell the difference between the class of retract rational varieties and the class of varieties that are p -retract rational for every prime p .

In this note we consider the notion of p -retract rationality in the context of algebraic tori. The birational geometry of tori has been intensely studied by Voskresenskii [29–31], Endo and Miyata [9, 10, 19], Colliot-Thélène and Sansuc [6, 7], Kunyavskii [13, 14], and many others; the standard reference on this material is [28].

If k is a field, an algebraic k -torus is determined by its character lattice, viewed as an integral representation of the absolute Galois group of k . Birational properties of algebraic tori frequently translate to properties of their character lattice, with the notable exception of rationality. The Zariski problem in the case of tori is known as Voskresenskii’s Conjecture; see [28, p. 68].

The main technical result of this paper is a criterion for p -retract rationality of a k -torus T in terms of its character lattice \widehat{T} ; see Proposition 3.1. As a consequence of this description we will show that retract rationality of tori is a p -local property in the following sense.

Theorem 1.1 *Let T be an algebraic torus. Then T is retract rational if and only if it is p -retract rational for every prime p .*

Let G be a linear algebraic group, and let V be a finite-dimensional generically free representation of G . There exists a dense G -invariant open subset U of V such that the geometric quotient U/G exists and $U \rightarrow U/G$ is a G -torsor. One may regard U/G as a variety approximating the classifying stack BG . By the no-name lemma [21, Lemma 2.1], the stable (retract, p -retract) rationality of U/G does not depend on the representation V , but only on G . We say that BG is stably (retract, p -retract) rational if so is U/G . In different but equivalent terminology, this means that the Noether problem for stable (retract, p -retract) rationality has an affirmative answer [11, §3]. Theorem 1.1 has the following consequence.

Corollary 1.2 *Let G be a group of multiplicative type over k . If BG is p -retract rational for every prime p , then it is retract rational.*

One might ask if the retract rationality of BG is a p -local property in the case when G is an arbitrary linear algebraic group. A positive answer would have profound implications for the Noether problem.

For example, if A_n denotes the alternating group on n elements, Merkurjev showed that the classifying stack BA_n is p -retract rational for every prime p [18, Theorem 6.1]. On the other hand, retract rationality of BA_n is a classical problem due to Hilbert, which remains open for every $n \geq 6$ [17] and [8, §4.7].

We also give examples of tori that are not p -retract rational at a specified finite set of primes.

Theorem 1.3 *Let S be a finite set of primes. There exists an algebraic torus T such that T is p -retract rational if and only if $p \notin S$.*

We note that any unirational k -variety X (in particular, any algebraic torus) is p -retract rational for all but at most finitely primes p . If X is unirational, one can find a generically finite dominant rational map $\varphi: \mathbb{A}^n \dashrightarrow X$ of some degree $d \geq 1$; see [22, Lemma 1] for infinite k , and [20] for arbitrary k . If $U \subseteq \mathbb{A}^n$ is a non-empty open subset such that φ is defined over U and is finite of degree d , the diagram

$$\begin{array}{ccc}
 & & U \\
 & \nearrow & \downarrow \\
 \mathbb{A}^n & \xrightarrow{\varphi} & X
 \end{array}$$

shows that X is p -retract rational for every prime p not dividing d . In the case of tori, the assertion also follows from Lemma 2.3(i) and Proposition 3.1.

The rest of the paper is structured as follows. In Section 2 we define p -invertible lattices and establish their basic properties. We use this notion to characterize p -retract rationality of tori in Section 3, and to prove Theorem 1.1 and Corollary 1.2 in Section 4. In Section 5 we prove Theorem 1.3, and deduce a consequence for the Noether problem in Corollary 5.3. In the appendix, we insert a proof of an auxiliary result due to Colliot-Thélène, but so far unpublished.

2 p -invertible Lattices

Throughout this paper we will denote by p a prime number. If T is a torus over a field k , we denote by \widehat{T} its character lattice. If M is a lattice, we set $M_{(p)} := M \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, where $\mathbb{Z}_{(p)}$ is the localization of \mathbb{Z} at the prime ideal (p) . If d is an integer, we denote by $\varphi_{M,d}: M \rightarrow M$ the endomorphism of multiplication by d .

Recall that if G is a profinite group, a G -lattice M is defined to be invertible (or permutation projective) if it is a direct summand of a permutation G -lattice.

Definition 2.1 Let G be a profinite group and let M be a G -lattice. We say that M is p -invertible (or p -permutation projective) if there exists a permutation G -lattice P such that $M_{(p)}$ is a direct summand of $P_{(p)}$. In other words, M is p -invertible if there exists a commutative diagram of G -lattices

$$\begin{array}{ccc}
 M & \xrightarrow{t} & P \\
 \searrow \varphi_{M,d} & & \downarrow \pi \\
 & & M
 \end{array}$$

for some d not divisible by p .

It follows from the definition that ι is injective. It is clear that if M is invertible, it is p -invertible for every prime p .

If G is a profinite group and M is a G -lattice, the G -action on M factors through a finite quotient G' . The next lemma assures us that M is p -invertible as a G -lattice if and only if it is p -invertible as a G' -lattice.

Lemma 2.2 *Let G be a profinite group and M be a G -lattice. Let H be a closed normal subgroup of G acting trivially on M . Then M is p -invertible as a G -lattice if and only if it is p -invertible as a G/H -lattice.*

Proof Assume that $\varphi_{M,d}$ factors through a permutation G -lattice P for some d not divisible by p . By [6, Lemme 2(i)], P^H is a permutation G/H -lattice. Since $\varphi_{M,d}$ factors through P^H , M is p -invertible as a G/H -lattice.

Conversely, assume that $\varphi_{M,d}$ factors through a permutation G/H -lattice P for some d not divisible by p . By [6, Lemme 2(iv)], P is also a permutation G -lattice, so M is p -invertible as a G -lattice as well. ■

Lemma 2.3 *Let G be a finite group and let M be a G -lattice.*

- (i) *If p is prime and $p \nmid |G|$, then M is p -invertible.*
- (ii) *If M is p -invertible for every prime p , then M is invertible.*
- (iii) *Assume that there exists a diagram*

$$\begin{array}{ccc}
 M & \xrightarrow{\iota} & P \\
 & \searrow \varphi & \downarrow \pi \\
 & & M',
 \end{array}$$

where P is a permutation G -lattice, φ is injective, and $\text{coker } \varphi$ is finite of order not divisible by p . Then M is p -invertible.

- (iv) *Let M and N be two stably equivalent G -lattices, and assume that M is p -invertible. Then N is p -invertible.*

Recall that two G -lattices M and N are said to be stably equivalent if there exist permutation G -lattices P and Q such that $M \oplus P \cong N \oplus Q$.

Proof (i) Fix an embedding $\iota: M \hookrightarrow P$, where P is a permutation G -lattice. The standard proof of Maschke’s Theorem shows the existence of a homomorphism of $\mathbb{Q}[G]$ -modules $\pi: P_{\mathbb{Q}} \rightarrow M_{\mathbb{Q}}$ such that $\pi \circ \iota = \text{id}_M$ with only $|G|$ in the denominator. In other words, $\pi' := \varphi_{M,|G|} \circ \pi$ is a well-defined homomorphism $P \rightarrow M$. Hence $\varphi_{M,|G|} = \pi' \circ \iota$ factors through P . If $p \nmid |G|$, this shows that M is p -invertible.

(ii) By assumption, for every prime p there exist an integer d_p not divisible by p , a permutation G -lattice P_p , and homomorphisms

$$f_p: M \rightarrow P_p \quad \text{and} \quad g_p: P_p \rightarrow M$$

such that $\varphi_{M,d_p} = g_p \circ f_p$. The d_p are coprime, hence we may write $1 = \sum_{i=1}^r a_i d_{p_i}$ for some integers a_i . Letting $P := \bigoplus_{i=1}^r P_{p_i}$, we have that $\text{id}_M = g \circ f$, where $f = (f_{p_i}): M \rightarrow P$ and $g = \sum a_i g_{p_i}: P \rightarrow M$.

((iii)) The map $\varphi_{(p)}: M_{(p)} \rightarrow M'_{(p)}$ is an isomorphism, and

$$\text{id}_{M_{(p)}} = \varphi_{(p)}^{-1} \circ \varphi_{(p)}$$

factors through $P_{(p)}$. Clearing denominators, this gives a $d \in \mathbb{Z}$ not divisible by p such that $\varphi_{M,d}$ factors through P .

((iv)) By assumption, $M \oplus P \cong N \oplus Q$ for some permutation G -lattices P and Q . There exists a permutation G -lattice R such that $M_{(p)}$ is a direct summand of $R_{(p)}$, hence $N_{(p)} \oplus Q_{(p)} \cong M_{(p)} \oplus P_{(p)}$ is a direct summand of $(R \oplus P)_{(p)}$, so $N_{(p)}$ is also a summand of $(R \oplus P)_{(p)}$. ■

If G is a finite group and M is a G -module, for every integer i we denote by $\widehat{H}^i(G, M)$ the Tate cohomology group of degree i ; see [16, §2.5] or [3, Chapter VI].

Lemma 2.4 *Let M be a p -invertible G -lattice. Then we have the following.*

- (i) $\widehat{H}^i(G', M)_{(p)} = 0$ for every subgroup G' of G , $i = \pm 1$.
- (ii) If F is a flasque G -lattice, then $\text{Ext}_G^1(F, M)_{(p)} = 0$.

Proof ((i)) Assume that $\varphi_{M,d}$ factors through the permutation G -lattice P , for some d not divisible by p . We have a short exact sequence

$$0 \longrightarrow M \xrightarrow{\varphi_{M,d}} M \longrightarrow A \longrightarrow 0,$$

where A is d -torsion. Consider the associated long exact sequence in cohomology

$$\dots \longrightarrow \widehat{H}^{i-1}(G', A) \longrightarrow \widehat{H}^i(G', M) \xrightarrow{\psi_i} \widehat{H}^i(G', M) \longrightarrow \widehat{H}^i(G', A) \longrightarrow \dots,$$

where $\psi_i := \widehat{H}^i(\varphi_{M,d})$. Since A is d -torsion, $\widehat{H}^i(G', A)$ is also d -torsion for every i , so $\widehat{H}^i(G', A)_{(p)} = 0$. It follows that $(\psi_i)_{(p)}$ is an isomorphism. On the other hand, ψ_i factors through $\widehat{H}^i(G', P)$, which vanishes for $i = \pm 1$, because a permutation G -lattice is both flasque and coflasque; so ψ_i is the zero map for $i = \pm 1$.

((ii)) If P is a permutation G -lattice, [6, Lemme 9] shows that $\text{Ext}_G^1(F, P) = 0$ for every flasque G -lattice F . The assertion $\text{Ext}_G^1(F, M)_{(p)} = 0$ can now be proved using the same argument as in part ((i)), with H^i replaced by Ext_G^1 . ■

Lemma 2.5 *Let G be a finite group, G_p a p -Sylow subgroup of G , and M a G -lattice. The following are equivalent.*

- (i) M is p -invertible as a G -lattice;
- (ii) M is p -invertible as a G_p -lattice;
- (iii) M is invertible as a G_p -lattice;
- (iv) $\text{Ext}_G^1(F, M)_{(p)} = 0$ for every flasque G -lattice F .

Proof ((iii)) \Rightarrow ((iv)). Let F be a flasque G -lattice, and consider the G -lattice $R := \text{Hom}_{\mathbb{Z}}(F, M)$. By [3, §III, Proposition 2.2], we have $\text{Ext}_G^1(F, M) = H^1(G, R)$. As G_p -lattices, M is invertible and F is flasque, so by [6, Lemme 1] we have $H^1(G_p, R) = 0$. The composition

$$H^1(G, R) \xrightarrow{\text{Res}} H^1(G_p, R) \xrightarrow{\text{Cores}} H^1(G, R)$$

is given by multiplication by $[G : G_p]$, so $\text{Ext}_G^1(F, M)$ is $[G : G_p]$ -torsion. Since G_p is a p -Sylow subgroup of G , p does not divide $[G : G_p]$, hence

$$\text{Ext}_G^1(F, M)_{(p)} = 0.$$

((iv) \Rightarrow (i)). Consider a flasque resolution of the G -lattice M

$$0 \longrightarrow M \longrightarrow P \longrightarrow F \longrightarrow 0.$$

Recall that this means that the sequence is exact, P is a permutation G -lattice, and F is a flasque G -lattice. We have

$$\text{Ext}_{\mathbb{Z}_{(p)}[G]}^1(F_{(p)}, M_{(p)}) \cong \text{Ext}_{\mathbb{Z}[G]}^1(F, M)_{(p)} = 0,$$

so the localized sequence $0 \rightarrow M_{(p)} \rightarrow P_{(p)} \rightarrow F_{(p)} \rightarrow 0$ splits, proving that $M_{(p)}$ is a direct summand of $P_{(p)}$. Therefore M is p -invertible.

The implication (i) \Rightarrow (ii) is obvious, and (ii) \Rightarrow (iii) follows from combining Lemma 2.3(i) and (ii). ■

3 The Character Lattice of a p -retract Rational Torus

Let k be a field, let X be a normal equi-dimensional affine k -scheme with normal projective completion \overline{X} , and let Z_1, \dots, Z_r be the irreducible components of the boundary $\overline{X} \setminus X$. Denote $\text{Div}_{\partial\overline{X}} \overline{X} := \bigoplus_{i=1}^r \mathbb{Z}Z_i$. We have the following exact sequence:

$$(3.1) \quad 0 \longrightarrow k'^* \longrightarrow k[X]^* \longrightarrow \text{Div}_{\partial\overline{X}} \overline{X}.$$

Here $k' := k[\overline{X}]$ is the algebraic closure of k in $k(X)$; it is finite-dimensional k -algebra. If X is integral, it is a field.

If G is a finite group and M is a G -lattice, we denote by $[M]^\text{fl}$ the flasque class of M ; see [6, Lemme 5], where it is denoted $\rho(M)$, or [16, §2.7]. If some G -lattice $F \in [M]^\text{fl}$ is invertible, or p -invertible, then the same is true for any $F' \in [M]^\text{fl}$; see Lemma 2.3(iv) for the case where M is p -invertible. When this happens, we say that $[M]^\text{fl}$ is invertible, or p -invertible, respectively.

By a theorem of Saltman [24, Theorem 3.14(a)], a k -torus T is retract rational if and only if $[\widehat{T}]^\text{fl}$ is invertible; see also [16, Lemma 9.5.4(b)]. We now prove a p -local version of this result. In the course of the proof, we will make use of [24, Theorem 2.3], hence we recall a piece of notation used there. If G is a finite group and $f: M \rightarrow M'$ is a homomorphism of G -lattices, we write $\eta(f) = [0]$ if there exists a diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow & & \downarrow g & & \\ 0 & \longrightarrow & M' & \longrightarrow & P' & \longrightarrow & E' & \longrightarrow & 0, \end{array}$$

where P and P' are permutation G -lattices, E and E' are invertible G -lattices, and g factors through a permutation G -lattice [24, pp. 174–175].

Proposition 3.1 *Let T be a torus over k . The following are equivalent.*

- (i) T is p -retract rational.
- (ii) $[\widehat{T}]^\text{fl}$ is p -invertible.

(iii) There exists a commutative diagram of $\text{Gal}(k)$ -lattices

$$(3.2) \quad \begin{array}{ccccccc} & & \widehat{T} & \longrightarrow & P & & \\ & & \downarrow \varphi_{\widehat{T},d} & & \downarrow & & \\ 0 & \longrightarrow & \widehat{T} & \longrightarrow & M & \longrightarrow & Q \longrightarrow 0, \end{array}$$

where d is not divisible by p , the bottom row is exact, and P and Q are permutation lattices.

Proof Let L be a splitting field of T , and set $G := \text{Gal}(L/k)$.

(ii) \Rightarrow (iii). Assume that the G -lattice $[\widehat{T}]^{\text{fl}}$ is p -invertible, and choose a flasque resolution $0 \rightarrow \widehat{T} \rightarrow P \rightarrow E \rightarrow 0$ of \widehat{T} , where E is p -invertible. There exists $d \in \mathbb{Z}$ not divisible by p such that $\varphi_{E,d}$ factors through a permutation G -lattice Q . The diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widehat{T} & \longrightarrow & P & \longrightarrow & E \longrightarrow 0 \\ & & \downarrow \varphi_{\widehat{T},d} & & \downarrow \varphi_{P,d} & & \downarrow \varphi_{E,d} \\ 0 & \longrightarrow & \widehat{T} & \longrightarrow & P & \longrightarrow & E \longrightarrow 0 \end{array}$$

then shows that $\eta(\varphi_{\widehat{T},d}) = [0]$. By [24, Theorem 2.3], a diagram of type (3.2) exists.

(iii) \Rightarrow (ii). If we have a diagram of type (3.2), then by [24, Theorem 2.3] we have $\eta(\varphi_{\widehat{T},d}) = [0]$. This means that there exists a diagram

$$(3.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \widehat{T} & \xrightarrow{\iota_1} & P_1 & \xrightarrow{\pi_1} & E_1 \longrightarrow 0 \\ & & \downarrow \varphi_{\widehat{T},d} & & \downarrow \rho_1 & & \downarrow \psi_1 \\ 0 & \longrightarrow & \widehat{T} & \xrightarrow{\iota_2} & P_2 & \xrightarrow{\pi_2} & E_2 \longrightarrow 0, \end{array}$$

where the rows are flasque resolutions, and ψ_1 factors through a permutation G -lattice Q' . By [6, Lemme 4], ι_1 factors through P_2 , so we have a diagram

$$(3.4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \widehat{T} & \xrightarrow{\iota_2} & P_2 & \xrightarrow{\pi_2} & E_2 \longrightarrow 0 \\ & & \parallel & & \downarrow \rho_2 & & \downarrow \psi_2 \\ 0 & \longrightarrow & \widehat{T} & \xrightarrow{\iota_1} & P_1 & \xrightarrow{\pi_1} & E_1 \longrightarrow 0. \end{array}$$

Let $\rho := \rho_2 \circ \rho_1$ and $\psi := \psi_2 \circ \psi_1$. The bottom row of (3.3) and the top row of (3.4) coincide. Joining the two previous diagrams along said rows, we obtain

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widehat{T} & \xrightarrow{\iota_1} & P_1 & \xrightarrow{\pi_1} & E_1 \longrightarrow 0 \\ & & \downarrow \varphi_{\widehat{T},d} & & \downarrow \rho & & \downarrow \psi \\ 0 & \longrightarrow & \widehat{T} & \xrightarrow{\iota_1} & P_1 & \xrightarrow{\pi_1} & E_1 \longrightarrow 0. \end{array}$$

We also have the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widehat{T} & \xrightarrow{\iota_1} & P_1 & \xrightarrow{\pi_1} & E_1 \longrightarrow 0 \\ & & \downarrow \varphi_{\widehat{T},d} & & \downarrow \varphi_{P_1,d} & & \downarrow \varphi_{E_1,d} \\ 0 & \longrightarrow & \widehat{T} & \xrightarrow{\iota_1} & P_1 & \xrightarrow{\pi_1} & E_1 \longrightarrow 0. \end{array}$$

Combining these two diagrams, it is immediate to deduce that $\varphi_{P_1,d} - \rho$ is zero on $\text{im } \iota_1$, and that the square

$$\begin{array}{ccc} P_1 & \xrightarrow{\pi_1} & E_1 \\ \varphi_{P_1,d} - \rho \downarrow & & \downarrow \varphi_{E_1,d} - \psi \\ P_1 & \xrightarrow{\pi_1} & E_1 \end{array}$$

commutes. Since $\varphi_{P_1,d} - \rho$ is zero on $\text{im } \iota_1 = \ker \pi_1$, there exists $f: E_1 \rightarrow P_1$ making the diagram

$$\begin{array}{ccc} P_1 & \xrightarrow{\pi_1} & E_1 \\ \varphi_{P_1,d} - \rho \downarrow & \swarrow f & \downarrow \varphi_{E_1,d} - \psi \\ P_1 & \xrightarrow{\pi_1} & E_1 \end{array}$$

commute.

Recall that ψ_1 factors through a permutation G -lattice Q' . Since ψ factors through ψ_1 , it also factors through Q' . We thus have commutative triangles

$$\begin{array}{ccc} E_1 & \longrightarrow & Q' \\ & \searrow \psi & \downarrow \\ & & E_1 \end{array} \quad \begin{array}{ccc} E_1 & \xrightarrow{f} & P_1 \\ & \searrow \varphi_{E_1,d} - \psi & \downarrow \pi_1 \\ & & E_1 \end{array}$$

Therefore, $\varphi_{E_1,d} = (\varphi_{E_1,d} - \psi) + \psi$ factors through $P_1 \oplus Q'$, which implies that $E_1 \in [\widehat{T}]^{\text{fl}}$ is p -invertible.

((iii) \Rightarrow (i)). By [24, Theorem 2.3], the existence of a diagram of G -lattices of type (3.2) implies that $\eta(\varphi_{\widehat{T},d}) = [0]$. By [24, Theorem 3.14(b)], this implies that the ring homomorphism $f_d: L[\widehat{T}]^G \rightarrow L[\widehat{T}]^G$ induced by $\varphi_{\widehat{T},d}$ factors rationally, that is, there exist an integer $n \geq 1$, a non-zero polynomial $w \in k[x_1, \dots, x_n]$, a non-zero $u \in L[\widehat{T}]^G$, and maps $\alpha: L[\widehat{T}]^G \rightarrow k[x_1, \dots, x_n][1/w]$ and $\beta: k[x_1, \dots, x_n][1/w] \rightarrow L[\widehat{T}]^G[1/u]$ such that $f_d = \beta \circ \alpha$; see [24, Definition 3.2]. Geometrically, this means that the map $T \rightarrow T$ given by $t \mapsto t^d$ factors rationally through \mathbb{A}^n . Since the degree of this map is coprime to p , we conclude that T is p -retract rational.

((i) \Rightarrow (iii)). Assume that we are given a diagram

$$\begin{array}{ccc} & & X \\ & \swarrow s & \downarrow \varphi \\ \mathbb{A}^n & \dashrightarrow \pi & T, \end{array}$$

where X is a variety and φ is a finite-dominant map of degree d not divisible by p . We have an induced map $\pi^*: k[T] \rightarrow k[x_1, \dots, x_n][1/w]$, where $w \neq 0$ is a polynomial in x_1, \dots, x_n .

Let $\psi := \varphi_L: X_L \rightarrow T_L$. We have a G -homomorphism $\psi^*: L[T]^* \rightarrow L[X]^*$ and a G -equivariant norm map $\psi_*: L[X]^* \rightarrow L[T]^*$, induced by the norm map of field theory $L(X)^* \rightarrow L(T)^*$. The composition $\psi_* \circ \psi^*$ is given by $s \mapsto s^d$.

We have $L[T]^*/L^* = \widehat{T}$. Let k' be the algebraic closure of k in $k(X)$; it is a finite extension of k . By (3.1), $N' := L[X]^*/(k' \otimes L)^*$ is a G -sublattice of $\text{Div}_{\partial \overline{X}_L} \overline{X}_L$, where \overline{X}

is a normal projective compactification of X . We obtain the following commutative diagrams of G -equivariant maps with exact rows:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & L^* & \longrightarrow & L[T]^* & \longrightarrow & \widehat{T} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \psi^* & & \downarrow \psi^* & & \\
 0 & \longrightarrow & (k' \otimes L)^* & \longrightarrow & L[X]^* & \longrightarrow & N' & \longrightarrow & 0
 \end{array}$$

and

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & L^* & \longrightarrow & L[T]^* & \longrightarrow & \widehat{T} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow \psi_* & & \uparrow \psi_* & & \\
 0 & \longrightarrow & (k' \otimes L)^* & \longrightarrow & L[X]^* & \longrightarrow & N' & \longrightarrow & 0.
 \end{array}$$

The composition $\psi_* \circ \psi^*: \widehat{T} \rightarrow \widehat{T}$ equals $\varphi_{\widehat{T},d}$.

Let $P := L[x_1, \dots, x_n][1/w]^*/L^*$. Write $w = c \cdot w_1^{a_1} \cdots w_r^{a_r}$, where $c \in L^*$, the w_i are irreducible polynomials in $L[x_1, \dots, x_n]$, and w_i is not a scalar multiple of w_j when $i \neq j$. The cosets of the w_i modulo L^* freely generate P as a \mathbb{Z} -module, and they are permuted by the G -action, hence P is a permutation G -lattice.

The ring map $\pi_L^*: L[T] \rightarrow L[x_1, \dots, x_n][1/w]$ induces a group homomorphism $L[T]^* \rightarrow L[x_1, \dots, x_n][1/w]^*$, hence an injective G -homomorphism $\pi': \widehat{T} \rightarrow P$. The restriction of $s: X \rightarrow \mathbb{A}^n$ to the domain of definition of π gives a map $k[x_1, \dots, x_n][1/w] \rightarrow k[X][1/u]$, for some non-zero $u \in k[X]$. This in turn induces a homomorphism of G -lattices

$$P \longrightarrow N := L[X][1/u]^*/(k' \otimes L)^*.$$

We have an exact sequence $0 \rightarrow N' \rightarrow N \rightarrow R \rightarrow 0$ for some G -lattice R . We have constructed the commutative diagram

$$(3.5) \quad \begin{array}{ccccccc}
 & & \widehat{T} & \longrightarrow & P & & \\
 & & \downarrow \psi^* & & \downarrow & & \\
 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & R \longrightarrow 0.
 \end{array}$$

The norm map $L(X)^* \rightarrow L(T)^*$ induces a commutative square

$$\begin{array}{ccc}
 L[X]^* & \longrightarrow & L[X][1/u]^* \\
 \downarrow \psi_* & & \downarrow \psi_* \\
 L[T]^* & \longrightarrow & L[T][1/\psi_*(u)]^*.
 \end{array}$$

Since $u \in k[X]$, we have $\psi_*(u) \in k[T]$. We obtain a commutative square

$$\begin{array}{ccc}
 L[X]^*/(k' \otimes L)^* & \longrightarrow & L[X][1/u]^*/(k' \otimes L)^* \\
 \downarrow \psi_* & & \downarrow \psi_* \\
 L[T]^*/L^* & \longrightarrow & L[T][1/\psi_*(u)]^*/L^*.
 \end{array}$$

Recall that $\widehat{T} = L[T]^*/L^*$, $N' = L[X]^*/(k' \otimes L)^*$, and

$$N = L[X][1/u]^*/(k' \otimes L)^*,$$

hence the above diagram is

$$\begin{array}{ccc} N' & \longrightarrow & N \\ \downarrow \psi_* & & \downarrow \psi_* \\ \widehat{T} & \longrightarrow & M, \end{array}$$

where we define $M := L[T][1/\psi_*(u)]^*/L^*$. The homomorphism $\widehat{T} \rightarrow M$ is injective, and its cokernel is $Q = \text{Div}_{Z_L} T_L$, where Z is the zero locus of $\psi_*(u)$ inside T . In other words, Q is the permutation G -lattice generated by the irreducible factors of $\psi_*(u)$ in $L[T]$. By construction, we have a commutative diagram with exact rows:

$$(3.6) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widehat{T} & \longrightarrow & M & \longrightarrow & Q & \longrightarrow & 0. \end{array}$$

Combining (3.5) and (3.6). gives a diagram of type (3.2). By construction, $\psi_* \circ \psi^* = \varphi_{\widehat{T},d}$, and P and Q are permutation G -lattices, hence ((iii)) follows. ■

Corollary 3.2 *Let T be a k -torus with splitting field L and $G := \text{Gal}(L/k)$. For a prime p , denote by G_p a p -Sylow subgroup of G , and let $k_p := L^{G_p}$ be the fixed field of G_p . Then T is p -retract rational over k if and only T_{k_p} is retract rational over k_p .*

Proof The character lattice of T_{k_p} is \widehat{T} viewed as a G_p -lattice via restriction. Fix a G -lattice $F \in [\widehat{T}]^{\text{fl}}$. A flasque resolution of M as a G -lattice is, in particular, a flasque resolution of M as a G_p -lattice; hence $F \in [\widehat{T}_{k_p}]^{\text{fl}}$ when F is viewed as a G_p -lattice.

By Proposition 3.1, T is p -retract rational if and only if F is p -invertible. By Lemma 2.5, this is the same as F being invertible as a G_p -lattice. By [16, Lemma 9.5.4(b)], this is equivalent to the retract rationality of T_{k_p} . ■

4 Proof of Theorem 1.1 and Corollary 1.2

Proof of Theorem 1.1 If T is retract rational, it is also p -retract rational for every prime p . Conversely, if T is p -retract rational for every p , then by Proposition 3.1, $[\widehat{T}]^{\text{fl}}$ is p -invertible for every p . By Lemma 2.3(ii), $[\widehat{T}]^{\text{fl}}$ is invertible. By [16, Lemma 9.5.4(b)], T is retract rational. ■

Let G be a group of multiplicative type. Embedding G in some quasi-split torus S , one obtains a short exact sequence

$$(4.1) \quad 1 \longrightarrow G \longrightarrow S \longrightarrow T \longrightarrow 1,$$

where T is a torus and S is a quasi-split torus. Since S is quasi-split, it is an open subset of some generically free linear G -representation. By definition, BG is retract rational (p -retract rational) if and only if T is; see the introduction. Recall that the rationality

(p -retract rationality) of T does not depend on the choice of the embedding $G \hookrightarrow S$, by the no-name lemma [21, Lemma 2.1]. The map $S \rightarrow T$ is a versal G -torsor, in the sense of [18].

Proof of Corollary 1.2 Let G be a group of multiplicative type, and consider a sequence of type (4.1) for G . By assumption, T is p -retract rational for every prime p , thus it is retract rational by Theorem 1.1. This means precisely that BG is retract rational, as desired. ■

5 Proof of Theorem 1.3

Let G be a finite group. Recall that the augmentation map $\epsilon: \mathbb{Z}[G] \rightarrow \mathbb{Z}$ is defined by $\epsilon(g) = 1$ for every $g \in G$. Set $I_G := \ker \epsilon$ and let $J_G := I'_G$ be the dual of I_G . Equivalently, J_G is the cokernel of the norm map $N: \mathbb{Z} \rightarrow \mathbb{Z}[G]$ given by $N(1) = \sum_{g \in G} g$. If L/k is a Galois extension of Galois group G , then J_G is isomorphic to the character lattice of the norm-one torus $R_{L/k}^{(1)}(\mathbb{G}_m)$.

The following is a p -local version of [6, Proposition 2].

Proposition 5.1 *Let G be a finite group and let G_p be a p -Sylow subgroup of G . The following conditions are equivalent.*

- (i) G_p is cyclic.
- (ii) Every flasque G -lattice is p -invertible.
- (iii) $[J_G]^{\text{fl}}$ is p -invertible.

Proof By definition, $[J_G]^{\text{fl}}$ is flasque, so ((ii)) implies ((iii)).

((iii)) \Rightarrow ((i)). By Lemma 2.5, $[J_G]^{\text{fl}}$ is invertible as a G_p - p -lattice. By [6, Proposition 2], G_p is cyclic.

((i)) \Rightarrow ((ii)). By Lemma 2.5, we may assume that G is a cyclic p -group. In this case, by [6, Proposition 2] every flasque G -lattice is invertible, hence p -invertible. ■

Proof of Theorem 1.3 Let L/k be a finite Galois extension. By Proposition 3.1 and Proposition 5.1 the torus $R_{L/k}^{(1)}(\mathbb{G}_m)$ is p -retract rational if and only if the p -Sylow subgroups of $\text{Gal}(L/k)$ are cyclic.

Let S be a finite set of primes. Set $G := \prod_{p \in S} (\mathbb{Z}/p\mathbb{Z})^2$, and consider a Galois field extension L/k such that $\text{Gal}(L/k) \cong G$. It is well known that such extensions exist, even for $k = \mathbb{Q}$; see [15, Chapter VI, Exercise 23]. We conclude that $T := R_{L/k}^{(1)}(\mathbb{G}_m)$ is p -retract rational if and only if $p \notin S$, as claimed. ■

Lemma 5.2 *Let T be a k -torus, and let T' be its dual. For every prime p , BT is p -retract rational if and only if T' is p -retract rational.*

Proof Let L be the splitting field of T , $G = \text{Gal}(L/k)$, G_p a p -Sylow subgroup of G , and $k_p = L^{G_p}$ the fixed field of G_p . By Corollary 3.2, T' is p -retract rational if and only if T'_{k_p} is retract rational over k_p . Consider an exact sequence $1 \rightarrow T \rightarrow S \rightarrow Q \rightarrow 1$, where S is quasi-split. Since BT is p -retract rational if and only if Q is, from Corollary 3.2 we deduce that BT is p -retract rational if and only if BT_{k_p} is retract

rational over k_p . By Proposition A.1, this is equivalent to T'_{k_p} being retract rational over k_p . ■

Corollary 5.3 *For every finite set of primes S , there exists an algebraic torus T such that the classifying stack BT is p -retract rational if and only if $p \notin S$.*

Proof Combine Theorem 1.3 and Lemma 5.2. ■

A Appendix

The following result was stated by Merkurjev. The proof given below develops a message from Colliot-Thélène to Merkurjev (11 October, 2015).

Proposition A.1 *Let T be an algebraic k -torus, and let T' be its dual. Then we have the following.*

- (i) *BT is stably rational if and only if T' is.*
- (ii) *BT is retract rational if and only if T' is.*

Proof If T' is stably rational, by [28, Theorem 2, p. 52] there exists a short exact sequence $1 \rightarrow S_1 \rightarrow S_2 \rightarrow T' \rightarrow 1$ where S_1 and S_2 are quasi-split. Recall that a quasi-split torus is isomorphic to its dual and is an open subset of some affine space, hence it is a rational variety. The dual sequence gives a versal T -torsor $S'_2 \rightarrow S'_1$ with rational base, hence BT is stably rational.

If T' is retract rational, by [16, Lemma 9.5.4(b)] the flasque class of \widehat{T}' is invertible, hence we have a sequence $1 \rightarrow U \rightarrow S \rightarrow T' \rightarrow 1$ where S is quasi-split and U is an invertible torus, *i.e.*, there exists an isomorphism of tori $U \times Q \cong P$, where Q is a torus and P is a quasi-split torus. Dualization gives $U' \times Q' \cong P'$, hence U' is retract rational. The dual $1 \rightarrow T \rightarrow S' \rightarrow U' \rightarrow 1$ of the previous sequence yields a versal T -torsor $S' \rightarrow U'$ over a retract rational base, hence BT is retract rational.

Before proving the converse, we need some preparation. Let L be the splitting field of T and let $G := \text{Gal}(L/k)$. Choose a coflasque resolution of \widehat{T}

$$(A.1) \quad 0 \longrightarrow \widehat{Q} \longrightarrow \widehat{R} \longrightarrow \widehat{T} \longrightarrow 0,$$

that is, \widehat{R} is a permutation G -lattice and \widehat{Q} is coflasque, and a flasque resolution of \widehat{Q}

$$(A.2) \quad 0 \longrightarrow \widehat{Q} \xrightarrow{\alpha} \widehat{S} \xrightarrow{\alpha'} \widehat{F} \longrightarrow 0.$$

Since $R \rightarrow Q$ is a versal T -torsor, BT is stably rational, respectively, retract rational, if and only if Q is. Applying [28, Theorem 2, p. 52], respectively, [16, Lemma 9.5.4(b)], to sequence (A.2), this is in turn equivalent to \widehat{F} being stably permutation, respectively, invertible.

(ii) Assume that BT is stably rational. By (A.1), this implies that Q is stably rational. Since $\widehat{F} \in [\widehat{Q}]^{\text{fl}}$, by [28, Theorem 2, p. 52] there is a permutation G -lattice \widehat{S}_1 such that $\widehat{F} \oplus \widehat{S}_1$ is a permutation G -lattice. Then the sequence

$$0 \longrightarrow \widehat{Q} \xrightarrow{(\alpha, 0)} \widehat{S} \oplus \widehat{S}_1 \xrightarrow{(\alpha', \text{id})} \widehat{F} \oplus \widehat{S}_1 \longrightarrow 0$$

is another flasque resolution of \widehat{Q} . We may thus assume that \widehat{F} is a permutation G -lattice. By [6, Lemme 1(viii)], it follows that $\widehat{S} \cong \widehat{Q} \oplus \widehat{F}$, and we obtain $\widehat{S} \cong \widehat{Q}' \oplus \widehat{F}$ by dualization. Consider the sequence

$$0 \longrightarrow \widehat{T}' \xrightarrow{\beta} \widehat{R}' \xrightarrow{\beta'} \widehat{Q}' \longrightarrow 0$$

dual to (A.1). Then the sequence

$$0 \longrightarrow \widehat{T}' \xrightarrow{(\beta, 0)} \widehat{R}' \oplus \widehat{F} \xrightarrow{(\beta', \text{id})} \widehat{Q}' \oplus \widehat{F} \longrightarrow 0$$

is also exact. By [28, Theorem 2, p. 52], we conclude that T' is stably rational.

((ii)) We have seen that BT is retract rational if and only if \widehat{F} is invertible. Assume that BT is retract rational. Applying [6, Lemme 1(vii)'] to (A.2), we have $\widehat{S} \cong \widehat{Q} \oplus \widehat{F}$, hence $\widehat{S}' \cong \widehat{Q}' \oplus \widehat{F}'$, showing that \widehat{Q}' is invertible. Since $\widehat{Q}' \in [\widehat{T}']^{\text{fl}}$, by [16, Lemma 9.5.4(b)] we conclude that T' is retract rational. ■

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