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ON THE HYPERSTABILITY OF THE DRYGAS FUNCTIONAL EQUATION ON A RESTRICTED DOMAIN

JEDSADA SENASUKH and SATIT SAEJUNG ✉

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Abstract

We prove hyperstability results for the Drygas functional equation on a restricted domain (a certain subset of a normed space). Our results are more general than the ones proposed by Aiemsomboon and Sintunavarat ['Two new generalised hyperstability results for the Drygas functional equation', *Bull. Aust. Math. Soc.* **95** (2017), 269–280] and our proof does not rely on the fixed point theorem of Brzdęk as was the case there. A characterisation of the Drygas functional equation in terms of its asymptotic behaviour is given. Several examples are given to illustrate our generalisations. Finally, we point out a misleading statement in the proof of the second result in the paper by Aiemsomboon and Sintunavarat and propose its correction.

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1. Introduction

The stability of functional equations seems to originate from the following question of Ulam [14] concerning homomorphisms between two groups.

> *Let* $(H,+)$ *be a group and let* $(G,+,d)$ *be a metric group. Given* $\varepsilon > 0$, *does there exist* $\delta > 0$ *such that if a function* $f : H \to G$ *satisfies the inequality*
>
> $$d(f(x+y), f(x)+f(y)) < \delta \quad \text{for all } x, y \in H,$$
>
> *then there exists a homomorphism* $F : H \to G$ *such that*
>
> $$d(F(x), f(x)) < \varepsilon \quad \text{for all } x \in H?$$

The functional equation

$$f(x+y) = f(x) + f(y)$$

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is known as the *Cauchy functional equation* and many stability results for it (in the sense of Hyers–Ulam–Rassias) have been studied in various ways (see [4, 7] and the references therein). The *hyperstability result* seems to have been first published in [3] concerning ring homomorphisms. We refer to [5, 6] for the hyperstability of the Cauchy functional equation. The stability and hyperstability of other functional equations have also been considered (see [4]).

To obtain a characterisation of a quasi-inner product space, Drygas [8] considered the functional equation

$$f(x) + f(y) - f(x-y) - 2\Big(f\Big(\frac{x+y}{2}\Big) - f\Big(\frac{x-y}{2}\Big)\Big) = 0, \tag{1.1}$$

where $f : X \to \mathbb{R}$ (the set of real numbers) and X is a real or complex vector space. By replacing y by $-y$ in (1.1), we obtain

$$2\Big(f\Big(\frac{x+y}{2}\Big) - f\Big(\frac{x-y}{2}\Big)\Big) - f(x+y) + f(x) + f(-y) = 0. \tag{1.2}$$

From (1.1) and (1.2),

$$f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y) = 0. \tag{1.3}$$

The functional equation (1.3) is known as the *Drygas functional equation*. Ebanks *et al.* [9] gave a general solution of the Drygas functional equation as follows.

Theorem E [9]. *Let G be a commutative group and let K be a commutative field (of characteristic different from two). Suppose that $f : G \to K$ satisfies the Drygas functional equation*

$$f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y) = 0 \quad \textit{for all } x, y \in G.$$

Then f is of the form

$$f(x) = A(x) + H(x, x) \quad \textit{for all } x \in G,$$

where $A : G \to K$ is a homomorphism and $H : G \times G \to K$ is a symmetric bihomomorphism (that is, H is additive in each variable and $H(x, y) = H(y, x)$ for all $x, y \in G$).

Before discussing the hyperstability of the Drygas functional equation on a restricted subset of a normed space, we first recall the precise definition of the functional equation as follows.

Definition 1.1. Let X and Y be two normed spaces and let $\varnothing \neq D \subset X$. We say that a function $f : D \to Y$ is *Drygas on D* if

$$f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y) = 0 \quad \text{for all } x, y \in D \text{ with } -y, x \pm y \in D.$$

In particular, if $D = X$, then we simply say that f is *Drygas*.

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Inspired by the works of Hyers [10], Aoki [2], Rassias [12] and Brzdęk [5, 6], Piszczek and Szczawińska [11] applied Brzdęk's fixed point theorem to prove a hyperstability result for the Drygas functional equation on a restricted subset of a normed space. From now on, we use the following notation: for a given f, we define

$$\Delta_f(x, y) := f(x + y) + f(x - y) - 2f(x) - f(y) - f(-y).$$

We also let $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of all positive integers and of all integers, respectively.

Theorem PS [11]. *Let X and Y be a normed space and a Banach space, respectively. Suppose that D is a nonempty subset of $X\backslash\{0\}$ such that:*

(D1) *D is symmetric with respect to zero, that is, $x \in D$ if and only if $-x \in D$; and*
(D2) *there exists $M \in \mathbb{N}$ such that $mx \in D$ for all $x \in D$ and for all integers $m \geq M$.*

Let $c \geq 0$ and $p < 0$ be given. Suppose that $f : D \to Y$ satisfies the inequality

$$\|\Delta_f(x, y)\| \leq c(\|x\|^p + \|y\|^p) \quad \text{for all } x, y \in D \text{ with } x \pm y \in D. \tag{1.4}$$

Then f is Drygas on D.

As mentioned in [11, Example 4], Condition (D2) cannot be removed. Aiemsomboon and Sintunavarat [1] obtained some hyperstability results by replacing the expression $c(\|x\|^p + \|y\|^p)$ in (1.4) by $h(x) + h(y)$ (see [1, Theorem 2.1]) and $h_1(x)h_2(y)$ (see [1, Theorem 2.2]) together with some additional assumptions. To state their result, we first define the following notation. Suppose that X is a normed space and that D is a nonempty subset of $X\backslash\{0\}$ satisfying (D1) and (D2). For a function $h : D \to [0, \infty)$ and for each integer m with $|m| \geq M$, let

$$\lfloor h\rceil(m) := \inf\{t \geq 0 : h(mx) \leq th(x) \text{ for all } x \in D\}.$$

In particular, $h(mx) \leq \lfloor h\rceil(m)h(x)$ for all $x \in D$.

Theorem AS. *Let X, Y, and D be defined as in Theorem PS. Suppose that $f : D \to Y$ and $\varphi : D \times D \to [0, \infty)$ are two functions satisfying the inequality*

$$\|\Delta_f(x, y)\| \leq \varphi(x, y) \quad \text{for all } x, y \in D \text{ with } x \pm y \in D.$$

Suppose that one of the following assumptions is true.

(1) *$\varphi(x, y) := h(x) + h(y)$ for all $x, y \in D$, where $h : D \to [0, \infty)$ is a function such that:*

   (W_1) *$M_1 := \{m \geq M : 2\lfloor h\rceil(m + 1) + \lfloor h\rceil(m) + \lfloor h\rceil(-m) + \lfloor h\rceil(2m + 1) < 1\}$ is an infinite set; and*
   (W_2) *$\lim_{m\to\infty}\lfloor h\rceil(m) = 0$ and $\lim_{m\to\infty}\lfloor h\rceil(-m) = 0$ (see [1, Theorem 2.1]).*

(2) *$\varphi(x, y) := h_1(x)h_2(y)$ for all $x, y \in D$, where $h_1, h_2 : X \to [0, \infty)$ are functions such that:*

(W$'_1$) $M_2 := \{m \geq M : 2\lfloor h_1\rceil(m+1)\lfloor h_2\rceil(m+1) + \lfloor h_1\rceil(m)\lfloor h_2\rceil(m) + \lfloor h_1\rceil(-m)\lfloor h_2\rceil(-m) + \lfloor h_1\rceil(2m+1)\lfloor h_2\rceil(2m+1) < 1\}$ *is an infinite set;*

(W$'_2$) $\lim_{m\to\infty}\lfloor h_1\rceil(\pm m)\lfloor h_2\rceil(\pm m) = 0$*; and*

(W$'_3$) $\lim_{m\to\infty}\lfloor h_1\rceil(m)$ *or* $\lim_{m\to\infty}\lfloor h_2\rceil(m) = 0$ *(see* [1, Theorem 2.2]*).*

Then f is Drygas on D.

In this paper, we prove, without using the fixed point theorem of Brzdęk [7], some hyperstability results for the Drygas functional equation on a restricted domain. The condition on our domain is more general than the one proposed by Piszczek and Szczawińska [11]. The main results of Aiemsomboon and Sintunavarat [1] can be derived from our main results. Moreover, the hyperstability of the inhomogeneous Drygas functional equation is also considered. Finally, we point out that the proof of [1, Theorem 2.2] is not correct.

2. Main results

Throughout this paper, we make the following assumptions.

- X and Y are normed spaces.
- $\varnothing \neq D \subset X\backslash\{0\}$ satisfies Conditions (D1) of Theorem PS and

  (D2*) For each $x \in D$ there exists $m_x \in \mathbb{N}$ such that $mx \in D$ for all integers $m \geq m_x$.
- $\mathbb{D} := \{(x, y) \in D \times D : x \pm y \in D\}$.

Remark 2.1. From Condition (D1), it is easy to see that a function $f : D \to Y$ is Drygas on D if and only if $\Delta_f(x, y) = 0$ for all $(x, y) \in \mathbb{D}$ if and only if $\Delta_f(-x, -y) = 0$ for all $(x, y) \in \mathbb{D}$.

The following example shows that the class of nonempty subsets of X satisfying Conditions (D1) and (D2) is a *proper* subclass of that satisfying Conditions (D1) and (D2*).

Example 2.2. We consider the Banach space l_∞ of bounded real sequences $(x_n)_{n=1}^\infty$ equipped with the supremum norm $\|(x_n)_{n=1}^\infty\| := \sup_n |x_n| < \infty$. For each $n \in \mathbb{N}$, we define

$$D_n := \{\pm e_n\} \cup \{\pm m e_n : m \geq n\},$$

where $e_k := (\delta_n^{(k)})_{n=1}^\infty$ and

$$\delta_n^{(k)} := \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{if } n \neq k. \end{cases}$$

It is easy to see that $D := \bigcup_{n=1}^\infty D_n$ satisfies (D1) and (D2*). To see that D fails Condition (D2), we note that, for each $M \in \mathbb{N}$, we have $e_{M+1} \in D$ and $Me_{M+1} \notin D$.

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2.1. Hyperstability of the Drygas functional equation on a restricted domain.

THEOREM 2.3. *Suppose that $f : D \to Y$ is a function such that*

$$\lim_{m\to\infty} \Delta_f(mx, my) = 0 \quad \text{for all } (x, y) \in \mathbb{D}.$$

Suppose that one of the following conditions is satisfied.

(a) $\lim_{m\to\infty} \Delta_f((m + 1)x, mx) = 0$ *for all* $x \in D$.
(b) $\lim_{m\to\infty} \Delta_f(mx, (m + 1)x) = 0$ *for all* $x \in D$.
(c) $\lim_{m\to\infty} \Delta_f(x, mx) = 0$ *for all* $x \in D$.

Then f is Drygas on D.

PROOF. Let $(x, y) \in \mathbb{D}$ be given. It follows from Conditions (D1) and (D2*) that there exists $M \in \mathbb{N}$ such that

$$\{\pm(m - 1)x, \pm(m - 1)y, \pm mx, \pm my, \pm m(x + y), \pm m(x - y)\} \subset D$$

for all integers $m \geq M$. Let $m \geq M$ be an integer. Then

$$\begin{aligned}
\|\Delta_f(x, y)\| \leq{}& 2\|\Delta_f((m + 1)x, mx)\| + \|\Delta_f((m + 1)y, my)\| + \|\Delta_f(-(m + 1)y, -my)\| \\
&+ \|\Delta_f((m + 1)(x + y), m(x + y))\| + \|\Delta_f((m + 1)(x - y), m(x - y))\| \\
&+ \|\Delta_f((2m + 1)x, (2m + 1)y)\| + 2\|\Delta_f((m + 1)x, (m + 1)y)\| \\
&+ \|\Delta_f(mx, my)\| + \|\Delta_f(-mx, -my)\|; \qquad \text{(a*)} \\
\|\Delta_f(-x, -y)\| \leq{}& 2\|\Delta_f(mx, (m + 1)x)\| + \|\Delta_f(my, (m + 1)y)\| + \|\Delta_f(-my, -(m + 1)y)\| \\
&+ \|\Delta_f(m(x + y), (m + 1)(x + y))\| + \|\Delta_f(m(x - y), (m + 1)(x - y))\| \\
&+ \|\Delta_f((2m + 1)x, (2m + 1)y)\| + 2\|\Delta_f((m + 1)x, (m + 1)y)\| \\
&+ \|\Delta_f(mx, my)\| + \|\Delta_f(-mx, -my)\|; \qquad \text{(b*)} \\
2\|\Delta_f(x, y)\| \leq{}& 2\|\Delta_f(x, mx)\| + \|\Delta_f(y, my)\| + \|\Delta_f(-y, -my)\| \\
&+ \|\Delta_f(x + y, m(x + y))\| + \|\Delta_f(x - y, -m(x - y))\| \\
&+ \|\Delta_f(-(m - 1)x, -(m - 1)y)\| + \|\Delta_f(-(m - 1)x, -(m - 1)y)\| \\
&+ \|\Delta_f(-mx, -my)\| + \|\Delta_f(mx, my)\|. \qquad \text{(c*)}
\end{aligned}$$

If (□) holds, where □ = a, b, c, then letting $m \to \infty$ in the corresponding inequality (□*) together with Remark 2.1 implies that f is Drygas on D. This completes the proof. □

The following example shows that Conditions (a), (b) or (c) in Theorem 2.3 cannot be omitted.

EXAMPLE 2.4. Let $X = Y := \mathbb{R}$ and $D := \mathbb{R}\backslash\{0\}$. Note that D satisfies Conditions (D1) and (D2*). Let $f : D \to Y$ be defined by

$$f(x) := \frac{1}{x} \quad \text{for all } x \in D.$$

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We note that, for each $(x, y) \in \mathbb{D}$, each $z \in D$ and each $m \in \mathbb{N}$,

$$
\begin{aligned}
|\Delta_f(mx, my)| &= \frac{1}{m}\left|\frac{1}{x+y} + \frac{1}{x-y} - \frac{2}{x}\right|,\\
|\Delta_f((m+1)z, mz)| &= \left|\frac{1}{(2m+1)z} + \frac{1}{z} - \frac{2}{(m+1)z}\right|,\\
|\Delta_f(mz, (m+1)z)| &= \left|\frac{1}{(2m+1)z} - \frac{1}{z} - \frac{2}{mz}\right|,\\
|\Delta_f(z, mz)| &= \left|\frac{1}{(m+1)z} - \frac{1}{(m-1)z} - \frac{2}{z}\right|.
\end{aligned}
$$

It follows that:

- $\lim_{m\to\infty} |\Delta_f(mx, my)| = 0$;
- $\lim_{m\to\infty} |\Delta_f((m+1)z, mz)| = \lim_{m\to\infty} |\Delta_f(mz, (m+1)z)| = 1/|z|$; and
- $\lim_{m\to\infty} |\Delta_f(z, mz)| = 2/|z|$.

It is easy to see that f is not Drygas on D.

Corollary 2.5. *Let $\varphi : D \times D \to [0, \infty)$ be a function such that*

$$\varphi(x, y) = \varphi(x, -y) \quad \text{and} \quad \lim_{m\to\infty} \varphi(mx, my) = 0 \quad \text{for all } (x, y) \in \mathbb{D}. \tag{2.1}$$

Suppose that $f : D \to Y$ satisfies the inequality

$$\|\Delta_f(x, y)\| \le \varphi(x, y) \quad \text{for all } (x, y) \in \mathbb{D}.$$

Suppose that one of the following conditions is satisfied.

(a) $\lim_{m\to\infty} \varphi((m+1)x, mx) = 0$ *for all $x \in D$.*
(b) $\lim_{m\to\infty} \varphi(mx, (m+1)x) = 0$ *for all $x \in D$.*
(c) $\lim_{m\to\infty} \varphi(x, mx) = 0$ *for all $x \in D$.*

Then f is Drygas on D.

Based on the notion of the asymptotic behaviour of the Cauchy functional equation given by Skof [13, Teorema 3], we obtain the following characterisation of the Drygas functional equation on a restricted domain.

Corollary 2.6. *Suppose that $f : D \to Y$ is a function. Then the following statements are equivalent.*

(1) $\lim_{(x,y)\in\mathbb{D}, \|x\|+\|y\|\to\infty} \|\Delta_f(x, y)\| = 0$.
(2) *f is Drygas on D.*

Proof. The implication (2) ⇒ (1) is trivial. We now suppose that (1) holds. To prove (2), let $(x, y) \in \mathbb{D}$ and $z \in D$ be given. For a sufficiently large $m \in \mathbb{N}$, we note that $(mx, my), (mx, (m+1)x) \in \mathbb{D}$. Moreover,

$$\lim_{m\to\infty} (\|mx\| + \|my\|) = \lim_{m\to\infty} (\|mz\| + \|(m+1)z\|) = \infty.$$

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The statement (1) implies that

$$\lim_{m\to\infty} \|\Delta_f(mx, my)\| = \lim_{m\to\infty} \|\Delta_f(mz, (m+1)z)\| = 0.$$

It follows from Theorem 2.3 that f is Drygas on D. This completes the proof. □

2.2. Theorem AS(1) as a consequence of Corollary 2.5. We state the following hyperstability result for inhomogeneous Drygas function equations on a restricted domain satisfying Conditions (D1) and (D2).

Corollary 2.7. *Suppose that $\emptyset \neq E \subset X \setminus \{0\}$ satisfies Conditions (D1) and (D2). Let $h_1, h_2 : E \to [0, \infty)$ be functions such that $\lim_{m\to\infty} \lfloor h_1 \rceil(m) = \lim_{m\to\infty} \lfloor h_2 \rceil(m) = 0$. Suppose that $f : E \to Y$ and $c : E \times E \to Y$ are functions satisfying the inequality*

$$\|\Delta_f(x, y) - c(x, y)\| \le h_1(x) + h_2(y) \quad \text{for all } x, y \in E \text{ with } x \pm y \in E.$$

Suppose that there is a function $g : E \to Y$ satisfying the equation

$$\Delta_g(x, y) = c(x, y) \quad \text{for all } x, y \in E \text{ with } x \pm y \in E.$$

Then $\Delta_f(x, y) = c(x, y)$ for all $x, y \in E$ with $x \pm y \in E$.

Proof. Define a function $\varphi : E \times E \to [0, \infty)$ by

$$\varphi(x, y) := h_1(x) + h_2(y) + h_2(-y) \quad \text{for all } x, y \in E.$$

We see that $\varphi(x, y) = \varphi(x, -y) \ge h_1(x) + h_2(y)$ and

$$\|\Delta_{\tilde{f}}(x, y)\| \le \varphi(x, y) \quad \text{for all } x, y \in E \text{ with } x \pm y \in E,$$

where $\tilde{f} : E \to Y$ is defined by

$$\tilde{f}(x) := f(x) - g(x) \quad \text{for all } x \in E.$$

In fact,

$$\begin{aligned} \Delta_{\tilde{f}}(x, y) &:= \tilde{f}(x + y) + \tilde{f}(x - y) - 2\tilde{f}(x) - \tilde{f}(y) - \tilde{f}(-y) \\ &= \Delta_f(x, y) - \Delta_g(x, y) \\ &= \Delta_f(x, y) - c(x, y). \end{aligned}$$

For a sufficiently large $m \in \mathbb{N}$:

- $\varphi(mx, my) \le \lfloor h_1 \rceil(m) h_1(x) + \lfloor h_2 \rceil(m)(h_2(y) + h_2(-y))$ for all $x, y \in E$ with $x \pm y \in E$; and
- $\varphi(mz, (m+1)z) \le \lfloor h_1 \rceil(m) h_1(z) + \lfloor h_2 \rceil(m+1)(h_2(z) + h_2(-z))$ for all $z \in E$.

This implies that φ satisfies (2.1) and Condition (b) of Corollary 2.5. Consequently, $\tilde{f}$ is Drygas on E and hence the result follows. □

Remark 2.8. If $h_1 = h_2 = h$, then we immediately obtain [1, Corollary 2.3] and hence Theorem 2.1 of [1] by letting $c(x, y) = 0$ and $g(x) = 0$.

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Before moving on, we remark on the Condition (W_2) of Theorem AS(1):
(W_2) $\lim_{m\to\infty}\lfloor h\rceil(m) = 0$ and $\lim_{m\to\infty}\lfloor h\rceil(-m) = 0$.

It is easy to see that $\lim_{m\to\infty}\lfloor h\rceil(m) = 0 \iff \lim_{m\to\infty}\lfloor h\rceil(-m) = 0$. In fact, we note that $\lfloor h\rceil(m) \le \lfloor h\rceil(-1)\lfloor h\rceil(-m) \le \lfloor h\rceil^2(-1)\lfloor h\rceil(m)$. Moreover, it follows from Condition (W_2) that $M_1 := \{m \ge M : 2\lfloor h\rceil(m+1) + \lfloor h\rceil(m) + \lfloor h\rceil(-m) + \lfloor h\rceil(2m+1) < 1\}$ is an infinite set. In particular, Condition (W_1) in Theorem AS(1) is superfluous.

Remark 2.9. According to Corollary 2.7, we note that $\lfloor h\rceil(\cdot)$ is not necessarily defined if E satisfies only Condition (D2*) in place of Condition (D2).

The following example shows that our Corollary 2.5 is a genuine generalisation of [1, Theorem 2.1].

Example 2.10. Let $\varphi : (\mathbb{R}\backslash\{0\}) \times (\mathbb{R}\backslash\{0\}) \to [0,\infty)$ be a function defined by

$$\varphi(x,y) := \frac{1}{1+|x+y|} + \frac{1}{1+|x-y|} \quad \text{for all } x,y \in \mathbb{R}\backslash\{0\}.$$

Then there are no functions $h_1, h_2 : \mathbb{R}\backslash\{0\} \to [0,\infty)$ such that:

(a) $\varphi(x,y) \le h_1(x) + h_2(y)$ for all $x,y \in \mathbb{R}\backslash\{0\}$ with $x \pm y \in \mathbb{R}\backslash\{0\}$; and
(b) $\lim_{m\to\infty}\lfloor h_1\rceil(m) = \lim_{m\to\infty}\lfloor h_2\rceil(m) = 0$.

We first note that:

- $\lim_{m\to\infty}\varphi(xm, my) = 0$ for all $x,y \in \mathbb{R}\backslash\{0\}$ with $x \pm y \in \mathbb{R}\backslash\{0\}$; and
- $\lim_{m\to\infty}\varphi(x, mx) = 0$ for all $x \in \mathbb{R}\backslash\{0\}$.

Suppose that there are two functions h_1, h_2 such that (a) and (b) hold. Choose $x_0 \in \mathbb{R}\backslash\{0\}$. For a large $m \in \mathbb{N}$, it follows from (a) that

$$\begin{aligned} 0 < \frac{1}{1+|x_0|} &\le \varphi(mx_0, (m+1)x_0) \\ &\le h_1(mx_0) + h_2((m+1)x_0) \\ &\le \lfloor h_1\rceil(m)h_1(x_0) + \lfloor h_2\rceil(m+1)h_2(x_0). \end{aligned}$$

It follows from (b) that $\lim_{m\to\infty}\varphi(mx_0, (m+1)x_0) = 0$, which is impossible.

2.3. An error in the proof of Theorem 2.2 of [1] and its correction. The proof of Theorem AS(2) (see [1, Theorem 2.2 and its proof]) is based on the fixed point theorem of Brzdęk [7]. The authors of [1] only made use of Conditions (W_1') and (W_2') until the very last line of the proof to obtain the following inequality

$$\|F_m(x) - f(x)\| \le \frac{\lfloor h_1\rceil(m+1)\lfloor h_2\rceil(m)}{1 - 2\alpha(m+1) - \alpha(m) - \alpha(-m) - \alpha(2m+1)},$$

where $\alpha(n) := \lfloor h_1\rceil(n)\lfloor h_2\rceil(n)$. The authors claim that

$$\lim_{m\to\infty} \frac{\lfloor h_1\rceil(m+1)\lfloor h_2\rceil(m)}{1 - 2\alpha(m+1) - \alpha(m) - \alpha(-m) - \alpha(2m+1)} = 0$$

as a consequence of Conditions (W_2') and (W_3'). Unfortunately, it is *not* true. We show that there exist two functions $h_1, h_2 : \mathbb{Z}\setminus\{0\} \to [0,\infty)$ such that:

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- $\lim_{m\to\infty} \lfloor h_1 \rceil(\pm m)\lfloor h_2 \rceil(\pm m) = 0$;
- $\lim_{m\to\infty} \lfloor h_1 \rceil(m) = 0$; and
- $\lim_{m\to\infty} \lfloor h_1 \rceil(m+1)\lfloor h_2 \rceil(m) \neq 0$.

The construction is as follows. Let $\mathbb{P}$ be the set of all primes. For convenience, we write $\mathbb{P} := \{p_1, p_2, \ldots\}$, where $p_1 < p_2 < \cdots$. We first observe that if $h : \mathbb{P} \to [0, \infty)$ is given and

$$h(m) := h(p_1)^{\alpha_1} h(p_2)^{\alpha_2} \cdots h(p_l)^{\alpha_l}, \tag{2.2}$$

where $m := p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_l^{\alpha_l}$ and $\alpha_1, \ldots, \alpha_l$ are nonnegative integers, then $h : \mathbb{N} \to [0, \infty)$ is *completely multiplicative*, that is, $h(mn) = h(m)h(n)$ for all $m, n \in \mathbb{N}$.

Proposition 2.11. *Suppose that* $h : \mathbb{N} \to [0, 1)$ *is a completely multiplicative function. If* $\lim_{m\to\infty} h(p_m) = 0$, *then* $\lim_{m\to\infty} h(m) = 0$.

Proof. Let $\varepsilon > 0$. We choose $N \in \mathbb{N}$ such that $h(p_m) < \varepsilon$ for all $m > N$. We also choose M such that $h(p_j^M) < \varepsilon$ for all $j = 1, 2, \ldots, N$. Let $m \geq (p_1 p_2 \cdots p_N)^M$ be an integer. We consider the following two cases.

Case 1: p_k is a factor of m for some $k > N$. Since $h : \mathbb{N} \to [0, 1)$ is completely multiplicative, we have $h(m) < h(p_k) < \varepsilon$.

Case 2: $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_N^{\alpha_N}$, where $\alpha_1, \alpha_2, \ldots, \alpha_N$ are nonnegative integers. Since $m \geq (p_1 p_2 \cdots p_N)^M$, we have $\alpha_j \geq M$ for some $j = 1, 2 \ldots, N$. In particular, this gives $h(m) \leq h(p_j^M) < \varepsilon$. The proof is finished. □

Proposition 2.12. *There exist two completely multiplicative functions* h_1, h_2 *mapping* $\mathbb{N} \to [0, \infty)$ *such that:*

- $\lim_{m\to\infty} h_1(m)h_2(m) = 0$;
- $\lim_{m\to\infty} h_1(m) = 0$; *and*
- $\lim_{m\to\infty} h_1(m+1)h_2(m) \neq 0$.

Proof. We first construct inductively a strictly increasing sequence $\{i_n\}$ of positive integers. Suppose that $i_1 := 1$. If i_m is already defined, then let i_{m+1} be the largest integer j such that

$$p_j \mid (p_1 p_2 \cdots p_{i_m} + 1).$$

Obviously, $i_{m+1} > i_m$. Moreover,

$$p_1 p_2 \cdots p_{i_m} + 1 = p_{i_m+1}^{\alpha_{i_m+1}} p_{i_m+2}^{\alpha_{i_m+2}} \cdots p_{i_{m+1}}^{\alpha_{i_{m+1}}},$$

where $\alpha_{i_m+1}, \alpha_{i_m+2}, \ldots, \alpha_{i_{m+1}}$ are nonnegative integers and $\alpha_{i_{m+1}} \geq 1$. We note that

$$p_{i_m}^{i_m} > p_1 p_2 \cdots p_{i_m} + 1 = p_{i_m+1}^{\alpha_{i_m+1}} p_{i_m+2}^{\alpha_{i_m+2}} \cdots p_{i_{m+1}}^{\alpha_{i_{m+1}}} > p_{i_m}^{\alpha_{i_m+1}+\alpha_{i_m+2}+\cdots+\alpha_{i_{m+1}}}$$

and hence

$$i_m > \alpha_{i_m+1} + \alpha_{i_m+2} + \cdots + \alpha_{i_{m+1}}.$$

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For convenience, we write

$$(m] := \{i_m + 1, i_m + 2, \ldots, i_{m+1}\}.$$

For each $j \in (2m-1]$, we define

$$h_1(p_j) := \left(\frac{1}{2m^2}\right)^{(i_{2m+1}-i_{2m})i_{2m}},$$
$$h_2(p_j) := (2m)^{(i_{2m+1}-i_{2m})i_{2m}}.$$

For each $j \in (2m]$, we define

$$h_1(p_j) := \left(\frac{1}{2m}\right)^{i_{2m}-i_{2m-1}},$$
$$h_2(p_j) := 1.$$

Note that:

- $\lim_{m\to\infty} h_1(p_m)h_2(p_m) = 0$; and
- $\lim_{m\to\infty} h_1(p_m) = 0$.

Moreover,

$$\begin{aligned} &h_1(p_1p_2\cdots p_{i_{2m}}+1)h_2(p_1p_2\cdots p_{i_{2m}}) \\ &\quad\geq h_1\left(p_{i_{2m}+1}^{i_{2m}}p_{i_{2m}+2}^{i_{2m}}\cdots p_{i_{2m+1}}^{i_{2m}}\right)h_2\left(p_{i_{2m-1}+1}p_{i_{2m-1}+2}\cdots p_{i_{2m}}\right) \\ &\quad= \left(\frac{1}{2m}\right)^{(i_{2m}-i_{2m-1})i_{2m}(i_{2m+1}-i_{2m})}(2m)^{(i_{2m+1}-i_{2m})i_{2m}(i_{2m}-i_{2m-1})} = 1. \end{aligned}$$

By defining $h : \mathbb{N} \to [0,1)$ as in (2.2), it is easy to see from Proposition 2.12 that:

- $\lim_{m\to\infty} h_1(m)h_2(m) = 0$;
- $\lim_{m\to\infty} h_1(m) = 0$; and
- $\lim_{m\to\infty} h_1(m+1)h_2(m) \neq 0$. □

For $h_1, h_2 : \mathbb{N} \to [0,\infty)$ in Proposition 2.12, we now define completely multiplicative functions $\widetilde{h_1}, \widetilde{h_2} : \mathbb{Z}\backslash\{0\} \to [0,\infty)$ by

$$\widetilde{h_i}(m) = \widetilde{h_i}(-m) := h_i(m) \quad \text{for all } m \in \mathbb{N} \text{ and for all } i = 1, 2.$$

It is easy to see that $D := \mathbb{Z}\backslash\{0\}$ satisfies Conditions (D1) and (D2). We note that $\lfloor\widetilde{h_i}\rceil(m) = \widetilde{h_i}(m)$ for all $m \in \mathbb{N}$. Hence $\widetilde{h_1}$ and $\widetilde{h_2}$ are our candidates.

Finally, we propose the following correction of Theorem AS(2), which is a consequence of our Corollary 2.5 where $\varphi(x,y) := h_1(x)(h_2(y) + h_2(-y))$.

Corollary 2.13. *Suppose that $\varnothing \neq E \subset X \setminus \{0\}$ satisfies Conditions (D1) and (D2). Let $h_1, h_2 : E \to [0,\infty)$ be functions such that one of the following conditions is satisfied.*

(1) $\lim_{m\to\infty}\lfloor h_1\rceil(m)\lfloor h_2\rceil(m) = \lim_{m\to\infty}\lfloor h_1\rceil(m+1)\lfloor h_2\rceil(m) = 0.$
(2) $\lim_{m\to\infty}\lfloor h_1\rceil(m)\lfloor h_2\rceil(m) = \lim_{m\to\infty}\lfloor h_1\rceil(m)\lfloor h_2\rceil(m+1) = 0.$

Suppose that $f : E \to Y$ satisfies

$$\|\Delta_f(x, y)\| \le h_1(x)h_2(y) \quad \text{for all } x, y \in E \text{ with } x \pm y \in E.$$

Then f is Drygas on E.

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JEDSADA SENASUKH, Department of Mathematics,
Faculty of Science, Khon Kaen University,
Khon Kaen 40002, Thailand
e-mail: senasukh@kkumail.com

SATIT SAEJUNG, Department of Mathematics,
Faculty of Science, Khon Kaen University,
Khon Kaen 40002, Thailand
and
Research Center for Environmental and Hazardous Substance Management,
Khon Kaen University, Khon Kaen, 40002, Thailand
e-mail: saejung@kku.ac.th