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THE TUKEY ORDER ON COMPACT SUBSETS OF SEPARABLE METRIC SPACES

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Abstract. One partially ordered set, Q, is a *Tukey quotient* of another, P, if there is a map (a *Tukey quotient*) $\phi : P \to Q$ carrying cofinal sets of P to cofinal sets of Q. Two partial orders which are mutual Tukey quotients of each other are said to be *Tukey equivalent*. Let $\mathcal{D}_{\mathfrak{c}}$ be the partially ordered set of Tukey equivalence classes of directed sets of size $\leq \mathfrak{c}$. It is shown that $\mathcal{D}_{\mathfrak{c}}$ contains an antichain of size $2^{\mathfrak{c}}$, and so has size $2^{\mathfrak{c}}$. The elements of the antichain are of the form $\mathcal{K}(M)$, the set of compact subsets of a separable metrizable space M, ordered by inclusion. The order structure of such $\mathcal{K}(M)$'s under Tukey quotients is investigated. Relative Tukey quotients are introduced. Applications are given to function spaces and to the complexity of weakly countably determined Banach spaces and Gul'ko compacta.

§1. Introduction. If there is a map $\phi : P \to Q$, where P and Q are partial orders, such that ϕ maps cofinal sets of P to cofinal sets of Q, then Q is said to be a *Tukey quotient* of P, denoted $P \geq_T Q$, and ϕ is called a *Tukey quotient*. Two partial orders, P and Q, which are mutual Tukey quotients of each other, $P \geq_T Q$ and $Q \geq_T P$, are said to be *Tukey equivalent*, abbreviated $P =_T Q$. Introduced to study Moore–Smith convergence in topology [19, 26], Tukey quotients and equivalence are fundamental notions of order theory, and are being actively investigated, especially in connection with partial orders arising naturally in analysis and topology [6–10, 14, 18, 20–23].

A partially ordered set is *directed* if every two elements have an upper bound. For a cardinal κ, let $\mathcal{D}_\kappa$ be the collection of Tukey equivalence classes of directed sets of size $\leq \kappa$. Note that $\mathcal{D}_\kappa$ is partially ordered by Tukey quotients. A key contribution to the theory of Tukey equivalence of directed sets was made by Todorcevic in [25]. He showed that consistently there are just five members of $\mathcal{D}_{\omega_1}$ (namely, $\mathbf{1}$, ω, ω_1, $\omega \times \omega_1$ and $[\omega_1]^{<\omega}$). Todorcevic also established, in ZFC, that for all regular κ, the collection $\mathcal{D}_{\kappa^{\aleph_0}}$ has size at least 2^κ, because it contains an antichain of size 2^κ. Under CH, then, there are $2^{\mathfrak{c}}$-many directed sets of size ω_1. This pair of results gives a rather dramatic answer to Isbell's question [12] as to the size of $\mathcal{D}_{\omega_1}$, 'maybe 5, maybe $2^{\mathfrak{c}}$, it depends on your set theory'.

Left unresolved is the size of $\mathcal{D}_{\mathfrak{c}}$. Evidently $|\mathcal{D}_{\mathfrak{c}}| \leq 2^{\mathfrak{c}}$. From Todorcevic's second result it is at least 2^κ for any regular $\kappa \leq \mathfrak{c}$, and hence may consistently be $2^{\mathfrak{c}}$. Dobrinen et al [6] have also consistently constructed other $2^{\mathfrak{c}}$-sized families of

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directed sets of size continuum. Note, however, that standard Easton forcing gives a model where for each $\alpha \in \omega_1$ not a limit, $2^{\aleph_\alpha} = \aleph_{\omega_1+\alpha}$, and so $\sup\{2^\kappa : \kappa \leq \mathfrak{c},\ \kappa$ is regular$\} < 2^{\mathfrak{c}}$. Here we show in ZFC that $\mathcal{D}_{\mathfrak{c}}$ contains antichains of size $2^{\mathfrak{c}}$.

Our directed sets are all of the form $\mathcal{K}(M)$ where M is a separable metrizable space, and $\mathcal{K}(M)$ is the set of compact subsets of M ordered by inclusion. Write $\mathcal{M}$ for the set of (homeomorphism classes of) separable metrizable spaces, and $\mathcal{K}(\mathcal{M})$ for the collection of Tukey equivalence classes of $\mathcal{K}(M)$'s where M is in $\mathcal{M}$. Then $\mathcal{K}(\mathcal{M})$ is a subposet of $\mathcal{D}_{\mathfrak{c}}$ under Tukey quotients. Beyond our main result constructing a $2^{\mathfrak{c}}$-sized antichain in $\mathcal{K}(\mathcal{M})$ we investigate the order structure of $\mathcal{K}(\mathcal{M})$, including results on the initial part of $\mathcal{K}(\mathcal{M})$, its additivity, cofinality, maximal size of embedded chains, and other partially ordered sets which embed in $\mathcal{K}(\mathcal{M})$.

The paper is concluded with some applications of our results to function spaces of separable metrizable spaces with the pointwise or compact-open topologies, and to the complexity of the classes of weakly countably determined Banach spaces and Gul'ko compacta that arise in functional analysis.

For these applications we are obliged to extend our results on the Tukey ordering of $\mathcal{K}(M)$'s to a more general setting which is of interest in its own right. Let P' be a subset of a partially ordered set P. We can study the relative properties of P' in P. Call a subset C of P *cofinal for P' in P* (or a relative cofinal set) if for every p' from P' there is a c from C such that $p' \leq c$. Observe that when $P' = P$ relative cofinal sets for P' in P are simply cofinal sets for P. As an example of a natural pair of a partially ordered set and subset, identifying a point x in a separable metric space M with the singleton, $\{x\}$, we can think of M as a subset of $\mathcal{K}(M)$, and in this context a collection $\mathcal{C}$ of compact subsets of M is cofinal for M in $\mathcal{K}(M)$ if it is a compact cover of M.

Given two pairs of partially ordered sets with subsets, (P', P) and (Q', Q), a map $\phi : P \to Q$ is a *relative Tukey quotient* if ϕ takes subsets of P cofinal for P' to sets cofinal for Q' in Q, and we write $(P', P) \geq_T (Q', Q)$. Note that $(P, P) \geq_T (Q, Q)$ if and only if $P \geq_T Q$. So no ambiguity arises if we abbreviate $(P, P) \geq_T (Q', Q)$ to $P \geq_T (Q', Q)$, and similarly when $Q' = Q$. In the context of our directed sets, $(M, \mathcal{K}(M)) \geq_T (N, \mathcal{K}(N))$ means that there is a map of $\mathcal{K}(M)$ to $\mathcal{K}(N)$ taking (compact) covers to (compact) covers, while $\mathcal{K}(M) \geq_T (N, \mathcal{K}(N))$ signifies the existence of a map taking cofinal families in $\mathcal{K}(M)$ to compact covers of N. Taking relative Tukey quotients is transitive on pairs $(M, \mathcal{K}(M))$, consequently $(\mathcal{M}, \mathcal{K}(\mathcal{M}))$, the set of relative Tukey equivalence classes of $(M, \mathcal{K}(M))$'s for M in $\mathcal{M}$, is a partially ordered set. Our results on the order structure of $\mathcal{K}(M)$ transfer to that of $(\mathcal{M}, \mathcal{K}(\mathcal{M}))$.

Observe that if $\mathcal{K}(M) \geq_T \mathcal{K}(N)$ or if $(M, \mathcal{K}(M)) \geq_T (N, \mathcal{K}(N))$, then $\mathcal{K}(M) \geq_T (N, \mathcal{K}(N))$. We will construct a $2^{\mathfrak{c}}$-sized family $\mathcal{A}$ of separable metrizable spaces such that if M and N are distinct members of $\mathcal{A}$ then $\mathcal{K}(M) \not\geq_T (N, \mathcal{K}(N))$ and $\mathcal{K}(N) \not\geq_T (M, \mathcal{K}(M))$. The preceding observation gives the claimed $2^{\mathfrak{c}}$-sized antichain in $\mathcal{K}(\mathcal{M})$, and also one in $(\mathcal{M}, \mathcal{K}(\mathcal{M}))$, and further the stronger properties of the family $\mathcal{A}$ are applied in our applications to function spaces. We will see that the relation $(M, \mathcal{K}(M)) \geq_T (N, \mathcal{K}(N))$ is naturally associated with the classification of weakly countably determined Banach spaces and Gul'ko compacta, and our results on chains will be applied here.

§**2. Background Material.** Let P' be a subset of P and Q' a subset of Q, where P and Q are partially ordered sets (posets). We record here, with only the sketchiest of proof, basic results on relative properties of P' in P, and relative Tukey quotients $(P', P) \geq_T (Q', Q)$. Most are natural extensions from the nonrelative case. Full proofs can be found in [15]. We also give basic material on the poset $\mathcal{K}(X)$ of compact subsets of a space X ordered by set inclusion, and the pair of X and $\mathcal{K}(X)$. We write $[\mathcal{K}(M)]_T$ (respectively, $[(M, \mathcal{K}(M)]_T)$ for the (relative) Tukey equivalence class of $\mathcal{K}(M)$ (respectively, $(M, \mathcal{K}(M))$). All topological spaces are Tychonoff. Denote by $\mathcal{X}$ the class of homeomorphism classes of all (Tychonoff) spaces.

2.1. Relative Tukey Order. There is a dual form of relative Tukey quotients. Call $\psi : Q' \to P'$ a relative Tukey map from (Q', Q) to (P', P) if and only if for any $U \subseteq Q'$ unbounded in Q, $\psi(U) \subseteq P'$ is unbounded in P. Taking the contrapositive $\psi : Q' \to P'$ is a relative Tukey map from (Q', Q) to (P', P) if and only if for any subset B of P' bounded in P, $\psi^{-1}(B) \subseteq Q'$ is bounded in Q.

Lemma 2.1. *There is a relative Tukey quotient ϕ from (P', P) to (Q', Q) if and only if there is a relative Tukey map ψ from (Q', Q) to (P', P).*

Proof. Suppose a relative Tukey quotient $\phi : P \to Q$ is given and let $q \in Q'$. Then there is $p_q \in P'$ such that whenever $p \geq p_q$, $\phi(p) \geq q$. Define $\psi : Q' \to P'$ by setting $\psi(q) = p_q$. This ψ is a relative Tukey map.

Now suppose a relative Tukey map $\psi : Q' \to P'$ is given. For each $p \in P$, let $Q_p = \{q \in Q' : \psi(q) \leq p\}$. Then $\psi(Q_p)$ is bounded in P, so Q_p is bounded by some $q_p \in Q$. Let $\phi(p) = q_p$. Then ϕ is a relative Tukey quotient. ⊣

Recall that a poset P is Dedekind complete if and only if every subset of P with an upper bound has a least upper bound. If we assume Q is Dedekind complete in the second part of the above argument, then $\phi(p) = q_p$ can be taken to be the least upper bound of the set Q_p, and this ensures that ϕ is order-preserving. This gives the first part of the following lemma. The second part follows easily from the fact that P' is cofinal for itself in P.

Lemma 2.2. *If $(P', P) \geq_T (Q', Q)$ and Q is Dedekind complete then there is a Tukey quotient witnessing this that is order-preserving.*

Conversely, if $\phi : P \to Q$ is order-preserving and $\phi(P')$ is cofinal for Q' in Q, then ϕ is a relative Tukey quotient of (P', P) to (Q', Q).

The following result is easy to prove using the definition of a Tukey quotient (a map taking cofinal sets to cofinal sets), but when combined with the result above is highly convenient.

Lemma 2.3. *If C is a cofinal set of a poset P then C and P are Tukey equivalent.*

The following lemma is straightforward from the definitions:

Lemma 2.4.

(1) *Suppose $P_1' \subseteq P_2' \subseteq P_2 \subseteq P_1$ and $Q_2' \subseteq Q_1' \subseteq Q_1 \subseteq Q_2$. Then $(P_1', P_1) \geq_T (Q_1', Q_1)$ implies $(P_2', P_2) \geq_T (Q_2', Q_2)$.*
(2) *If P' is directed and Q and R are Dedekind compete then $(P', P) \geq_T (Q', Q)$ and $(P', P) \geq_T (R', R)$ implies $(P', P) \geq_T (Q' \times R', Q \times R)$.*
(3) *Whenever $(P', P) \geq_T (Q' \times R', Q \times R)$ we have $(P', P) \geq_T (Q', Q)$ and $(P', P) \geq_T (R', R)$.*

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2.2. Cofinality, Additivity and Calibres. Define the *cofinality* of P' in P to be $\text{cof}(P', P) = \min\{|C| : C$ is cofinal for P' in $P\}$. Define the *additivity* of P' in P to be $\text{add}(P, P') = \min\{|S| : S \subseteq P'$ and S has no upper bound in $P\}$. Then $\text{cof}(P) = \text{cof}(P, P)$ and $\text{add}(P) = \text{add}(P, P)$ coincide with the usual notions of cofinality and additivity of a poset.

LEMMA 2.5. *If* $(P', P) \geq_T (Q', Q)$ *then* $\text{cof}(P', P) \geq \text{cof}(Q', Q)$, *and* $\text{add}(P', P) \leq \text{add}(Q', Q)$.

The first part is immediate using a Tukey quotient of (P', P) to (Q', Q). The second part is clear using a Tukey map of (Q', Q) to (P', P).

Let $\kappa \geq \mu \geq \lambda$ be cardinals. We say that P' has *calibre* (κ, λ, μ) in P if for every κ-sized subset S of P' there is a λ-sized subset S_0 such that every μ-sized subset S_1 of S_0 has an upper bound in P. When $P' = P$ this coincides with the standard definition of calibre of a poset.

LEMMA 2.6. *If* $(P', P) \geq_T (Q', Q)$, P' *has calibre* (κ, λ, μ) *in* P *and* κ *is regular, then* Q' *has calibre* (κ, λ, μ) *in* Q.

2.3. Powers, Embedding Well-Orders.

LEMMA 2.7. *Let* P *be a poset and* $Q \subseteq P$. *If* $\kappa < add(Q)$, *then* $(Q, P) \geq_T (Q^\kappa, P^\kappa)$.

Further, if Q *is directed then the following are equivalent*:

(1) $(Q, P) \geq_T \omega$, (2) $(Q, P) \geq_T (Q \times \omega, P \times \omega)$, *and* (3) *the additivity of* (Q, P) *is* $\aleph_0$.

LEMMA 2.8. *For a directed poset* P *without a largest element, the ordinal* $\text{add}(P)$ (*order*) *embeds in* P.

2.4. The Poset $\mathcal{K}(X)$ and the Tukey relation of X in $\mathcal{K}(X)$. Let X be any topological space, and $\mathcal{K}(X)$ the set of compact subsets of X (including the empty set) ordered by inclusion. We consider pairs $(S, \mathcal{K}(X))$, with particular attention to the cases $S = \mathcal{I}(X) = X$, $S = \mathcal{F}(X) =$ all finite subsets of X, and $S = \mathcal{K}(X)$. Recall that $\mathcal{K}(X)$ has a natural topology, the Vietoris topology, and for Tychonoff X, $\mathcal{K}(X)$ is a Tychonoff space. Then each $\mathcal{K}(X)$ is an element of $\mathcal{X}$ and call a class map $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ a $\mathcal{K}$-operator if for every X in $\mathcal{X}$ we have $X \subseteq \mathcal{S}(X) \subseteq \mathcal{K}(X)$. For any $\mathcal{K}$-operator $\mathcal{S}$ the relative Tukey relation is transitive on pairs $(\mathcal{S}(M), \mathcal{K}(M))$, so write $(\mathcal{S}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$ for the poset of all relative Tukey equivalence classes, $[(\mathcal{S}(M), \mathcal{K}(M))]_T$ for M from $\mathcal{M}$. Note that $\mathcal{K}(X)$ is Dedekind complete. Hence we may assume (and typically do) that our Tukey quotients are order-preserving.

Before looking at order properties let us recall that for a separable metrizable space M, the space $\mathcal{K}(M)$ is also separable metrizable (the Hausdorff metric is a compatible metric). If M is Polish (separable and completely metrizable) then $\mathcal{K}(M)$ is Polish. Since X embeds as a closed subspace in $\mathcal{K}(X)$ the converses to the previous two statements also hold. Consequently every $\mathcal{K}$-operator maps $\mathcal{M}$ into $\mathcal{M}$. Two additional properties of $\mathcal{K}(X)$:

LEMMA 2.9. *Let* X *be a space.*

(1) *For any* K *in* $\mathcal{K}(X)$, *the set* $\downarrow K = \{L \in \mathcal{K}(X) : L \subseteq K\}$ *is a compact subset of* $\mathcal{K}(X)$.
(2) *For any compact subset* $\mathcal{K}$ *of* $\mathcal{K}(X)$, *its union*, $\bigcup \mathcal{K}$, *is a compact subset of* X.

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For any space Z abbreviate $\mathcal{K}(\mathcal{K}(Z))$ to $\mathcal{K}^2(Z)$.

LEMMA 2.10. *Let Z be a space, and $\mathcal{S}$ a $\mathcal{K}$-operator. Then $\mathcal{K}(Z) = (\mathcal{K}(Z), \mathcal{K}(Z)) =_T (\mathcal{S}(\mathcal{K}(Z)), \mathcal{K}^2(Z)) =_T (\mathcal{K}^2(Z), \mathcal{K}^2(Z)) = \mathcal{K}^2(Z)$.*

Hence for spaces X and Y, $\mathcal{K}(X) \geq_T \mathcal{K}(Y)$ if and only if $(\mathcal{S}(\mathcal{K}(X)), \mathcal{K}^2(X)) \geq_T (\mathcal{S}(\mathcal{K}(Y)), \mathcal{K}^2(Y))$.

PROOF. The second Tukey equivalence follows from the first by taking $\mathcal{S} = \mathcal{K}$, so we need to prove that for any $\mathcal{K}$-operator $\mathcal{S}$ we have $\mathcal{K}(Z) =_T (\mathcal{S}(\mathcal{K}(Z)), \mathcal{K}^2(Z))$. First define $\phi_1 : \mathcal{K}^2(Z) \to \mathcal{K}(Z)$ by $\phi_1(\mathcal{K}) = \bigcup \mathcal{K}$. Then ϕ_1 is order-preserving, and $\phi_1(\mathcal{S}(\mathcal{K}(Z))) \supseteq \phi_1(\mathcal{K}(Z)) = \mathcal{K}(Z)$. Thus, by Lemma 2, $(\mathcal{S}(\mathcal{K}(Z)), \mathcal{K}^2(Z)) \geq_T (\mathcal{K}(Z), \mathcal{K}(Z))$.

For the reverse Tukey quotient define $\phi_2 : \mathcal{K}(Z) \to \mathcal{K}^2(Z)$ by $\phi_2(K) = \downarrow K$. Then ϕ_2 is order-preserving. It suffices to show that $\phi_2(\mathcal{K}(Z))$ is cofinal in $\mathcal{K}^2(Z)$. But take any $\mathcal{K}$ a compact subset of $\mathcal{K}(Z)$. Then $K = \bigcup \mathcal{K}$ is a compact subset of Z, and $\phi(K) = \downarrow \bigcup \mathcal{K} \supseteq \mathcal{K}$. ⊣

THEOREM 2.11. *Let $\mathcal{S}$ be a $\mathcal{K}$-operator. Then there is an order embedding, $\Phi = \Phi_{\mathcal{S}}$, of $\mathcal{K}(\mathcal{M})$ into $(\mathcal{S}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$ such that $\Phi(\mathcal{K}(\mathcal{M}))$ is cofinal in $(\mathcal{S}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$. Hence $\mathcal{K}(\mathcal{M}) =_T (\mathcal{S}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$.*

In particular, $\mathcal{K}(\mathcal{M})$, $(\mathcal{M}, \mathcal{K}(\mathcal{M}))$, and $(\mathcal{F}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$ are all Tukey equivalent.

PROOF. Fix the $\mathcal{K}$-operator $\mathcal{S}$ and define $\Phi : \mathcal{K}(\mathcal{M}) \to (\mathcal{S}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$ by $\Phi([\mathcal{K}(M)]_T) = [(\mathcal{S}(\mathcal{K}(M)), \mathcal{K}^2(M))]_T$. By the preceding lemma, $\mathcal{K}(M)$ and $\mathcal{K}(M')$ are in the same Tukey class if and only if $(\mathcal{S}(\mathcal{K}(M)), \mathcal{K}^2(M))$ and $(\mathcal{S}(\mathcal{K}(M')), \mathcal{K}^2(M'))$ are in the same relative Tukey class. Hence Φ is well-defined.

By the preceding lemma, $\mathcal{K}(M) \geq_T \mathcal{K}(M')$ if and only if $(\mathcal{S}(\mathcal{K}(M)), \mathcal{K}^2(M)) \geq_T (\mathcal{S}(\mathcal{K}(M')), \mathcal{K}^2(M'))$. Hence Φ is an order embedding.

Take any member, $[(\mathcal{S}(M), \mathcal{K}(M))]_T$ of $(\mathcal{S}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$. By the preceding lemma, $(\mathcal{S}(\mathcal{K}(M)), \mathcal{K}^2(M)) \geq_T (\mathcal{K}(M), \mathcal{K}(M))$, and, since $(\mathcal{K}(M), \mathcal{K}(M)) \geq_T (\mathcal{S}(M), \mathcal{K}(M))$, we have $(\mathcal{S}(\mathcal{K}(M)), \mathcal{K}^2(M)) \geq_T (\mathcal{S}(M), \mathcal{K}(M))$. Thus $[\mathcal{K}(M)]_T$ is in $\mathcal{K}(\mathcal{M})$ and $\Phi([\mathcal{K}(M)]_T) \geq_T [(\mathcal{S}(M), \mathcal{K}(M))]_T$, and $\Phi(\mathcal{K}(\mathcal{M}))$ is cofinal in $(\mathcal{S}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$. By Lemma 2.3, $\Phi(\mathcal{K}(\mathcal{M})) =_T (\mathcal{S}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$ and since Φ is order embedding we have $\mathcal{K}(\mathcal{M}) =_T (\mathcal{S}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$. ⊣

We give variants and dual versions of a relative Tukey quotient of $(\mathcal{S}(X), \mathcal{K}(X))$ to $(\mathcal{S}(Y), \mathcal{K}(Y))$. The special case given in the Corollary 2.13 and subsequent lemmas play a key role when we consider Gul'ko compacta.

LEMMA 2.12. *Fix two spaces X and Y and the $\mathcal{K}$-operator $\mathcal{S}$. The following are equivalent*:

(1) *there is a relative Tukey quotient, ϕ, of $(\mathcal{S}(X), \mathcal{K}(X))$ to $(\mathcal{S}(Y), \mathcal{K}(Y))$,*
(2) *there is a map $\phi' : \mathcal{S}(X) \to \mathcal{K}(Y)$ such that $\phi'(\mathcal{S}(X))$ is cofinal for $\mathcal{S}(Y)$, and if K is a compact subset of X then $\overline{\bigcup \phi'(\downarrow K \cap \mathcal{S}(X))}$ is compact,*
(3) *there is a relative Tukey map, ψ, of $(\mathcal{S}(Y), \mathcal{K}(Y))$ into $(\mathcal{S}(X), \mathcal{K}(X))$, and*
(4) *there is a map $\psi' : \mathcal{S}(Y) \to \mathcal{S}(X)$ such that if K is a compact subset of X then $\overline{\bigcup \psi'^{-1}(\downarrow K)}$ is compact.*

PROOF. Lemma 2.1 asserts that (1) and (3) are equivalent. Lemma 2.2 gives the equivalence of (1) and (2).

Noting that a subset B of X is bounded in $\mathcal{K}(X)$ if and only if it has compact closure, we see that conditions (3) and (4) are the contrapositives of each other. ⊣

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COROLLARY 2.13. *Fix two spaces X and Y. The following are equivalent*:

(1) *there is a relative Tukey quotient, ϕ, of $(X, \mathcal{K}(X))$ to $(Y, \mathcal{K}(Y))$,*
(2) *there is a map $\phi' : X \to \mathcal{K}(Y)$ such that $\phi'(X)$ is a cover of Y, and if K is a compact subset of X then $\overline{\bigcup\{\phi'(x) : x \in K\}}$ is compact,*
(3) *there is a relative Tukey map, ψ, of $(Y, \mathcal{K}(Y))$ into $(X, \mathcal{K}(X))$, and*
(4) *there is a map $\psi' : Y \to X$ such that if K is a compact subset of X then $\overline{\psi'^{-1}(K)}$ is compact.*

The next few lemmas are existence and preservation results for Tukey quotients on $\mathcal{K}(X)$ and $(X, \mathcal{K}(X))$.

Call a $\mathcal{K}$-operator, $\mathcal{S}$, *productive* if for any pair of spaces X and Y we have $(\mathcal{S}(X \times Y), \mathcal{K}(X \times Y)) =_T (\mathcal{S}(X), \mathcal{K}(X)) \times (\mathcal{S}(Y), \mathcal{K}(Y))$. Note that many natural $\mathcal{K}$-operators, including the identity, $\mathcal{F}$ and $\mathcal{K}$, are productive.

LEMMA 2.14. *Let $\mathcal{S}$ be a productive $\mathcal{K}$-operator. Let X be any space and C a compact space. Then $(\mathcal{S}(X), \mathcal{K}(X)) =_T (\mathcal{S}(X \times C), \mathcal{K}(X \times C))$.*

PROOF. By hypothesis $(\mathcal{S}(X \times C), \mathcal{K}(X \times C)) =_T (\mathcal{S}(X), \mathcal{K}(X)) \times (\mathcal{S}(C), \mathcal{K}(C))$. So it suffices to show $(\mathcal{S}(X), \mathcal{K}(X)) \times (\mathcal{S}(C), \mathcal{K}(C))$ is Tukey equivalent to $(\mathcal{S}(X), \mathcal{K}(X))$. Tukey quotients witnessing this are obtained by defining $\phi_1(K, L) = K$ and $\phi_2(K) = (K, C)$. ⊣

LEMMA 2.15. *Let A be a closed subspace of a space X. Let S be a subset of $\mathcal{K}(X)$. Then $(S, \mathcal{K}(X)) \geq_T (S \cap \mathcal{K}(A), \mathcal{K}(A))$.*

In particular, $(X, \mathcal{K}(X)) \geq_T (A, \mathcal{K}(A))$, $(\mathcal{F}(X), \mathcal{K}(X)) \geq_T (\mathcal{F}(A), \mathcal{K}(A))$, $\mathcal{K}(X) \geq_T \mathcal{K}(A)$, and $\mathcal{K}(X) \geq_T (A, \mathcal{K}(A))$.

PROOF. Define $\phi : \mathcal{K}(X) \to \mathcal{K}(A)$ by $\phi(K) = K \cap A$. Since A is closed, $K \cap A$ is in $\mathcal{K}(A)$. Clearly, ϕ is order-preserving. So to show that ϕ is the required relative Tukey quotient it suffices to show that $\phi(S)$ is cofinal for $S \cap \mathcal{K}(A)$. But this is clear since for any $K \in \mathcal{K}(A) \subseteq \mathcal{K}(X)$, $K = \phi(K)$. ⊣

Any continuous function $f : X \to Y$ induces a continuous function $\mathcal{K}f : \mathcal{K}(X) \to \mathcal{K}(Y)$ defined by $\mathcal{K}f(K) = f(K)$. A map $f : X \to Y$ is said to be *compact covering* if for every compact subset L of Y there is a compact subset K of X such that $f(K) \supseteq L$. Note that f is compact-covering if and only if $\mathcal{K}f$ is a surjection.

LEMMA 2.16. *Let $f : X \to Y$ be a continuous map. Let S be a subset of $\mathcal{K}(X)$. Then $(S, \mathcal{K}(X)) \geq_T (\mathcal{K}f(S), \mathcal{K}(Y))$.*

If f is surjective, then $(X, \mathcal{K}(X)) \geq_T (Y, \mathcal{K}(Y))$, $(\mathcal{F}(X), \mathcal{K}(X)) \geq_T (\mathcal{F}(Y), \mathcal{K}(Y))$, and $\mathcal{K}(X) \geq_T (Y, \mathcal{K}(Y))$. If f is compact covering, then $\mathcal{K}(X) \geq_T \mathcal{K}(Y)$.

The next lemma essentially says that Wadge reductions between compacta induce Tukey equivalence.

LEMMA 2.17. *Suppose X is a subspace of a compact K, Y is a subspace of a compact L, and $f : K \to L$ is continuous such that $X = f^{-1}Y$. For any subset S of $\mathcal{K}(X)$ we have $(S, \mathcal{K}(X)) =_T (\mathcal{K}f(S), \mathcal{K}(Y))$.*

In particular, $(X, \mathcal{K}(X)) =_T (Y, \mathcal{K}(Y))$, $(\mathcal{F}(X), \mathcal{K}(X)) =_T (\mathcal{F}(Y), \mathcal{K}(Y))$ and $\mathcal{K}(X) =_T \mathcal{K}(Y)$.

Lemma 2.18. *If M is a separable metrizable space then there is a zero-dimensional space M_0 such that $\mathcal{K}(M) =_T \mathcal{K}(M_0)$, $(M, \mathcal{K}(M)) =_T (M_0, \mathcal{K}(M_0))$, and $(\mathcal{F}(M), \mathcal{K}(M)) =_T (\mathcal{F}(M_0), \mathcal{K}(M_0))$.*

Proof. The space M is homeomorphic to a subspace of the Hilbert cube, $[0, 1]^\omega$. So we assume that M is in fact a subspace of $[0, 1]^\omega$. Fix a continuous surjection of the Cantor set, $\{0, 1\}^\omega$ to $[0, 1]^\omega$, and set $M_0 = f^{-1}M$. Then M_0 is zero-dimensional, and the preceding lemma immediately yields the desired conclusion. ⊣

Lemma 2.19. *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of spaces. Then $\mathcal{K}(\prod_{\lambda\in\Lambda} X_\lambda) =_T \prod_{\lambda\in\Lambda} \mathcal{K}(X_\lambda)$.*

Proof. The two maps $K \mapsto (\pi_\lambda(K))_{\lambda\in\Lambda}$ and $(K_\lambda)_{\lambda\in\Lambda} \mapsto \prod_{\lambda\in\Lambda} K_\lambda$ are the required Tukey quotients. ⊣

To clear the way to apply Lemma 2.7 we make some additivity calculations. A space X is ω-bounded if and only if whenever $\{x_n : n \in \omega\}$ is a sequence in X, then $\overline{\{x_n : n \in \omega\}}$ is compact. A space X is strongly ω-bounded if and only if whenever $\{K_n : n \in \omega\}$ is a countable family of compact subsets of X, then $\overline{\bigcup\{K_n : n \in \omega\}}$ is compact. Every metrizable ω-bounded space is compact.

Lemma 2.20. *Let X be a space. Then:*

(1) *The additivity of $\mathcal{K}(X)$ is $\aleph_0$ if and only if X is not strongly ω-bounded.*
(2) *The additivity of $(X, \mathcal{K}(X))$ is $\aleph_0$ if and only if the additivity of $(\mathcal{F}(X), \mathcal{K}(X))$ is $\aleph_0$ if and only if X is not ω-bounded.*

In particular, if X is metrizable then the additivity of $\mathcal{K}(X)$ is $\aleph_0$ if and only if additivity of $(X, \mathcal{K}(X))$ is $\aleph_0$ if and only if additivity of $(\mathcal{F}(X), \mathcal{K}(X))$ is $\aleph_0$ if and only if X is not compact.

§3. The Structure of $\mathcal{K}(\mathcal{M})$.

We will first establish various order properties of $\mathcal{K}(\mathcal{M})$, namely, size, cofinality, additivity and calibres. We will achieve this through determining exactly what subsets of $\mathcal{K}(\mathcal{M})$ are bounded. Next we will construct an antichain of size $2^\mathfrak{c}$ in $\mathcal{K}(\mathcal{M})$ and use a similar construction to embed different posets into $\mathcal{K}(\mathcal{M})$. Both of these arguments depend heavily on the Lemma 3.1, which gives an equivalent condition to the existence of Tukey quotients. Note that through Theorem 2.11 some of these results transfer immediately to $(\mathcal{M}, \mathcal{K}(\mathcal{M}))$ and $(\mathcal{F}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$. However, all theorems will be proven directly for $\mathcal{K}(\mathcal{M})$, $(\mathcal{M}, \mathcal{K}(\mathcal{M}))$ and $(\mathcal{F}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$.

3.1. Key Lemma. If X is a separable metrizable space and C is a subset of $\mathcal{K}(X)^2$ then for each $K \in \mathcal{K}(X)$ let $C[K] = \{L \in \mathcal{K}(X) : (K, L) \in C\}$ and for each $S \subseteq \mathcal{K}(X)$ let $C[S] = \bigcup\{C[K] : K \in S\}$.

Lemma 3.1. *Let M and N be subspaces of a separable metrizable space X. Note that $\mathcal{K}(M)$ and $\mathcal{K}(N)$ are subspaces of $\mathcal{K}(X)$. Let S be a subset of $\mathcal{K}(M)$ and T be a subset of $\mathcal{K}(N)$.*

If $(S, \mathcal{K}(M)) \geq_T (T, \mathcal{K}(N))$ then there is a closed subset C of $\mathcal{K}(X)^2$ such that $C[\mathcal{K}(M)]$ is contained in $\mathcal{K}(N)$ and $C[S] \supseteq T$.

In the case when X is compact, a (strengthened) converse also holds: if there is a closed subset C of $\mathcal{K}(X)^2$ such that $C[\mathcal{K}(M)] \subseteq \mathcal{K}(N)$ and $C[S]$ is cofinal for T in $\mathcal{K}(N)$ then $(S, \mathcal{K}(M)) \geq_T (T, \mathcal{K}(N))$.

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PROOF. To start fix an order-preserving map ϕ of $\mathcal{K}(M)$ to $\mathcal{K}(N)$ witnessing the relative Tukey quotient $(S, \mathcal{K}(M)) \geq_T (T, \mathcal{K}(N))$. Let $C_0 = \{(K, L) \in \mathcal{K}(X)^2 : K \in \mathcal{K}(M) \text{ and } L \subseteq \phi(K)\}$. Let $C = \overline{C_0}$. Then C is closed in $\mathcal{K}(X)^2$.

To verify that $C[\mathcal{K}(M)] \subseteq \mathcal{K}(N)$ we need to show that if (K, L') is in C, where K is in $\mathcal{K}(M)$, then L' is in $\mathcal{K}(N)$. As $\mathcal{K}(M)^2$ is metrizable, there is a sequence, $(K_n, L_n)_n$ on C_0 converging to (K, L'). Note that for each n we have that $L_n \subseteq \phi(K_n)$. Let $K_\infty = \{K\} \cup \bigcup\{K_n : n \in \omega\}$. Then K_∞ is compact and contains every K_n. So for each n we see that $L_n \subseteq \phi(K_\infty)$. Since $\downarrow\phi(K_\infty)$ is compact, the limit, L', of the L_n's is in $\downarrow\phi(K_\infty) \subseteq \mathcal{K}(N)$.

Take any L in T, and pick K from S such that $L \subseteq \phi(K)$. Then (K, L) is in C_0, and clearly $L \in C[S]$. Thus $C[S] \supseteq T$.

Now suppose X is compact and C is a closed subset of $\mathcal{K}(X)^2$ such that $C[\mathcal{K}(M)] \subseteq \mathcal{K}(N)$ and $C[S]$ is cofinal for T in $\mathcal{K}(N)$. Define $\phi : \mathcal{K}(M) \to \mathcal{K}(N)$ by $\phi(K) = \bigcup \pi_2(C \cap (\downarrow K \times \mathcal{K}(X)))$. Since π_2 is continuous and C, $\downarrow K$ and $\mathcal{K}(X)$ are all compact, $\pi_2(C \cap (\downarrow K \times \mathcal{K}(X)))$ is a compact subset of $\mathcal{K}(X)$, and $\phi(K)$ is indeed an element of $\mathcal{K}(N)$. We show that ϕ is the desired relative Tukey quotient. Clearly ϕ is order preserving. Hence it remains to show that $\phi(S)$ is cofinal for T in $\mathcal{K}(N)$.

Take any L in T. By hypothesis on C there is a K in S and L' in $C[K]$ such that $L \subseteq L'$. Then (K, L') is in $C \cap (\downarrow K \times \mathcal{K}(X))$, and by definition of ϕ we have, as desired, that $L \subseteq L' \subseteq \phi(K)$. ⊣

We record the most useful instances of the above lemma.

COROLLARY 3.2. *Let M and N be subspaces of $[0,1]^\omega$. Then the following equivalences hold*:

1. $\mathcal{K}(M) \geq_T \mathcal{K}(N)$ *if and only if there is a closed subset C of $\mathcal{K}([0,1]^\omega)^2$ such that $C[\mathcal{K}(M)] = \mathcal{K}(N)$*,
2. $(\mathcal{F}(M), \mathcal{K}(M)) \geq_T (\mathcal{F}(N), \mathcal{K}(N))$ *if and only if there is a closed subset C of $\mathcal{K}([0,1]^\omega)^2$ such that $\mathcal{F}(N) \subseteq C[\mathcal{F}(M)]$*,
3. $(M, \mathcal{K}(M)) \geq_T (N, \mathcal{K}(N))$ *if and only if there is a closed subset C of $\mathcal{K}([0,1]^\omega)^2$ such that $\bigcup C[\mathcal{K}(M)] = N = \bigcup C[M]$, and*
4. $\mathcal{K}(M) \geq_T (N, \mathcal{K}(N))$ *if and only if there is a closed subset C of $\mathcal{K}([0,1]^\omega)^2$ such that $\bigcup C[\mathcal{K}(M)] = N$.*

3.2. Initial Structure. The initial structures of $(\mathcal{M}, \mathcal{K}(\mathcal{M}))$, $(\mathcal{F}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$ and $\mathcal{K}(\mathcal{M})$ are connected with the projective hierarchy. Recall that a separable metrizable space M is *analytic* (denoted $\mathbf{\Sigma}^1_1$) if it is the continuous image of a Polish space, *co-analytic* (denoted $\mathbf{\Pi}^1_1$) if it is the complement in a Polish space of an analytic set, and for $n \geq 2$, the set M is $\mathbf{\Sigma}^1_n$ if the continuous image of a $\mathbf{\Pi}^1_{n-1}$ set, and $\mathbf{\Pi}^1_n$ if the complement in a Polish space of something in $\mathbf{\Sigma}^1_n$. The $\mathbf{\Sigma}^1_n$ classes are preserved by some relative quotients.

LEMMA 3.3. *Suppose $\mathcal{S}$ is a $\mathcal{K}$-operator such that $\mathcal{S}(M)$ is $\mathbf{\Sigma}^1_n$ when M is $\mathbf{\Sigma}^1_n$. If $(\mathcal{S}(M), \mathcal{K}(M)) \geq_T (\mathcal{S}(N), \mathcal{K}(N))$ and M is $\mathbf{\Sigma}^1_n$, then so is N.*

PROOF. Suppose $(\mathcal{S}(M), \mathcal{K}(M)) \geq_T (\mathcal{S}(N), \mathcal{K}(N))$. We may assume that M and N are subspaces of $[0,1]^\omega$. According to Lemma 3.1, there is a closed set C such

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that $S(N) = \bigcup C[S(M)]$. If M is $\mathbf{\Sigma}^1_n$ then, by hypothesis, $\mathcal{S}(M)$ is $\mathbf{\Sigma}^1_n$, and then by a standard calculation, using C closed, so is $\bigcup C[\mathcal{S}(M)]$, which is $\mathcal{S}(N)$. As a closed subspace of $\mathcal{K}(N)$, and hence $\mathcal{S}(N)$, we see N is also $\mathbf{\Sigma}^1_n$. ⊣

The above lemma applies to the cases $\mathcal{S}(M) = M$ and $\mathcal{S}(M) = \mathcal{F}(M)$. But the hypothesis does not hold for $\mathcal{S}(M) = \mathcal{K}(M)$. Indeed $\mathbb{Q}$ is analytic but $\mathcal{K}(\mathbb{Q})$ is co-analytic but not analytic. Further, as $\mathcal{K}(\mathbb{Q}) \geq_T \mathcal{K}(N)$ where $N = \mathcal{K}(\mathbb{Q})$ (Lemma 2.10), Tukey quotients, $\mathcal{K}(M) \geq_T \mathcal{K}(N)$, do not, in general, preserve $\mathbf{\Sigma}^1_1$.

THEOREM 3.4. *Below M and N are separable metrizable spaces.*

1. *The minimum Tukey equivalence class in $(\mathcal{M}, \mathcal{K}(\mathcal{M}))$ is $[(\mathbf{1}, \mathcal{K}(\mathbf{1}))]_T$, and $(M, \mathcal{K}(M))$ is in this class if and only if M is compact.*
2. *It has a unique successor, $[(\omega, \mathcal{K}(\omega))]_T$, which consists of all $(M, \mathcal{K}(M))$ where M is σ-compact but not compact.*
3. *This has $[(\omega^\omega, \mathcal{K}(\omega^\omega))]_T = \{(M, \mathcal{K}(M)) : M \text{ is analytic but not } \sigma\text{-compact}\}$ as a successor.*
4. *However it is consistent that there is a co-analytic N which is not σ-compact such that $(N, \mathcal{K}(N)) \not\geq_T (\omega^\omega, \mathcal{K}(\omega^\omega))$.*
5. *In general, if $(M, \mathcal{K}(M)) \geq_T (N, \mathcal{K}(N))$ and M is $\mathbf{\Sigma}^1_n$, then so is N.*

PROOF. Claim (1) is trivial, and (5) is Lemma 3.3 with $S(M) = M$. For (2) note that if M is not compact then it contains a closed copy of ω, and so there is quotient $(M, \mathcal{K}(M)) \geq_T (\omega, \mathcal{K}(\omega))$. Conversely, if ϕ witnesses $(\omega, \mathcal{K}(\omega)) \geq_T (M, \mathcal{K}(M))$ then $\phi(\omega)$ is a countable cover of M by compacta.

Claim (3) relies on a result of Hurewicz implying that every analytic set which is not σ-compact contains a closed copy of the irrationals. Suppose that M is not σ-compact but $(\omega^\omega, \mathcal{K}(\omega^\omega)) \geq_T (M, \mathcal{K}(M))$. By part (5) we know M is analytic. Hence ω^ω embeds as a closed set in M, so $(\omega^\omega, \mathcal{K}(\omega^\omega))$ and $(M, \mathcal{K}(M))$ are Tukey equivalent, and thus there is nothing in the Tukey order strictly between $(\omega, \mathcal{K}(\omega))$ and $(\omega^\omega, \mathcal{K}(\omega^\omega))$.

Assume $\omega_1 < \mathfrak{d}$ and 'there is a co-analytic subset N of $\mathbb{R}$ of size ω_1'. Then in this model the claim in (4) holds. For if ϕ is a Tukey quotient of $(M, \mathcal{K}(M))$ to $(\omega^\omega, \mathcal{K}(\omega^\omega))$, then $\phi(M)$ is a compact cover of ω^ω of size $\leq \omega_1$. But $\mathfrak{d}$ is the minimal size of a compact cover of ω^ω. ⊣

An almost identical proof gives an almost identical result for the initial structure of $(\mathcal{F}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$: (1) the minimum Tukey equivalence class in $(\mathcal{F}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$ is $[(\mathcal{F}(\mathbf{1}), \mathcal{K}(\mathbf{1}))]_T$, and $(\mathcal{F}(M), \mathcal{K}(M))$ is in this class if and only if M is compact; (2) it has a unique successor, $[(\mathcal{F}(\omega), \mathcal{K}(\omega))]_T$, which consists of all $(\mathcal{F}(M), \mathcal{K}(M))$ where M is σ-compact but not compact; (3) this has $[(\mathcal{F}(\omega^\omega), \mathcal{K}(\omega^\omega))]_T = \{(\mathcal{F}(M), \mathcal{K}(M)) : M \text{ is analytic but not } \sigma\text{-compact}\}$ as a successor; (4) however it is consistent that there is a co-analytic N which is not σ-compact such that $(\mathcal{F}(N), \mathcal{K}(N)) \not\geq_T (\mathcal{F}(\omega^\omega), \mathcal{K}(\omega^\omega))$.

Since, by Lemma 2.10, the Tukey relation $\mathcal{K}(M) \geq_T \mathcal{K}(N)$ is a special case of the relative relation $(M', \mathcal{K}(M')) \geq_T (N', \mathcal{K}(N'))$ we can also recover the initial structure of $\mathcal{K}(\mathcal{M})$.

Corollary 3.5 (Christensen [5], Fremlin [8]). *Below M' and N' are separable metrizable spaces.*

(1) *The minimum Tukey equivalence class in $\mathcal{K}(\mathcal{M})$ is $[\mathbf{1}]_T$, and $\mathcal{K}(M')$ is in this class if and only if M' is compact.*
(2) *It has a unique successor, $[\omega]_T$, which consists of all $\mathcal{K}(M')$ where M' is locally compact but not compact.*
(3) *This has $[\omega^\omega]_T = \{\mathcal{K}(M') : M'$ is Polish$\}$ as a unique successor.*

Proof. According to Lemma 2.10, $\mathcal{K}(M) \geq_T \mathcal{K}(N)$ if and only if $(\mathcal{K}(M), \mathcal{K}(\mathcal{K}(M))) \geq_T (\mathcal{K}(N), \mathcal{K}(\mathcal{K}(N)))$. Now apply the preceding theorem to $M = \mathcal{K}(M')$ and recall that $\mathcal{K}(M')$ is compact if and only if M' is compact, $\mathcal{K}(M')$ is σ-compact if and only if M' is locally compact, and Christensen showed that $\mathcal{K}(M')$ is analytic if and only if M' is Polish.

The reason $[\omega^\omega]_T$ is the unique successor of $[\omega]_T$ in $\mathcal{K}(\mathcal{M})$ is that if M' is not locally compact, then it contains a closed copy of the metric fan $\mathbb{F}$, and it is straightforward to check that $\mathcal{K}(\omega^\omega) =_T \mathcal{K}(\mathbb{F})$. ⊣

3.3. Cofinal Structure.

Down Sets and Cardinality.

Lemma 3.6. *Fix a separable metrizable space M. Let $\mathcal{R}$ and $\mathcal{S}$ be $\mathcal{K}$-operators. Then*

(1) $D_{\mathcal{R},\mathcal{S}} = \{N \in \mathcal{M} : (\mathcal{R}(M), \mathcal{K}(M)) \geq_T (\mathcal{S}(N), \mathcal{K}(N))\}$ *has size* $\mathfrak{c}$.
If $\mathcal{S}$ is productive, then
(2) $T_{\mathcal{R},\mathcal{S}} = \{N \in \mathcal{M} : (\mathcal{R}(M), \mathcal{K}(M)) =_T (\mathcal{S}(N), \mathcal{K}(N))\}$ *has size either* 0 *or* $\mathfrak{c}$, *and*
(3) $T_{\mathcal{S}}(M) = T_{\mathcal{S}} = \{N \in \mathcal{M} : (\mathcal{S}(M), \mathcal{K}(M)) =_T (\mathcal{S}(N), \mathcal{K}(N))\}$ *has size* $\mathfrak{c}$.

Proof. Note that $D_{\mathcal{R},\mathcal{S}} \subseteq D_{\mathcal{I},\mathcal{S}} = \{N \in \mathcal{M} : (M, \mathcal{K}(M)) \geq_T (\mathcal{K}(N), \mathcal{K}(N))\}$. So first we show $|D_{\mathcal{I},\mathcal{K}}| \leq \mathfrak{c}$. We can assume, without loss of generality (replacing M with a homeomorphic copy, if necessary), that M is a subspace of $[0,1]^\omega$. Take any separable metrizable N such that $\mathcal{K}(M) \geq_T (N, \mathcal{K}(N))$. Again we can assume N is a subspace of $[0,1]^\omega$, and so by Lemma 3.1, we have $N = \bigcup C[\mathcal{K}(M)]$ for some closed $C \subseteq \mathcal{K}([0,1]^\omega)^2$. Since there are at most $\mathfrak{c}$-many closed subsets of the separable metrizable space $\mathcal{K}([0,1]^\omega)^2$ we have the claimed upper bound.

Since for any $\mathcal{R}$ and $\mathcal{S}$, we clearly have $(\mathcal{R}(M), \mathcal{K}(M)) \geq_T (\mathcal{S}(\mathbf{1}), \mathcal{K}(\mathbf{1}))$, and $(\mathcal{S}(\mathbf{1}), \mathcal{K}(\mathbf{1})) =_T (\mathcal{K}(\mathbf{1}), \mathcal{K}(\mathbf{1}))$, the set $D_{\mathcal{R},\mathcal{S}}$ contains $T_{\mathcal{K}}(\mathbf{1})$. So, noting that $\mathcal{K}$ is productive, the proof of (1) is complete once we prove claim (3).

Now assume $\mathcal{S}$ is productive, and prove claim (2). Suppose $T_{\mathcal{R},\mathcal{S}}$ is not empty, say it contains N. We show it has size $\mathfrak{c}$. According to Lemma 2.18 there is a zero-dimensional separable metrizable space N_0 such that $(\mathcal{S}(N), \mathcal{K}(N)) =_T (\mathcal{S}(N_0), \mathcal{K}(N_0))$. Without loss of generality, then, we assume N is zero-dimensional.

It is well known that there is a continuum sized family, $\mathcal{C}$, of pairwise nonhomeomorphic continua (compact, connected, metrizable spaces). Then for any C from $\mathcal{C}$, Lemma 2.14 tells us that $(\mathcal{S}(N), \mathcal{K}(N)) =_T (\mathcal{S}(N \times C), \mathcal{K}(N \times C))$. Since N is zero-dimensional the connected components of $N \times C$ are the sets $\{x\} \times C$, for x in N, which are all homeomorphic to C.

For distinct C and C' from $\mathcal{C}$, any homeomorphism of $N \times C$ with $N \times C'$ must carry connected components of $N \times C$ to connected components to $N \times C'$, which

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is impossible since C and C' are not homeomorphic. Hence the $N \times C$'s, for C in $\mathcal{C}$, are distinct (pairwise nonhomeomorphic) members of each of $T_{\mathcal{R},\mathcal{S}}$.

Since M is always in $T_{\mathcal{S}}$, this latter set is never empty and so must have size $\mathfrak{c}$. This gives claim (3). ⊣

The first option of part (2) of the preceding result, that $T_{\mathcal{R},\mathcal{S}}$ can have size 0, can not be eliminated (at least consistently). Let $\mathcal{R} = \mathcal{I}$ and $\mathcal{S} = \mathcal{K}$. Assume $\omega_1 < \mathfrak{d}$. Let M be a subspace of $\mathbb{R}$ of size ω_1. Note that all compact subsets of M are countable, so it is not σ-compact. We show there is no separable metrizable space N such that $(M, \mathcal{K}(M)) =_T \mathcal{K}(N)$, in other words, $T_{\mathcal{R},\mathcal{S}}$ is empty. For if ϕ_1 is a Tukey quotient of $(M, \mathcal{K}(M))$ to $\mathcal{K}(N)$, then $\phi_1(M)$ is a cofinal collection in $\mathcal{K}(N)$ of size $\leq \omega_1$. If N were not locally compact then $\omega^\omega =_T \mathcal{K}(\omega^\omega) \leq_T \mathcal{K}(N)$, and $\mathrm{cof}(\mathcal{K}(N)) \geq \mathrm{cof}(\omega^\omega) = \mathfrak{d}$. So under $\omega_1 < \mathfrak{d}$, the space N must be locally compact, and $\mathcal{K}(N) =_T \omega$. But now a Tukey quotient of ω to $(M, \mathcal{K}(M))$ forces M to be σ-compact, which it is not.

Since there are $2^{\mathfrak{c}}$ homeomorphism classes of separable metrizable spaces, but each (relative) Tukey equivalence classes, $T_{\mathcal{S}}$ of productive $\mathcal{K}$-operators contains just $\mathfrak{c}$-many elements, we immediately deduce:

Corollary 3.7. *Let $\mathcal{S}$ be a productive $\mathcal{K}$-operator. Then $|(\mathcal{S}(\mathcal{M}), \mathcal{K}(\mathcal{M}))| = 2^{\mathfrak{c}}$. In particular, $\mathcal{K}(\mathcal{M})$, $(\mathcal{F}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$, and $(\mathcal{M}, \mathcal{K}(\mathcal{M}))$ all contain exactly $2^{\mathfrak{c}}$ elements.*

Bounded Sets; Cofinality, Additivity and Calibres. Let M and N be separable metrizable spaces, and $\mathcal{C}$ a family of subspaces of M such that $|\mathcal{C}| \leq |N|$. We define the *join* of $\mathcal{C}$ (along N) as follows. Index (with repeats, if necessary) $\mathcal{C} = \{C_y : y \in N\}$. Define, $J(\mathcal{C}) = J_N(\mathcal{C}) = \bigcup\{C_y \times \{y\} : y \in N\}$, considered as a subspace of $M \times N$. The join operation on $\mathcal{C}$ gives an upper bound for $\mathcal{C}$.

Lemma 3.8. *For each $C = C_y$ from $\mathcal{C}$, the subspace $C_y \times \{y\}$ is a closed subspace of $J_N(\mathcal{C})$ homeomorphic to C_y. Hence, by Lemma 2.15, for any $\mathcal{K}$-operator $\mathcal{S}$, and every C in $\mathcal{C}$: $(\mathcal{S}(J(\mathcal{C})), \mathcal{K}(J(\mathcal{C}))) \geq_T (\mathcal{S}(C), \mathcal{K}(C))$.*

Lemma 3.9. *Let $\mathcal{S}$ be a productive $\mathcal{K}$-operator. A subset of $(\mathcal{S}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$ is bounded if and only if it has size $\leq \mathfrak{c}$.*

Proof. Suppose first that $\mathcal{C}$ is a $\leq \mathfrak{c}$-sized subset of $(\mathcal{S}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$. Pick a representative M_c, a subspace of $[0,1]^\omega$, for each $c \in \mathcal{C}$. Let $M = [0,1]^\omega$ and $N = [0,1]$. Then the observation immediately above says $J(\mathcal{C})$ works as an upper bound of $\mathcal{C}$ in $(\mathcal{S}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$.

For the converse suppose the subset $\mathcal{C}$ of $(\mathcal{S}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$ has an upper bound $(\mathcal{S}(M), \mathcal{K}(M))$. Then $\mathcal{C}$ is a subset of $D'_{\mathcal{S}} = \{[\mathcal{K}(N)]_T : (\mathcal{S}(M), \mathcal{K}(M)) \geq_T (\mathcal{S}(N), \mathcal{K}(N))\}$. Since the set $D_{\mathcal{S},\mathcal{S}}$ of Lemma 3.6 has size $\leq \mathfrak{c}$, so does $D'_{\mathcal{S}}$. ⊣

The following result is immediate from Lemma 3.9.

Corollary 3.10.

(1) $\mathrm{add}(\mathcal{K}(\mathcal{M})) = \mathfrak{c}^+$. *Also,* $\mathrm{add}(\mathcal{F}(\mathcal{M}), \mathcal{K}(\mathcal{M})) = \mathfrak{c}^+ = \mathrm{add}(\mathcal{M}, \mathcal{K}(\mathcal{M}))$.
(2) $\mathrm{cof}(\mathcal{K}(\mathcal{M})) = 2^{\mathfrak{c}}$. *Also,* $\mathrm{cof}(\mathcal{F}(\mathcal{M}), \mathcal{K}(\mathcal{M})) = 2^{\mathfrak{c}} = \mathrm{cof}(\mathcal{M}, \mathcal{K}(\mathcal{M}))$.
(3) $\mathcal{K}(\mathcal{M})$ *(respectively,* $(\mathcal{M}, \mathcal{K}(\mathcal{M}))$ *and* $(\mathcal{F}(\mathcal{M}), \mathcal{K}(\mathcal{M})))$ *has calibre* (κ, λ, μ) *if and only if* $\mu \leq \mathfrak{c}$;

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Antichains.

THEOREM 3.11. *Let B be a $\mathfrak{c}$-sized totally imperfect subset of* $[0,1]$. *There is a $2^{\mathfrak{c}}$-sized family, $\mathcal{A}$, of subsets of B such that for distinct M and N from $\mathcal{A}$ we have $\mathcal{K}(M) \not\geq_T (N, \mathcal{K}(N))$ and $\mathcal{K}(N) \not\geq_T (M, \mathcal{K}(M))$.*

Hence $\{[\mathcal{K}(M)]_T : M \in \mathcal{A}\}$ is an antichain in $\mathcal{K}(\mathcal{M})$ of size $2^{\mathfrak{c}}$. Further $\{[(M, \mathcal{K}(M))]_T : M \in \mathcal{A}\}$ and $\{[(\mathcal{F}(M), \mathcal{K}(M))]_T : M \in \mathcal{A}\}$ are $2^{\mathfrak{c}}$-sized antichains in $(\mathcal{M}, \mathcal{K}(\mathcal{M}))$ and $(\mathcal{F}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$, respectively.

PROOF. Let $[0,1] = I$ and let B be a $\mathfrak{c}$-sized subset of I which is totally imperfect (contains no uncountable compact subsets), for example a Bernstein set. We construct a $\mathfrak{c}$-sized M_s inside B for each $s \in \mathfrak{c}$. Then for each $S \subseteq \mathfrak{c}$ we let $M_S = \bigcup_{s \in S} M_s$ and show that $S_2 \nsubseteq S_1$ and $S_1 \nsubseteq S_2$ imply $(M_{S_1}, \mathcal{K}(M_{S_1})) \not\leq_T \mathcal{K}(M_{S_2})$. This will complete the proof.

Let $\mathcal{D} = \{(s, C) : \ s \in \mathfrak{c}, \ C \text{ is a closed subset of } \mathcal{K}(I)^2\}$. Enumerate $\mathcal{D} = \{p_\alpha : \alpha \in \mathfrak{c}\}$ so that each element is repeated $\mathfrak{c}$-many times. Let $p_\alpha = (s_\alpha, C_\alpha)$.

We will construct $M_{\alpha,s}$ for each $\alpha \in \mathfrak{c}$ and $s \in \mathfrak{c}$, and then let $M_s = \bigcup_{\alpha \in \mathfrak{c}} M_{\alpha,s}$. We will also construct Out_α for each $\alpha \in \mathfrak{c}$ and set $Out_{<\alpha} = \bigcup_{\beta<\alpha} Out_\beta$ and $Out_{\leq\alpha} = \bigcup_{\beta\leq\alpha} Out_\beta$. Define $M_{<\alpha,s}$ and $M_{\leq\alpha,s}$ similarly.

For each stage β following will be true:

(1) $Out_{\leq\beta}$ is disjoint from $M_{\leq\beta,s}$ for each $s \in \mathfrak{c}$;
(2) $|M_{\leq\beta,s}| \leq |\beta|$ for each $s \in \mathfrak{c}$ and $|Out_{\leq\beta}| \leq |\beta|$;
(3) for each $s \in \mathfrak{c}$, $s \notin \beta$ implies $M_{\leq\beta,s} = \emptyset$ and $s \in \beta$ implies $M_{\beta,s} \setminus \bigcup_{t \in \mathfrak{c}} M_{<\beta,t} \neq \emptyset$;
(4) if $s_\beta \in \beta$ there are two cases:
   (a) either for each $K \in \mathcal{K}(B \setminus Out_{<\beta})$ we have $\bigcup C_\beta[K] \subseteq \bigcup_{s\in\mathfrak{c}} M_{<\beta,s} \cup K$;
   (b) or there is $K_\beta \in \mathcal{K}(B \setminus Out_{<\beta})$ such that $\bigcup C_\beta[K_\beta] \setminus (\bigcup_{s\in\mathfrak{c}} M_{<\beta,s} \cup K_\beta) \neq \emptyset$, and in this case $Out_\beta \cap (\bigcup C_\beta[\mathcal{K}_\beta] \setminus (\bigcup_{s\in\mathfrak{c}} M_{<\beta,s} \cup K_\beta)) \neq \emptyset$ and $K_\beta \subseteq M_{\beta,s_\beta}$.

Now suppose the conditions are true for all $\beta < \alpha$.

Step 1: If $s_\alpha \notin \alpha$, set $Out_\alpha = \emptyset$ and proceed to Step 2.

If $s_\alpha \in \alpha$ consider two cases. Case 1: if for each $K \in \mathcal{K}(B \setminus Out_{<\alpha})$ we have $\bigcup C_\alpha[K] \subseteq \bigcup_{s\in\mathfrak{c}} M_{<\alpha,s} \cup K$, then let $Out_\alpha = \emptyset$. Case 2: there is $K_\alpha \in \mathcal{K}(B \setminus Out_{<\alpha})$ such that $\bigcup C_\alpha[K_\alpha] \setminus (\bigcup_{s\in\mathfrak{c}} M_{<\alpha,s} \cup K_\alpha) \neq \emptyset$. Pick $a_\alpha \in \bigcup C_\alpha[K_\alpha] \setminus (\bigcup_{s\in\mathfrak{c}} M_{<\alpha,s} \cup K_\alpha)$ and let $Out_\alpha = \{a_\alpha\}$.

Step 2: For each $s \notin \alpha$ set $M_{\alpha,s} = \emptyset$. Let $M'_{\alpha,s_\alpha} = \emptyset$ if no K_α was chosen and let $M'_{\alpha,s_\alpha} = K_\alpha$ if it was. Since only at most α-many $M_{<\alpha,s}$'s are nonempty and those that are nonempty have size at most $|\alpha|$, $B \setminus (Out_{\leq\alpha} \cup \bigcup_{s\in\mathfrak{c}} M_{<\alpha,s} \cup M'_{\alpha,s_\alpha})$ is $\mathfrak{c}$-sized. Pick $|\alpha|$-many distinct points of $B \setminus (Out_{\leq\alpha} \cup \bigcup_{s\in\mathfrak{c}} M_{<\alpha,s} \cup M'_{\alpha,s_\alpha})$ and list them $\{x_{\alpha,s} : s \in \alpha\}$. Now for each $s \in \alpha$, if $s \neq s_\alpha$ let $M_{\alpha,s} = \{x_{\alpha,s}\}$ and for $s = s_\alpha$, let $M_{\alpha,s_\alpha} = \{x_{\alpha,s_\alpha}\} \cup M'_{\alpha,s_\alpha}$.

Since K_α is countable all conditions are satisfied. Condition 3 implies that each M_s is $\mathfrak{c}$-sized. Moreover, note that each M_s contains a $\mathfrak{c}$-sized subset that is disjoint from all other M_t's. So if $S_1 \nsubseteq S_2$, $M_{S_1} \setminus M_{S_2}$ is $\mathfrak{c}$-sized.

We need to show that $S_2 \nsubseteq S_1$ and $S_1 \nsubseteq S_2$ imply $M_{S_1} \not\leq_T \mathcal{K}(M_{S_2})$. Suppose $S_2 \nsubseteq S_1$, $S_1 \nsubseteq S_2$ and pick $s \in S_2 \setminus S_1$. Take any closed subset C of $\mathcal{K}(I)^2$. Then there is $\alpha \in \mathfrak{c}$ such that $(s, C) = p_\alpha$ and $s \in \alpha$ (this is why we need $\mathfrak{c}$-repetitions).

Suppose for each $K \in \mathcal{K}(B\backslash Out_{<\alpha})$ we have $\bigcup C[K] \subseteq \bigcup_{t\in\mathfrak{c}} M_{<\alpha,t} \cup K$ then $\bigcup C[\mathcal{K}(M_{S_2})] \subseteq M_{S_2} \cup \bigcup_{t\in\mathfrak{c}} M_{<\alpha,t}$. Then if $M_{S_1} = \bigcup C[\mathcal{K}(M_{S_2})]$ we get $M_{S_1}\backslash M_{S_2} \subseteq \bigcup_{t\in\mathfrak{c}} M_{<\alpha,t}$, which is $<\mathfrak{c}$-sized. So, $M_{S_1} = \bigcup C[\mathcal{K}(M_{S_2})]$ contradicts $S_1 \nsubseteq S_2$.

Now suppose there is $K_\alpha \in \mathcal{K}(B\backslash Out_{<\alpha})$ such that $\bigcup C[K_\alpha]\backslash(\bigcup_{t\in\mathfrak{c}} M_{<\alpha,t} \cup K_\alpha) \neq \emptyset$. Then at stage α we made sure that $K_\alpha \in \mathcal{K}(M_s) \subseteq \mathcal{K}(M_{S_2})$ so $a_\alpha \in \bigcup C[K_\alpha] \subseteq \bigcup C[\mathcal{K}(M_{S_2})]$; but $a_\alpha \in Out_\alpha$ and therefore it misses all M's, namely it misses M_{S_1}. So $\bigcup C[\mathcal{K}(M_{S_2})]\backslash M_{S_1} \neq \emptyset$. ⊣

Embeddings. It is interesting to see what other posets embed in $\mathcal{K}(\mathcal{M})$. We were motivated by papers by Knight, McCluskey, McMaster and Watson [13, 17] that studied what posets embed into $\mathcal{P}(\mathbb{R})$ ordered by homeomorphic embeddability. Note that any poset that does embed in $\mathcal{K}(\mathcal{M})$ must have the property that the set of predecessors of any element has size no more than $\mathfrak{c}$. For example, $\mathcal{P}(\mathbb{R})$ does not embed in any of $\mathcal{K}(\mathcal{M})$, $(\mathcal{M}, \mathcal{K}(\mathcal{M}))$, or $(\mathcal{F}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$ (while, interestingly, $\mathcal{P}(\mathbb{R})$ embeds into $\mathcal{P}(\mathbb{R})$ ordered by homeomorphic embeddability). On the other hand, it is immediate from Lemmas 2.7 and 3.9 as well as Corollary 3.10 that:

Corollary 3.12. *$\mathfrak{c}^+$ is the largest ordinal that embeds in $\mathcal{K}(\mathcal{M})$ (respectively, $(\mathcal{M}, \mathcal{K}(\mathcal{M}))$ or $(\mathcal{F}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$).*

A natural question is: does every poset of size $\leq \mathfrak{c}$ embed in $\mathcal{K}(\mathcal{M})$? We develop some machinery demonstrating that two natural partial orders of size continuum, the real line and $\mathcal{P}(\omega)$, do order embed in $\mathcal{K}(\mathcal{M})$, $(\mathcal{M}, \mathcal{K}(\mathcal{M}))$ and $(\mathcal{F}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$.

Theorem 3.13. *Let B be a $\mathfrak{c}$-sized totally imperfect subset of $[0,1]$. Let $B_y = B \times \{y\}$ for each $y \in [0,1]$.*

Suppose $\mathcal{S} \subseteq \mathcal{P}([0,1])$ has size at most $\mathfrak{c}$. Then for each y in $[0,1]$ there is a subspace M_y of B_y such that, whenever $y \notin S \in \mathcal{S}$, writing M_S for $\bigcup_{y\in S} M_y$, we have $\mathcal{K}(M_S) \ngeq_T (M_y, \mathcal{K}(M_y))$.

Proof. Let $[0,1] = I$. We write $B_S = \bigcup_{y\in S} B_y$ for each $S \subseteq I$. The construction will ensure that every M_y has size $\mathfrak{c}$.

Let $\mathcal{D} = \{(S, C) : C$ is a closed subset of $\mathcal{K}(I^2)^2$, $S \in \mathcal{S}\}$. Enumerate $\mathcal{D} = \{p_\alpha : \alpha \in \mathfrak{c}\}$. Let $p_\alpha = (S_\alpha, C_\alpha)$.

We will construct $M_{\alpha,y}$ for each $\alpha \in \mathfrak{c}$ and $y \in I$, and then let $M_y = \bigcup_{\alpha\in\mathfrak{c}} M_{\alpha,y}$. We will also construct Out_α for each $\alpha \in \mathfrak{c}$ and set $Out_{<\alpha} = \bigcup_{\beta<\alpha} Out_\beta$, and $Out_{\leq\alpha} = \bigcup_{\beta\leq\alpha} Out_\beta$. Define $M_{<\alpha,y}$ and $M_{\leq\alpha,y}$ similarly.

For each stage β following will be true:

(1) $Out_{\leq\beta}$ is disjoint from $M_{\leq\beta,y}$ for each $y \in I$;
(2) $|M_{\leq\beta,y}| \leq |\beta|$ for each $y \in I$ and $|Out_{\leq\beta}| \leq |\beta|$;
(3) $M_{\beta,y}\backslash M_{<\beta,y}$ is nonempty for each $y \in I$;
(4) there are two cases:
  (a) either $\bigcup C_\beta[\mathcal{K}(B_{S_\beta}\backslash Out_{<\beta})] \subseteq \bigcup_{y\in I} M_{<\beta,y} \cup B_{S_\beta}$;
  (b) or there is $K_\beta \in \mathcal{K}(B_{S_\beta}\backslash Out_{<\beta})$ such that $\bigcup C_\beta[K_\beta]\backslash(\bigcup_{y\in I} M_{<\beta,y} \cup B_{S_\beta}) \neq \emptyset$, and in this case $Out_\beta \cap (\bigcup C_\beta[\mathcal{K}_\beta]\backslash(\bigcup_{y\in I} M_{<\beta,y} \cup B_{S_\beta})) \neq \emptyset$ and $K_\beta \cap B_y \subseteq M_{\beta,y}$ for each $y \in S_\beta$.

Now suppose the conditions are true for all $\beta < \alpha$. If $\bigcup C_\alpha[\mathcal{K}(B_{S_\alpha}\backslash Out_{<\alpha})] \subseteq \bigcup_{y\in I} M_{<\alpha,y} \cup B_{S_\alpha}$, then let $Out_\alpha = \emptyset$. Otherwise, there is $K_\alpha \in \mathcal{K}(B_{S_\alpha}\backslash Out_{<\alpha})$ such that $\bigcup C_\alpha[K_\alpha]\setminus(\bigcup_{y\in I} M_{<\alpha,y} \cup B_{S_\alpha}) \neq \emptyset$. Pick $a_\alpha \in \bigcup C_\alpha[K_\alpha]\setminus(\bigcup_{y\in I} M_{<\alpha,y} \cup B_{S_\alpha})$ and let $Out_\alpha = \{a_\alpha\}$.

Now for each $y \in I$ pick $x_{\alpha,y} \in B_y\setminus(M_{<\alpha,y} \cup Out_{\leq\alpha})$. For $y \notin S_\alpha$ let $M_{\alpha,y} = \{x_{\alpha,y}\}$; for $y \in S_\alpha$, if no K_α was chosen let $M_{\alpha,y} = \{x_{\alpha,y}\}$ and if K_α was chosen let $M_{\alpha,y} = \{x_{\alpha,y}\} \cup (K_\alpha \cap B_y)$.

Then $K_\alpha \cap B_y$ is countable for each $y \in I$, since B_y is a closed subset of B_{S_α} and all compact subsets of B_y are countable. So all conditions are satisfied. Condition (3) implies that each M_y is $\mathfrak{c}$-sized.

We need to show that $y \notin S \in \mathcal{S}$ implies $(M_y, \mathcal{K}(M_y)) \not\leq_T \mathcal{K}(M_S)$. Take any closed subset C of $\mathcal{K}(I^2)^2$. Then there is $\alpha \in \mathfrak{c}$ such that $(S, C) = p_\alpha$. Suppose $\bigcup C[\mathcal{K}(B_S\setminus Out_{<\alpha})] \subseteq \bigcup_{x\in I} M_{<\alpha,x} \cup B_S$ is the case. Then since $\bigcup C[\mathcal{K}(M_S)] \subseteq \bigcup C[\mathcal{K}(B_S\setminus Out_{<\alpha})] \subseteq \bigcup_{x\in I} M_{<\alpha,x} \cup B_S$ and $M_{\alpha,y}\setminus M_{<\alpha,y} \neq \emptyset$ (which implies that $M_{\alpha,y}\setminus(\bigcup_{x\in I} M_{<\alpha,x} \cup B_S) \neq \emptyset$) we have $M_y \setminus \bigcup C[\mathcal{K}(M_S)] \neq \emptyset$.

Now suppose there is $K_\alpha \in \mathcal{K}(B_S\setminus Out_{<\alpha})$ such that $\bigcup C[K_\alpha]\setminus \bigcup_{x\in I} M_{<\alpha,x} \cup B_S \neq \emptyset$. Then at stage α we made sure that $K_\alpha \in \mathcal{K}(M_S)$ and $a_\alpha \in \bigcup C[K_\alpha]$ but $a_\alpha \in Out_\alpha$ so it misses all M's, namely it misses M_y. So $\bigcup C[\mathcal{K}(M_S)]\setminus M_y \neq \emptyset$. ⊣

Corollary 3.14. *There is a copy of* $([0,1]^\omega, \leq)$ *in* $\mathcal{K}(\mathcal{M})$, $(\mathcal{M}, \mathcal{K}(\mathcal{M}))$ *and* $(\mathcal{F}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$. *So* $(\mathbb{Q}, \leq)$, $(\mathbb{R}, \leq)$ *and* $([0,1], \leq)$ *are also embedded.*

Proof. Let $\mathcal{S} = \{S_x = \bigcup_{n\in\omega}[\frac{1}{2^{2n+1}}, x(n)] : x \in \Pi_{n\in\omega}[\frac{1}{2^{2n+1}}, \frac{1}{2^{2n}}]\}$ in Theorem 3.13. Then, for $x, y \in \Pi_{n\in\omega}[\frac{1}{2^{2n+1}}, \frac{1}{2^{2n}}]$, $x \not\leq y$ implies that there is $n \in \omega$ such that $x(n) > y(n)$ and therefore $x(n) \notin S_y$. So, by Theorem 3.13, we get $(M_{x(n)}, \mathcal{K}(M_{x(n)})) \not\leq_T \mathcal{K}(M_{S_y})$. But since $M_{x(n)}$ is a closed subset of M_{S_x} we get $(M_{S_x}, \mathcal{K}(M_{S_x})) \not\leq_T \mathcal{K}(M_{S_y})$, which implies $\mathcal{K}(M_{S_x}) \not\leq_T \mathcal{K}(M_{S_y})$, $(M_{S_x}, \mathcal{K}(M_{S_x})) \not\leq_T (M_{S_y}, \mathcal{K}(M_{S_y}))$ and $(\mathcal{F}(M_{S_x}), \mathcal{K}(M_{S_x})) \not\leq_T (\mathcal{F}(M_{S_y}), \mathcal{K}(M_{S_y}))$. However, if $x \leq y$, S_x is a closed subset of S_y and therefore $\mathcal{K}(M_{S_x}) \leq_T \mathcal{K}(M_{S_y})$, $(M_{S_x}, \mathcal{K}(M_{S_x})) \leq_T (M_{S_y}, \mathcal{K}(M_{S_y}))$ and $(\mathcal{F}(M_{S_x}), \mathcal{K}(M_{S_x})) \leq_T (\mathcal{F}(M_{S_y}), \mathcal{K}(M_{S_y}))$. ⊣

Corollary 3.15. *There is a copy of* $\mathcal{P}(\omega)$ *in* $\mathcal{K}(\mathcal{M})$, $(\mathcal{M}, \mathcal{K}(\mathcal{M}))$ *and* $(\mathcal{F}(\mathcal{M}), \mathcal{K}(\mathcal{M}))$. *Hence every countable partial order embeds.*

Proof. Let $N = \{\frac{1}{n+1} : n \in \omega\}$ and $\mathcal{S} = \mathcal{P}(N)$ in Theorem 3.13. As in Corollary 3.14 if $S_1 \not\subseteq S_2$ then $(M_{S_1}, \mathcal{K}(M_{S_1})) \not\leq_T \mathcal{K}(M_{S_2})$. Since N is discrete, $S_1 \subseteq S_2$ implies M_{S_1} is a closed subset of M_{S_2}, so we get $\mathcal{K}(M_{S_1}) \leq_T \mathcal{K}(M_{S_2})$, and the relative versions, as well. ⊣

§4. Applications.

4.1. Function Spaces. For any space X let $C(X)$ be the set of all real-valued continuous functions on X. Let $\mathbf{0}$ be the constant zero function on X. For any function f from $C(X)$, subset S of X and $\epsilon > 0$ let $B(f, S, \epsilon) = \{g \in C(X) : |f(x) - g(x)| < \epsilon\ \forall x \in S\}$. Write $C_p(X)$ for $C(X)$ with the pointwise topology (so basic neighborhoods of an f in $C_p(X)$ have the form $B(f, F, \epsilon)$ where F is finite and $\epsilon > 0$). Write $C_k(X)$ for $C(X)$ with the compact-open topology (so basic neighborhoods of an f in $C_k(X)$ have the form $B(f, K, \epsilon)$ where K is compact and $\epsilon > 0$).

The spaces $C_p(X)$ and $C_k(X)$ are connected to $\mathcal{K}(X)$. For $C_k(X)$ this is evident from the definition of the basic open sets, and the connection is very tight and topological.

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Let Z be a space, and z a point in Z. Write $\mathcal{T}_z^Z$ for the family of all neighborhoods of z in Z ordered by reverse inclusion.

LEMMA 4.1. *For any space X we have that $\mathcal{K}(X)\times\omega$ is Tukey equivalent to $\mathcal{T}_{\mathbf{0}}^{C_k(X)}$, where $\mathbf{0}$ is the constant zero function.*

If X is not strongly ω-bounded, then $\mathcal{K}(X)$ is Tukey equivalent to $\mathcal{T}_{\mathbf{0}}^{C_k(X)}$.

PROOF. Observe first that $\mathcal{B} = \{B(\mathbf{0}, K, 1/n) : K \in \mathcal{K}(X),\ n \in \omega\setminus\{0\}\}$ is cofinal in $\mathcal{T}_{\mathbf{0}}^{C_k(X)}$. It is easy to check that $B(\mathbf{0}, K', 1/n') \subseteq B(\mathbf{0}, K, 1/n)$ if and only if $K \subseteq K'$ and $n \le n'$, and hence $\mathcal{B}$ is clearly Tukey equivalent to $\mathcal{K}(X) \times \omega$. Now recall (Lemma 2.3) that if C is a cofinal subset of a directed set P then P and C are Tukey equivalent.

When X is not strongly ω-bounded, $\mathcal{K}(X)$ has countable additivity (Lemma 2.20), and $\mathcal{K}(X) =_T \mathcal{K}(X) \times \omega$ (Lemma 2.7(2)). ⊣

The next lemma is immediate.

LEMMA 4.2. *If f is a continuous open surjection of X to Y, then for any x from X, we have $\mathcal{T}_x^X \ge_T \mathcal{T}_{f(x)}^Y$.*

Similarly, if Y embeds in X then, for any y from Y, we have $\mathcal{T}_y^X \ge_T \mathcal{T}_y^Y$.

Recalling that $C_k(Y)$ is homogeneous, so $\mathcal{T}_{\mathbf{0}}^{C_k(Y)} =_T \mathcal{T}_f^{C_k(Y)}$ for every f from $C_k(Y)$, we combine the previous two lemmas.

PROPOSITION 4.3. *Suppose X and Y are spaces such that either there is a continuous open surjection of $C_k(X)$ onto $C_k(Y)$ or $C_k(Y)$ embeds in $C_k(X)$.*

Then $\mathcal{K}(X) \times \omega \ge_T \mathcal{K}(Y) \times \omega$, and if neither X nor Y are strongly ω-bounded spaces then $\mathcal{K}(X) \ge_T \mathcal{K}(Y)$.

The connection between $\mathcal{K}(X)$ and $C_p(X)$ is more indirect, and associated with the linear topological structure. The weak dual of $C_p(X)$ is denoted $L_p(X)$. The space X embeds in $L_p(X)$ as a closed subspace which is a Hamel basis. Let $\hat{X} = \bigoplus_{n\in\omega}(X^n \times \mathbb{R}^n)$. There is a natural continuous map $p : \hat{X} \to L_p(X)$, namely $p((x_1, \dots, x_n), (\lambda_1, \dots, \lambda_n)) = \sum_{i=1}^n \lambda_i x_i$. As X is a Hamel basis, p is surjective. (See [2] for proofs of all these claims about $C_p(X)$ and $L_p(X)$.)

PROPOSITION 4.4. *Let X and Y be spaces.*

(1) *If X is not strongly ω-bounded and there is a linear embedding of $C_p(Y)$ into $C_p(X)$ then $(\mathcal{F}(X), \mathcal{K}(X)) \ge_T (\mathcal{F}(Y), \mathcal{K}(Y))$.*

(2) *If X and Y are metrizable and there is a continuous linear surjection of $C_p(X)$ onto $C_p(Y)$ then (a) $\mathcal{K}(X) \ge_T \mathcal{K}(Y)$ and (b) $(\mathcal{F}(X), \mathcal{K}(X)) \ge_T (\mathcal{F}(Y), \mathcal{K}(Y))$.*

PROOF. For claim (1), suppose $\psi : C_p(Y) \to C_p(X)$ is a linear embedding. Then the dual map $\psi^* : L_p(X) \to L_p(Y)$ is a continuous linear surjection. Since Y is a closed subspace of $L_p(Y)$, combining the map p from $\hat{X}$ onto $L_p(X)$, ψ^* and tracing down onto Y, it follows that $(\mathcal{F}(\hat{X}), \mathcal{K}(\hat{X})) \ge_T (\mathcal{F}(Y), \mathcal{K}(Y))$. We verify that $(\mathcal{F}(\hat{X}), \mathcal{K}(\hat{X}) =_T (\mathcal{F}(X), \mathcal{K}(X))$.

Since X embeds as a closed set in $\hat{X}$, evidently $(\mathcal{F}(\hat{X}), \mathcal{K}(\hat{X})) \ge_T (\mathcal{F}(X), \mathcal{K}(X))$. The reverse Tukey quotient also holds. To see this first define $\phi_1 : \mathcal{K}(X)\times\omega \to \mathcal{K}(\hat{X})$ by $\phi_1(K, n) = \bigoplus_{m\le n}(K^m \times [-n, n]^m)$. Then it is straightforward to verify ϕ_1 is a relative Tukey quotient of $(\mathcal{F}(X) \times \omega, \mathcal{K}(X) \times \omega)$ to $(\mathcal{F}(\hat{X}), \mathcal{K}(\hat{X}))$. As X is not

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ω-bounded, $\mathcal{F}(X)$ has countable additivity in $\mathcal{K}(X)$ (Lemma 2.20), so according to Lemma 2.7, we have $(\mathcal{F}(X), \mathcal{K}(X)) \geq_T (\mathcal{F}(X) \times \omega, \mathcal{K}(X) \times \omega)$. Combining these two relative quotients gives the claim.

For claim (2a) apply Lemma 3.1 of [4], and see the discussion on page 881. Claim (2b) needs a little more explanation although we again use [4]. Let ψ be a continuous linear surjection of $C_p(X)$ onto $C_p(Y)$. For any y in Y, let ψ_y be the element of $L_p(X)$ obtained by setting $\psi_y(f) = \psi(f)(y)$. As X is a Hamel basis for $L_p(X)$ there is a finite set, $\operatorname{supp} y = \{x_1, \ldots, x_n\}$, of elements of X such that ψ_y is a linear combination of the x_i's (with all coefficients nonzero). Lemma 3.1 of [4] says that if K is a compact subset of X then the set $\phi(K) = \{y \in Y : \operatorname{supp} y \subseteq K\}$ is a compact subset of Y. Clearly ϕ is an order-preserving map from $\mathcal{K}(X)$ to $\mathcal{K}(Y)$. To establish claim (2b) we show $\phi(\mathcal{F}(X))$ is cofinal for $\mathcal{F}(Y)$ in $\mathcal{K}(Y)$. Take any finite subset G of Y. Set $F = \bigcup_{y \in G} \operatorname{supp} y$. Then F is a finite subset of X, and clearly, by definition of ϕ, we have $G \subseteq \phi(F)$. ⊣

Propositions 4.3 and 4.4, along with the $2^{\mathfrak{c}}$-sized antichain of Theorem 3.11 directly imply the following.

THEOREM 4.5. *There is a $2^{\mathfrak{c}}$-sized family $\mathcal{A}$ of separable metrizable spaces such that*:

(1) *if M, N are distinct elements of $\mathcal{A}$, then $C_k(M)$ is not the continuous open image of $C_k(N)$ and does not embed in $C_k(N)$, and*
(2) *if M, N are distinct elements of $\mathcal{A}$, then $C_p(M)$ is not the continuous linear image of $C_p(N)$ and does not linearly embed in $C_p(N)$.*

Marciszewski in his article in [11] gave an example of a $\mathfrak{c}$-sized family of compact metrizable spaces such that if M, N are distinct elements of the family then $C_p(M)$ and $C_p(N)$ are not linearly homeomorphic.

In [4] it is shown that if M is completely metrizable, N is metrizable and there is a continuous linear surjection of $C_p(M)$ onto $C_p(N)$ then N is completely metrizable. Combining Lemma 3.3 and Lemma 4.4(1) and (2b) yields a related result.

THEOREM 4.6. *Suppose M and N are separable metrizable. If M is $\mathbf{\Sigma}^1_n$, and either (1) $C_p(M)$ linearly embeds in $C_p(N)$ or (2) $C_p(N)$ is the continuous linear image of $C_p(M)$, then N is $\mathbf{\Sigma}^1_n$.*

Marciszewski and Pelant [16] proved part (2) by different methods. The first part is new.

4.2. Inequivalent Banach Spaces and Gul'ko Compacta. The primary aim of this section is to create large—of size $\mathfrak{c}^+$—families of weakly countably determined Banach spaces and Gul'ko compacta of strictly increasing complexity (see Theorem 4.7). Our objective will be attained by constructing a special embedding of the ordinal $\mathfrak{c}^+$ in $(\mathcal{M}, \mathcal{K}(\mathcal{M}))$ (see Lemma 3.12). The other ideas appearing in this section are mostly due to Aviles [3]. We will quote his results that we use with the minimal detail necessary for our purposes.

A Banach space X is said to be *weakly countably determined* if there exists a family $\{K_s : s \in \omega^{<\omega}\}$ of weak*-compact subsets of X^{**}, and a set $M \subseteq \omega^\omega$ such that $X = \bigcup_{x \in M} \bigcap_{n \in \omega} K_{x|n}$. Weakly countably determined Banach spaces were introduced by Vasak [27], the special case when $M = \omega^\omega$ (or equivalently, M analytic) was earlier studied by Talagrand [24] under the name *weakly $\mathcal{K}$-analytic*. Every weakly compactly generated Banach space is weakly $\mathcal{K}$-analytic, and so weakly countably determined. A topological space K is *Gul'ko compact* if it is compact and the Banach space $C(K)$ (with the supremum norm) is weakly countably determined.

Clearly the complexity of a weakly countably determined Banach space is connected to the complexity of the separable metrizable spaces M appearing in the definition. We will see that in fact the connection is directly with the order-complexity of $(M, \mathcal{K}(M))$ as measured by Tukey quotients. Let $\mathcal{C}$ be a class of separable metrizable spaces. A Banach space X is said to be *weakly $\mathcal{C}$-determined* if there exists a family $\{K_s : s \in \omega^{<\omega}\}$ of weak*-compact subsets of X^{**}, and a set $M \subseteq \omega^\omega$, $M \in \mathcal{C}$ such that $X = \bigcup_{x \in M} \bigcap_{n \in \omega} K_{x|n}$. Call a compact space K $\mathcal{C}$-Gul'ko compact if $C(K)$ is weakly $\mathcal{C}$-determined. The *weight* of a space is the minimal size of a base.

Theorem 4.7. *There are $\mathfrak{c}^+$-sized families, $\{K_\alpha : \alpha < \mathfrak{c}^+\}$, $\{\mathcal{C}_\alpha : \alpha < \mathfrak{c}^+\}$ of compacta of weight $\mathfrak{c}$ and of separable metrizable spaces, respectively, such that for $\beta < \alpha < \mathfrak{c}^+$:*

(1) *K_α is $\mathcal{C}_\alpha$-Gul'ko but not $\mathcal{C}_\beta$-Gul'ko, or, equivalently,*
(2) *the Banach space $X_\alpha = C(K_\alpha)$ is weakly $\mathcal{C}_\alpha$-determined but not weakly $\mathcal{C}_\beta$-determined.*

There are, up to homeomorphism, $2^\mathfrak{c}$ many compact spaces of weight $\mathfrak{c}$. So if $2^\mathfrak{c} = \mathfrak{c}^+$ then the above theorem gives maximal families of inequivalent Gul'ko compacta, and weakly countably determined Banach spaces, of weight $\mathfrak{c}$.

The families $\mathcal{C}_\alpha$ appearing in Theorem 4.7 will be 'nice'. A class $\mathcal{C}$ of separable metrizable spaces will be called *nice* if it is closed under the following operations: closed subspaces, continuous images, countable products, and Wadge reduction, that is, if $f : A \to B$ is a continuous function between Polish spaces, $C \subseteq B$, $C \in \mathcal{C}$, then $f^{-1}(C) \in \mathcal{C}$.

The following definitions will be useful. Let p be a free filter on a set Γ, and define $X(p) = \Gamma \cup \{*\}$ to have the topology where every point of Γ is isolated, and the neighborhoods of $*$ are all $\{*\} \cup S$ where $S \in p$. For an x in $\mathbb{R}^\Gamma$, let the support of x be the set $\{\gamma \in \Gamma : x_\gamma \neq 0\}$. For any subspace S of $\mathbb{R}^\Gamma$, the set $S \times \{0\}$ is a subspace of $\mathbb{R}^{\Gamma \cup \{*\}}$ homeomorphic to S. We will identify these two sets.

Lemma 4.8. *Let K be a compact subspace of $\mathbb{R}^\Gamma$ such that every x in K has countable support. Let D be a separable metrizable space. The following are equivalent:*

(1) *there is a function $f : \Gamma \to D$ (a* determining function*) such that for all x in K, compact $C \subseteq D$ and $\epsilon > 0$ the set $\{\gamma \in f^{-1}C : |x_\gamma| > \epsilon\}$ is finite,*
(2) *there is a filter p on Γ such that $(D, \mathcal{K}(D)) \geq_T (X(p), \mathcal{K}(X(p)))$ and $K \subseteq C_p(X(p)) \subseteq \mathbb{R}^{\Gamma \cup \{*\}}$.*

Proof. We show (1) implies (2). Fix a determining function $f : \Gamma \to D$ and suppose K satisfies the condition in (1). We consider K as a subspace of $\mathbb{R}^{\Gamma \cup \{*\}}$. Define a filter p by saying that S contained in Γ has complement in p if and only if $S \cap f^{-1}C$ is finite for all compact $C \subseteq D$. For C in $\mathcal{K}(D)$ write $C_f = f^{-1}C \cup \{*\}$. Then the topology on $X(p)$ is the finest so that all points of Γ are isolated and each C_f is compact. Hence a real-valued function on $X(p)$ is continuous if (and only if) its restriction to every C_f is continuous. It easily follows from the condition on elements x of K that $K \subseteq C_p(X(p))$.

Fix any d_0 in D. Define $\psi : X(p) \to D$ by $f(*) = d_0$ and $\psi(\gamma) = f(\gamma)$. Then for any compact $C \subseteq D$, we have $\overline{\psi^{-1}C} \subseteq C_f$. Since C_f is compact in $X(p)$, so is $\overline{\psi^{-1}C}$. It follows from our characterizations of relative Tukey maps, Corollary 2.13 (1) $\iff$ (4), that $(D, \mathcal{K}(D)) \geq_T (X(p), \mathcal{K}(X(p)))$, as required for (2).

Now suppose (2) holds. Fix the filter p. Again apply Corollary 2.13 (1) $\iff$ (4) and fix a map $\psi : X(p) \to D$ such that for every compact subset C of D, the set $\overline{\psi^{-1}C}$ is compact. Define $f : \Gamma \to D$ to be the restriction of ψ to Γ. We know $K \subseteq C_p(X(p))$.

Take any x from K. Take any compact $C \subseteq D$. Then $C_f = f^{-1}C \cup \{*\}$ is compact as a closed subset of $X(p)$ contained in $\overline{\psi^{-1}C} \cup \{*\}$ which is compact. As x is continuous its restriction to C_f, a super-sequence, is also continuous, and this easily gives the condition on x required by (1). ⊣

The equivalence of (1) and (2) in the following result is precisely Theorem 7 from [3], while equivalence of (2) and (3) is immediate from the preceding Lemma.

Proposition 4.9. *Let $\mathcal{C}$ be a nice class, and K be a compact subspace of $\mathbb{R}^\Gamma$ such that every x in K has countable support. The following are equivalent:*

(1) *K is $\mathcal{C}$-Gul'ko,*
(2) *there is a D in $\mathcal{C}$ and a determining function $f : \Gamma \to D$ for K,*
(3) *there is a D in $\mathcal{C}$ and a filter p on Γ such that $(D, \mathcal{K}(D)) \geq_T (X(p), \mathcal{K}(X(p)))$ and $K \subseteq C_p(X(p)) \subseteq \mathbb{R}^\Gamma$.*

In [3], Aviles further develops a 'machine' for creating interesting compacta that were introduced by Argyros, Arvanitakis, and Mercourakis in [1]. The machine takes as input a subspace M (A in [3]) of ω^ω and generates (a) a space $\hat{M} = wf(M) \times \omega^\omega$, where $wf(M)$ is a subspace of the Cantor set, (b) a set $\mathcal{T} = \mathcal{T}_M$ of size $\mathfrak{c}$ and (c) a compact subset K_M of $\{0,1\}^{\mathcal{T}}$ such that every x in K_M has countable support. Note that K_M satisfies the hypothesis of Proposition 4.9. Since K_M is a subspace of $\{0,1\}^{\mathcal{T}}$ and $|\mathcal{T}| = \mathfrak{c}$ it has weight no more than $\mathfrak{c}$. When M is uncountable it is straightforward to verify that K_M has weight equal to $\mathfrak{c}$. Using the equivalence of Lemma 4.8, Theorems 16 and 17 of [3] give two key properties of the machine:

(1) there is a filter p on $\mathcal{T}$ such that $(\hat{M}, \mathcal{K}(\hat{M})) \geq_T (X(p), \mathcal{K}(X(p)))$ and $K \subseteq C_p(X(p))$, but
(2) there is no filter p on $\mathcal{T}$ such that both $(M, \mathcal{K}(M)) \geq_T (X(p), \mathcal{K}(X(p)))$ and $K \subseteq C_p(X(p))$.

We now prove claim (1) of Theorem 4.7. The second claim immediately follows from the definitions. Note that in the proof the family $\{(O_\alpha, \mathcal{K}(O_\alpha)) : \alpha < \mathfrak{c}^+\}$ is a subset of $(\mathcal{M}, \mathcal{K}(\mathcal{M}))$ order isomorphic to $\mathfrak{c}^+$.

Proof. Define $O_0 = \omega^\omega$. Apply the machine to $M = O_0$ to get $I_0 = \hat{O}_0$ and $K_0 = K_{O_0}$. Let $\mathcal{C}_0$ be the smallest family containing I_0 and closed under the operations defining a 'nice class' (closed subspaces, continuous images, countable products and Wadge reduction). Then $\mathcal{C}_0$ is a nice class, it has size $\mathfrak{c}$, and K_0 is $\mathcal{C}_0$-Gul'ko.

Now for $\alpha < \mathfrak{c}^+$ inductively define a pair of separable metrizable spaces O_α and I_α, compact K_α and nice family $\mathcal{C}_\alpha$ of size $\mathfrak{c}$, as follows.

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Let $\mathcal{C}_{<\alpha} = \bigcup\{\mathcal{C}_\beta : \beta < \alpha\}$. It has size $\mathfrak{c}$. Set $O_\alpha = J(\mathcal{C}_{<\alpha})$ as defined in Section 3.3. Apply the machine to $M = O_\alpha$ to get $I_\alpha = \hat{O}_\alpha$ and $K_\alpha = K_{O_\alpha}$. Let $\mathcal{C}_\alpha$ be the smallest nice class containing $\mathcal{C}_{<\alpha}$ and I_α. Then $\mathcal{C}_\alpha$ has size $\mathfrak{c}$.

Now suppose $\beta < \alpha < \mathfrak{c}^+$. Since $I_\alpha = \hat{O}_\alpha$ is in $\mathcal{C}_\alpha$, from the first property of the machine and Proposition 4.9 we have that K_α is $\mathcal{C}_\alpha$-Gul'ko. But for every D in C_β, we have $(O_\alpha, \mathcal{K}(O_\alpha)) \geq_T (D, \mathcal{K}(D))$ ($\mathcal{C}_\beta \subseteq \mathcal{C}_{<\alpha}$, key property of the J construction, and definition of O_α), hence by transitivity of $\geq_T$ on $(\mathcal{M}, \mathcal{K}(\mathcal{M}))$, and the second property of the machine, there is no filter p on $\mathcal{T}$ such that both $(D, \mathcal{K}(D)) \geq_T (X(p), \mathcal{K}(X(p)))$ and $K_\alpha \subseteq C_p(X(p))$. From Proposition 4.9 we deduce that K_α is not $\mathcal{C}_\beta$-Gul'ko. ⊣

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