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RANK k VECTORS IN SYMMETRY CLASSES OF TENSORS*

BY
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1. **Introduction.** Let F be a field, G a subgroup of S_m, the symmetric group of degree m, and χ a linear character on G, i.e., a homomorphism of G into the multiplicative group of F. Let $V_1, \ldots, V_m$ be vector spaces over F such that $V_i = V_{\sigma(i)}$ for $i=1, \ldots, m$ and for all $\sigma \in G$. If W is a vector space over F, then a m-multilinear function $f: \mathrm{X}_{i=1}^m V_i \to W$ is said to be *symmetric with respect to G and* χ *if*

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(m)}) = \chi(\sigma) f(x_1, \ldots, x_m)$$

for any $\sigma \in G$ and for arbitrary $x_i \in V_i$. A pair (P, μ) consisting of a vector space P over F and a m-multilinear function $\mu: \mathrm{X}_{i=1}^m V_i \to P$, symmetric with respect to G and χ, is a *symmetry classes of tensors* over $V_1, \ldots, V_m$ associated with G and χ if the following universal factorization property is satisfied: for any vector space U over F and any m-multilinear function $f: \mathrm{X}_{i=1}^m V_i \to U$, symmetric with respect to G and χ, there exists a unique linear mapping $g: P \to U$ such that $f = g\mu$.

The symmetry class over $V_1, \ldots, V_m$ associated with G and χ always exists and is unique up to vector space isomorphism (see [11], [12]). We shall denote such a space by $(V_1, \ldots, V_m)_\chi(G)$. If $V_1 = \cdots = V_m = V$, then such a space is usually denoted by $V_\chi^m(G)$ [11]. The vector $\mu(x_1, \ldots, x_m)$ is called *decomposable* and is denoted by $x_1 * \cdots * x_m$. The most familiar symmetry classes are the tensor, Grassmann and symmetric spaces.

Let $T_i: V_i \to V_i$ be linear mappings such that $T_i = T_{\sigma(i)}$ for $i=1, \ldots, m$ and for all $\sigma \in G$. Then

$$\phi: (x_1, \ldots, x_m) \to T_1 x_1 * \cdots * T_m x_m$$

is symmetric with respect to G and χ and hence induces a unique linear mapping $K(T_1, \ldots, T_m)$ on $(V_1, \ldots, V_m)_\chi(G)$ such that

$$K(T_1, \ldots, T_m) x_1 * \cdots * x_m = T_1 x_1 * \cdots * T_m x_m.$$

$K(T_1, \ldots, T_m)$ is called the *associated transformation* of $T_1, \ldots, T_m$. When $T_1 = \cdots = T_m = T$, we shall denote $K(T_1, \ldots, T_m)$ simply by $K(T)$ [9, 11].

A non-zero vector in $(V_1, \ldots, V_m)_\chi(G)$ is said to have *rank k* if it is the sum of k but not less than k non-zero decomposable elements in $(V_1, \ldots, V_m)_\chi(G)$. The set of all rank k vectors in $(V_1, \ldots, V_m)_\chi(G)$ is denoted by $R_k((V_1, \ldots, V_m)_\chi(G))$.

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In this paper we prove that (i) the rank of each vector in $(V_1, \ldots, V_m)_\chi(G)$ is unchanged if we extend $V_1, \ldots, V_m$; (ii) for each rank k vector in $(V_1, \ldots, V_m)_\chi(G)$ and each orbit 0 of G there associates a unique subspace of V_i where $i \in 0$; (iii) if there is an orbit 0 of G such that $|0| \geq 2$, dim $V_j \geq |0| + 2(k-1)$ where $j \in 0$, then $(V_1, \ldots, V_m)_\chi(G)$ has a basis consisting of rank k vectors. (i) and (ii) generalize two results of Lim [8]. We also give some criteria for determining the rank of a vector in $(V_1, \ldots, V_m)_\chi(G)$. From (i) and (ii) we obtain an application on intersections of symmetry classes and an application on equalities of two associated transformations.

2. **Properties of rank** k **vectors.** Throughout this section, let $(V_1, \ldots, V_m)_\chi(G)$ denote a symmetry class of tensors over $V_1, \ldots, V_m$ associated with a subgroup G of S_m and a linear character χ on G.

For any vectors $z_1, \ldots, z_n$ in a vector space Z, let $\langle z_1, \ldots, z_n \rangle$ denote the subspace of Z spanned by $z_1, \ldots, z_n$.

LEMMA 1. *Let* $x_1 + \cdots + x_k = y_1 + \cdots + y_q \in R_k((V_1, \ldots, V_m)_\chi(G))$ *where* $x_i = x_{i1} * \cdots * x_{im}$, $y_n = y_{n1} * \cdots * y_{nm}$ *for each* $i = 1, \ldots, k$ *and* $n = 1, \ldots, q$. *Then for each orbit* 0 *of* G,

$$\sum_{i=1}^{k} \langle x_{id} : d \in 0 \rangle \subseteq \sum_{n=1}^{q} \langle y_{nd} : d \in 0 \rangle.$$

Proof. Suppose that for some j, $1 \leq j \leq k$,

$$\langle x_{jd} : d \in 0 \rangle \nsubseteq \sum_{n=1}^{q} \langle y_{nd} : d \in 0 \rangle.$$

Then for some $s \in 0$, $x_{js} \notin \sum_{n=1}^{q} \langle y_{nd} : d \in 0 \rangle$.

Consider the associated transformation $K(T_1, \ldots, T_m)$ on $(V_1, \ldots, V_m)_\chi(G)$ where $T_i = T_{\sigma(i)}$ for all $\sigma \in G$, $i = 1, \ldots, m$ and $T_1 \ \ldots, T_m$ are defined as follows:

If $i \in 0$, $T_i : V_i \to V_i$ is a linear mapping such that $T_i(x_{js}) = 0$ and $T_i \mid \sum_{n=1}^{q} \langle y_{nd} : d \in 0 \rangle$ is the identity mapping.

If $i \notin 0$, $T_i : V_i \to V_i$ is the identity mapping.

We have $K(T_1, \ldots, T_m)(\sum_{i=1}^{k} x_i) = K(T_1, \ldots, T_m)(\sum_{n=1}^{q} y_n)$. Since $K(T_1, \ldots, T_m)y_n = y_n$ for $n = 1, \ldots, q$ and

$$K(T_1, \ldots, T_m)x_j = T_1 x_{j1} * \cdots * T_m x_{jm} = 0,$$

it follows that

$$K(T_1, \ldots, T_m)(x_1 + \cdots + x_{j-1} + x_{j+1} + \cdots + x_k) = y_1 + \cdots + y_q \in R_k((V_1, \ldots, V_m)_\chi(G)).$$

This is a contradiction since the left hand side is a vector of rank less than k or the zero vector. Hence

$$\langle x_{jd} : d \in 0 \rangle \subseteq \sum_{n=1}^{q} \langle y_{nd} : d \in 0 \rangle$$

for each $j=1, \ldots, k$. Hence

$$\sum_{i=1}^{k}\langle x_{id}: d \in 0\rangle \subseteq \sum_{n=1}^{q}\langle y_{nd}: d \in 0\rangle.$$

THEOREM 1. *Let $x_1+\cdots+x_k=y_1+\cdots+y_k \in R_k((V_1, \ldots, V_m)_\chi(G))$ where $x_j=x_{j1} * \cdots * x_{jm}$ and $y_j=y_{ji} * \cdots * y_{jm}$ for each $j=1, \ldots, k$. Then for each orbit 0 of G,*

$$\sum_{j=1}^{k}\langle x_{jd}: d \in 0\rangle = \sum_{j=1}^{k}\langle y_{jd}: d \in 0\rangle.$$

Proof. This follows immediately from Lemma 1.

COROLLARY 1. *Suppose that $x_1 * \cdots * x_m=y_1 * \cdots * y_m \in V_\chi^m(G)$ and $x_1 * \cdots * x_m \neq 0$. Then $\langle x_1, \ldots, x_m\rangle=\langle y_1, \ldots, y_m\rangle$.*

This corollary generalizes a lemma of Marcus and Minc [11].

EXAMPLE. Let $\otimes^m V$ denote the mth tensor product space of a vector space V. Let $z \in \otimes^m V$ be a rank k vector. Then for any non-zero vector $v \in V$, $v \otimes z$ is of rank k in $\otimes^{m+1} V$. To prove this, we first note that $v \otimes z \neq 0$. Suppose $v \otimes z=y_1+\cdots+y_n \in R_n(\otimes^{m+1} V)$ where $y_i=y_{i1}\otimes\cdots\otimes y_{i(m+1)}$, $1 \leq i \leq n$. Clearly $n \leq k$. By Lemma 1, $\langle v\rangle \supseteq \langle y_{11}, \ldots, y_{n1}\rangle$. This implies that $y_{i1}=\lambda_i v$ for some non-zero scalars λ_i. Hence $v\otimes z=v\otimes(\sum_{i=1}^n \lambda_i y_{i2}\otimes\cdots\otimes y_{i(m+1)})$. Thus $z=\sum_{i=1}^n \lambda_i y_{i2}\otimes\cdots\otimes y_{i(m+1)} \in R_k(\otimes^m V)$. This shows that $n=k$. Hence $v\otimes z \in R_k(\otimes^{m+1} V)$.

DEFINITION. Let $z=z_1+\cdots+z_k$ be a *rank k vector* in $(V_1, \ldots, V_m)_\chi(G)$ where $z_j=z_{j1} * \cdots * z_{jm}$, $1 \leq j \leq k$. For each orbit 0 of G, we define $0(z)$ to be the subspace $\sum_{j=1}^k \langle z_{jd}: d \in 0\rangle$.

THEOREM 2. *Let $U_1, \ldots, U_m$ be subspaces of $V_1, \ldots, V_m$ respectively such that $U_i=U_{\sigma(i)}$ for $i=1, \ldots, m$ and for all $\sigma \in G$. Then*

$$R_k((U_1, \ldots, U_m)_\chi(G)) \subseteq R_k((V_1, \ldots, V_m)_\chi(G)).$$

Proof. Let $y \in R_k((U_1, \ldots, U_m)_\chi(G))$. For each orbit 0 of G and each $r \in 0$, $0(y) \subseteq U_r$. Suppose

$$y = \sum_{j=1}^{n} y_j \in R_n((V_1, \ldots, V_m)_\chi(G))$$

where y_j is a decomposable element for each j. Then $n \leq k$. According to Lemma 1, we have for each 0 of G,

$$\sum_{j=1}^{n} 0(y_j) \subseteq 0(y) \subseteq U_r$$

where $r \in 0$. If $n<k$, then the rank of y is less than k in $(U_1, \ldots, U_m)_\chi(G)$ which is a contradiction. Therefore $n=k$ and $y \in R_k((V_1, \ldots, V_m)_\chi(G))$.

THEOREM 3. *Let $x \in R_k((V_1, \ldots, V_m)_\chi(G))$. Let $y=y_1 * \cdots * y_m \neq 0$. If for some orbit 0, there is a $s \in 0$ such that $y_s \notin 0(x)$, then $x+y$ is of rank k or $k+1$.*

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Proof. If $x+y=0$, then $x=-y$. This implies $k=1$. By Theorem 1, $y_s \in 0(x)$, a contradiction.

If $x+y=\sum_{j=1}^{n} z_j$ is of rank n where $1 \leq n < k$, then $x=\sum_{j=1}^{n} z_j - y$. This implies that $n=k-1$ since x is of rank k. By Theorem 1,

$$0(x) = 0(z_1)+\cdots+0(z_{k-1})+0(y).$$

Hence $y_s \in 0(x)$ which is a contradiction.

Therefore $x+y$ is of rank k or $k+1$.

THEOREM 4. *Let x be a rank k vector in $(V_1, \ldots, V_m)_\chi(G)$. Let $y=y_1 * \cdots * y_m \neq 0$. If for some orbit 0 of G, there are $d, q \in 0$ such that y_d, y_q are linearly independent and*

$$\langle y_d, y_q \rangle \cap 0(x) = \{0\},$$

then $x+y$ is of rank $k+1$.

Proof. By Theorem 3, $x+y$ is of rank k or $k+1$. Assume that $x+y=z$ is of rank k where $z=\sum_{j=1}^{k} z_j$ and $z_j=z_{j1} * \cdots * z_{jm}$, $1 \leq j \leq k$. Since $y=-x+z$, it follows from Lemma 1 that

$$0(y) \subseteq 0(x)+0(z).$$

If $0(z) \subseteq 0(x)$, then $0(y) \subseteq 0(x)$, which is a contradiction to the hypothesis. Hence $0(z) \nsubseteq 0(x)$. Thus for some $s \in 0$ and some $1 \leq r \leq k$, $z_{rs} \notin 0(x)$. We have either $\langle y_d \rangle + (\langle z_{rs} \rangle + 0(x))$ or $\langle y_q \rangle + (\langle z_{rs} \rangle + 0(x))$ is a direct sum. We may assume that $\langle y_d \rangle + (\langle z_{rs} \rangle + 0(x))$ is a direct sum.

Let $g_s : V_s \to V_s$ be a linear mapping such that

$$g_s(y_d) = 0, \qquad g_s(z_{rs}) = 0$$

and

$$g_s|_{0(x)} = \text{identity mapping}.$$

Let $g_i : V_i \to V_i$ be the identity mapping if $i \notin 0$ and $g_i = g_s$ if $i \in 0$. Then

$$K(g_1, \ldots, g_m)(x+y) = K(g_1, \ldots, g_m)z = x = K(g_1, \ldots, g_m)\left(\sum_{j \neq r} z_j\right).$$

Since x is of rank k and $K(g_1, \ldots, g_n)(\sum_{j \neq r} z_j)$ is either the zero vector or of rank $<k$, we obtain a contradiction. Hence $x+y$ is of rank $k+1$.

THEOREM 5. *Let x be a rank k vector in $(V_1, \ldots, V_m)_\chi(G)$. Let y be a non-zero decomposable element. If there are two orbits 0_1 and 0_2 of G such that*

$$0_1(y) \nsubseteq 0_1(x) \quad \textit{and} \quad 0_2(y) \nsubseteq 0_2(x),$$

then $x+y$ is of rank $k+1$.

Proof. Let $y=y_1 * \cdots * y_m$. Choose $d \in 0$ such that $y_d \notin 0_1(x)$. Let $x+y=z$. By Theorem 3, z is of rank k or $k+1$. Suppose $z=\sum_{j=1}^{k} z_j$ is of rank k where z_j is a decomposable element for each j.

Let $g_d: V_d \to V_d$ be a linear mapping such that $g_d(y_d)=0$ and $g_d \mid_{0_1(x)}=$identity mapping. Let $g_s: V_s \to V_s$ be the linear mapping such that $g_s=g_d$ if $s \in 0_1$, and g_s is the identity mapping if $s \notin 0_1$. Let $K(g_1, \ldots, g_m)z_j=z'_j$, $1 \leq j \leq k$. Then

$$K(g_1, \ldots, g_m)(x+y) = x = K(g_1, \ldots, g_m)z = \sum_{j=1}^{k} z'_j.$$

In view of Theorem 1, $0_2(x)=\sum_{j=1}^{k} 0_2(z'_j)$. Since $g_s: V_s \to V_s$ is the identity mapping if $s \in 0_2$, it follows that $0_2(z_j)=0_2(z'_j)$, $1 \leq j \leq k$. Hence

$$0_2(x) = \sum_{j=1}^{k} 0_2(z_j) = 0_2(z).$$

Since $y=-x+z$, it follows from Lemma 1 that

$$0_2(y) \subseteqq 0_2(x)+0_2(z) = 0_2(x)\cdot$$

This contradicts the hypothesis. Hence $x+y$ is of rank $k+1$.

LEMMA 2. *Let* $x=x_1 * \cdots * x_m \in (V_1, \ldots, V_m)_\chi(G)$. *If* $x=0$ *then* $\dim\langle x_i: i \in 0\rangle < |0|$ *for some orbit* 0 *of* G *where* $|0|$ *denotes the number of elements in* 0.

Proof. Suppose that $\dim\langle x_i: i \in 0\rangle=|0|$ for all orbits 0 of G. For each j, let $f_j: V_j \to F$ be a linear map such that $f_j(x_j)=1$, $f_j(x_d)=0$ for all d where $j \neq d$ and j, d belong to the same orbit of G. Since

$$f:(w_1, \ldots, w_m) \to \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{m} f_{\sigma(i)}(w_i), \qquad w_i \in V_i,$$

is symmetric with respect to G and χ, there exists a linear mapping $h:(V_1, \ldots, V_m)_\chi(G) \to F$ such that

$$h(w_1 * \cdots * w_m) = f(w_1, \ldots, w_m).$$

Since $f_{\sigma(j)}(x_j)=1$ if and only if $\sigma(j)=j$, it follows that $\prod_{j=1}^{m} f_{\sigma(j)}(x_j)=0$ if $\sigma \neq 1$. Hence

$$f(x_1, \ldots, x_m) = \chi(1) \prod_{j=1}^{m} f_j(x_j) = 1.$$

Therefore $h(x_1 * \cdots * x_m)=1$. This is a contradiction since $x_1 * \cdots * x_m=0$. Hence the proof is complete.

THEOREM 6. *Let* $x_j=x_{j1} * \cdots * x_{jm}$, $j=1, \ldots, k$, *be* k *decomposable elements in* $(V_1, \ldots, V_m)_\chi(G)$. *If for each orbit* 0,

$$\dim\left(\sum_{j=1}^{k} \langle x_{jd}: d \in 0\rangle\right) = |0|\, k,$$

then $\sum_{j=1}^{k} x_j$ *is of rank* k.

Proof. This follows from Lemma 2, Theorem 4 and Theorem 5 by induction.

REMARK. Taking $G=S_m$, $\chi=$"sign of permutation" character in Theorem 1, Theorem 2 and Theorem 6 we obtain Theorem 3, Theorem 5 and Theorem 6 in [8] respectively.

LEMMA 3. *Let $U_1, \ldots, U_t$ be vector spaces over the same field such that* $\dim U_i \geq m_i$ *where m_i is a positive integer for each i. Then $(\otimes^{m_1} U_1)\otimes\cdots\otimes(\otimes^{m_t} U_t)$ has a basis consisting of decomposable elements of the form*

$$(x_{11}\otimes\cdots\otimes x_{1m_1})\otimes\cdots\otimes(x_{t1}\otimes\cdots\otimes x_{tm_t})$$

in which $x_{i1}, \ldots, x_{im_i}$ are linearly independent for each i.

Proof. It suffices to show that the set of all decomposable elements $x_{i1}\otimes\cdots\otimes x_{im_i}$ such that $x_{i1}, \ldots, x_{im_i}$ are linearly independent in U_i spans $\otimes^{m_i} U_i$. This can be shown easily by induction on m_i.

LEMMA 4. *$(V_1, \ldots, V_m)_\chi(G)$ has a basis consisting of decomposable elements v such that* $\dim 0(v)=|0|$ *for each orbit* 0 *of G provided* $\dim V_j \geq |0|$ *for $j \in 0$.*

Proof. Let $0_1, \ldots, 0_t$ be all the orbits of G. In view of Lemma 3 and the canonical isomorphism between $V_1\otimes\cdots\otimes V_m$ and $(\otimes^{|0_1|} V_{j_1})\otimes\cdots\otimes(\otimes^{|0_t|} V_{j_t})$ where $j_1 \in 0_1, \ldots, j_t \in 0_t$, $V_1\otimes\cdots\otimes V_m$ has a basis consisting of decomposable elements $v_1\otimes\cdots\otimes v_m$ in which $\dim\langle v_j : j \in 0\rangle=|0|$ for each orbit 0.

Since the mapping $f: V_1\otimes\cdots\otimes V_m \to (V_1, \cdots, V_m)_\chi(G)$ such that

$$f(v_1\otimes\cdots\otimes v_m) = v_1 * \cdots * v_m, \qquad v_i \in V_i,$$

is onto, it follows that $(V_1, \ldots, V_m)_\chi(G)$ has a basis consisting of decomposable elements v such that $\dim 0(v)=|0|$ for each orbit 0.

THEOREM 7. *Suppose for each orbit 0 of G,* $\dim V_j \geq |0|$ *where $j \in 0$. Then $(V_1, \ldots, V_m)_\chi(G)$ has a basis consisting of rank k vectors if one of the following conditions holds:*

(i) *There is an orbit 0_1 such that $|0_1| \geq 2$ and* $\dim V_r \geq |0_1|+2(k-1)$, $r \in 0_1$.

(ii) *There are two orbits 0_1 and 0_2 such that*

$$\dim V_r \geq |0_1|+k-1, \qquad r \in 0_1,$$

$$\dim V_s \geq |0_2|+k-1, \qquad s \in 0_2.$$

Proof. Case (i). The result is trivial when $k=1$. Let $k \geq 2$. Let J be the set of all decomposable elements v such that $\dim 0(v)=|0|$ for each orbit 0. Let $x=x_1*\cdots*x_m$ be an element of J. We shall show that there are two rank k vectors A and B such that $x=A-B$.

Let $0_1=\{j_1, \ldots, j_s\}$. Then $x_{j_1}, \ldots, x_{j_s}$ are linearly independent vectors. Choose vectors $u_1, \ldots, u_{2(k-1)}$ such that

$$x_{j_1}, \ldots, x_{j_s}, u_1, \ldots, u_{2(k-1)}$$

are linearly independent. Let $y=y_1*\cdots*y_m$ such that $y_i=x_i$ for $i\neq j_2$, $y_{j_2}=x_{j_2}+u_1$. Let $z=z_1*\cdots*z_m$ where $z_i=x_i$ for $i\neq j_2$, $z_{j_2}=u_1$. Then $x=y-z$. Let $w=w_1*\cdots*w_m$ where $w_i=x_i$ for $i\neq j_1$, $i\neq j_2$ and $w_{j_1}=x_{j_2}$, $w_{j_2}=u_2$. If $k\geq 3$, then for each positive integer $p\leq k-2$, let $v_p=v_{p1}*\cdots*v_{pm}$ such that $v_{pi}=x_i$ for $i\neq j_1$,

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$i \neq j_2$ and $v_{pj_1}=u_{2p+1}$, $v_{pj_2}=u_{2p+2}$. Finally let

$$A = y+w+\sum_{p=1}^{k-2} v_p,$$

$$B = z+w+\sum_{p=1}^{k-2} v_p.$$

Then $x=A-B$.

In view of Lemma 2 and Theorem 4, A and B are both of rank k. Since J spans $(V_1, \ldots, V_m)_\chi(G)$ (Lemma 4), it follows that the set of all rank k vectors spans $(V_1, \ldots, V_m)_\chi(G)$. This proves case (i).

Case (ii) can be proved similarly by applying Lemma 4 and Theorem 5.

Corollary 2 was proved by Brawley [2] using matrix language.

COROLLARY 2. *Let U and V be two vector spaces over the same field. Then $U\otimes V$ has a basis consisting of rank k vectors for each $k \leq \min\{\dim U, \dim V\}$.*

COROLLARY 3. *Let $\Lambda^2 U$ be the second Grassmann space over a vector space U. Then $\Lambda^2 U$ has a basis consisting of rank k vectors if $2k \leq \dim U$.*

EXAMPLE. Let U be a finite dimensional vector space over an algebraically closed field of characteristic 0. Let $U^{(m)}$ be the mth symmetric product space of U with decomposable elements denoted by $u_1 \ldots u_m$, $u_i \in U$. For each $u \in U$, let $u^m = \overbrace{u \ldots u}^{\text{m times}}$. Let $y_1, \ldots, y_n$ be n linearly independent vectors in U. In view of Propositions 9 and 10 of [5],

$$y_1^m + y_2^m = z_1 \cdots z_m$$

for some z_i where $\langle z_1, \ldots, z_m\rangle = \langle y_1, y_2\rangle$. Hence Theorem 4 and Theorem 5 imply that $y_1^m + \cdots + y_n^m$ is of rank $[(n+1)/2]$. Since $\{u^m : u \in U\}$ spans $U^{(m)}$ [1; p. 131,] it is easily shown that $U^{(m)}$ has a basis consisting of rank k vectors if $\dim U \geq 2k-1$.

THEOREM 8. *Let x, y and z be three non-zero decomposable elements of $(V_1, \ldots, V_m)_\chi(G)$. Let $0_1, \ldots, 0_t$ be all the orbits of G. If $x+y=z$, then for all i, $0_i(x)=0_i(y)$, except possibly for one value j of i, in which case*

$$\dim 0_j(x) \leq \dim(0_j(x) \cap 0_j(y))+1.$$

Proof. Suppose that there exist distinct s and q such that

$$0_s(x) \neq 0_s(y), \qquad 0_q(x) \neq 0_q(y).$$

We may assume $0_s(x) \nsubseteq 0_s(y)$. Let $x = x_1 * \cdots * x_m$. Choose $d \in 0_s$ such that $x_d \notin 0_s(y)$.

Let $T_d : V_d \to V_d$ be a linear mapping such that $T_d(x_d)=0$ and $T_d \mid_{0_s(y)}$ is the identity mapping. Let $T_n : V_n \to V_n$ be the identity mapping if $n \notin 0_s$ and $T_n = T_d$ if $n \in 0_s$. Then

$$K(T_1, \ldots, T_m)(x+y) = K(T_1, \ldots, T_m)z = y.$$

Since T_n is the identity mapping if $n \notin 0_s$, by Theorem 1, $0_q(y)=0_q(z)$. In view of Lemma 1,

$$0_q(x) \subseteq 0_q(y)+0_q(z) = 0_q(y)$$

Therefore $0_q(y) \nsubseteq 0_q(x)$. Let $z=z_1 * \cdots * z_m$. Choose $r \in 0_q$ such that $z_r \notin 0_q(x)$. Let $f_r: V_r \to V_r$ be a linear mapping such that $f_r(z_r)=0$ and $f_r \mid_{0_q(x)}$ is the identity mapping. Let $f_n: V_n \to V_n$ be the identity mapping if $n \notin 0_q$ and $f_n=f_r$ if $n \in 0_q$. Then

$$K(f_1, \ldots, f_m)(x+y) = x+K(f_1, \ldots, f_m)y = K(f_1, \ldots, f_m)z = 0.$$

Therefore $K(f_1, \ldots, f_m)y=-x$. Since f_n is the identity mapping for $n \in 0_s$, it follows from Theorem 1 that $0_s(x)=0_s(y)$, which is impossible.

Hence there is possibly only one j such that $0_j(x) \neq 0_j(y)$.

Now assume that such a j exists and

$$\dim 0_j(x) > 1+\dim(0_j(x) \cap 0_j(y)).$$

Then it is not hard to see that there are linearly independent vectors x_d, x_p, where $d, p \in 0_j$ such that $0_j(y) \cap \langle x_d, x_p \rangle = \{0\}$. By Theorem 4, $x+y$ is of rank 2. This contradicts the hypothesis. Hence the proof is complete.

The above theorem contains the known facts in tensor, Grassmann and symmetric spaces as special cases. See Lemma 3.1 [14], Lemma 5 [3] and Theorem 1.14 [4].

3. **Applications.** As an application of Theorem 1 and Theorem 2, we prove the following theorem which generalizes the result concerning intersection of tensor products in [6, section 1.15].

THEOREM 9. *Let U_i and W_i be subspaces of V_i where $U_i=U_{\sigma(i)}$, $W_i=W_{\sigma(i)}$, $V_i=V_{\sigma(i)}$ for $i=1, \ldots, m$ and for all $\sigma \in G$. Then*

$$(U_1, \ldots, U_m)_\chi(G) \cap (W_1, \ldots, W_m)_\chi(G) = (U_1 \cap W_1, \ldots, U_m \cap W_m)_\chi(G).$$

Proof. Clearly

$$(U_1 \cap W_1, \ldots, U_m \cap W_m)_\chi(G) \subseteq (U_1, \ldots, U_m)_\chi(G) \cap (W_1, \ldots, W_m)_\chi(G).$$

Let z be a non-zero vector of $(U_1, \ldots, U_m)_\chi(G) \cap (W_1, \ldots, W_m)_\chi(G)$. Then $z \in R_k((V_1, \ldots, V_m)_\chi(G))$ for some positive integer k. By Theorem 2,

$$z \in R_k((U_1, \ldots, U_m)_\chi(G)) \cap R_k((W_1, \ldots, W_m)_\chi(G)).$$

Hence

$$z = x_1+\cdots+x_k = y_1+\cdots+y_k$$

for some $x_i \in R_1((U_1, \ldots, U_m)_\chi(G))$ and some $y_i \in R_1((W_1, \ldots, W_m)_\chi(G))$, $1 \leq i \leq m$. By Theorem 1, for each orbit 0 of G, we have

$$\sum_{i=1}^{k} 0(x_i) = \sum_{i=1}^{k} 0(y_i) \subseteq W_q \cap U_q, \qquad q \in 0.$$

Therefore $z \in (U_1 \cap W_1, \ldots, U_m \cap W_m)_\chi(G)$. This completes the proof.

As an application of Theorem 1, we prove the following generalization of a result of Marcus [9].

THEOREM 10. *Let* $K(f_1, \ldots, f_m)$, $K(g_1, \ldots, g_m)$ *be two non-zero associated transformations on* $(V_1, \ldots, V_m)_\chi(G)$. *Suppose that* (i) *for each orbit* 0 *of* G *and each* $i \in 0$, *rank* $f_i > |0|$ *or* (ii) $\chi \equiv 1$. *Then* $K(f_1, \ldots, f_m) = K(g_1, \ldots, g_m)$ *if and only if* $f_i = \lambda_i g_i$ *for some scalars* λ_i *with* $\lambda_1 \lambda_2 \ldots \lambda_m = 1$.

Proof. The sufficiency of the theorem is trivial. We proceed to prove the necessity.

Case (i). Let $0_1 = \{j_1, j_2, \ldots, j_k\}$ be any orbit of G and $s \in 0_1$. Let $v_1 \in V_s$ such that $f_s(v_1) = z_1 \neq 0$. Let $z_1, \ldots, z_{k+1}$ be $k+1$ linearly independent vectors in the range space of f_s. Choose $v_i \in V_s$ such that $f_s(v_i) = z_i$, $i \geq 2$.

By the hypothesis on the rank of f_j, we are able to choose for each $1 \leq i \leq k$ a decomposable element $y_{i1} * \cdots * y_{im}$ such that

$$\{y_{ij_1}, \ldots, y_{ij_k}\} = \{v_1, \ldots, v_{k+1}\} - \{v_{i+1}\}$$

and

$$\dim\langle f_l(y_{il}) : l \in 0_r\rangle = |0_r|, \qquad r \geq 2,$$

where $0_2, \ldots, 0_t$ are the other orbits of G. In view of Lemma 2,

$$K(f_1, \ldots, f_m)(y_{i1} * \cdots * y_{im}) = K(g_1, \ldots, g_m)(y_{i1} * \cdots * y_{im}) \neq 0.$$

By Theorem 1,

$$0_1(f_1(y_{i1}) * \cdots * f_m(y_{im})) = 0_1(g_1(y_{i1}) * \cdots * g_m(y_{im})).$$

Hence

$$\langle z_1, \ldots, \hat{z}_{i+1}, \ldots, z_{k+1}\rangle = \langle g_s(v_1), \ldots, \widehat{g_s(v_{i+1})}, \ldots, g_s(v_{k+1})\rangle, \qquad i = 1, \ldots, k.$$

This implies that

$$(1) \qquad \bigcap_{i=1}^{k} \langle z_1, \ldots, \hat{z}_{i+1}, \ldots, z_{k+1}\rangle = \bigcap_{i=1}^{k} \langle g_s(v_1), \ldots, \widehat{g_s(v_{i+1})}, \ldots, g_s(v_{k+1})\rangle$$

Since $z_1, \ldots, z_{k+1}$ are linearly independent, the left hand side of (1) is $\langle z_1\rangle$. Since the right hand side of (1) contains $\langle g_s(v_1)\rangle$ and $g_s(v_1) \neq 0$, it follows that $\langle z_1\rangle = \langle f_s(v_1)\rangle = \langle g_s(v_1)\rangle$. This shows that the rank of $g_s \geq k+1$. By symmetry, $g_s(u) \neq 0$ implies that $\langle g_s(u)\rangle = \langle f_s(u)\rangle$. Hence $\langle f_s(v)\rangle = \langle g_s(v)\rangle$ for all $v \in V_s$. This implies that $f_s = \lambda_s g_s$ for some scalar λ_s. Clearly $\lambda_1 \ldots \lambda_m = 1$.

Case (ii). $\chi \equiv 1$. Let $u_1 \in V_1, \ldots, u_m \in V_m$ such that $f_i(u_i) \neq 0$ and $u_i = u_{\sigma(i)}$ for all i and for all $\sigma \in G$. Then

$$K(f_1, \ldots, f_m)(u_1 * \cdots * u_m) = K(g_1, \ldots, g_m)(u_1 * \cdots * u_m).$$

Since $\chi \equiv 1$, $f_1(u_1) * \cdots * f_m(u_m) = g_1(u_1) * \cdots * g_m(u_m) \neq 0$. Theorem 1 implies that $\langle f_i(u_i)\rangle = \langle g_i(u_i)\rangle$, $i = 1, \ldots, m$. Similarly if $w_i \in V_i$, $g_i(w_i) \neq 0$ and $w_i = w_{\sigma(i)}$ for all i and $\sigma \in G$, then $\langle g_i(w_i)\rangle = \langle f_i(w_i)\rangle$. Hence $\langle f_i(v_i)\rangle = \langle g_i(v_i)\rangle$ for all $v_i \in V_i$. This implies that $f_i = \lambda_i g_i$ for some scalar λ_i, $1 \leq i \leq m$. Clearly $\lambda_1 \ldots \lambda_m = 1$.

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