
The Min Mean-Weight Cycle in a Random Network

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The mean weight of a cycle in an edge-weighted graph is the sum of the cycle's edge weights divided by the cycle's length. We study the minimum mean-weight cycle on the complete graph on n vertices, with random i.i.d. edge weights drawn from an exponential distribution with mean 1. We show that the probability of the min mean weight being at most c/n tends to a limiting function of c which is analytic for $c \leq 1/e$, discontinuous at $c = 1/e$, and equal to 1 for $c > 1/e$. We further show that if the min mean weight is $\leq 1/(en)$, then the length of the relevant cycle is $\Theta_p(1)$ (i.e., it has a limiting probability distribution which does not scale with n), but that if the min mean weight is $> 1/(en)$, then the relevant cycle almost always has mean weight $(1 + o(1))/(en)$ and length at least $(2/\pi^2 - o(1)) \log^2 n \log \log n$.

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1. Introduction

Many combinatorial optimization problems have been studied when the input is a complete (directed or undirected) graph with independent random weights on the edges. This line of work has been active since the mid-1980s for problems such as the minimum spanning tree [11, 14], shortest path [13, 18, 20, 16, 17], travelling salesman path [12], minimum weight perfect matching (the assignment problem) [2, 23, 25], spanners [6], and Steiner tree [5, 3]. In this paper, we study the *minimum mean-weight cycle*.

Given a directed graph with arc weights, the minimum mean-weight cycle problem is that of finding a cycle with minimum mean weight. The mean weight of a cycle is the ratio between its total weight and its number of arcs. The min mean-weight cycle problem, and

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the closely related minimum ratio cycle problem (where each arc has a cost and a transit time, and the mean ratio of a cycle is the total cost divided by the total transit time), have applications in areas ranging from discrete event systems and computer-aided design to graph theory; see Dasdan [8] for a detailed discussion and references. An experimental study of various algorithms for min mean cycle can be found in [15], including experiments on random graphs. An algorithm by Young, Tarjan and Orlin [27] emerges as particularly efficient. Their algorithm is based on the parametric shortest path problem, which is the problem of finding shortest paths in graphs where the edge costs are of the form $w_{i,j} + \lambda$, where each $w_{i,j}$ is constant and λ is a parameter that varies. This problem is well-defined when λ is at least

$$- \min_{\text{cycle } C} \frac{\sum_{ij \in C} w_{i,j}}{|C|},$$

but when λ is below this value there is a negative cycle, so the problem becomes ill-defined. The authors of [27] conjectured that their algorithm is faster on average than in the worst case, by a factor of n ; analysing the structure of the min mean cycle is an intermediate step towards that conjecture.

In this paper, we study the min mean-weight cycle in the complete graph on n vertices, with random i.i.d. edge weights drawn from an exponential distribution with mean 1, so that $\mathbb{P}[w_e > x] = e^{-x}$. We do this for both the directed complete graph, which is relevant to the experiments of Young, Tarjan and Orlin [27] and subsequent experiments, and for the undirected complete graph, so that we can more readily compare our results to earlier work on cycles in the random graph $G_{n,p}$ [19, 10].

The min *max*-weight cycle has been studied by Janson [19] and others [10]. One way to instantiate the random graph $G_{n,p}$ is to start with the undirected complete graph with i.i.d. exponential edge weights, and put each edge in $G_{n,p}$ if its weight is smaller than $\log 1/(1-p)$ (or if we instead use weights that are uniform in $[0, 1]$, the edge is included if its weight is smaller than p). As the parameter p is increased from 0 to 1, the first cycle to appear is the min *max*-weight cycle. Janson [19] gives formulas for when that cycle occurs (*i.e.*, its max weight), and for its length distribution: the probability that the min *max*-weight cycle has max weight less than c/n tends as $n \rightarrow \infty$ to a continuous function of c , which is analytic and increases from 0 to 1 as c increases from 0 to 1, is non-analytic but continuous at $c = 1$, and equals 1 for $c > 1$ (see Figure 1). The limiting length distribution (see Table 1) is completely supported on finite values (*i.e.*, which do not grow with n), but this distribution has a fat tail which gives it an infinite expected value. (For finite n , the expected length is order $n^{1/6}$ [10].)

We find that the min *mean*-weight cycle has a qualitatively different behaviour: the probability that the min mean-weight cycle has mean weight at most c/n tends as $n \rightarrow \infty$ to a function of c which is piecewise analytic, but which is *discontinuous* at $c = 1/e$ (see Figure 1). More precisely, the mean weight of the min mean-weight cycle is with constant probability within an interval $(1/e, 1/e + o(1))/n$, where the $o(1)$ term goes to 0 as $n \rightarrow \infty$. Furthermore, the limiting length distribution of the min mean-weight cycle is not supported on finite values. In other words, the probability that the min mean-weight cycle has length k tends to a positive limiting value p_k , but $\sum_k p_k < 1$ (see Table 1). What

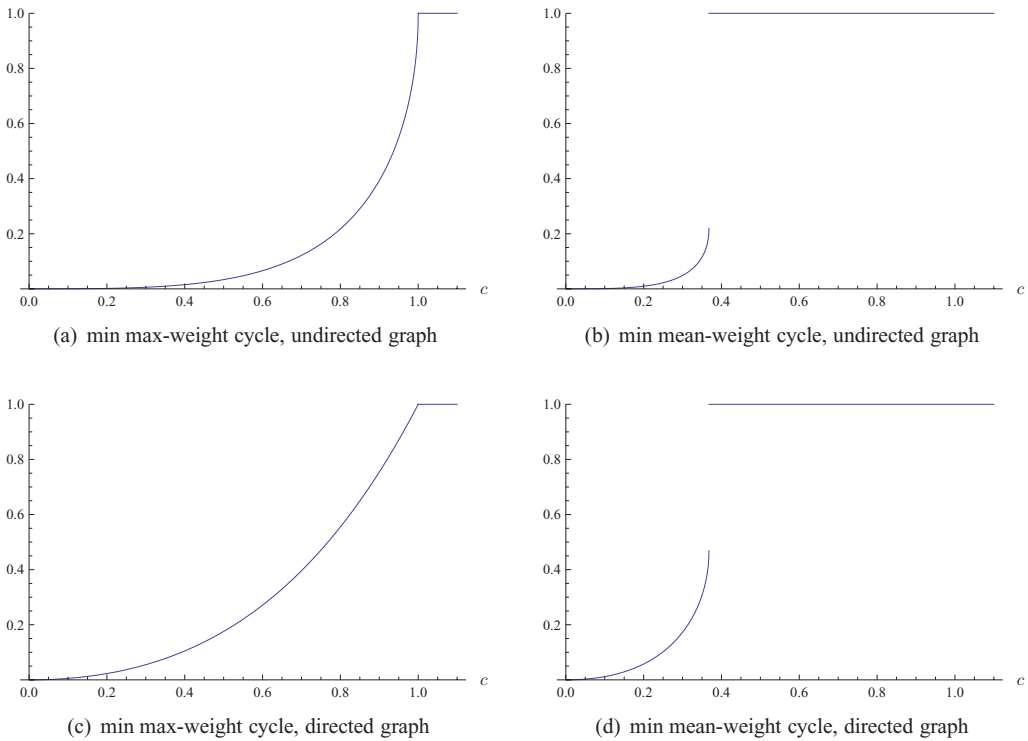


Figure 1. (Colour online) The probability that the min max/mean-weight cycle in the undirected/directed complete graph has max/mean weight less than c/n . (Panel (a) was computed by Janson [19].) For the min max-weight cycle, the function is non-analytic at $c = 1$ but continuous. For the min mean-weight cycle, the function is discontinuous at $c = 1/e$.

this means is that with constant probability the cycle has length order 1, and with constant probability the cycle has a length which is a function of n tending to infinity.

It is natural to ask what this function of n is. The behaviour of the long cycles is complicated, and we do not conjecture a value for the true answer. The best that we could prove is that the length of the min mean-weight cycle is almost always either $O(1)$ or else at least $(2/\pi^2 - o(1)) \log^2 n \log \log n$.

In the related problem of finding the maximum length path whose mean weight is at most c/n , Aldous [1] found that there is a transition point at $c = 1/e$, where for fixed $c < 1/e$ the length is $o(n)$, and for fixed $c > 1/e$ the length is order n . Recently Ding [9] studied the behaviour of this path length when c is at or near $1/e$, and proved that the length exhibits a transition at $c = 1/e + \Theta(1/\log^2 n)$, with unspecified constants. By comparison, we prove that with probability $1 - o(1)$ the min mean-weight cycle has mean weight at most $(1/e + (\pi^2 + o(1))/(2e \log^2 n))/n$, but we do not know if the $O(1/\log^2 n)$ correction term is sharp.

Whether the complete graph is directed or undirected will affect the length distribution and the max/mean weight of the min max/mean-weight cycle, but each of the qualitative behaviours discussed above is unaffected by whether the graph is directed or undirected

Table 1. The limiting length distribution of the min max/mean-weight cycle. The leftmost column is due to Janson [19]. Here $T(c) = \sum_{k=1}^{\infty} k^{k-1} c^k / k!$ is the ‘tree function’. For the min max-weight cycle, the length distribution is supported on finite values, while for the min mean-weight cycle, a constant fraction of the probability mass $(1 - \sum_k p_k)$ drifts off to infinity. The size of the jumps at the discontinuities in Figure 1 is $1 - \sum_k p_k$.

	Undirected graph min max cycle	Directed graph min max cycle	Undirected graph min mean cycle	Directed graph min mean cycle
p_2	—	0.281718	—	0.116616
p_3	0.121608	0.154845	0.035248	0.061750
p_4	0.084915	0.098900	0.022796	0.039132
p_5	0.063827	0.068937	0.016229	0.027417
p_6	0.050329	0.050915	0.012283	0.020485
p_7	0.041047	0.039195	0.009701	0.016005
p_8	0.034331	0.031129	0.007905	0.012923
p_9	0.029280	0.025334	0.006598	0.010701
p_{10}	0.025365	0.021027	0.005613	0.009039
p_{100}	0.000921	0.000264	0.000165	0.000238
p_k	$\frac{1}{2} \int_0^1 c^{k-1} \sqrt{1-c} e^{\frac{c}{2} + \frac{c^2}{4}} dc$	$\int_0^1 c^{k-1} (1-c) e^c dc$	$\frac{k^k}{2k!} \int_0^{\frac{1}{c}} c^{k-\frac{1}{2}} e^{\frac{c}{2} + \frac{c^2}{2}} \frac{dc}{\sqrt{T(c)}}$	$\frac{k^k}{k!} \int_0^{\frac{1}{c}} \frac{c^k e^c}{T(c)} dc$
p_k	$(1+o(1)) \frac{e^{3/4} \sqrt{\pi}}{4k^{3/2}}$	$(1+o(1)) \frac{e}{k^2}$	$(1+o(1)) \frac{e^{-\frac{1}{2} + \frac{1}{2e} + \frac{1}{2e^2}}}{\sqrt{8\pi k^{3/2}}}$	$(1+o(1)) \frac{e^{-1 + \frac{1}{e}}}{\sqrt{2\pi k^{3/2}}}$
p_k	$\frac{0.938071 \dots + o(1)}{k^{3/2}}$	$\frac{2.71828 \dots + o(1)}{k^2}$	$\frac{0.155598 \dots + o(1)}{k^{3/2}}$	$\frac{0.212023 \dots + o(1)}{k^{3/2}}$
$\sum_k p_k$	1	1	$1 - e^{-\frac{1}{2} + \frac{1}{2e} + \frac{1}{2e^2}}$ = 0.219946...	$1 - e^{-1 + \frac{1}{e}}$ = 0.468536...

(as shown in Figure 1 and Table 1), though the exponent characterizing the fatness of the tail of the length distribution does change for the min max-weight cycle.

We call a cycle *c-light* if its mean weight is $< c/n$. We start with an elementary calculation of the expected number of *c-light* cycles of length k . Then we show that for $c \leq 1/e$, the set of light cycles is well approximated by a Poisson process with intensity given by the first-moment computation. For $c > 1/e$, the number of *c-light* cycles diverges. Given this Poisson approximation, it is straightforward to do the computations illustrated in Figure 1 and Table 1. A key difference between the min mean-weight cycle and min max-weight cycle is that the expected number of *c-light* cycles is finite at the critical value $c = 1/e$, while for the max-weight cycles, the expected number of light cycles diverges at the critical value of c . As we will explain, the finite expected number of light cycles at the critical value of c is what leads to the discontinuity in the curves in Figure 1 and it is also why $\sum_k p_k < 1$. With probability tending to $1 - \sum_k p_k$ the min mean-weight cycle is long (has length tending to infinity with n) and has mean weight $(1/e + o(1))/n$; analysing its length is difficult because the Poisson approximation breaks down in this regime, but we bound it below by $(2/\pi^2 - o(1)) \log^2 n \log \log n$.

2. Review of the tree function

Because it plays a key role in the formulas for min mean-weight cycles in the subcritical regime, *i.e.*, for weight $< 1/(en)$, we briefly review the *tree function* and the closely related Lambert W function. The tree function T is the exponential generating function for rooted spanning trees. Recalling Cayley’s formula that there are k^{k-1} rooted spanning trees on k nodes, we have

$$T(z) = \sum_{k=1}^{\infty} k^{k-1} \frac{z^k}{k!}.$$

From Stirling’s formula

$$\sqrt{2\pi k} \frac{k^k}{e^k} \leq k! \leq \sqrt{2\pi k} \frac{k^k}{e^k} e^{1/(12k)},$$

this sum converges when $|z| \leq 1/e$. Using techniques from the theory of generating functions, one can see that

$$T(z) = z e^{T(z)} \tag{2.1}$$

(see, *e.g.*, [26, Proposition 5.3.1]). It is straightforward to check that $T(1/e) = 1$. Near this critical point, using (2.1), one can deduce

$$T\left(\frac{1-\delta}{e}\right) = 1 - \sqrt{2\delta} + O(\delta). \tag{2.2}$$

The Lambert W function is defined by the equation

$$z = W(z) e^{W(z)}.$$

This is a multivalued function, but the principal branch is defined so that $W(z) = -T(-z)$ when $|z| \leq 1/e$, and by analytic continuation elsewhere. The tree function figures prominently in the analysis of random graphs near the phase transition (see, *e.g.*, [21]), and the Lambert W function is an important function in applied mathematics; for further background see [7].

3. The expected number of light cycles

Given $c > 0$, say that a directed or undirected k -cycle C or k -path P is c -light if its mean weight $w(C)/k$ is at most c/n .

Lemma 3.1. *With exponential edge weights, if $0 \leq c_1 \leq c_2$ then*

$$\mathbb{P}[k\text{-cycle or } k\text{-path is } c_2\text{-light but not } c_1\text{-light}] \begin{cases} \sim \frac{k^k}{k!} \frac{c_2^k - c_1^k}{n^k} & \text{if } k = o(n), \\ \leq \frac{k^k}{k!} \frac{c_2^k - c_1^k}{n^k} & \text{for any } k. \end{cases}$$

Proof. The weight $w(C)$ of a k -cycle or k -path C is distributed as the sum of k independent exponential random variables, that is, according to the Gamma distribution with shape

parameter k , which has density function

$$\phi : x \mapsto e^{-x} x^{k-1} / \Gamma(k), \tag{3.1}$$

where Γ is the gamma function, which is $\Gamma(k) = (k - 1)!$ for positive integers k . Thus

$$\mathbb{P}[c_1 k/n < w(C) \leq c_2 k/n] = \int_{c_1 k/n}^{c_2 k/n} \phi(x) dx.$$

Now $e^{-k/n} \leq 1$, and when $k = o(n)$ we have $e^{-k/n} = 1 - o(1)$. □

Lemma 3.2. *Let N_k denote the number of directed k -cycles in the complete graph on n vertices. For $k \geq 2$ we have*

$$N_k \begin{cases} \sim \frac{n^k}{k} & \text{if } k = o(\sqrt{n}), \\ \leq \frac{n^k}{k} & \text{for any } k. \end{cases}$$

(For $k \geq 3$, the number of undirected k -cycles is of course $\frac{1}{2}N_k$.)

Proof.

$$\frac{kN_k}{n^k} = \frac{n(n-1)\cdots(n-k+1)}{n^k} = \prod_{i=0}^{k-1} \left(1 - \frac{i}{n}\right) = \exp\left[-\frac{k^2}{2n} + O\left(\frac{k}{n}\right) + O\left(\frac{k^3}{n^2}\right) + \cdots\right],$$

which is $1 - o(1)$ when $k = o(\sqrt{n})$, and at most 1 in all cases. □

Theorem 3.3. *For the directed complete graph, let $Z_c^{(k)}$ denote the number of directed k -cycles with mean weight less than c/n , and let $Z_c = \sum_k Z_c^{(k)}$ denote the total number of c -light directed cycles. If $0 \leq c_1 \leq c_2$ then*

$$\mathbb{E}[Z_{c_2}^{(k)} - Z_{c_1}^{(k)}] = \begin{cases} \sim \frac{(c_2^k - c_1^k)k^k}{k \times k!} & \text{if } k = o(\sqrt{n}), \\ \leq \frac{(c_2^k - c_1^k)k^k}{k \times k!} & \text{for any } k, \end{cases} \tag{3.2}$$

and

$$\lim_{n \rightarrow \infty} \mathbb{E}[Z_c] = \begin{cases} \sum_{2 \leq k < \infty} \frac{(ck)^k}{k \times k!} = T(c) - c & \text{for } c \leq 1/e, \\ \infty & \text{for fixed } c > 1/e. \end{cases} \tag{3.3}$$

For the undirected complete graph, the expected number of undirected c -light k -cycles is

$$\frac{1}{2} \mathbb{E}[Z_c^{(k)}], \quad \text{for } k \geq 3,$$

and the total expected number of c -light cycles is

$$\lim_{n \rightarrow \infty} \mathbb{E}[\# \text{ undirected } c\text{-light cycles}] = \begin{cases} \frac{T(c) - c - c^2}{2} & \text{for } c \leq 1/e, \\ \infty & \text{for fixed } c > 1/e. \end{cases} \tag{3.4}$$

Proof. Equation (3.2) is immediate from Lemmas 3.1 and 3.2. For large k , by Stirling’s formula the expression in (3.2) is asymptotic to

$$\frac{(c_2^k - c_1^k)e^k}{\sqrt{2\pi k^{3/2}}}.$$

Thus for fixed $c > 1/e$, the expected number Z_c of c -light cycles tends to ∞ . For $c \leq 1/e$, note that $\mathbb{E}[Z_c] \leq T(c) - c$, and that $\mathbb{E}[Z_c] \geq \sum_{k \leq k_0} \mathbb{E}[Z_c^{(k)}]$, whose summands converge to the $k \leq k_0$ terms for the series for $T(c) - c$. Taking the $k_0 \rightarrow \infty$ limit then yields (3.3). The formulas for the undirected complete graph follow immediately from the formulas for the directed complete graph. □

4. Poisson approximation for short light cycles

Next we show that the short c -light cycles are well approximated by a Poisson process. Here ‘short’ means length at most $L_0 = \log n / (2 \log \log n)$, though for our main results it would suffice to prove this Poisson approximation for any L_0 which tends to infinity as $n \rightarrow \infty$. For $c \leq 1/e$, we know from the first-moment bounds in Theorem 3.3 that with high probability there are no cycles of length $\omega(1)$. It will then follow that the set of all c -light cycles is well approximated by a Poisson process when $c \leq 1/e$.

For our purposes, the most convenient method to show Poisson approximation is the Chen–Stein method, as formulated by Arratia, Goldstein and Gordon [4, Theorem 2].

Theorem 4.1 ([4]). *Let $\{X_\alpha : \alpha \in I\}$ be a finite set of indicator random variables of dependent events, and let $\{Y_\alpha : \alpha \in I\}$ be a set of mutually independent Poisson random variables such that $\mathbb{E}[Y_\alpha] = \mathbb{E}[X_\alpha]$ for each α . For each α let B_α be a subset of I , which is interpreted as the ‘neighbourhood of α ’. Then the total variation distance between the dependent Bernoulli process $(X_\alpha)_{\alpha \in I}$ and the independent Poisson process $(Y_\alpha)_{\alpha \in I}$ is at most*

$$2 \sum_{\alpha \in I} \sum_{\beta \in B_\alpha} \mathbb{E}[X_\alpha] \mathbb{E}[X_\beta] + 2 \sum_{\alpha \in I} \sum_{\substack{\beta \in B_\alpha \\ \beta \neq \alpha}} \mathbb{E}[X_\alpha X_\beta] + \sum_{\alpha \in I} \mathbb{E}[|\mathbb{E}[X_\alpha | \{X_\beta : \beta \notin B_\alpha\}] - \mathbb{E}[X_\alpha]|]. \tag{4.1}$$

With a suitable choice of the neighbourhood sets B_α , the third term above can easily be made zero, and analysing the first two terms above is manageable.

Theorem 4.2. *Suppose c_0 is fixed and $L_0 = \log n / (2 \log \log n)$. For both the directed and undirected complete graphs with exponential edge weights, for any ε , for sufficiently large n , the collection of c_0 -light cycles with length at most L_0 is within total variation distance ε of a Poisson process whose intensity is the expected number of such cycles. In particular, except with probability ε , for all $k \leq L_0$ and all $c \leq c_0$, the number of c -light cycles of length k equals the number of points in the corresponding region of the Poisson process.*

Proof. We divide the interval $(0, c_0]$ into subintervals of length Δ . To apply Theorem 4.1, let I denote the set of pairs (C, c) , where C is a k -cycle (directed or undirected) with

$k \leq L_0$, and $(c - \Delta, c]$ is one of the subintervals. Let $X_{C,c}$ be the indicator random variable for cycle C being c -light (i.e., a cycle with mean weight at most c/n), but not $(c - \Delta)$ -light.

Let $B_{C,c}$ denote the subset of pairs $(C', c') \in I$ for which cycles C and C' have at least one edge in common. The variables $\{X_{C',c'} : (C', c') \notin B_{C,c}\}$ only depend on edges that are disjoint from cycle C , so conditioning on them has no effect on the weight of cycle C . Thus

$$\mathbb{E}[X_{C,c} | \{X_{C',c'} : (C', c') \notin B_{C,c}\}] - \mathbb{E}[X_{C,c}] = 0,$$

and so the third term in (4.1) is zero.

Next we observe that the first two terms of (4.1) are unaffected by the subdivision of the interval $(0, c_0]$. We can define $X_C = \sum_c X_{C,c}$, where the sum is over the right endpoints of the intervals in the subdivision of $(0, c_0]$, which are still Bernoulli random variables, and define B_C to be the set of cycles that have at least one edge in common with C . Then

$$\sum_C \sum_{C' \in B_C} \mathbb{E}[X_C] \mathbb{E}[X_{C'}] = \sum_{C,c} \sum_{(C',c') \in B_{C,c}} \mathbb{E}[X_{C,c}] \mathbb{E}[X_{C',c'}],$$

and similarly for the second term. Therefore we work with the X_C and B_C .

For the first term of (4.1), we write

$$\sum_C \sum_{C' \in B_C} \mathbb{E}[X_C] \mathbb{E}[X_{C'}] = \sum_{k \leq L_0} \sum_{\ell \leq L_0} q_k q_\ell N_k N_\ell \mathbb{P}[k\text{-cycle and } \ell\text{-cycle share an edge}],$$

where q_k is the probability that a k -cycle is c_0 -light. The expected number of edge intersections between a k -cycle and an ℓ -cycle is $k\ell/n^2$ for the directed case, and $k\ell/\binom{n}{2}$ for the undirected case, so they intersect with probability at most $k\ell/\binom{n}{2}$. Thus the first term is at most

$$\frac{4}{n(n-1)} \left[\sum_{k \leq L_0} q_k N_k k \right]^2.$$

But from Theorem 3.3, $q_k N_k \leq (c_0 e)^k / \sqrt{2\pi k^3}$. So the first term of (4.1) is bounded by

$$O\left(\frac{(c_0 e)^{2L_0}}{n^2}\right) = \frac{1}{n^{2-o(1)}},$$

which tends to 0 as $n \rightarrow \infty$.

For the second term of (4.1), we consider all possible pairs of distinct non-edge-disjoint cycles C, C' of I . Let k be the number of edges common to C and to C' , $k + \ell$ be the length of C and $k + m$ be the length of C' . We let w/n denote the total weight of the edges shared by C and C' , let v/n denote the total weight of edges in C but not C' , and let x/n denote the total weight of edges in C' but not C . The probability that both cycles are c_0 -light is

$$\mathbb{E}[X_C X_{C'}] = \iiint_{\substack{w+v < (k+\ell)c_0 \\ w+x < (k+m)c_0}} \frac{e^{-w/n} (w/n)^{k-1}}{\Gamma(k)} \frac{e^{-v/n} (v/n)^{\ell-1}}{\Gamma(\ell)} \frac{e^{-x/n} (x/n)^{m-1}}{\Gamma(m)} \frac{dw dv dx}{n^3}.$$

We can bound the $e^{-w/n}$, $e^{-v/n}$, and $e^{-x/n}$ terms by 1:

$$\mathbb{E}[X_C X_{C'}] \leq \frac{1}{n^{k+\ell+m}} \iiint_{\substack{w+v < (k+\ell)c_0 \\ w+x < (k+m)c_0}} \frac{w^{k-1} v^{\ell-1} x^{m-1}}{\Gamma(k) \Gamma(\ell) \Gamma(m)} dw dv dx.$$

We enlarge the domain of integration to the set of (w, v, x) for which $w < (k + \ell)c_0$, $v < (k + \ell)c_0$, and $x < (k + m)c_0$, so that the triple integral has a product form that can be evaluated explicitly:

$$\begin{aligned} \mathbb{E}[X_C X_{C'}] &\leq \frac{1}{n^{k+\ell+m}} \frac{((k + \ell)c_0)^k}{k!} \frac{((k + \ell)c_0)^\ell}{\ell!} \frac{((k + m)c_0)^m}{m!} \\ &\leq \frac{c_0^{k+\ell+m}}{n^{k+\ell+m}} (k + \ell)^{k+\ell} (k + m)^m \\ &\leq \frac{(c_0 L_0)^{2L_0}}{n^{k+\ell+m}}. \end{aligned}$$

We now count the number of cycle pairs (C, C') which are distinct and have at least one edge in common given k, ℓ, m .

Suppose $C' \setminus C$ consists of $i \geq 1$ paths. There are at most L_0^i possibilities for the lengths m_1, m_2, \dots, m_i of the paths of $C' \setminus C$. With those lengths specified, we can list the $k + \ell$ vertices of C in order from some arbitrary starting point, specify where along C each path of $C' \setminus C$ starts and ends, and specify the $m_j - 1$ vertices of each path. Thus, for either the directed or undirected setting, the number of such configurations is at most $n^{k+\ell} L_0^i L_0^{2i} n^{m-i}$.

Altogether the number of overlapping cycles (C, C') is bounded by

$$\sum_{i=1}^{L_0} \frac{L_0^{3i}}{n^i} n^{k+\ell+m} \leq 2 \frac{L_0^3}{n} n^{k+\ell+m},$$

provided $L_0 \leq \sqrt[3]{n/2}$. There are at most L_0 choices for each of k, ℓ , and m . The second term of (4.1) is then bounded by $4c_0^{2L_0} L_0^{2L_0+6}/n$ (when $L_0 \leq \sqrt[3]{n/2}$). When $L_0 = \frac{1}{2} \log n / \log \log n$, we have

$$\begin{aligned} \frac{4}{n} c_0^{2L_0} L_0^{2L_0+6} &\leq \frac{4}{n} \exp[(\log \log n - \log \log \log n)(\log n / \log \log n + 6) \\ &\quad + \log c_0 \log n / \log \log n] \\ &= 4 \exp[6(\log \log n - \log \log \log n) \\ &\quad - \log n \log \log \log n / \log \log n + \log c_0 \log n / \log \log n], \end{aligned}$$

which tends to 0 as $n \rightarrow \infty$. □

5. Below the critical point: short light cycles

Given the Poisson approximation result in Theorem 4.2 and the first-moment estimate in Theorem 3.3, it is straightforward to derive the formulas for the mean weight of the min mean-weight cycle (shown in Figure 1), and the probability that the length of the cycle is

k for any fixed k (in Table 1). Similar computations were done by Janson [19] for the min max-weight cycle.

5.1. Weight of the cycle

Theorem 5.1. *For the directed complete graph, for fixed c , there is a cycle with mean weight $\leq c/n$ with probability*

$$\lim_{n \rightarrow \infty} \mathbb{P}[\exists \text{ cycle with mean weight } \leq c/n] = \begin{cases} 1 - \exp[-T(c) + c] & c \leq 1/e, \\ 1 & c > 1/e, \end{cases}$$

while for the undirected complete graph the probability is

$$\lim_{n \rightarrow \infty} \mathbb{P}[\exists \text{ cycle with mean weight } \leq c/n] = \begin{cases} 1 - \exp[(-T(c) + c + c^2)/2] & c \leq 1/e, \\ 1 & c > 1/e. \end{cases}$$

Proof. For $c \leq 1/e$, by the first moment estimate, with probability $1 - o(1)$ there are no c -light cycles with length $> L_0 = \log n / (2 \log \log n)$. By the Poisson approximation, there is a c -light cycle of length $\leq L_0$ with probability $\exp[-\mu] + o(1)$, where

$$\mu = (1 + o(1)) \sum_{k=2}^{L_0} k^{k-1} c^k / k! = T(c) - c + o(1)$$

for the directed complete graph, and $\mu = (T(c) - c - c^2)/2 + o(1)$ for the undirected complete graph. For fixed $c > 1/e$, the sum

$$(1 + o(1)) \sum_{k=2}^{L_0} k^{k-1} c^k / k!$$

tends to infinity with n , and the Poisson approximation still holds, so with probability $1 - o(1)$ there is a c -light cycle. □

So the finiteness of $T(c) - c$ and $(T(c) - c - c^2)/2$ at $c = 1/e$ accounts for the discontinuities in the curves in Figure 1. Recalling the behaviour of the tree function near $c = 1/e$, we see that these curves for the min mean-weight cycle have a square-root plus constant behaviour to the left of the critical point.

5.2. Length of the cycle

Theorem 5.2. *Suppose k is fixed as $n \rightarrow \infty$. For the directed complete graph, for $k \geq 2$,*

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{P}[\text{min mean-weight cycle has length } k] \\ &= \lim_{n \rightarrow \infty} \mathbb{P}[\text{min mean-weight cycle has length } k \text{ and weight } \leq \frac{k}{e}] \\ &= \int_0^{1/e} \frac{c^{k-1} k^k}{k!} e^{-T(c)+c} dc. \end{aligned}$$

For the undirected complete graph, for $k \geq 3$,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{P}[\text{min mean-weight cycle has length } k] \\ &= \lim_{n \rightarrow \infty} \mathbb{P}[\text{min mean-weight cycle has length } k \text{ and weight } \leq \frac{k}{e}] \\ &= \int_0^{1/e} \frac{c^{k-1} k^k}{2k!} e^{(-T(c)+c+c^2)/2} dc. \end{aligned}$$

Proof. We subdivide the interval $[0, 1/e]$ into subintervals of width Δ , and let $[c, c + \Delta]$ be one of these subintervals. By Theorem 3.3, the expected number of k -cycles which are $(c + \Delta)$ -light but not c -light is

$$(1 + o(1)) \frac{(kc^{k-1}\Delta)k^k}{k \times k!} + O(\Delta^2)$$

in the directed setting, and half that in the undirected setting (for $k \geq 3$), where the $o(1)$ term goes to 0 for fixed k when $\Delta \rightarrow 0$ and $n \rightarrow \infty$. Using Poisson approximation for cycles of length at most $L_0 = \log n / (2 \log \log n)$ and mean weight $\leq 1/e$ (Theorem 4.2), the fact that it is unlikely that there is any cycle with mean weight $\leq 1/e$ and length more than L_0 (Theorem 3.3), and the probability that there is a cycle with mean weight $\leq c/n$ (Theorem 5.1), we see that the probability that the min mean-weight cycle has length k and weight between c and $c + \Delta$ is

$$(1 + o(1)) \frac{c^{k-1} k^k}{k!} \times e^{-T(c)+c} \Delta + O(\Delta^2)$$

in the directed setting, and

$$(1 + o(1)) \frac{1}{2} \frac{c^{k-1} k^k}{k!} \times e^{(-T(c)+c+c^2)/2} \Delta + O(\Delta^2)$$

in the undirected setting, where the $o(1)$ terms go to 0 uniformly in c for fixed k when $n \rightarrow \infty$ and $\Delta \rightarrow 0$. Summing these expressions over the subintervals of $[0, 1/e]$ and taking the $\Delta \rightarrow 0$ limit gives the integral expressions for the probability that the min mean-weight cycle has length k and mean weight $\leq 1/e$.

Next we consider the possibility that the min mean-weight cycle has length k and mean weight $> 1/e$. Suppose $0 < \delta < 1$. With probability tending to 1 as $n \rightarrow \infty$, there is a $(1 + \delta/k)/e$ -light cycle. But the expected number of k -cycles that are $(1 + \delta/k)/e$ -light but not $1/e$ -light tends to 0 as $\delta \rightarrow 0$. So the probability that the min mean-weight cycle has length k and mean weight $> 1/e$ tends to 0 as $n \rightarrow \infty$. □

The formulas in Table 1 are rewritten slightly using equation (2.1) to write $e^{-T(c)} = c/T(c)$. Theorem 5.1 and Theorem 5.2 imply

$$\lim_{n \rightarrow \infty} \mathbb{P}[\text{min mean-weight cycle has mean weight } > 1/e] > 0$$

but

$$\sum_k \lim_{n \rightarrow \infty} \mathbb{P}[\text{min mean-weight cycle has length } k \text{ and mean weight } > 1/e] = 0.$$

There is no contradiction of course. In Section 6 we further investigate the length of the min mean-weight cycle when its mean weight is $> 1/e$.

5.3. Tail behaviour of the length distribution

We can approximate the large- k behaviour of the probability p_k that the min mean-weight cycle has length k (sending n to infinity first and then k). We make the substitution $c = (1 - \delta)/e$ to obtain, for the directed complete graph,

$$p_k = \frac{e^{-k} k^k}{k!} e^{1/e} \int_0^1 (1 - \delta)^k \frac{e^{-\delta/e}}{T((1 - \delta)/e)} \frac{d\delta}{e}.$$

The integrand is approximately $e^{-k\delta}$ for small δ , and large δ contribute negligibly, so the integral is approximately $1/(ke)$, and so for large k

$$p_k = (1 + o(1)) \frac{e^{-1+1/e}}{\sqrt{2\pi}} k^{-3/2}.$$

For the undirected complete graph, a similar computation yields

$$p_k = (1 + o(1)) \frac{e^{-1/2+1/(2e)+1/(2e^2)}}{2\sqrt{2\pi}} k^{-3/2}.$$

By comparison, Janson [19] shows that for the min max-weight cycle on the undirected complete graph, the expected number of cycles with max weight at most c/n is $\frac{1}{2}(\log \frac{1}{1-c} - c - c^2/2)$, so the probability that such a cycle exists is $1 - (1 - c)^{1/2} e^{c/2+c^2/4}$ which has its threshold at 1, and so

$$p_k = \frac{1}{2} \int_0^1 c^{k-1} (1 - c)^{1/2} e^{c/2+c^2/4} dc,$$

which for large k is

$$p_k = (1 + o(1)) \frac{\sqrt{\pi}}{4} e^{3/4} k^{-3/2}.$$

The computations for the min max-weight cycle on the directed complete graph are similar, though it is perhaps surprising that unlike the previous three cases, the asymptotics of p_k in this case are $p_k = \Theta(k^{-2})$. More specifically, we have

$$p_k = \int_0^1 c^{k-1} (1 - c) e^c dc,$$

Letting $c = 1 - \delta/k$, the integrand is for small δ asymptotically

$$e^{-\delta} \frac{\delta}{k} e \frac{d\delta}{k},$$

so we see that the integral is asymptotically e/k^2 .

6. Above the critical point: long light cycles

Recall that a cycle C is c -light if $\text{weight}(C) \leq \text{length}(C)c/n$. We say that a cycle C is A -uniformly c -light if C is c -light, and in addition, for every subpath P of C ,

$$\text{weight}(P) \leq [\text{length}(P) + A] \frac{c}{n}.$$

In the directed complete graph, let $Z_{L,\delta}$ denote the number of $(1 + \delta)/e$ -light cycles of length between $L - 1/\delta$ and L , and let $Y_{\delta,A}^{L_1,L_2}$ denote the number of A -uniformly $(1 + \delta)/e$ -light cycles of length L for which $L_1 < L \leq L_2$. We will eventually choose the parameters δ, L_1, L_2 , and A so that

$$\begin{aligned} \delta &= \Theta(1/\log^2 n), \\ L_1, L_2 &= \omega(\log^2 n \log \log n), \\ A &\approx \log n. \end{aligned}$$

We aim to show that with high probability such cycles exist.

We use a theorem of Komlós, Major and Tusnády [22, Theorem 1] from the second of two papers they wrote relating random walk to Brownian motion.

Theorem 6.1 ([22]). *Suppose that X_1, X_2, \dots are i.i.d. random variables with expected value 0, variance 1, and finite exponential moments, i.e., their density function $f(x)$ satisfies*

$$\int e^{tx} f(x) dx < \infty, \quad \text{for } |t| \leq t_0 > 0.$$

Then these random variables can be coupled to i.i.d. standard normal random variables Y_1, Y_2, \dots such that for any λ there are constants K_1 and K_2 for which

$$\mathbb{P} \left[\max_{k \leq n} \left| \sum_{i=1}^k X_i - \sum_{i=1}^k Y_i \right| > K_1 \log n + x \right] < K_2 e^{-\lambda x}.$$

Lemma 6.2. *Let C be a particular cycle of length L , and let W be its (random) weight. By definition, cycle C is (nW/L) -light. The probability that C is A -uniformly (nW/L) -light is independent of W and n , and is at least*

$$\exp[-(\pi^2/2 + o(1))L/A^2],$$

where the $o(1)$ term tends to 0 as $L/A^2 \rightarrow \infty$ and $L/A^3 \rightarrow 0$.

Proof. The edge weights W_1, \dots, W_L are distributed as L independent exponential random variables with mean 1. The total weight is $W = \sum_{i=1}^L W_i$. Because the edge weights are exponential random variables, conditioning on the total weight W does not affect the joint distribution of relative weights W_i/W , which are distributed as the arc lengths of the arcs between L uniformly random points placed on a circle of unit circumference. The cycle is by definition (nW/L) -light. The cycle is A -uniformly (nW/L) -light when

$$\sum_{i=1}^{\ell} W_{a+i \bmod L} \leq (\ell + A) \frac{\sum_{i=1}^L W_i}{L} \tag{6.1}$$

for each $a, \ell \in \{1, \dots, L\}$, where $a + i \bmod L$ is interpreted as a value in the range $1, \dots, L$. This property is invariant under uniform scalings of the edge weights, so whether or not the cycle is also A -uniformly (nW/L) -light is solely a function of these random arc lengths, independent of the total weight W and n .

Let $X_i = W_i - 1$. Suppose that $0 < \varepsilon \leq 1/4$ and

$$0 \leq \sum_{i=1}^L X_i \leq \varepsilon A, \tag{6.2}$$

and that for each $k \in \{1, 2, \dots, L\}$,

$$-\frac{1-\varepsilon}{2}A \leq \sum_{i=1}^k X_i \leq \frac{1-\varepsilon}{2}A. \tag{6.3}$$

Equation (6.2) implies $\sum_{i=1}^L W_i \geq L$. When $a + \ell \leq L$, (6.3) implies $\sum_{i=1}^\ell W_{a+i} \leq \ell + (1 - \varepsilon)A$, which then implies (6.1) when $a + \ell \leq L$. If $a + \ell > L$, then

$$\sum_{i=1}^\ell W_{a+i \bmod L} = \sum_{i=1}^L W_i - \sum_{i=1+a+\ell \bmod L}^a W_i \leq \varepsilon A + (1 - \varepsilon)A + \ell$$

by (6.2) and (6.3), so again we have (6.1).

Now X_i has zero mean, unit variance, and finite exponential moments, so Theorem 6.1 implies that the partial sums $\sum_{i=1}^k X_i$ are well approximated by a standard Brownian motion. It is known how to compute using Fourier analysis the probability that a standard Brownian motion B_t stays within an interval up through time T . If the interval is $[-a, a]$, then this probability is

$$\Theta\left(\exp\left[-\frac{\pi^2 T}{8 a^2}\right]\right)$$

(see, e.g., [24, pp. 216–218]). Conditional upon the Brownian motion remaining within this interval, its final position within the interval at time T has a well-behaved distribution which for large T/a^2 converges to a sine function, since the other Fourier coefficients decay more rapidly.

We set $T = L$ and $a = A(1 - 2\varepsilon)/2$; we see that the Brownian motion B_t stays within the interval $\pm A(1 - 2\varepsilon)/2$ and ends within the interval $(\varepsilon A/3, 2\varepsilon A/3)$ with probability at least

$$\Theta\left(\varepsilon \times \exp\left[-\frac{1}{(1 - 2\varepsilon)^2} \frac{\pi^2 L}{2 A^2}\right]\right). \tag{6.4}$$

If this event occurs, and the partial sums $\sum_{i=1}^k X_k$ are within $\varepsilon A/3$ of the Brownian motion, then equations (6.3) and (6.2) hold.

By assumption $L/A^3 \rightarrow 0$, so let us suppose $L/A^3 \leq 1$. Then $A \geq L^{1/3}$. By assumption $L/A^2 \rightarrow \infty$, so $L \rightarrow \infty$. Let us take

$$\varepsilon = 6 \max(L/A^3, K_1 \log L/A, e^{-L^{1/2}/A}),$$

which by our assumptions tends to 0, and we can suppose that it is at most $1/4$ as assumed above.

In the KMT theorem, we choose $\lambda = 20$ and $x = L/A^2$. By our choice of ε , the deviation $K_1 \log L + x$ is smaller than $\varepsilon A/3$. The probability that the Brownian motion and the random walk are not within $\varepsilon A/3$ of one another is at most $K_2 e^{-20L/A^2}$. Now $20 > \pi^2/(2(1 - 2\varepsilon)^2)$, so provided L/A^2 is sufficiently large, even if we condition on the unlikely event that the Brownian motion stays within the interval and ends within an even smaller interval, it is still extremely likely that the random walk does not deviate more than $\varepsilon A/3$ from the Brownian motion. Thus, the probability that the cycle is A -uniformly c -light can be bounded below by an expression of the form (6.4). Since $\varepsilon \geq e^{-L^{1/2}/A}$ and $L/A^2 \rightarrow \infty$, the factor of ε in (6.4) can be absorbed into the exponent where it becomes $\log \varepsilon = o(L/A^2)$, and since $\varepsilon \rightarrow 0$, (6.4) can be further rewritten as $\exp[-(\pi^2/2 + o(1))L/A^2]$. □

Lemma 6.3. *For the directed complete graph, the expected number of A -uniformly $(1 + \delta)/e$ -light directed cycles with length more than L_1 and at most L_2 is*

$$\mathbb{E}[Y_{\delta,A}^{L_1,L_2}] = \sum_{L=L_1+1}^{L_2} \frac{(1 + \delta)^L}{L^{3/2}} \exp[-(\pi^2/2 + o(1))L/A^2],$$

where the $o(1)$ term tends to 0 as $L_2^2/n \rightarrow 0$, $L_1/A^2 \rightarrow \infty$, and $L_2/A^3 \rightarrow 0$. (For the undirected graph, the expected value is half as large.)

Proof. Essentially immediate from the first moment estimates for c -light cycles (Theorem 3.3) and Lemma 6.2. Since $L/A^2 \rightarrow \infty$, the factor of $(1 + o(1))/\sqrt{2\pi}$ gets absorbed into the $o(1)$ term in the exponent. □

Lemma 6.4. *If $L_2^3 e^A/n \leq 1/2$, then*

$$\frac{\text{Var}[Y_{\delta,A}^{L_1,L_2}]}{\mathbb{E}[Y_{\delta,A}^{L_1,L_2}]} \leq 1 + \frac{2L_2^3 e^A (1 + \delta)^{L_2}}{n}.$$

(This same bound holds for both the directed and undirected complete graph.)

Proof. For any cycle C of length between L_1 and L_2 , let U_C denote the indicator of the event that C is uniformly light (within this proof, ‘uniformly light’ means A -uniformly $(1 + \delta)/e$ -light). Let C_1, C_2 be two cycles whose lengths are both in the range $(L_1, L_2]$. If C_1 and C_2 have no edges in common then U_{C_1} and U_{C_2} are independent. So we have

$$\begin{aligned} \text{Var}[Y_{\delta,A}^{L_1,L_2}] &= \sum_{C_1} \sum_{C_2} (\mathbb{E}[U_{C_1} U_{C_2}] - \mathbb{E}[U_{C_1}]\mathbb{E}[U_{C_2}]) \\ &= \sum_{C_1} \sum_{C_2: |C_2 \cap C_1| \geq 1} (\mathbb{E}[U_{C_1} U_{C_2}] - \mathbb{E}[U_{C_1}]\mathbb{E}[U_{C_2}]) \\ &\leq \sum_{C_1} \sum_{C_2: |C_2 \cap C_1| \geq 1} \mathbb{E}[U_{C_1} U_{C_2}] \\ &= \sum_{C_1} \mathbb{E}[U_{C_1}] \sum_{C_2: |C_2 \cap C_1| \geq 1} \mathbb{E}[U_{C_2} | U_{C_1} = 1]. \end{aligned}$$

We partition the inner sum into sub-sums depending on the overlaps between C_1 and C_2 . If $C_2 = C_1$, then of course $\mathbb{E}[U_{C_2}|U_{C_1} = 1] = 1$. Otherwise, $C_2 \setminus C_1$ consists of a collection of disjoint paths – say there are i of them, and their lengths are m_1, m_2, \dots, m_i . Let $m = \sum_j m_j < L_2$. To specify the j th path, we can specify its start and end points on C_1 , as well as the internal vertices, so there are $\leq L_2^i n^{m_j-1}$ possible such subpaths. Hence the number of such C_2 is at most $L_2^{2i} n^{m-i}$. Conditional on the weights in C_1 , the probability that C_2 is uniformly light is at most the probability that for each j , the j th subpath $C_2 \setminus C_1$ has weight at most $w_j := (m_j + A)(1 + \delta)/(en)$. The probability that the j th subpath is light enough is at most

$$\frac{w_j^{m_j}}{m_j!} \leq \frac{[(1 + \delta)(1 + A/m_j)]^{m_j}}{n^{m_j} \sqrt{2\pi m_j}} \leq \frac{e^A (1 + \delta)^{m_j}}{n^{m_j}},$$

and so the conditional probability that C_2 is uniformly light is at most $e^{Ai} (1 + \delta)^m / n^m$. We see that the contribution to $\sum_{C_2:|C_2 \cap C_1| \geq 1} \mathbb{E}[U_{C_2}|U_{C_1} = 1]$ from the case where $C_2 \setminus C_1$ consists of i paths of lengths m_1, \dots, m_i is

$$\sum_{C_2: C_2 \setminus C_1 = i \text{ paths of lengths } m_1, \dots, m_i} \mathbb{E}[U_{C_2}|U_{C_1}] \leq \left[\frac{L_2^2 e^A}{n} \right]^i (1 + \delta)^m.$$

Since each $m_j \leq L_2$, the total contribution from cases where $C_2 \setminus C_1$ consists of i paths is

$$\sum_{C_2: C_2 \setminus C_1 = i \text{ paths}} \mathbb{E}[U_{C_2}|U_{C_1}] \leq \left[\frac{L_2^3 e^A}{n} \right]^i (1 + \delta)^m.$$

Since by assumption $L_2^3 e^A \leq n/2$, we can sum over i to obtain

$$\sum_{i \geq 1} \sum_{C_2: C_2 \setminus C_1 = i \text{ paths}} \mathbb{E}[U_{C_2}|U_{C_1}] \leq 2 \frac{L_2^3 e^A}{n} (1 + \delta)^m.$$

Combining this with the case $C_2 = C_1$, and using $m \leq L_2$, we obtain

$$\sum_{C_2: |C_2 \cap C_1| \geq 1} \mathbb{E}[U_{C_2}|U_{C_1} = 1] \leq 1 + 2 \frac{L_2^3 e^A}{n} (1 + \delta)^{L_2}. \quad \square$$

We now have the ingredients to prove our upper bound on the minimum mean weight.

Theorem 6.5. *For both the directed and undirected complete graph, with probability $1 - o(1)$ the mean weight of the minimum mean-weight cycle is at most*

$$1 + \frac{\pi^2/2 + o(1)}{\log^2 n} \cdot \frac{1}{en}.$$

Proof. We want to show that with high probability there are $(1 + \delta)/e$ -light cycles, and we do this by showing that in fact there are A -uniformly $(1 + \delta)/e$ -light cycles with high probability. For any real-valued random variable Y we have $\mathbb{P}[Y = 0] \leq \text{Var}[Y]/\mathbb{E}[Y]^2$.

We choose the parameters A , L_1 , and L_2 so that $\text{Var}[Y_{\delta,A}^{L_1,L_2}] \ll \mathbb{E}[Y_{\delta,A}^{L_1,L_2}]^2$, which will imply that with high probability $Y_{\delta,A}^{L_1,L_2} > 0$, i.e., that there is an A -uniformly $(1 + \delta)/e$ -light cycle with size between L_1 and L_2 .

We gather our constraints, which are the same for both the directed and undirected complete graph:

$$\begin{aligned}
 L_1/A^2 &\gg 1, \\
 L_2/A^3 &\ll 1, \\
 \sum_{L=L_1+1}^{L_2} \frac{(1 + \delta)^L}{L^{3/2}} \exp[-(\pi^2/2 + o(1))L/A^2] &\gg 1 + \frac{2L_2^3 e^A (1 + \delta)^{L_2}}{n}, \\
 \frac{2L_2^3 e^A}{n} &\leq 1, \\
 L_2^2 &\ll n.
 \end{aligned}$$

From the third constraint above we need $\delta > (1 + o(1))\pi^2/(2A^2)$, since otherwise the left-hand side would be smaller than one. From the fourth constraint we need $A \leq \log n$, so we need

$$\delta \geq \frac{\pi^2 + o(1)}{2 \log^2 n}.$$

In order to make δ this small, we need $A = (1 + o(1)) \log n$ and $L_2 \gg \log^2 n \log \log n$, since otherwise the left-hand side of the third constraint would be smaller than one. The second constraint then requires $L_2 \ll \log^3 n$. Choosing L_1 and L_2 to include many cycle lengths L has the advantage of increasing $\mathbb{E}[Y_{\delta,A}^{L_1,L_2}]$, but this increase is dwarfed by the exponential factors, and ultimately only affects the $o(1)$ term, so we may as well take $L_1 = L_2 - 1$.

We can make δ nearly as small as this bound by picking the following parameter values. Let $\varepsilon > 0$ be an arbitrarily small positive constant. Then

$$\begin{aligned}
 A &= (1 - \varepsilon) \log n, \\
 \delta &= \frac{\pi^2/2 + 13\varepsilon}{\log^2 n}, \\
 L_2 &= \frac{1}{\varepsilon} \log^2 n \log \log n, \\
 L_1 &= L_2 - 1.
 \end{aligned}$$

It is straightforward to verify that for sufficiently small fixed ε , the above values satisfy the preceding constraints for all sufficiently large n . □

The upper bound on the mean weight of the minimum mean-weight cycle is a key ingredient in bounding its length from below in the supercritical regime.

Lemma 6.6. *If $\delta \ll 1$, then with high probability there are no cycles of length at most*

$$\frac{\log \delta^{-1}}{2\delta}$$

that are $(1 + \delta)/e$ -light but not $1/e$ -light. (This holds for both the directed and undirected complete graph.)

Proof. Using the inequality version of the first-moment estimate (3.2) and Stirling's formula, the expected number of cycles of length $\leq L_0$ that are $(1 + \delta)/e$ -light but not $1/e$ -light is at most

$$\sum_{k=1}^{L_0} \frac{1}{\sqrt{2\pi k^{3/2}}} [(1 + \delta)^k - 1].$$

For $k \leq 1/\delta$ we can use the bound

$$(1 + \delta)^k - 1 \leq (e - 1)\delta k,$$

and find that the first $1/\delta$ terms (if there are that many) sum up to at most

$$\frac{e - 1}{\sqrt{2\pi}} \int_0^{1/\delta} k^{-1/2} \delta \, dk = \frac{e - 1}{\sqrt{\pi/2}} \delta^{1/2} = o(1),$$

since by assumption $\delta \ll 1$.

For the remaining terms we use the bound

$$(1 + \delta)^k - 1 \leq e^{\delta k},$$

and group the terms into blocks of $\lceil 1/\delta \rceil$ terms (except that the last block may have fewer terms). For the block containing k , the block sum is $O(e^{\delta k}/(\delta k^{3/2}))$. These block-sum bounds increase geometrically, so the total is

$$O\left(\frac{e^{\delta L_0}}{\delta L_0^{3/2}}\right).$$

Taking $L_0 = (\log \delta^{-1})/(2\delta)$, we have $e^{\delta L_0} = \delta^{-1/2}$, so the above bound is $O(1/(\delta L_0)^{3/2}) = O(1/(\log \delta^{-1})^{3/2}) = o(1)$. \square

We can now bound from below the length of the supercritical minimal mean-weight cycle.

Theorem 6.7. *For both the directed and undirected complete graph, conditional upon the min mean-weight cycle having mean weight $> 1/(en)$, with probability $1 - o(1)$ its length is at least*

$$(2/\pi^2 - o(1)) \log^2 n \log \log n.$$

Proof. Immediate from Theorem 6.5 and Lemma 6.6, together with the fact (Theorem 5.1) that the event being conditioned on has probability bounded away from 0. \square

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