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A CLOSED-FORM GARCH VALUATION MODEL FOR POWER EXCHANGE OPTIONS WITH COUNTERPARTY RISK

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In this paper, a discrete-time framework is proposed to value power exchange options with counterparty default risk, where counterparty risk is considered in a reduced-form setting and the variance processes of the underlying assets are captured by GARCH processes. In addition, the proposed model allows for the correlation between the intensity of default and the variances of the underlying assets by breaking down the total risk into systematic and idiosyncratic components. By dint of measure-change techniques and characteristic functions, we obtain the closed-form pricing formula for the value of power exchange options with counterparty default risk. Finally, numerical results are presented to show the power exchange option values.

Keywords: default risk, GARCH, idiosyncratic risk, power exchange options, systematic risk
JEL classification: G13

1. INTRODUCTION

The over-the-counter (OTC) derivatives market has grown considerably in the last decades. The notional amounts of all the OTC contracts reached almost $542,435 billion at the end of June 2017 and the largest amount has reached $710,338 billion over the past 10 years. The OTC derivatives mainly involve foreign exchange contracts, interest rate contracts, equity-linked contracts, commodity contracts, and credit default swaps (CDS). The largest part of this market is interest rate contracts, which comprise 76–81% of the OTC derivative market. Although the equity-linked contracts have only a 10% ratio, the corresponding notional amounts are about $7,000 billion.

In this paper, we are interested in the counterparty risk that may stem from the OTC derivatives markets. Counterparty risk largely springs from the creditworthiness of

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an institution such as banks, broker dealers, or other non-banking institutions in the financial system. In addition, the risk causes cumulative losses to the financial system from a counterparty that fails to deliver on its OTC derivative obligation. The financial market turmoil from 2007 onwards has emphasized the importance of counterparty risk. Therefore, one must take the credit exposure seriously when discussing credit-sensitive OTC derivatives such as forwards, swaps and options. For example, Jarrow and Yu [16] present the pricing formulae of defaultable bonds and CDS including default intensities dependent on the default of a counterparty. In their model, firms have correlated defaults due not only to an exposure to common risk factors, but also to firm-specific risks. Leung and Kwok [21] perform valuation of CDS with counterparty risk with inter-dependent default correlation. Yoon and Kim [29] study the European vulnerable options under constant and stochastic interest rates model using double Mellin transforms. Carr and Ghamami [5] consider risk-neutral valuation of a contingent claim under bilateral counterparty risk. Crepey [8,9] develop a backward stochastic differential equations approach to the valuation and hedging of bilateral counterparty risk on OTC derivatives. Wang [27] presents a valuation model for vulnerable European call options with counterparty default risk at the exercise time. In a word, the counterparty default risk has been one of the risks that participants in OTC markets have to face.

The main idea of this paper is to propose a discrete-time framework to value power exchange options, which are widely prevailing in OTC markets, with counterparty default risk at any time prior to maturity of the option. The power exchange option is a European option with payoff

$$(\mu_1 S_1^{\alpha_1}(T) - \mu_2 S_2^{\alpha_2}(T))^+$$

at maturity T, which means to exchange the value $\mu_1 S_1^{\alpha_1}(T)$ of asset S_1 for the value $\mu_2 S_2^{\alpha_2}(T)$ of asset S_2. Here α_1, α_2, μ_1, and μ_2 are positive constants.

Power exchange options are a generation of Fischer–Margrabe exchange options (see, e.g., Fischer [13] and Margrabe [22]) and power options (see, e.g., Tompkins [25]), both of which have voluminously useful applications in the field of compensation design and in hedging nonlinear risks. Blenman and Clark [2] explicitly solve the valuation of European power exchange options under the assumption that the underlying assets follow geometric Brownian motions without default risk. Wang [26] extends the framework of Blenman and Clark [2] to deal with the pricing problem of power exchange options with correlated jump risk. Wang et al. [28] investigate the valuation of power exchange options not only with jump risk but also counterparty default risk under the assumption that default occurs only at maturity. In this paper, we focus on counterparty default risk that may occur prior to the maturity of the option.

Generally speaking, there are two approaches for credit risk modeling: the structural model and the reduced-form one. The structural model seriously considers the problem what exactly triggers the default event and assumes that the default is triggered when assets or some function thereof hit or fall below some certain boundary. The pioneering work by Black and Scholes [1] and Merton [23] describes a default happening if the value of a firm's asset is below the debt obligations at maturity. Johnson and Stulz [17] first propose the structural model for pricing European options with default risk, which only occurs at the maturity of the option. Klein [20] extends the result of Johnson and Stulz [17] to consider the Black–Scholes options with connection between the option's underlying asset and the assets of the counterparty. Without regard to the problem of what exactly triggers the default event, the reduced-form model assumes Poisson-type arrivals of defaults with an exogenously given intensity. Hull and White [15] develop a general reduced-form default risk model for pricing

European and American options, in which the underlying asset and the option issuer's defaults are assumed to be independent. Duffie and Singleton [11] present the valuation of defaultable bonds and credit-spread options subject to default risk in a reduced-form model. Bo et al. [3] investigates a stochastic portfolio optimization problem with default risk under a reduced-form framework. Fard [12] proposes a jump-diffusion model to price European vulnerable options with credit risk in a reduced-form model.

In this paper, we develop a pricing model for power exchange options with default risk in a reduced-form setting and the time-varying variances of the underlying assets driven by GARCH processes. GARCH processes are first proposed by Bollerslev [4] and then used to price options by Duan [10], Heston and Nandi [14], Christoffersen et al. [6] and many other studies. Moreover, there is a connection between the variance processes and the intensity of default in the proposed model. Compared with the earlier studies, this paper has three main features. First, this paper is the first try to consider the valuation of power exchange option in a GARCH reduced-form model. Second, the proposed model allows the correlation between the intensity of default and the variances of the two underlying assets. Lastly, the closed-form valuation formula of power exchange options is obtained.

The remainder of the paper is organized as follows. In Section 2, we present the model and derive a closed-form pricing formula for power exchange options. Section 3 presents the numerical results. The conclusion is summarized in Section 4. The detailed proofs are shown in the appendix.

2. THE FRAMEWORK OF VALUING POWER EXCHANGE OPTIONS

In this section, we present the pricing framework in detail and derive a closed-form pricing formula for power exchange options. Moreover, the proposed model considers counterparty default risk in a reduced-form setting and allows for the correlation between the variances of the two underlying assets and the default intensity. To connect the variance processes of the two underlying assets and the default intensity, we start with the dynamic of the market index, standing for a common risk factor.

2.1. Model Description

Throughout this paper, we suppose $\mathbb{P}$ is the physical probability measure on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the dynamic of the market index $M(t)$ satisfies,

$$\begin{cases} \ln M(t) = \ln M(t-1) + r + (\lambda_m - \frac{1}{2})h_m(t) + \sqrt{h_m(t)}Z_m(t), \\ h_m(t) = w_m + b_m h_m(t-1) + a_m(Z_m(t-1) - c_m\sqrt{h_m(t-1)})^2, \end{cases} \tag{1}$$

where $w_m > 0, b_m > 0, a_m > 0, c_m > 0$, r is the risk-free interest rate, λ_m is the market price of risk, and $Z_m(t)$ is a standard normal disturbance which reflects the shocks to the return. $h_m(t)$ is the conditional variance of the return between time $t-1$ and t which is known from the information set at time $t-1$, appearing in the mean as a return premium, i.e.,

$$\mathbb{E}_t^{\mathbb{P}}\left[\frac{M(t)}{M(t-1)}\right] = e^{r+\lambda_m h_m(t)},$$

where $\mathbb{E}_t^{\mathbb{P}}[\cdot]$ means the conditional expectation $\mathbb{E}^{\mathbb{P}}[\cdot|\mathcal{F}_t]$ under measure $\mathbb{P}$ given information at time t. This GARCH process is first adopted by Heston and Nandi [14] to value index options and has been extended by Christoffersen et al. [6] and Christoffersen et al. [7]. Here

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we use Heston and Nandi [14] as the benchmark model and further adopt it to connect the dynamics of the two underlying assets.

Based on the postulate of the market index, we assume that shocks to the returns of the underlying assets consist of two parts: idiosyncratic shocks and common ones. Following Wang [27], we suppose the prices of underlying assets under measure $\mathbb{P}$ satisfy the following processes,

$$\begin{cases} \ln S_i(t) = \ln S_i(t-1) + r + (\lambda_i - \frac{1}{2})h_i(t) + \sqrt{h_i(t)}Z_i(t) \\ \qquad\qquad + (\beta_i\lambda_m - \frac{1}{2}\beta_i^2)h_m(t) + \beta_i\sqrt{h_m(t)}Z_m(t), \\ h_i(t) = w_i + b_i h_i(t-1) + a_i(Z_i(t-1) - c_i\sqrt{h_i(t-1)})^2, \end{cases} \tag{2}$$

where $w_i > 0, b_i > 0, a_i > 0, c_i > 0,\ S_i(t),\ i = 1, 2$ represents the price of underlying asset i at the close of day t, and $Z_i(t)$ is a standard normal disturbance reflecting the idiosyncratic shocks to the return of the underlying asset $S_i(t)$. $Z_m(t)$ is defined in Eq. (1), reflecting the common shocks to returns of the underlying assets and is assumed to be independent with $Z_i(t)$. $h_i(t)$ is the conditional variance of the log return of underlying assets between time $t-1$ and t corresponding to idiosyncratic shocks, which is also known at the end of day $t-1$, and λ_i denotes the market price of risk, stemming from idiosyncratic shocks. β_i figures the impact of common shocks on the return of underlying asset $S_i(t)$ and the value of β_i can be shown in the following way,

$$\begin{aligned} &\frac{\mathrm{Cov}_{t-1}(\ln((M(t))/(M(t-1))), \ln((S_i(t))/(S_i(t-1))))}{\mathrm{Var}_{t-1}(\ln((M(t))/(M(t-1))))} \\ &\quad = \frac{\mathrm{Cov}_{t-1}(\sqrt{h_m(t)}Z_m(t), \sqrt{h_i(t)}Z_i(t) + \beta_i\sqrt{h_m(t)}Z_m(t))}{\mathrm{Var}_{t-1}(\sqrt{h_m(t)}Z_m(t))} \\ &\quad = \beta_i, \end{aligned} \tag{3}$$

where $\mathrm{Cov}_{t-1}(\cdot,\cdot)$ means the covariance under measure $\mathbb{P}$ and the fact that $Z_i(t)$ and $Z_m(t)$ are independent is used in the second equation. Clearly, the total conditional variance of $\ln S_i(t)$ is given by $h_i(t) + \beta_i^2 h_m(t)$, consisting of conditional variance of the log return of underlying assets corresponding to idiosyncratic shocks and common ones.

For valuation purposes, we determine an equivalent martingale measure in the following. Inspired by the affine structure of the pricing kernel in Christoffersen et al. [7], we define the following conditional Radon–Nikodym derivative,

$$\begin{aligned} L(t+1) &:= \left.\frac{\mathrm{d}Q}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_t} \\ &= \frac{\begin{array}{c}\exp\{\theta_m\sqrt{h_m(t+1)}Z_m(t+1) + \theta_1\sqrt{h_1(t+1)}Z_1(t+1) \\ + \theta_2\sqrt{h_2(t+1)}Z_2(t+1)\}\end{array}}{\begin{array}{c}\mathbb{E}_t^{\mathbb{P}}[\exp\{\theta_m\sqrt{h_m(t+1)}Z_m(t+1) + \theta_1\sqrt{h_1(t+1)}Z_1(t+1) \\ + \theta_2\sqrt{h_2(t+1)}Z_2(t+1)\}]\end{array}}, \end{aligned} \tag{4}$$

where $h_i(t+1),\ i = m, 1, 2$ are the conditional variances of the log return of market index and underlying assets, known at time t and $Z_m(t+1)$, $Z_1(t+1)$, $Z_2(t+1)$ are the normal shocks to returns of the market index, the underlying asset S_1 and S_2. To make sure that Q is an equivalent martingale measure, we introduce Proposition 2.1.

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PROPOSITION 2.1: *The measure Q is an equivalent martingale measure if and only if*

$$\theta_m = -\lambda_m, \quad \theta_1 = -\lambda_1, \quad \theta_2 = -\lambda_2.$$

Moreover, $Z_i(t) + \lambda_i\sqrt{h_i(t)}$ is a standard normal distribution under measure Q, for $i = m, 1, 2$.

Proof. The proof is similar to that of Proposition 2.1 in Wang [27]. We give the detail in the appendix.

Based on the above proposition, the risk-neutral dynamics of the market index and the underlying assets are given as follows.

PROPOSITION 2.2: *The dynamic of the market index $M(t)$ under Q meets the following form,*

$$\begin{cases} \ln M(t) = \ln M(t-1) + r - \frac{1}{2}h_m(t) + \sqrt{h_m(t)}Z_m^*(t), \\ h_m(t) = w_m + b_m h_m(t-1) + a_m(Z_m^*(t-1) - (c_m + \lambda_m)\sqrt{h_m(t-1)})^2, \end{cases} \tag{5}$$

where $Z_m^*(t) := Z_m(t) + \lambda_m\sqrt{h_m(t)}$ is a standard normal variable under measure Q.

The dynamic of the underlying assets $S_i(t),\ i = 1, 2$ under Q satisfy the following forms,

$$\begin{cases} \ln S_i(t) = \ln S_i(t-1) + r - \frac{1}{2}h_i(t) + \sqrt{h_i(t)}Z_i^*(t) - \frac{1}{2}\beta_i^2 h_m(t) + \beta_i\sqrt{h_m(t)}Z_m^*(t), \\ h_i(t) = w_i + b_i h_i(t-1) + a_i(Z_i^*(t-1) - (c_i + \lambda_i)\sqrt{h_i(t-1)})^2, \end{cases} \tag{6}$$

where $Z_i^*(t) := Z_i(t) + \lambda_i\sqrt{h_i(t)}$ is a standard normal variable under measure Q.

Proof. Girsanov's theorem (see page 190 in Karatzas [18]) immediately gives us that $Z_i^*(t) := Z_i(t) + \lambda_i\sqrt{h_i(t)},\ i = m, 1, 2$. Further, $Z_m^*(t)$, $Z_1^*(t)$ and $Z_2^*(t)$ are independent standard normal variables under Q because $Z_m(t)$, $Z_1(t)$, and $Z_2(t)$ are independent under $\mathbb{P}$.

2.2. Counterparty Default Risk

In this subsection, we consider counterparty default risk in a reduced form model, in which the default event is governed by a specified intensity process. Let τ be a random variable which stands for the first jump time of a doubly stochastic Poisson process (Cox process) with intensity $\Lambda(t)$. Suppose that the intensity $\Lambda(t)$ satisfies the following process under the risk-neutral measure Q,

$$\Lambda(t+1) = w_\lambda + b_\lambda\Lambda(t) + a_\lambda(Z_m^*(t))^2 + c_\lambda Z_\lambda^2(t), \tag{7}$$

where $w_\lambda > 0$, $b_\lambda \geq 0$, $c_\lambda \geq 0$, $a_\lambda \geq 0$, and $Z_\lambda(t)$ is a standard normal variable under Q independent of $Z_m^*(t)$ and $Z_i^*(t)$, two independent standard normal variables under Q defined in Proposition 2.2. It should be noted that the intensity process is also affected by $Z_m^*(t)$, and hence the correlation between the variances of the two underlying assets and the default intensity is captured in the proposed model. The proposed framework is more realistic, since all assets are exposed to systematic risk.

2.3. Valuation of Power Exchange Options

Now, we are in position to value power exchange options with default risk in the proposed framework.

Due to the possible default of the counterparty, the payoff of power exchange options depends on whether the default event occurs or not during the lifetime of the options. Consequently, the payoff of power exchange options is composed of two parts. If there is no default before the maturity T, the payoff of power exchange options can be expressed as $I(\tau > T)(\mu_1 S_1^{\alpha_1}(T) - \mu_2 S_2^{\alpha_2}(T))^+$, where τ denotes the default time and $I(\tau > T)$ indicates that there is no default events before the maturity T. If the default event occurs during the lifetime of the option, only a proportion $\alpha \in (0,1)$ of its market value can be recovered. In the circumstances, the payoff of power exchange options equals $\alpha \mathbb{E}^Q[e^{-r(T-\tau)}(\mu_1 S_1^{\alpha_1}(T) - \mu_2 S_2^{\alpha_2}(T))^+ | \mathscr{F}_\tau]$, where $\mathbb{E}^Q[e^{-r(T-\tau)}(\mu_1 S_1^{\alpha_1}(T) - \mu_2 S_2^{\alpha_2}(T))^+ | \mathscr{F}_\tau]$ stands for the value of power exchange options at time τ and α is the recover rate. Therefore, the value C^* of a power exchange option with the possibility of default prior to the maturity is

$$\begin{aligned} C^* &= e^{-rT}\mathbb{E}^Q[I(\tau > T)(\mu_1 S_1^{\alpha_1}(T) - \mu_2 S_2^{\alpha_2}(T))^+] \\ &\quad + \mathbb{E}^Q[I(0 \le \tau \le T)\alpha e^{-r\tau}\mathbb{E}^Q[e^{-r(T-\tau)}(\mu_1 S_1^{\alpha_1}(T) - \mu_2 S_2^{\alpha_2}(T))^+ | \mathscr{F}_\tau]] \\ &= e^{-rT}\mathbb{E}^Q[I(\tau > T)(\mu_1 S_1^{\alpha_1}(T) - \mu_2 S_2^{\alpha_2}(T))^+] \\ &\quad + \alpha e^{-rT}\mathbb{E}^Q[I(0 \le \tau \le T)(\mu_1 S_1^{\alpha_1}(T) - \mu_2 S_2^{\alpha_2}(T))^+]. \end{aligned} \tag{8}$$

Note the fact that $I(0 \le \tau \le T) = 1 - I(\tau > T)$, we can rewrite the above equation C^* as follows,

$$\begin{aligned} C^* &= e^{-rT}\mathbb{E}^Q[I(\tau > T)(\mu_1 S_1^{\alpha_1}(T) - \mu_2 S_2^{\alpha_2}(T))^+] \\ &\quad + \alpha e^{-rT}\mathbb{E}^Q[I(0 \le \tau \le T)(\mu_1 S_1^{\alpha_1}(T) - \mu_2 S_2^{\alpha_2}(T))^+] \\ &= (1-\alpha)e^{-rT}\mathbb{E}^Q[I(\tau > T)(\mu_1 S_1^{\alpha_1}(T) - \mu_2 S_2^{\alpha_2}(T))^+] \\ &\quad + \alpha e^{-rT}\mathbb{E}^Q[(\mu_1 S_1^{\alpha_1}(T) - \mu_2 S_2^{\alpha_2}(T))^+] \\ &:= (1-\alpha)e^{-rT}I_1 + \alpha e^{-rT}I_2, \end{aligned} \tag{9}$$

where I_1 and I_2 are given by

$$I_1 = \mathbb{E}^Q[I(\tau > T)(\mu_1 S_1^{\alpha_1}(T) - \mu_2 S_2^{\alpha_2}(T))^+], \tag{10}$$

$$I_2 = \mathbb{E}^Q[(\mu_1 S_1^{\alpha_1}(T) - \mu_2 S_2^{\alpha_2}(T))^+]. \tag{11}$$

From the above equations, it is clear that the closed-form formula of the power exchange option price C^* in Eq. (9) can be derived through the explicit form of I_1 and I_2. Therefore, we now turn to calculate the explicit solutions of I_1 and I_2. In order to compute I_1 and I_2, we adopt the moment generating function method in this paper. Suppose that $x_1(T) := \ln S_1(T)$ and $x_2(T) := \ln S_2(T)$, then the characteristic function $f(0;T,\phi_1,\phi_2,\phi_3)$ of $x_1(T)$, $x_2(T)$ and $\sum_{s=1}^{T}\Lambda(s)$ is defined as follows,

$$f(0;T,\phi_1,\phi_2,\phi_3) = \mathbb{E}^Q[e^{\phi_1 x_1(T) + \phi_2 x_2(T) + \phi_3 \sum_{s=1}^{T}\Lambda(s)}].$$

By inverting the characteristic function, we can compute the probabilities and hence obtain the explicit solutions of I_1 and I_2 shown in Proposition 2.4. The following proposition gives the explicit form of the conditional characteristic function $f(t;T,\phi_1,\phi_2,\phi_3)$,

$$f(t;T,\phi_1,\phi_2,\phi_3) = \mathbb{E}_t^Q[e^{\phi_1 x_1(T) + \phi_2 x_2(T) + \phi_3 \sum_{s=1}^{T}\Lambda(s)}].$$

PROPOSITION 2.3: *The conditional moment generating function of* $x_1(T) := \ln S_1(T)$, $x_2(T) := \ln S_2(T)$ *and* $\sum_{s=1}^{T} \Lambda(s)$, *meets the following form,*

$$f(t;T,\phi_1,\phi_2,\phi_3) = \exp\{\phi_1 x_1(t) + \phi_2 x_2(t) + \phi_3 \sum_{s=1}^{t} \Lambda(s) + A(t) + B_1(t)h_1(t+1) + B_2(t)h_2(t+1) + B_3(t)h_m(t+1) + B_4(t)\Lambda(t+1)\},$$

where $A(t)$ *and* $B_i(t)$, $i = 1,2,3,4$ *are the abbreviations of* $A(t;T,\phi_1,\phi_2,\phi_3)$ *and* $B_i(t;T,\phi_1,\phi_2,\phi_3)$ *for convenience and are given by*

$$\begin{aligned}
A(t) &= (\phi_1+\phi_2)r + A(t+1) + w_1 B_1(t+1) + w_2 B_2(t+1) + w_m B_3(t+1) + w_\lambda B_4(t+1) \\
&\quad - \frac{1}{2}\ln(1-2a_1 B_1(t+1)) - \frac{1}{2}\ln(1-2a_2 B_2(t+1)) \\
&\quad - \frac{1}{2}\ln(1-2(a_m B_3(t+1) + a_\lambda B_4(t+1))) - \frac{1}{2}\ln(1-2c_\lambda B_4(t+1)), \\
B_1(t) &= b_1 B_1(t+1) - \frac{1}{2}\phi_1 + \phi_1(c_1+\lambda_1) - \frac{1}{2}(c_1+\lambda_1)^2 + \frac{(1/2)(\phi_1-(c_1+\lambda_1))^2}{1-2a_1 B_1(t+1)}, \\
B_2(t) &= b_2 B_2(t+1) - \frac{1}{2}\phi_2 + \phi_2(c_2+\lambda_2) - \frac{1}{2}(c_2+\lambda_2)^2 + \frac{(1/2)(\phi_2-(c_2+\lambda_2))^2}{1-2a_2 B_2(t+1)}, \\
B_3(t) &= b_m B_3(t+1) - \frac{1}{2}\phi_1\beta_1^2 - \frac{1}{2}\phi_2\beta_2^2 + a_m B_3(t+1)(c_m+\lambda_m)^2 \\
&\quad + \frac{(\phi_1\beta_1 + \phi_2\beta_2 - 2a_m B_3(t+1)(c_m+\lambda_m))^2}{2(1-2(a_m B_3(t+1) + a_\lambda B_4(t+1)))}, \\
B_4(t) &= \phi_3 + b_\lambda B_4(t+1),
\end{aligned}$$

and these coefficients can be obtained recursively using the terminal conditions $A(T) = B_1(T) = B_2(T) = B_3(T) = B_4(T) = 0$.

Proof. See the detail in the appendix.

Based on the above moment generating function $f(0;T,\phi_1,\phi_2,\phi_3)$, we can obtain the power exchange option price in (8) in the following proposition.

PROPOSITION 2.4: *The price of power exchange options in (8) is given by*

$$C^* = (1-\alpha)e^{-rT}(\mu_1\Pi_1(0;T) - \mu_2\Pi_2(0;T)) + \alpha e^{-rT}(\mu_1\Pi_3(0;T) - \mu_2\Pi_4(0;T)), \quad \mathbf{(12)}$$

where the closed form of $f(0;T,\phi_1,\phi_2,\phi_3)$ *is derived in Proposition 2.3 and* $\Pi_i(t;T)$, $i = 1,2,3,4$ *are given as follows,*

$$\begin{aligned}
\Pi_1(0;T) &= \frac{1}{2} f(0;T,\alpha_1,0,-1) \\
&\quad + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi_1 \ln((\mu_2)/(\mu_1))} f(0;T,i\phi_1\alpha_1+\alpha_1,-i\phi_1\alpha_2,-1)}{i\phi_1}\right] d\phi_1,
\end{aligned}$$

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$$\Pi_2(0;T) = \frac{1}{2} f(0;T,0,\alpha_2,-1) + \frac{1}{\pi}\int_0^{\infty} Re\left[\frac{e^{-i\phi_1 \ln((\mu_2)/(\mu_1))} f(0;T,i\phi_1\alpha_1,-i\phi_1\alpha_2+\alpha_2,-1)}{i\phi_1}\right] \mathrm{d}\phi_1,$$

$$\Pi_3(0;T) = \frac{1}{2} f(0;T,\alpha_1,0,0) + \frac{1}{\pi}\int_0^{\infty} Re\left[\frac{e^{-i\phi_1 \ln((\mu_2)/(\mu_1))} f(0;T,i\phi_1\alpha_1+\alpha_1,-i\phi_1\alpha_2,0)}{i\phi_1}\right] \mathrm{d}\phi_1,$$

$$\Pi_4(0;T) = \frac{1}{2} f(0;T,0,\alpha_2,0) + \frac{1}{\pi}\int_0^{\infty} Re\left[\frac{e^{-i\phi_1 \ln((\mu_2)/(\mu_1))} f(0;T,i\phi_1\alpha_1,-i\phi_1\alpha_2+\alpha_2,0)}{i\phi_1}\right] \mathrm{d}\phi_1.$$

Proof. See the appendix.

In virtue of the explicit solution of the generating function, we have obtained the closed-form valuation formula of the power exchange options under the proposed GARCH framework with counterparty default risk in a reduced-form model. In the case of $\mu_1 = \mu_2 = 1$ and $\alpha_1 = \alpha_2 = 1$, the derived pricing formula reduces to exchange options with default risk and adding another condition $S_2(T) = K$, it reduces to European call options with default risk. The closed-form pricing formula of power options and European put options with default risk can similarly be derived employing the explicit solution of the generating function.

3. NUMERICAL RESULTS

In this section, we investigate the prices of power exchange options with counterparty risk under the proposed framework. For comparison, we also report the values of power exchange options without counterparty risk. In addition, we study the effects of the power exponents on prices.

To obtain the prices, we use the following values of the parameters: $\lambda_m = 1.576\text{E}+00$, $w_m = 3.000\text{E}-15$, $b_m = 8.500\text{E}-01$, $a_m = 3.921\text{E}-06$, $c_m = 1.755\text{E}+02$, $\lambda_1 = 1.017\text{E}+00$, $w_1 = 9.319\text{E}-11$, $b_1 = 9.497\text{E}-01$, $a_1 = 1.874\text{E}-05$, and $c_1 = 4.385\text{E}-04$. These parameters are borrowed from Wang [27], where the parameters are estimated using maximum likelihood and daily closing prices for the Standard and Poor's 500 index and Microsoft Corporation stocks for the period from January 3, 1995 to December 31, 2009. In addition, the initial levels of the variance for the S&P 500 index and the stock price are set to be the historical variances calculated from the returns data mentioned above, i.e., $h_m(1) = 7.596E-03$ and $h_s(1) = 3.383E-02$, and let $\beta_1 = 1.20$. Moreover, the parameter values for two underlying assets are set to be equal for simplicity. Without loss of generality, the annual risk-free interest rate is set to 0.02, and we set the current price of the stock to be 1, as the values of power exchange options do not depend on the current value of the S&P 500 index. For the parameters of the intensity, we set $w_\lambda = 8.637\text{E}-07$, $b_\lambda = 9.949\text{E}-01$, $a_\lambda = 1.372\text{E}-10$, and $c_\lambda = 1.372\text{E}-10$. The corresponding cumulative default probabilities are plotted in Figure 1, using the derived formula of $1 - f(0;T,0,0,-1)$.

To show the efficiency of the pricing method, here we present cpu times for obtaining values of power exchange options with $\alpha_1 = \alpha_2 = 2.0$ and default risk. The cpu times on

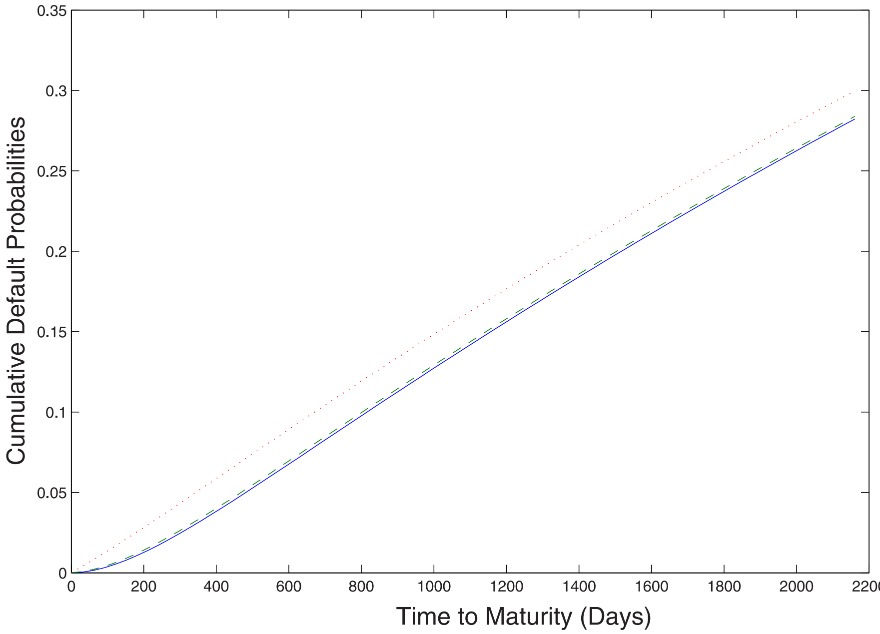

FIGURE 1. Cumulative default probabilities. The solid, dashed, and dotted lines correspond to cumulative default probabilities for $\lambda(1) = 1.275\mathrm{E}-06$, $\lambda(1) = 1.275\mathrm{E}-05$, and $\lambda(1) = 1.275\mathrm{E}-04$, respectively.

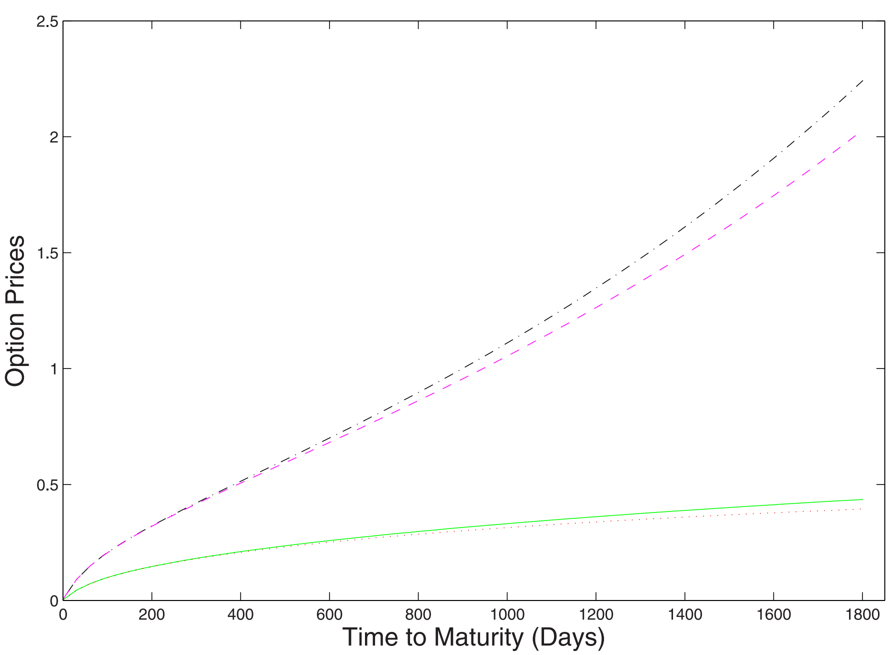

FIGURE 2. Option prices against time to maturity. The dotted and solid lines correspond to option prices with default risk and no default when $\alpha_1 = \alpha_2 = 1.0$. The dashed and dot-dashed lines correspond to option prices with default risk and no default when $\alpha_1 = \alpha_2 = 2.0$. The initial levels of the default is $\lambda(1) = 1.275\mathrm{E}-06$.

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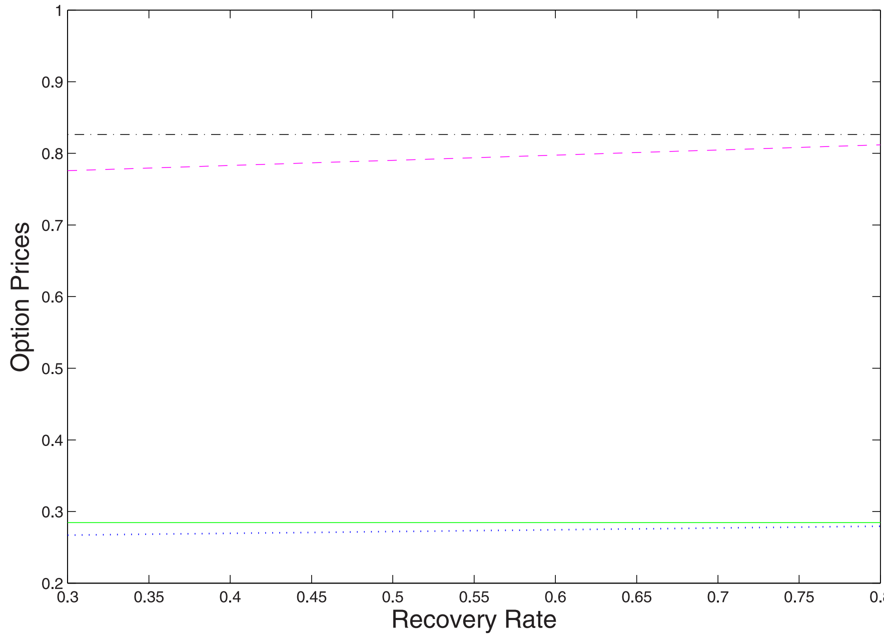

FIGURE **3.** Option prices against recovery rate. The dotted and solid lines correspond to option prices with default risk and no default when $\alpha_1 = \alpha_2 = 1.0$. The dashed and dot-dashed lines correspond to option prices with default risk and no default when $\alpha_1 = \alpha_2 = 2.0$. The initial levels of the default is $\lambda(1) = 1.275\text{E} - 06$.

a Core I5 1.7 GHz personal computer are 1.0931, 2.7801, and 5.6680 seconds for $T = 1.0$, $T = 2.0$ and $T = 5.0$, respectively. Figure 2 depicts the values of power exchange options under alternative maturities. It can be seen that counterparty risk has little impacts when the maturity is less than 1 year. Additionally, the impacts of counterparty risk become pronounced as the maturity rises and the power exponents enhance these impacts. Figure 3 presents option prices against recovery rate. Recovery rate only affects the payoff of the options when default happens, hence the values of the options without counterparty risk are not affected. Increasing the recovery rate from 0.30 to 0.80, values of the options with $\alpha_1 = \alpha_2 = 1.0$ change from 0.2671 to 0.2795, while those of the options with $\alpha_1 = \alpha_2 = 2.0$ are 0.7759 and 0.8119, respectively. The effects are enhanced by the power exponents. In addition, the more likely the counterparty is to default, the stronger the effects of the power exponents are.

Figures 4 and 5 plot option prices against the value of β_1. As shown in Figures 4 and 5, option prices first decrease and then increase as the market beta of the underlying asset S_1 rises. The market β of the underlying asset S_1 affects the total risk of S_1 and the correlation between two underlying assets as well. An increase in the market β of the underlying asset S_1 corresponds to a stronger correlation between the two underlying assets. The stronger correlation ensures that the values of two assets move in the same direction more likely, reducing option prices. On the other hand, a higher total risk of the underlying asset S_1 enhances option prices. Hence, the U-shaped curves appear. However, it should be noted that the values of the market β of the underlying asset S_1 corresponding to minimum option prices are quite different in Figures 4 and 5, which are affected by the power exponents.

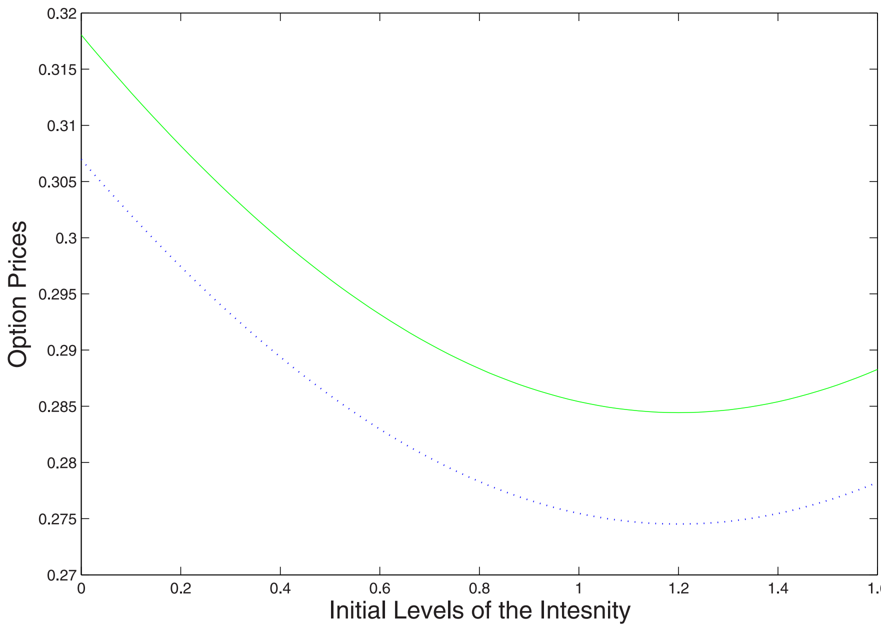

FIGURE 4. Option prices against the value of β_1. The dotted and solid lines correspond to option prices with default risk and no default when $\alpha_1 = \alpha_2 = 1.0$. The initial levels of the default is $\lambda(1) = 1.275\text{E} - 06$.

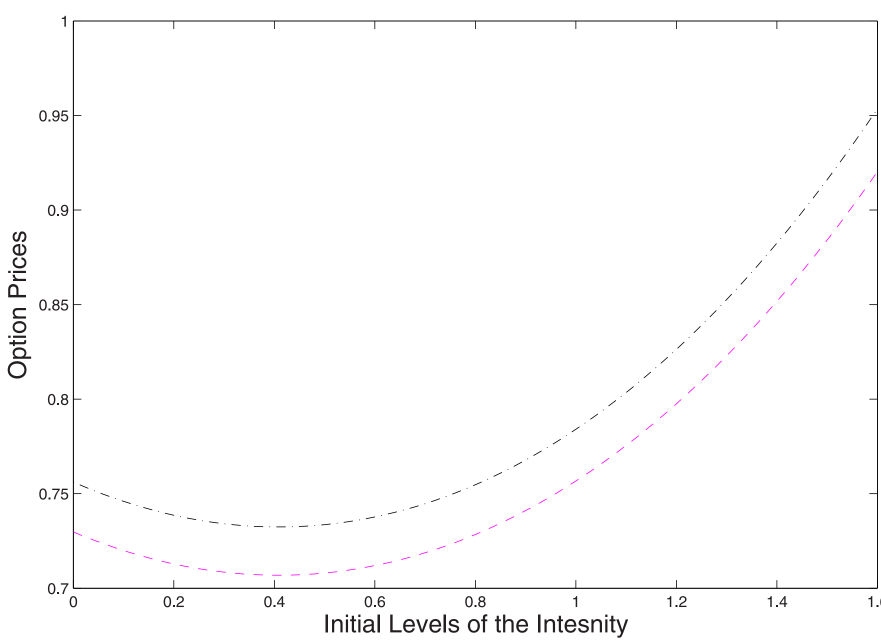

FIGURE 5. Option prices against the value of β_1. The dotted and solid lines correspond to option prices with default risk and no default when $\alpha_1 = \alpha_2 = 2.0$. The initial levels of the default is $\lambda(1) = 1.275\text{E} - 06$.

4. CONCLUSION

This article presents a closed-form valuation formula of power exchange options in a discrete-time framework, where counterparty risk is considered in a reduced-form setting and the variance processes of the underlying assets are captured by GARCH processes. In virtue of measure-change techniques and characteristic functions, we obtain the closed-form valuation formula for power exchange options, which involves the pricing formula for exchange options, power options, and vanilla call/put options as special cases. Finally, we present numerical results on power exchange option values.

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APPENDIX

Proof of Proposition 2.1: It is widely known the expected returns of $M(t)$ and $S_i(t),\ i=1,2$ under measure Q should be the risk-free interest rate, i.e.,

$$\mathbb{E}_t^{\mathbb{P}}\left[L(t+1)\frac{M(t+1)}{M(t)}\right]=e^r, \tag{A.1}$$

$$\mathbb{E}_t^{\mathbb{P}}\left[L(t+1)\frac{S_i(t+1)}{S(t)}\right]=e^r, \tag{A.2}$$

where $\mathbb{E}_t^{\mathbb{P}}[\cdot]$ means the conditional expectation under measure $\mathbb{P}$ given the information at time t and $L(t)$ is the Radon–Nikodym derivative defined in (4).

Using the expression of $L(t+1)$ and $M(t)$ defined in (4) and (1), we can obtain

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{P}}\left[L(t+1)\frac{M(t+1)}{M(t)}\right]\\
&=\frac{\mathbb{E}_t^{\mathbb{P}}[e^{\theta_m\sqrt{h_m(t+1)}Z_m(t+1)+\theta_1\sqrt{h_1(t+1)}Z_1(t+1)+\theta_2\sqrt{h_2(t+1)}Z_2(t+1)}((M(t+1))/(M(t)))]}{\mathbb{E}_t^{\mathbb{P}}[e^{\theta_m\sqrt{h_m(t+1)}Z_m(t+1)+\theta_1\sqrt{h_1(t+1)}Z_1(t+1)+\theta_2\sqrt{h_2(t+1)}Z_2(t+1)}]}\\
&=\frac{\mathbb{E}_t^{\mathbb{P}}[e^{\theta_m\sqrt{h_m(t+1)}Z_m(t+1)}((M(t+1))/(M(t)))]}{\mathbb{E}_t^{\mathbb{P}}[e^{\theta_m\sqrt{h_m(t+1)}Z_m(t+1)}]}\\
&=\frac{\mathbb{E}_t^{\mathbb{P}}[e^{\theta_m\sqrt{h_m(t+1)}Z_m(t+1)+r+(\lambda_m-(1/2))h_m(t+1)+\sqrt{h_m(t+1)}Z_m(t+1)}]}{\mathbb{E}_t^{\mathbb{P}}[e^{\theta_m\sqrt{h_m(t+1)}Z_m(t+1)}]}\\
&=\frac{e^{1/2(\theta_m+1)^2h_m(t+1)+r+(\lambda_m-(1/2))h_m(t+1)}}{e^{1/2\theta_m^2h_m(t+1)}}\\
&=e^{(\theta_m+\lambda_m)h_m(t+1)+r},
\end{aligned}$$

where we have used the fact that $Z_i(t+1),\ i=1,2$ is independent of $Z_m(t+1)$ given the information at time t in the second equality. Therefore, from (A.1), it is clear that $\theta_m=-\lambda_m$. In a similar way, we can have $\theta_i=-\lambda_i,\ i=1,2$. And $Z_i^*(t):=Z_i(t)+\lambda_i\sqrt{h_i(t)}$ is standard normal distribution under measure Q, for $i=m,1,2$ from the Girsanov's theorem. ■

Proof of Proposition 2.3: Recall the definition of $x_1(t) := \ln S_1(t)$, $x_2(t) := \ln S_2(t)$ and $f(t;T,\phi_1,\phi_2,\phi_3)$:

$$f(t;T,\phi_1,\phi_2,\phi_3) = \mathbb{E}_t^Q\left[e^{\phi_1 x_1(T)+\phi_2 x_2(T)+\phi_3\sum_{s=1}^{T}\Lambda(s)}\right].$$

Below, we will show that the moment generating function $f(t;T,\phi_1,\phi_2,\phi_3)$ has the log-linear form,

$$\begin{aligned}f(t;T,\phi_1,\phi_2,\phi_3) = &\exp\{\phi_1 x_1(t)+\phi_2 x_2(t)+\phi_3\sum_{s=1}^{t}\Lambda(s)+A(t)+B_1(t)h_1(t+1)\\ &+B_2(t)h_2(t+1)+B_3(t)h_m(t+1)+B_4(t)\Lambda(t+1)\}.\end{aligned}$$

Here and below, the notations $f(t)$, $A(t)$, and $B_i(t)$, $i=1,2,3,4$ are the abbreviations of $f(t;T,\phi_1,\phi_2,\phi_3)$, $A(t;T,\phi_1,\phi_2,\phi_3)$, and $B_i(t;T,\phi_1,\phi_2,\phi_3)$ for simplicity. At time T, $x_1(T)$, $x_2(T)$, and $\sum_{s=1}^{T}\Lambda(s)$ are known and it holds that $f(T)=\exp\{\phi_1 x(T)+\phi_2 x_2(T)+\phi_3\sum_{s=1}^{T}\Lambda(s)\}$, which in turn implies the following terminal conditions

$$A(T)=B_1(T)=B_2(T)=B_3(T)=B_4(T)=0.$$

In virtue of the law of iterated expectations, the form of $f(t)$ becomes

$$\begin{aligned}f(t) &= \mathbb{E}_t^Q[e^{\phi_1 x_1(T)+\phi_2 x_2(T)+\phi_3\sum_{s=1}^{T}\Lambda(s)}]\\ &= \mathbb{E}_t^Q[\mathbb{E}_{t+1}^Q[e^{\phi_1 x_1(T)+\phi_2 x_2(T)+\phi_3\sum_{s=1}^{T}\Lambda(s)}]]\\ &= \mathbb{E}_t^Q[f(t+1)]\\ &= \mathbb{E}_t^Q[\exp\{\phi_1 x_1(t+1)+\phi_2 x_2(t+1)+\phi_3\sum_{s=1}^{t+1}\Lambda(s)+A(t+1)+B_1(t+1)h_1(t+2)\\ &\quad +B_2(t+1)h_2(t+1)+B_3(t+1)h_m(t+1)+B_4(t+1)\Lambda(t+2)\}].\end{aligned}$$

Substituting the dynamics of $x_1(t+1)$, $x_2(t+1)$, $h_1(t+2)$, $h_2(t+2)$, and $\Lambda(t+2)$ in Eqs. (6) and (7) into the above equation and simplifying the results, we have

$$\begin{aligned}f(t) = \mathbb{E}_t^Q\Bigg[&\exp\{\phi_1 x_1(t)+\phi_1 r+\phi_2 x_2(t)+\phi_2 r+\phi_3\sum_{s=1}^{t}\Lambda(s)\\ &+A(t+1)+B_1(t+1)w_1+B_2(t+1)w_2\\ &+B_3(t+1)w_m+B_4(t+1)w_\lambda+h_1(t+1)\left(b_1B_1(t+1)-\frac{1}{2}\phi_1\right)\\ &+h_2(t+1)\left(b_2B_2(t+1)-\frac{1}{2}\phi_2\right)\\ &+h_m(t+1)\left(b_mB_3(t+1)-\frac{1}{2}\phi_1\beta_1^2-\frac{1}{2}\phi_2\beta_2^2\right)+\Lambda(t+1)(\phi_3+b_\lambda B_4(t+1))+\Psi\}\Bigg],\end{aligned}$$

where Ψ has the following form,

$$\Psi = \Psi_1+\Psi_2+\Psi_m+\Psi_\lambda,$$

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with Ψ_1, Ψ_2, Ψ_m, and Ψ_λ:

$$\begin{aligned}
\Psi_1 &= \phi_1\sqrt{h_1(t+1)}Z_1^*(t+1) + a_1B_1(t+1)(Z_1^*(t+1) - (c_1+\lambda_1)\sqrt{h_1(t+1)})^2,\\
\Psi_2 &= \phi_2\sqrt{h_2(t+1)}Z_2^*(t+1) + a_2B_2(t+1)(Z_2^*(t+1) - (c_2+\lambda_2)\sqrt{h_2(t+1)})^2,\\
\Psi_m &= (\phi_1\beta_1+\phi_2\beta_2)\sqrt{h_m(t+1)}Z_m^*(t+1) + a_mB_3(t+1)(Z_m^*(t+1) - (c_m+\lambda_m)\sqrt{h_m(t+1)})^2\\
&\quad + a_\lambda B_4(t+1)(Z_m^*(t+1))^2,\\
\Psi_\lambda &= c_\lambda B_4(t+1)Z_\lambda^2(t+1).
\end{aligned}$$

Now we calculate $\mathbb{E}_t^Q\Big[\exp\{\Psi_1+\Psi_2+\Psi_m+\Psi_\lambda\}\Big]$ in order to yield the solution of $f(t)$. Firstly,

$$\begin{aligned}
&\mathbb{E}_t^Q[\exp\{\Psi_1\}]\\
&\quad= \mathbb{E}_t^Q\left[\exp\left\{a_1B_1(t+1)(Z_1^*(t+1) - \left(c_1+\lambda_1-\frac{\phi_1}{2a_1B_1(t+1)}\right)\sqrt{h_1(t+1)})^2\right.\right.\\
&\qquad\left.\left. +\ a_1B_1(t+1)\left((c_1+\lambda_1)^2 - \left(c_1+\lambda_1-\frac{\phi_1}{2a_1B_1(t+1)}\right)^2\right)h_1(t+1)\right\}\right]\\
&\quad= \exp\left\{a_1B_1(t+1)\left((c_1+\lambda_1)^2 - \left(c_1+\lambda_1-\frac{\phi_1}{2a_1B_1(t+1)}\right)^2\right)h_1(t+1)\right\}\\
&\qquad\times\exp\left\{-\frac{1}{2}\ln(1-2a_1B_1(t+1)) + \frac{\begin{array}{c}a_1B_1(t+1)(c_1+\lambda_1-((\phi_1)/\\(2a_1B_1(t+1))))^2h_1(t+1)\end{array}}{1-2a_1B_1(t+1)}\right\},
\end{aligned}$$

where the fact that $\mathbb{E}e^{a(Z+b)^2} = e^{-(1/2)\ln(1-2a)+((ab^2)/(1-2a))}$ (Z is a standard normal variable) has been used in the last equality. Rearranging the above terms we get that

$$\begin{aligned}
\mathbb{E}_t^Q[\exp\{\Psi_1\}] &= \exp\left\{-\frac{1}{2}\ln(1-2a_1B_1(t+1))\right.\\
&\quad\left.+\left(\phi_1(c_1+\lambda_1) - \frac{1}{2}(c_1+\lambda_1)^2 + \frac{(1/2)(\phi_1-(c_1+\lambda_1))^2}{1-2a_1B_1(t+1)}\right)h_1(t+1)\right\}.
\end{aligned}$$

In a similar way, we can obtain

$$\begin{aligned}
\mathbb{E}_t^Q[\exp\{\Psi_2\}] &= \exp\left\{-\frac{1}{2}\ln(1-2a_2B_2(t+1))\right.\\
&\quad\left.+\left(\phi_2(c_2+\lambda_2) - \frac{1}{2}(c_2+\lambda_2)^2 + \frac{(1/2)(\phi_2-(c_2+\lambda_2))^2}{1-2a_2B_2(t+1)}\right)h_2(t+1)\right\},\\
\mathbb{E}_t^Q[\exp\{\Psi_m\}] &= \exp\left\{-\frac{1}{2}\ln[1-2(a_mB_3(t+1)+a_\lambda B_4(t+1))] + \Big[a_mB_3(t+1)(c_m+\lambda_m)^2\right.\\
&\quad\left.+\ \frac{(\phi_1\beta_1+\phi_2\beta_2-2a_mB_3(t+1)(c_m+\lambda_m))^2}{2(1-2(a_mB_3(t+1)+a_\lambda B_4(t+1)))}\Big]h_m(t+1)\right\},\\
\mathbb{E}_t^Q[\exp\{\Psi_\lambda\}] &= \exp\left\{-\frac{1}{2}\ln(1-2c_\lambda B_4(t+1))\right\}.
\end{aligned}$$

Therefore, $A(t)$, $B_1(t)$, $B_2(t)$, $B_3(t)$, and $B_4(t)$ can be obtained in the follows,

$$\begin{aligned} A(t) &= (\phi_1+\phi_2)r + A(t+1) + w_1 B_1(t+1) + w_2 B_2(t+1) + w_m B_3(t+1) + w_\lambda B_4(t+1) \\ &\quad - \frac{1}{2}\ln(1-2a_1 B_1(t+1)) - \frac{1}{2}\ln(1-2a_2 B_2(t+1)) \\ &\quad - \frac{1}{2}\ln(1-2(a_m B_3(t+1) + a_\lambda B_4(t+1))) - \frac{1}{2}\ln(1-2c_\lambda B_4(t+1)), \\ B_1(t) &= b_1 B_1(t+1) - \frac{1}{2}\phi_1 + \phi_1(c_1+\lambda_1) - \frac{1}{2}(c_1+\lambda_1)^2 + \frac{(1/2)(\phi_1-(c_1+\lambda_1))^2}{1-2a_1 B_1(t+1)}, \\ B_2(t) &= b_2 B_2(t+1) - \frac{1}{2}\phi_2 + \phi_2(c_2+\lambda_2) - \frac{1}{2}(c_2+\lambda_2)^2 + \frac{(1/2)(\phi_2-(c_2+\lambda_2))^2}{1-2a_2 B_2(t+1)}, \\ B_3(t) &= b_m B_3(t+1) - \frac{1}{2}\phi_1\beta_1^2 - \frac{1}{2}\phi_2\beta_2^2 + a_m B_3(t+1)(c_m+\lambda_m)^2 \\ &\quad + \frac{(\phi_1\beta_1+\phi_2\beta_2-2a_m B_3(t+1)(c_m+\lambda_m))^2}{2(1-2(a_m B_3(t+1)+a_\lambda B_4(t+1)))}, \\ B_4(t) &= \phi_3 + b_\lambda B_4(t+1). \end{aligned}$$

■

Proof of Proposition 2.4: In order to give the proof, first we define a new measure $\tilde{Q}$ as follows,

$$\tilde{Q}(A) := \frac{\mathbb{E}^Q[S_1^{\phi_1}(T)S_2^{\phi_2}(T)I(A)]}{\mathbb{E}^Q[S_1^{\phi_1}(T)S_2^{\phi_2}(T)]},$$

for any events $A \in \mathscr{F}_T$ and $I(\cdot)$ is an indicator function. The expression $\tilde{\Lambda}(t)$ means the process $\Lambda(t)$ under the new measure $\tilde{Q}$. Then we have the following result,

$$\begin{aligned} \mathbb{E}^Q[S_1^{\phi_1}(T)S_2^{\phi_2}(T)I(\tau>T)] &= \mathbb{E}^Q[S_1^{\phi_1}(T)S_2^{\phi_2}(T)]\mathbb{E}^{\tilde{Q}}[I(\tau>T)], \\ &= \mathbb{E}^Q[S_1^{\phi_1}(T)S_2^{\phi_2}(T)]\mathbb{E}^{\tilde{Q}}[e^{-\sum_{s=1}^T\tilde{\Lambda}(s)}], \\ &= \mathbb{E}^Q[S_1^{\phi_1}(T)S_2^{\phi_2}(T)]\frac{\mathbb{E}^Q[S_1^{\phi_1}(T)S_2^{\phi_2}(T)e^{-\sum_{s=1}^T\Lambda(s)}]}{\mathbb{E}^Q[S_1^{\phi_1}(T)S_2^{\phi_2}(T)]}, \\ &= \mathbb{E}^Q[e^{\phi_1 x_1(T)+\phi_2 x_2(T)-\sum_{s=1}^T\Lambda(s)}] = f(0;T,\phi_1,\phi_2,-1). \end{aligned} \quad \textbf{(A.3)}$$

Recall that $C^* = (1-\alpha)e^{-rT}I_1 + \alpha e^{-rT}I_2$ with I_1 in (10) and I_2 in (11). Simple calculation to I_1 yields

$$\begin{aligned} I_1 &= \mu_1\mathbb{E}^Q\left[I(\tau>T)S_1^{\alpha_1}(T)I\left(\frac{S_1^{\alpha_1}(T)}{S_2^{\alpha_2}(T)} \geqslant \frac{\mu_2}{\mu_1}\right)\right] - \mu_2\mathbb{E}^Q\left[I(\tau>T)S_2^{\alpha_2}(T)I\left(\frac{S_1^{\alpha_1}(T)}{S_2^{\alpha_2}(T)} \geqslant \frac{\mu_2}{\mu_1}\right)\right] \\ &:= \mu_1\Pi_1(0;T) - \mu_2\Pi_2(0;T). \end{aligned} \quad \textbf{(A.4)}$$

For term $\Pi_1(0;T)$ in (A.4), we define a new probability measure Q_1,

$$Q_1(A) := \frac{\mathbb{E}^Q[I(A)S_1^{\alpha_1}(T)I(\tau>T)]}{\mathbb{E}^Q[S_1^{\alpha_1}(T)I(\tau>T)]},$$

for any events $A \in \mathscr{F}_T$ and $I(\cdot)$ is an indicator function. In other way, we have

$$\mathbb{E}^{Q_1}[I(A)] = \frac{\mathbb{E}^Q[I(A)S_1^{\alpha_1}(T)I(\tau>T)]}{\mathbb{E}^Q[S_1^{\alpha_1}(T)I(\tau>T)]}.$$

Then with the definition of Q_1, one gets that

$$\begin{aligned} f_1(0;T,i\phi_1) &:= \mathbb{E}^{Q_1}[e^{i\phi_1(\alpha_1 x_1(T)-\alpha_2 x_2(T))}], \\ &= \frac{\mathbb{E}^{Q}[e^{i\phi_1(\alpha_1 x_1(T)-\alpha_2 x_2(T))}S_1^{\alpha_1}(T)I(\tau>T)]}{\mathbb{E}^{Q}[S_1^{\alpha_1}(T)I(\tau>T)]}, \\ &= \frac{\mathbb{E}^{Q}[e^{(i\phi_1\alpha_1+\alpha_1)x_1(T)-i\phi_1\alpha_2 x_2(T)}I(\tau>T)]}{\mathbb{E}^{Q}[S_1^{\alpha_1}(T)I(\tau>T)]}, \\ &= \frac{f(0;T,i\phi_1\alpha_1+\alpha_1,-i\phi_1\alpha_2,-1)}{f(0;T,\alpha_1,0,-1)}, \end{aligned}$$

where the last equation is deuced from Eq. (A.3), and the explicit form of $f(0;T,\phi_1,\phi_2,\phi_3)$ can be found in Proposition 2.3. Standard probability theory (see, e.g., Kendall and Stuart [19] and Shephard [24]) implies that the distribution $F_1(\alpha_1 x_1(T)-\alpha_2 x_2(T);x) := Q_1(\alpha_1 x_1(T)-\alpha_2 x_2(T)\leqslant x)$ with respect to the characteristic function $f_1(0;T,i\phi_1)$ is,

$$F_1(\alpha_1 x_1(T)-\alpha_2 x_2(T);x) = \frac{1}{2}-\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi_1 x}f_1(0;T,i\phi_1)}{i\phi_1}\right]\mathrm{d}\phi_1,$$

where $Re[\cdot]$ denotes the real part of a complex number. Therefore, we have that

$$\begin{aligned} \Pi_1(0;T) &= Q_1\left(\frac{S_1^{\alpha_1}(T)}{S_2^{\alpha_2}(T)}\geqslant\frac{\mu_2}{\mu_1}\right)\mathbb{E}^{Q}[S_1^{\alpha_1}(T)I(\tau>T)] \\ &= \left(1-F_1\left(\alpha_1 x_1(T)-\alpha_2 x_2(T);\ln\frac{\mu_2}{\mu_1}\right)\right)*f(0;T,\alpha_1,0,-1) \\ &= \frac{1}{2}f(0;T,\alpha_1,0,-1)+\frac{1}{\pi}\int_0^\infty \mathrm{Re} \\ &\quad\times\left[\frac{e^{-i\phi_1\ln((\mu_2)/(\mu_1))}f(0;T,i\phi_1\alpha_1+\alpha_1,-i\phi_1\alpha_2,-1)}{i\phi_1}\right]\mathrm{d}\phi_1, \end{aligned} \quad \textbf{(A.5)}$$

where we have used the expression of $f_1(0;T,i\phi_1)$ in the last equality.

In the similar way, we can obtain the expression of $\Pi_2(0;T)$ by defining a new measure Q_2,

$$Q_2(A) := \frac{\mathbb{E}^{Q}[I(A)S_2^{\alpha_2}(T)I(\tau>T)]}{\mathbb{E}^{Q}[S_2^{\alpha_2}(T)I(\tau>T)]},$$

for any events $A\in\mathscr{F}_T$. Under Q_2, one gets that

$$\begin{aligned} f_2(0;T,i\phi_1) &:= \mathbb{E}^{Q_2}[e^{i\phi_1(\alpha_1 x_1(T)-\alpha_2 x_2(T))}], \\ &= \frac{\mathbb{E}^{Q}[e^{i\phi_1(\alpha_1 x_1(T)-\alpha_2 x_2(T))}S_2^{\alpha_2}(T)I(\tau>T)]}{\mathbb{E}^{Q}[S_2^{\alpha_2}(T)I(\tau>T)]}, \\ &= \frac{f(0;T,i\phi_1\alpha_1,-i\phi_1\alpha_2+\alpha_2,-1)}{f(0;T,0,\alpha_2,-1)}. \end{aligned}$$

Therefore, it holds that

$$\begin{aligned} \Pi_2(0;T) &= Q_2\left(\frac{S_1^{\alpha_1}(T)}{S_2^{\alpha_2}(T)}\geqslant\frac{\mu_2}{\mu_1}\right)\mathbb{E}^{Q}[S_2^{\alpha_2}(T)I(\tau>T)] \\ &= \frac{1}{2}f(0;T,0,\alpha_2,-1)+\frac{1}{\pi}\int_0^\infty \mathrm{Re} \\ &\quad\times\left[\frac{e^{-i\phi_1\ln((\mu_2)/(\mu_1))}f(0;T,i\phi_1\alpha_1,-i\phi_1\alpha_2+\alpha_2,-1)}{i\phi_1}\right]\mathrm{d}\phi_1. \end{aligned} \quad \textbf{(A.6)}$$

As for I_2, we have that

$$I_2 = \mu_1 \Pi_3(0;T) - \mu_2 \Pi_4(0;T),$$

where

$$\Pi_3(0;T) = \frac{1}{2} f(0;T,\alpha_1,0,0) + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi_1 \ln((\mu_2)/(\mu_1))} f(0;T, i\phi_1\alpha_1 + \alpha_1, -i\phi_1\alpha_2, 0)}{i\phi_1}\right] \mathrm{d}\phi_1, \tag{A.7}$$

and

$$\Pi_4(0;T) = \frac{1}{2} f(0;T,0,\alpha_2,0) + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi_1 \ln((\mu_2)/(\mu_1))} f(0;T, i\phi_1\alpha_1, -i\phi_1\alpha_2 + \alpha_2, 0)}{i\phi_1}\right] \mathrm{d}\phi_1. \tag{A.8}$$

Therefore, the price of power exchange options is given by

$$C^* = (1-\alpha)e^{-rT}(\mu_1\Pi_1(0;T) - \mu_2\Pi_2(0;T)) + \alpha e^{-rT}(\mu_1\Pi_3(0;T) - \mu_2\Pi_4(0;T)),$$

with $\Pi_i(0;T),\ i = 1,2,3,4$ defined in Eqs. (A.5)–(A.8). ■