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R&D SPILLOVERS IN AN ENDOGENOUS GROWTH MODEL WITH PHYSICAL CAPITAL, HUMAN CAPITAL, AND VARIETIES

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There is a family of models with physical and human capital and R&D for which convergence properties have been discussed [Lutz G. Arnold, *European Economic Review* 44, 1599–1605 (2000); Manuel Gómez, *Studies in Nonlinear Dynamics and Econometrics* 9(1), Article 5 (2005)]. However, spillovers in R&D have been ignored in this context. We introduce spillovers in this model and derive the steady-state and stability properties. This new feature implies that the model is characterized by a system of four differential equations. A unique balanced growth path, along with a two-dimensional stable manifold, is obtained under simple and reasonable conditions. Transition is oscillatory toward the steady state for plausible values of parameters. We discovered that these features are due to the presence of the R&D spillovers externality in the decentralized equilibrium.

Keywords: Convergence, Stability, Spillovers, R&D

1. INTRODUCTION

Arnold (1998, 2000a) introduced a model with physical capital, human capital, and R&D and studied its convergence properties, without considering spillovers in the R&D technology. Funke and Strulik (2000) integrated three models as different stages of economic development and presented a model with physical capital, human capital, and R&D similar to that in Arnold (1998) as a developed country stage. Gómez (2005) showed that the convergence features of this model change dramatically when the predetermination of r is accounted for and fully characterized the model convergence properties and derived stability conditions. In particular, the model converges through a two-dimensional stable manifold with oscillatory dynamics. Before these contributions, human capital–based growth models and

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R&D-based growth models were studied separately. Benhabib and Perli (1994) showed that an endogenous-growth model with externalities in human-capital accumulation yields indeterminacy of equilibria. Ladrón-de-Guevara et al. (1997) studied the equilibrium dynamics of two extensions of the Uzawa–Lucas framework, in which they discovered multiple equilibrium in a model with leisure, but without externalities. Arnold (2000b, 2000c) demonstrated that the steady state of the Romer (1990) model of R&D is globally saddlepoint stable. Eicher and Turnovsky (2001) showed that a two-sector R&D-based nonscale growth model is represented by a two-dimensional stable saddlepath with oscillatory transitional dynamics. Arnold (2006) studied the stability properties of the Jones (1995) model and concluded that there was a unique balanced growth path on a two-dimensional stable manifold that showed monotonic transition for a broad range of parameters. These last two articles demonstrated that the system of differential equations to be analyzed in nonscale R&D growth models is of order four.

We add spillovers in R&D to the Arnold (2000a) model and derive the equilibrium and stability properties of the decentralized equilibrium of this model. It is worth noting that the existence and high magnitude of spillovers have been empirically proved by recent literature [e.g., Grilliches (1992); Porter and Stern (2000)]. Engelbrecht (1997) and Barrio-Castro (2002) concluded that there were statistically significant R&D spillovers in empirical specifications that included human capital, but models that jointly consider human–capital accumulation and R&D so far have not taken them into account. Additionally, we numerically solve our model for the transition path to derive the steady state and its stability properties.

The consideration of spillovers in the model provides a unique balanced growth path. We also compare the decentralized equilibrium transition path with the optimal one. In doing this we demonstrate that the spillovers' externality is the crucial feature driving the new properties described above. The system that describes the optimal growth path is simpler than the system that describes the decentralized equilibrium. However, it is also a stable equilibrium for reasonable values for the parameters.

In comparison with the Arnold (1998, 2000a) model, whose stability properties were studied in Gómez (2005), the consideration of spillovers increases the order of the differential equations system to four (as in the nonscale R&D growth models). Because of this, this article also constitutes an example of the analysis of high-dimensional systems of ODEs in economics and of the possible technical tools to use. In terms of results, it maintains the oscillatory convergence for typical calibrations. It is also shown that this model allows a lower and more reasonable value for the markup to fulfill the stability conditions in comparison to what is found in Gómez (2005).

In comparison with Jones's model, whose stability properties of the decentralized equilibrium have been studied in Arnold (2006), the introduction of the accumulation of human capital maintains the unique trajectory to the steady state.

In contrast to what occurred in the Jones model, we show that this model presents oscillatory convergence for typical and broad parameters' values.

In Section 2 we present the model. In Section 3 we present the balanced growth equilibrium and the stability analysis of the steady state. In Section 4 we calibrate the model, give some examples, and solve the model numerically. At the end of this section, we discuss the results and compare them with those from the transition path for the social planner equilibrium. In Section 5 we conclude.

2. THE MODEL

This section rebuilds the Arnold (1998, 2000a) model by accounting for the existence of spillovers in R&D.

2.1. Setup of the Model

Consider a closed economy inhabited by a constant population, normalized to one, of identical infinitely lived households that maximize the intertemporal utility function $\int_0^\infty \frac{C^{1-\theta}}{1-\theta} e^{-\rho t} dt$, $\rho > 0$, $\theta > 0$, where C denotes consumption, ρ is the time-discount rate, and θ is the relative–risk aversion coefficient, subject to the budget constraint and the knowledge accumulation technology. Human capital, H, can be devoted to production (H_Y), education (H_H), and R&D (H_n) and is calculated according to $\dot{H} = \xi H_H$. The accumulation of human capital is a nonmarket activity. R&D technology is the same as in Jones (1995), with no duplication effects in human capital, given by $\dot{n} = \epsilon H_n n^\phi$, where $0 < \phi < 1$ is the parameter that governs spillovers and n is the number of available varieties. $\phi > 0$ implies that to some extent the development of new varieties depends on the stock of previous available varieties: the "stand on the shoulders" effect. With $\phi = 0$, this would be the Arnold model described in Gómez (2005). The budget constraint faced by the household is $\dot{W} = rW + w(H - H_H) - C$, where r is the return per unit of aggregate wealth, W, and w the wage per unit of employed human capital, $H - H_H$. Let $g_z = \frac{\dot{z}}{z}$ denote the growth rate of any variable z. The first-order conditions for maximization of utility, using human capital accumulation and household restrictions, give

$$g_C = (r - \rho)/\theta, \tag{1}$$

$$g_w = r - \xi. \tag{2}$$

A single homogeneous final good Y is produced with Cobb–Douglas technology $Y = K^\beta D^\eta H_Y^{1-\beta-\eta}$, with $\beta > 0$, $\eta > 0$, and $\beta + \eta < 1$. K is physical capital and D is an index of differentiated goods given by $D = [\int_0^n x_i^\alpha di]^{1/\alpha}$, with $0 < \alpha < 1$, where x_i is the amount used for each one of the n intermediate goods and α governs the substitutability between varieties. The market for the final good is perfectly competitive and its price is normalized to one. Profit maximization gives the inverse factor demands

$$r = \frac{\beta Y}{K}, \tag{3}$$

$$w = \frac{(1-\beta-\eta)Y}{H_Y} \tag{4}$$

and the inverse demand for intermediate goods

$$P(i) = \frac{\eta Y}{n x^{\alpha}} x(i)^{\alpha-1}. \tag{5}$$

Each firm in the differentiated-goods sector owns a patent for selling its variety x_i. Researchers are granted infinitely lived patents (v). Free entry into R&D is assumed, so that $w/\epsilon = v n^{\phi}$ when innovations occur. Finally, no-arbitrage requires that the capital gain (from the patent) plus profits is equal to investing resources in the riskless asset $\dot{v} + \pi = rv \Leftrightarrow \dot{v}/v = r - \pi/v$. Producers act under monopolistic competition and maximization of operating profits $\pi_i = (p_{x_i} - 1)x_i$ gives $p_{x_i} = 1/\alpha$ and $\pi = (1-\alpha)\eta Y/n$.

Insertion of the differenced-goods equation into the resource constraint $\dot{K} = Y - \int_0^n x_i di - C$ simplifies it to

$$\dot{K} = (1-\alpha\eta)Y - C, \tag{6}$$

and subsquently the final-good production function may also be simplified to

$$Y^{1-\eta} = (\alpha\eta)^{\eta} K^{\beta} n^{\eta\frac{1-\alpha}{\alpha}} (u_1 H)^{1-\beta-\eta}, \tag{7}$$

where $u_1 = H_Y/H$ is the proportion of human capital employed in the final-good production. Similarly, we also denote $u_2 = H_n/H$ as the share of human capital allocated to research and $u_3 = H_H/H$ as the share of human capital allocated to human-capital accumulation.

2.2. The Dynamics of the Economy

The economy is characterized by the presence of physical-capital accumulation ($\dot{K} > 0$), human-capital accumulation ($\dot{H} > 0$), and R&D ($\dot{n} > 0$). We now derive the system that describes the dynamics of the economy. From (1) and (6) and then using (3), we obtain

$$g_{\chi} = \left(\frac{1}{\theta} - \frac{1-\alpha\eta}{\beta}\right) r + \chi - \frac{\rho}{\theta}, \tag{8}$$

where $\chi = C/K$. Departing from the growth rates versions of (7) and using (3), (4) and then replacing g_w by (2), we obtain

$$g_r = -\frac{1-\beta-\eta}{\beta}(r-\xi) + \frac{\eta}{\beta}\frac{1-\alpha}{\alpha} g_n \tag{9}$$

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Using (2), (4), and the free-entry condition and inserting the expression for profits into the nonarbitrage condition, we obtain

$$u_1 = \frac{(1-\beta-\eta)(\xi+\phi g_n)}{\epsilon(1-\alpha)\eta\psi}, \tag{10}$$

where the term ϕg_n measures the R&D spillovers externality. Using this last equation, the human capital accumulation function, and the definition $\psi = H/n^{1-\phi}$ (noting that $u_1+u_2+u_3=1$), we obtain the growth rate of ψ:

$$g_\psi = \xi\left\{1-\left[\frac{(1-\beta-\eta)(\xi+\phi g_n)}{(1-\alpha)\eta}+g_n\right]\frac{1}{\epsilon\psi}\right\}-(1-\phi)g_n. \tag{11}$$

We note that g_n is no longer given as a function of χ and r, which would be directly substituted into previous equations, as in previous contributions [e.g., Gómez (2005, p. 5)]. From (10), $g_{u_1} = (1-\phi)g_n - g_H + \phi\dot{g}_n/(\xi+\phi g_n)$. Then we note from (4) that $g_{u_1} = g_Y - g_w - g_H$. If $\phi = 0$ then $g_{u_1} = g_n - g_H$, as in Arnold (2000a), and one could write $g_n = g_r + \frac{1-\alpha\eta}{\beta}r - \chi - (r-\xi)$, which would be substituted into (9) and (11). This would mean that the locus $\dot{g_n} = 0$ was independent of other variables in the model, and Arnold (2000) could study convergence using two separate phase diagrams. However, with $\phi > 0$, this separability is lost. In this case, using (3), g_K from (6), and (2), we obtain the equation that describes the evolution of g_n:

$$\dot{g}_n = \frac{\xi+\phi g_n}{\phi}\left[g_r + \frac{1-\alpha\eta}{\beta}r - \chi - (r-\xi) - (1-\phi)g_n\right]. \tag{12}$$

The system composed of equations (8), (9), (11), and (12) describes the evolution of the economy. We can further simplify equation (12), replacing g_r from (9):

$$\dot{g}_n = \frac{\xi+\phi g_n}{\phi}\left\{\frac{(1-\alpha)\eta}{\beta}r - \chi + \frac{1-\eta}{\beta}\xi + \left[\frac{\eta}{\beta}\frac{1-\alpha}{\alpha} - (1-\phi)\right]g_n\right\}. \tag{13}$$

This is the differential equation that enters the system, when the model is compared to that in Arnold (2000).

3. BALANCED GROWTH EQUILIBRIUM

In this section, we first derive equations that describe the steady state and then study the convergence properties around the steady state.

3.1. The Steady State: Existence

In this section, we present the equations that describe the steady state of the model and demonstrate its existence.

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THEOREM 1. *Let $\xi > \rho$ and $\theta > 1$. There is one unique positive steady state of the model given by (r^*, χ^*, ψ^*, g_n^*), as follows:*

$$r^* = \frac{\theta\left[\frac{(1-\beta-\eta)}{\eta}\frac{\alpha}{1-\alpha}(1-\phi)+1\right]\xi-\rho}{\theta\left[\frac{(1-\beta-\eta)}{\eta}\frac{\alpha}{1-\alpha}(1-\phi)\right]+(\theta-1)}, \tag{14}$$

$$\chi^* = \left(\frac{1-\alpha\eta}{\beta}-\frac{1}{\theta}\right)r^*+\frac{\rho}{\theta}, \tag{15}$$

$$\psi^* = \frac{\xi\left[g_n^*(1-\alpha)\eta+(1-\beta-\eta)(\xi+\phi g_n^*)\right]}{(1-\alpha)\eta\epsilon[\xi-(1-\phi)g_n^*]}, \tag{16}$$

$$g_n^* = \frac{r^*(1-\theta)+\theta\xi-\rho}{\theta(1-\phi)}. \tag{17}$$

Proof. The shares of human capital to different sectors must be constant for an interior steady-state solution. In particular, the fact that the share in human-capital accumulation u_3^* is constant implies by the human capital–accumulation function that g_H^* is constant. With u_1^* and u_3^* constant, g_n^* and ψ^* must be constant, by the R&D function and (10). Thus $g_H^* = (1-\phi)g_n^*$. From constancy of ψ^* and u_1^* we can say that $g_Y^* = g_K^*$. This equality, the growth rates version of equation (7), and the constancy of g_n^*, g_H^*, and u_1^* imply that r^*, g_Y^*, and g_K^* are constant. Thus $\chi^* = (C/K)^*$ is constant [to see this divide (6) by K]. We now derive necessary and sufficient conditions for positivity. For $r^* > 0$ we reach $(A_1+1)\theta\xi > \rho$ and $\theta A_1 + (\theta-1) > 0$, where $A_1 = \frac{1-\beta-\eta}{\eta}\frac{\alpha}{1-\alpha}(1-\phi)$. For $\chi^* > 0$, we have $(A_1+1)\theta\xi > A_2\rho$ [where $A_2 = \frac{\theta(\frac{1-\alpha\eta}{\beta}-1)-(\theta(A_1+1)-1)}{(\theta\frac{1-\alpha\eta}{\beta}-1)}$], which (if $\theta \geq 1$) is always verified for $r^* > 0$.[1] For $\psi^* > 0$ we reach $(\theta-1)(A_1+1)\theta\xi + A_1\theta\rho > 0$, using that $r^* > 0$. Finally $g_n^* > 0$ implies $\xi > \rho$ if $r^* > 0$. This condition together with $\theta \geq 1$ simultaneously imply $r^* > 0$, $\psi^* > 0$ and $\chi^* > 0$. These two simple conditions are sufficient for a feasible steady state.

Positiveness of ψ^* is directly implied by the transversality condition on H. The transversality condition on human capital may be written as

$$\lim_{t\to\infty} e^{-\rho t}\lambda_2(t)H(t) = 0 \tag{18}$$

(λ_2 is the co-state of H), which converts to $(-\rho+\dot{\lambda}_2/\lambda_2+g_H) < 0$. As $\dot{\lambda}_2/\lambda_2 = \rho-\xi$ and $g_H^* = (1-\phi)g_n^*$, the transversality condition is equivalent to $\xi - (1-\phi)g_n^* > 0$, which is equivalent to $(\theta-1)(A_1+1)\theta\xi + A_1\theta\rho > 0$, stated above. ■

3.2. Stability

We will now analyze the dynamics of the model in the neighborhood of the steady state.

The analysis of the linearized system around the steady state will establish that for most reasonable values, the system has two eigenvalues with negative real parts. We also show that initial state conditions $K(0)$, $H(0)$, and $n(0)$ are sufficient to determine the initial point in the two-dimensional stable manifold; thus the balanced growth path is uniquely determined. However, as in Eicher and Turnovsky (2001, p. 95), because of the complexity of the model, we cannot rule out the case of instability.

Linearizing the system (8), (9), (11), and (12) around its steady state $(r^*, \chi^*, \psi^*, g_n^*)$ gives the following fourth-order system:

$$\begin{pmatrix} \dot{r} \\ \dot{\chi} \\ \dot{\psi} \\ \dot{g}_n \end{pmatrix} = \begin{pmatrix} -\frac{1-\beta-\eta}{\beta} r^* & 0 & 0 & \frac{\eta}{\beta}\frac{1-\alpha}{\alpha} r^* \\ \left(\frac{1}{\theta} - \frac{1-\alpha\eta}{\beta}\right)\chi^* & \chi^* & 0 & 0 \\ 0 & 0 & \xi - (1-\phi) g_n^* & -B_1 - (1-\phi)\psi^* \\ \frac{\xi+\phi g_n^*}{\phi}\frac{(1-\alpha)\eta}{\beta} & -\frac{\xi+\phi g_n^*}{\phi} & 0 & B_2 g_n^* - B_3 \frac{\xi}{\phi g_n^*} \end{pmatrix} \times \begin{pmatrix} r - r^* \\ \chi - \chi^* \\ \psi - \psi^* \\ g_n - g_n^* \end{pmatrix}, \tag{19}$$

where

$$B_1 = \xi\phi \frac{(1-\beta-\eta)}{(1-\alpha)\eta}\frac{1}{\epsilon} + \frac{\xi}{\epsilon};$$

$$B_2 = \left[\frac{\eta}{\beta}\frac{1-\alpha}{\alpha} - (1-\phi)\right];$$

$$B_3 = \left[\frac{(1-\alpha)\eta}{\beta} r^* - \chi^* + \frac{1-\eta}{\beta}\xi\right],$$

or $\dot{\mathbf{X}} = \mathbf{J}(\mathbf{X} - \mathbf{X}^*)$, where $\mathbf{J}$ is the Jacobian in (19). To demonstrate the conditions under which the system is completely stable and unstable we state the following theorem.

THEOREM 2. *If a feasible steady state exists, there are two possible trajectory solutions: there exists a unique steady state to which a unique path converges, or the steady state is unstable.*

The proof is found in four lemmas. The first two demonstrate that there are zero or two stable roots; i.e., the stable manifold is two-dimensional or it is unstable. The third presents a sufficient condition under which the solution is the first: two stable roots. The fourth demonstrates that, when there are two stable roots, the balanced growth path is uniquely determined by the state variables in the model.

LEMMA 1. *There are an even number of stable roots, zero (instability), two (stability), or four (indeterminacy).*

Proof. This is implied by the fact that, for a feasible steady state, the determinant of (19) is positive. That is,

$$-\left[B_2 g_n^* - B_3\frac{\xi}{\phi g_n^*}\right]\frac{1-\beta-\eta}{\beta} - \frac{\eta}{\beta}\frac{1-\alpha}{\alpha}\left[\frac{1}{\theta} - \frac{1-\eta}{\beta}\right]\frac{\xi+\phi g_n^*}{\phi} > 0. \quad \textbf{(20)}$$

We provide a proof. Using the fact that, by (13), in the steady state $B_3 = -B_2 g_n^*$, and by simplifying terms, the expression (20) turns out to be equal to the denominator of r^*, for a positive steady state ($r^* > 0$, $\chi^* > 0$, $\psi^* > 0$, $g_n^* > 0$):

$$\theta\left[\frac{(1-\beta-\eta)}{\eta}\frac{\alpha}{1-\alpha}(1-\phi)\right] + (\theta - 1) > 0, \quad \textbf{(21)}$$

which implies $\theta > 1$ as a sufficient condition. Thus, the system has an even number of stable roots, zero, two, or four, under the same conditions for a positive steady state. ■

LEMMA 2. *There are not four stable roots.*

Proof. We can rule out indeterminacy, as there is always a positive root, eliminating the possibility of having four roots with negative real parts. The case with four negative roots is excluded, as there is a root e_3:

$$e_3 = \xi - (1-\phi)g_n^*. \quad \textbf{(22)}$$

It was shown above that $\xi - (1-\phi)g_n^* > 0$ by the transversality condition on human capital accumulation (see the Proof of Theorem 1). ■

LEMMA 3. *A sufficient condition to rule out the instability outcome is* $1/\alpha < 1 + (1-\beta-\eta)/\eta$.

Proof. As by Lemmas 1 and 2 we remain with the possibility of zero or two stable roots, we need only to discover a sufficient condition for the existence of one stable root. By Lemmas 1 and 2, this is also a sufficient condition to obtain two stable roots, which guarantees stability. We use the Gershgorin Disc Theorem [e.g., Horn and Johnson (1985)] to determine this sufficient condition. Applying Corollary 1 of the Disc Theorem to matrix J in (19), we can see that for any positive real numbers d_1, d_2, d_3, d_4 the eigenvalues of J $e_1, \ldots, e_4$ are contained in the discs:

$$\left|z - \left(-\frac{1-\beta-\eta}{\beta}r^*\right)\right| \leq \frac{d_4}{d_1}\left|\frac{\eta}{\beta}\frac{1-\alpha}{\alpha}r^*\right|, \quad \textbf{(23)}$$

$$|z - \chi^*| \leq \frac{d_1}{d_2}\left|\left(\frac{1}{\theta} - \frac{1-\alpha\eta}{\beta}\right)\chi^*\right|, \quad \textbf{(24)}$$

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$$\left|z - \left(\xi - (1-\phi)g_n^*\right)\right| \le \frac{d_4}{d_3}\left|-B_1 - (1-\phi)\psi^*\right|, \tag{25}$$

$$\left|z - \left(B_2 g_n^* - B_3 \frac{\xi}{\phi g_n^*}\right)\right| \le \frac{d_1}{d_4}\left|\frac{\xi + \phi g_n^*}{\phi}\frac{(1-\alpha)\eta}{\beta}\right| + \frac{d_2}{d_4}\left|-\frac{\xi + \phi g_n^*}{\phi}\right|, \tag{26}$$

where $|\cdot|$ denotes the absolute value. Let $d_1 = d_2 = d_4 = 1$.[2] To obtain a sufficient condition for stability, we need only to look at the first disc with center in $-\frac{1-\beta-\eta}{\beta}r^*$ and radius $\frac{\eta}{\beta}\frac{1-\alpha}{\alpha}r^*$ and prove that it is all contained in the left half of the complex plane, so that there exists a negative eigenvalue within that circle. Signing the terms in (23), we obtain the sufficient condition,

$$-\frac{1-\beta-\eta}{\beta} + \frac{\eta}{\beta}\frac{1-\alpha}{\alpha} < 0, \tag{27}$$

which is easily converted into the meaningful expression in the lemmas.[3] ■

The conditions stated in Theorem 2 and in its lemmas are verified for a broad range of parameters, and particularly for sufficiently low markups, as indicated by (27).

LEMMA 4. *If the system of four differential equations has two stable roots, the state variables $K(0)$, $H(0)$, and $n(0)$ uniquely determine the starting point in the stable manifold.*

Proof. The initial values $\mathbf{X}(0)$ satisfy

$$\mathbf{X}(0) - \mathbf{X}^* = \sum_{i=1}^{2} \Omega_i \mathbf{b}_i, \tag{28}$$

where $\mathbf{b}_i$ are the eigenvectors corresponding to the two stable eigenvalues, and Ω_i are constants that can be determined. The system (28) comprises four equations into five unknowns $(C/K(0), r(0), g_n(0), \Omega_1(0), \Omega_2(0))$. Equations (3), (7), and (10) may be rewritten as

$$r(0) = \frac{(\alpha\eta)^{\frac{\eta}{1-\eta}} \left\{\frac{(1-\beta-\eta)[\xi + \phi g_n(0)]}{\epsilon(1-\alpha)\eta}\right\}^{\frac{1-\beta-\eta}{1-\eta}} n(0)^{\frac{(1-\phi)(1-\beta-\eta)+\eta\left(\frac{1-\alpha}{\alpha}\right)}{1-\eta}}}{K(0)^{\frac{1-\beta-\eta}{1-\eta}}}. \tag{29}$$

With this additional equation, we reach a system with five equations into five unknowns. This means that the stable variables select a specific starting point in the transition path. ■

This theorem establishes that there is a unique transition path to the steady state, under some reasonable conditions. This is a result similar to that in Eicher and Turnovsky (2001) and Arnold (2006), but in a model with human capital accumulation.

4. CALIBRATION AND ADJUSTMENT PATHS

In this section, we present calibration exercises and compute an adjustment path for a set of typical values of parameters. As in Eicher and Turnovsky (2001) and in Arnold (2006), we use calibration exercises to show that with the usual calibration parameters, we reach two complex conjugate stable eigenvalues and two unstable roots, showing that with high probability the model transition path would be uniquely determined. For the baseline calibration in Gómez (2005, p. 12), the sufficient condition for stability (27) is obtained for markups lower than 1.75 ($\alpha > 0.57$), thus for very reasonable values.

Remark 1. Experimentation with numerical values shows that there are two complex conjugate stable eigenvalues for a broad range of parameters (see Example 1), thus predicting oscillatory convergence to the steady state. However, it is possible to construct counterexamples (see Examples 2 and 3).

Example 1

For oscillatory transition through a stable path: with the benchmark calibration in Gómez (2005)—$\beta = 0.36$; $\eta = 0.36$; $\alpha = 0.4$; $\xi = 0.05$; $\rho = 0.023$; $\theta = 2$; $\delta = 0.1$—and $\phi = 0.4$, eigenvalues are 0.0448, 0.2487, $-0.0218 + 0.0740i$, and $-0.0218 - 0.0740i$. For the benchmark calibration in Funke and Strulik (2000)—$\beta = 0.36$; $\eta = 0.36$; $\alpha = 0.54$; $\xi = 0.05$; $\rho = 0.023$; $\theta = 2$; $\delta = 0.1$—and $\phi = 0.4$, eigenvalues are 0.049, 0.1753, $-0.0345 + 0.0640i$, and $-0.0345 - 0.0640i$. For the benchmark calibration in Gómez (2005) and $\phi = 0.8$, eigenvalues are 0.0477, 0.2216, $-0.0171 + 0.0542i$, and $-0.0171 - 0.0542i$.

Example 2

For monotonic transition through a stable path: with the first calibration in Example 1 but with $\alpha = 0.94$, eigenvalues are 0.0374, 0.0833, -0.0651, and -0.0560.

Example 3

For an unstable steady state: with the first calibration in Example 1 but with $\alpha = 0.20$, eigenvalues are 0.0474, 0.5433, $0.0006 + 0.0824$, and $0.0006 - 0.0824i$.

As a sensitivity analysis exercise, we have considered different values for substitutability between varieties (which also governs the markup) and for spillovers, maintaining other parameters as in the examples. Although variations in the value of spillovers maintain the stability and the oscillatory pattern (holding other parameters constant), extreme markup values make the difference regarding determinacy and monotonticity. For very high substitutability (low markup)—$\alpha \geq 0.94$—the balanced-growth path comes out to be determinate, but without the transition oscillatory pattern (two negative real roots). For very low substitutability—$\alpha \leq 0.2$ (high markup)—the steady state turns out to be unstable (the four roots come out to be positive). No drastic rise in the substitutability parameter α from those considered in the examples implies that there is a positive threshold for spillovers below which monotonics transitional dynamics arise (i.e., the stable roots are real). For instance, for $\alpha = 0.54$ the threshold is 0.02 and for $\alpha = 0.8$ the threshold is 0.22.

In the next few lines we describe the adjustment path of an economy calibrated with the first set of values presented in Example 1.[4] To integrate the fourth-order system of differential equations we use the method of backward integration described by Brunner and Strulik (2002). Figure 1 shows adjustment paths of the growth rate of physical capital, the interest rate, the shares of human capital allocated to each of the sectors, the human-capital and varieties growth rates, and the human capital–varieties ratio.[5]

The figures show oscillatory adjustment until the steady state is reached. Intuitively, the presence of spillovers increases the investment in R&D and the growth rates of per capita output, when compared to Gómez's (2005) results. Overall, there is an overshooting effect at the beginning of the transition path, compared with several oscillations as shown in Gómez (2005). Moreover, the presence of or increase in the spillovers parameter increases the number of years the economy takes to reach the steady state.

4.1. Transition in the Social Planner Solution

The equations that govern the transition path for the social planner solution are presented in the Appendix. Here, we compare the results with the ones we have presented for the decentralized equilibrium solution. For all parameters (except α), as in the first case of Example 1, with $\alpha \leq 0.48$ (with all positive real eigenvalues) and $\alpha \geq 0.63$ (with 2 positive real roots), the steady state turns out to be unstable. For $\alpha = 0.54$, the threshold of spillovers below which the system is stable is 0.5. The range of instability is then greater within the social planner solution than it was under the decentralized-equilibrium solution. For all ranges of parameters the two possible solutions are stability with oscillatory convergence and instability. When the equilibrium path is stable, it exhibits nonmonotonic behavior throughout the transition, growing through damped oscillations, much more pronounced than the oscillations presented in Figure 1.[6] Thus the externality causes softer oscillations in the decentralized equilibrium, as well as providing a wider range of parameters for which the equilibrium is stable.

4.2. Discussion

The model presented and studied here introduces spillovers in the R&D process into a model that already included the most studied sources of growth: physical and human capital and the increasing number of varieties (without spillovers). Thus, the importance of this model and its features depends crucially on the evidence of the existence of spillovers in the R&D process. In Romer's seminal model of endogenous technological change [Romer (1990)], the more resources the economy allocated to R&D, the more it would grow (the so-called scale

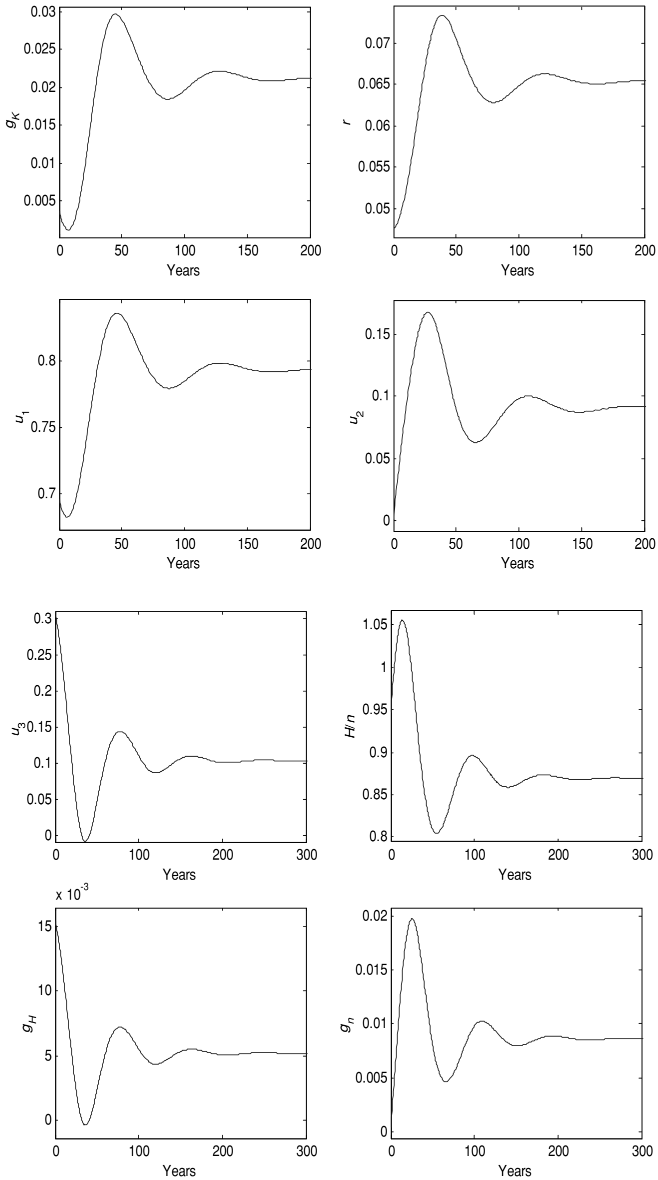

FIGURE 1. Transition paths for representative variables.

effect). The evidence on the simultaneity between increasing resources in R&D and stable TFP growth led Jones (1995) to model R&D with decreasing returns to the stock of knowledge. This assumption of $0 < \phi < 1$ is now included in a stream of literature called nonscale R&D-growth models. Some studies have supported the existence of high spillovers, calculating a rate of social return to R&D higher than the private one. Jones and Williams (2000), for instance, conclude that optimal R&D expenditure is at least four times higher than decentralized spending. A fruitful literature has regressed TFP growth on R&D stock and concluded that significant effects occur [Coe and Helpman (1995)]. Of particular interest for evaluating the empirical plausibility of our model with human capital accumulation are the extensions to the Coe and Helpman regressions provided by Engelbrecht (1997) and Barrio-Castro et al. (2002). Apart from a superior role for human capital reported in this last article, both conclude that a significant relationship exists between TFP and the stock of knowledge, even in the presence of human capital. Thus, the presence of spillovers in R&D not only is at the center of recent endogenous growth theory, but also has been empirically supported.

This model also predicts features that were not addressed by the earlier ones. The Arnold/Gómez model predicts oscillatory convergence, but the condition for stability implies implausibly high values for the markups. Norrbin (1993) presented markups for sectors in the United States, all below 1.7, whereas the condition for stability in Gómez (2005) implies a markup higher than 2. It is worth noting that in our model, for reasonable values of markups [between 1.2 and 1.4 in Norrbin (1993)], the uniquely determined oscillatory pattern through a two-dimensional stable manifold arises. For lower markup, the model continues to be stable but presents a monotonous trajectory to the steady state. Whether the evidence calls for monotonous or oscillatory convergence is an issue under discussion. Historical evidence of the stages of development of the most developed countries seems to suggest monotonic convergence [Maddison (2001, p. 74)]. Nevertheless, if one thinks of the experience of other less developed economies in the postwar period, one can argue that modern economies grow through cycles [Fiaschi and Lavezzi (2003, 2007)]. Our Figure 1 (and all paths resulting from calibration values in Example 1) resembles an initial fast-growing take-off and then a slowdown until the steady state, similar to what was suggested by Fiaschi and Lavezzi. The Eicher and Turnovsky (2001) model also predicts an overshooting of final values during the transition. The Jones/Arnold model also predicts monotonic or oscillatory convergence (but with a prevalence of monotonic convergence for most sets of parameters) and can rule out the case of an unstable steady state. This is not the case in Eicher and Turnovsky (2001), where instability cannot be ruled out. Their article also presents a sufficient condition for stability. For high markups in our model, the steady state is unstable. The dependence of our model's properties on the markup is interesting, given Gali's (1994) contribution on the crucial effects of markups in growth models. There are interesting policy implications from the sufficient conditions stated in Lemma 3: economies with relatively high shares of physical capital relative to the differentiated goods–sector share (high β/η), with relatively high markup (high $1/\alpha$) and high spillovers (high ϕ), tend to be unstable, thus showing diverging paths from the initial endowments.

5. CONCLUSION

The consideration of spillovers in a model with both R&D processes and human-capital accumulation has not been done previously. Nevertheless, two relevant empirical contributions [Engelbrecht (1997) and Barrio-Castro (2002)] have proved that in empirical specifications that include human capital, R&D spillovers are still high in value and present high statistical significance.

We introduce R&D spillovers into the endogenous growth model with physical capital, human capital, and varieties due to Arnold (1998, 2000a). We study the steady state and the convergence properties of the model. Furthermore, we solve it numerically to obtain a transition path to the steady state.

The dynamics of the model in the decentralized equilibrium are characterized by the behavior of a system of four differential equations. This article provides one more example of the analysis of a system of four ODEs in economics, and of the possible tools to implement.

The system can be either stable or unstable. Under mild conditions it proves to be characterized by a two-dimensional stable manifold and converges through a uniquely determined balanced growth path to the steady state. Thus, when compared to the model without spillovers, their presence increases the order of differential equations system to four, also providing a two-dimensional stable manifold. This model maintains the oscillatory convergence and allows for more reasonable markup values to fulfill stability conditions. We compare the decentralized equilibrium results with the social-planner ones and conclude that the internalization of the spillovers externality done by the social planner simplifies the dynamics of the model but also increases the range of parameters that lead to instability results. Thus, it is exactly the spillovers' externalities that imply our interesting results in the decentralized-equilibrium transitional dynamics.

When compared to a model without human capital accumulation [Jones (1995)], the stability properties also indicate a two-dimensional stable manifold. Two differences arise. First, we cannot rule out instability as can be done in Jones's model. Second, for most reasonable parameter sets, the transition to the steady state in the model presented here is oscillatory and not monotonic, as it is in Jones's model.

We found a crucial effect of the markup distortion in the convergence properties of the model. For high markups the steady state is unstable. For intermediate values, the steady state is stable and the transition oscillatory. For low markups, the steady state is stable and the transition monotonic.

NOTES

1. By the initial assumptions on parameters, $1 - \alpha\eta > \beta$ (see equation (7)]. For $\theta \geq 1$, $A_2 < 1$.
2. The eigenvalue e_3 stated in Lemma 2 can be recovered using the second disc (25) and setting $d_3 \rightarrow \infty$.

3. Other possible sufficient conditions can be achieved. Another intuitive condition is retrieved from disc (26) setting $d_1 = d_4 = 1$, $d_2 = 0$ and $d_3 \to \infty$:

$$\eta \frac{1-\alpha}{\beta}\left(1+\frac{1}{\alpha}\right) - (1-\phi) < 0.$$

As this is a more restrictive condition than (27), we have selected this one for presentation in the main text.

4. Results for adjustment paths that result from the different sets of parameter values presented above are available from the author upon request.

5. These figures can be compared with Figure 4 in Gómez (2005). Simulation made use of Matlab.

6. Figures can be supplied upon request. For space reasons they are not presented.

7. A complete analysis is available upon request. However, for space reasons, we are not presenting it in the article.

8. The reader can check that with $\phi = 0$ this converts into the matrix in equation (4.1) in Gómez (2005).

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APPENDIX: THE SOCIAL PLANNER'S SOLUTION

The formulation and the equilibrium solution of the social-planner problem for a similar model have been presented by Reis and Sequeira (2007), so we refer to them. To convert their model to ours just set $\delta = 0$ (in their notation). We will concentrate on presenting results for dynamics of the social planner equilibrium path and compare them with the results for the decentralized equilibrium presented in this article.[7] The externality that is internalized by the social planner is measured by the term ϕg_n in equation (10), which implies that the social planner allocates more human capital to R&D and less to the final good than does the market. It is exactly this term that determines in the decentralized equilibrium that the system dynamics includes four differential equations, one for g_n. As in the social planner solution this term does not exist, the dynamics will be characterized by three equations, resembling in part the analysis in Gómez (2005):

$$g_\chi = \left(\frac{1}{\theta} - \frac{1-\alpha\eta}{\beta}\right) M_{pK} + \chi - \frac{\rho}{\theta}, \tag{A.1}$$

$$g_{M_{pK}} = -\frac{1-\beta-\eta}{\beta}(M_{pK} - \xi) + \frac{\eta}{\beta}\frac{1-\alpha}{\alpha} g_n, \tag{A.2}$$

$$g_\psi = \xi \left\{1 - \left[\frac{(1-\beta-\eta)\xi}{(1-\alpha)\eta} + g_n\right]\frac{1}{\epsilon\psi}\right\} - (1-\phi) g_n, \tag{A.3}$$

where M_{pK} is the marginal productivity of physical capital (which is equal to r as there are no distortions in the final good market).

The Jacobian for this system is

$$\mathbf{J}=\begin{pmatrix} \left[-\frac{1-\beta-\eta}{\beta}+\left(\frac{\eta}{\beta}\right)^2\frac{(1-\alpha)^2}{\alpha}B_4\right]M_{pK}^* & -\frac{\eta}{\beta}\frac{1-\alpha}{\alpha}B_4M_{pK}^* & 0 \\ \left(\frac{1}{\theta}-\frac{1-\alpha\eta}{\beta}\right)\chi^* & \chi^* & 0 \\ 0 & 0 & \xi-(1-\phi)g_n^* \end{pmatrix}, \tag{A.4}$$

in which $B_4=\frac{1}{1-\phi-\frac{\eta}{\beta}\frac{1-\alpha}{\alpha}}$.[8] Using the same argument as in Gómez (2005), the necessary and sufficient conditions for stability (two eigenvalues with negative real parts) that are due to a positive determinant and a negative trace of $\mathbf{J}_2$ are the following:

$$\frac{1-\alpha}{\alpha}\frac{\eta}{1-\beta-\eta}\left(\frac{1}{\theta}-\frac{1-\eta}{\beta}\right)+\frac{\eta}{\beta}\frac{1-\alpha}{\alpha}<1-\phi; \tag{A.5}$$

$$\left[\left(1-\frac{1}{\theta}\right)+\eta\frac{1-\alpha}{\beta}\left(1+\frac{\eta}{\beta}\frac{1-\alpha}{\alpha}B_4\right)\right]M_{pK}^*+\frac{\rho}{\theta}<0. \tag{A.6}$$

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