J. Aust. Math. Soc. **89** (2010), 309–315
doi:10.1017/S1446788711001017

AUTOMORPHISMS OF THE UHF ALGEBRA THAT DO NOT EXTEND TO THE CUNTZ ALGEBRA

ROBERTO CONTI

(Received 23 March 2010; accepted 15 December 2010)

Communicated by G. A. Willis

Abstract

The automorphisms of the canonical core UHF subalgebra $\mathcal{F}_n$ of the Cuntz algebra $\mathcal{O}_n$ do not necessarily extend to automorphisms of $\mathcal{O}_n$. Simple examples are discussed within the family of infinite tensor products of (inner) automorphisms of the matrix algebras M_n. In that case, necessary and sufficient conditions for the extension property are presented. Also addressed is the problem of extending to $\mathcal{O}_n$ the automorphisms of the diagonal $\mathcal{D}_n$, which is a regular maximal abelian subalgebra with Cantor spectrum. In particular, it is shown that there exist product-type automorphisms of $\mathcal{D}_n$ that do not extend to (possibly proper) endomorphisms of $\mathcal{O}_n$.

2010 *Mathematics subject classification*: primary 46L40; secondary 46L05, 37B10.

Keywords and phrases: Cuntz algebra, endomorphism, automorphism, core UHF subalgebra, regular maximal abelian subalgebra.

1. Introduction

Among C^* algebras, the class of uniformly hyperfinite (UHF) algebras is perhaps one of the first nontrivial (that is, infinite-dimensional) examples that comes to mind and it was investigated in depth in the outstanding work by Glimm in the early 1960s [9]. Since that time, many more types of C^* algebras have been studied in depth, and our knowledge of the general theory of C^* algebras has increased greatly. Some time later, the Cuntz algebras $\mathcal{O}_n$ made their appearance [5]. These C^* algebras are quite different from UHF algebras as, for instance, they are traceless and purely infinite. However, it is well known that $\mathcal{O}_n$ contains canonically a copy of the UHF algebra of type n^∞ (that is, the C^*-algebraic tensor product of countably infinitely many copies of M_n, the $n \times n$ matrix algebra over $\mathbb{C}$) as the canonical core UHF subalgebra $\mathcal{F}_n$.

In contrast to the case of $\mathcal{F}_n$, the study of automorphisms of $\mathcal{O}_n$ is quite challenging. Typical problems range from the construction of explicit examples to global properties of the automorphism group $\mathrm{Aut}(\mathcal{O}_n)$ and some notable subgroups and quotients.

The group $\mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n)$ of all automorphisms of $\mathcal{O}_n$ leaving $\mathcal{F}_n$ globally invariant has sporadically appeared in the literature [3, 4, 6]. In particular, it turns out that $\mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n) = \mathrm{Aut}(\mathcal{O}_n) \cap \{\lambda_u \mid u \in \mathcal{U}(\mathcal{F}_n)\}$ and so the examples of permutation automorphisms of $\mathcal{O}_n$ exhibited in [4] restrict to automorphisms of $\mathcal{F}_n$ that are also easily seen to be outer. In this short note, we will add another piece of information by showing that the canonical restriction map

$$r : \mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n) \to \mathrm{Aut}(\mathcal{F}_n) \tag{1.1}$$

is not surjective. The kernel of this group homomorphism consists of the gauge automorphism, by [3, Corollary 4.10]. A few remarks are in order. First, it is plain that all inner automorphisms of $\mathcal{F}_n$ extend to inner automorphisms of $\mathcal{O}_n$. Second, it is well known (see, for example, [7, Corollary IV.5.8]) that every automorphism of $\mathcal{F}_n$ is approximately inner in the sense that it is the limit of inner automorphisms of $\mathcal{F}_n$ in the topology of pointwise norm-convergence, that is, $\mathrm{Aut}(\mathcal{F}_n) = \overline{\mathrm{Inn}}(\mathcal{F}_n)$. Therefore our result, in particular, shows the existence of sequences of inner automorphisms of $\mathcal{O}_n$ that converge pointwise in norm on $\mathcal{F}_n$ but not on $\mathcal{O}_n$. Moreover, it makes it clear that there are strict inclusions (see [1])

$$\mathrm{Inn}(\mathcal{F}_n) \subset r(\mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n)) \subset \mathrm{Aut}(\mathcal{F}_n). \tag{1.2}$$

Finally, we deduce that there are automorphisms of $\mathcal{F}_n$ that do not extend to $\mathcal{O}_n$ even as proper endomorphisms, as the latter possibility was ruled out in [3, Corollary 4.9]. In contrast, this last fact is not true for the canonical Cartan subalgebra $\mathcal{D}_n$ of $\mathcal{O}_n$ for there are proper endomorphisms of $\mathcal{O}_n$ that restrict to automorphisms of $\mathcal{D}_n$ (see [4]). Still, one can show the existence of automorphisms of $\mathcal{D}_n$ that do not extend to endomorphisms of $\mathcal{O}_n$.

We conclude this introductory section with a few words on notation (see [4]). For each integer $n \geq 2$, the Cuntz algebra $\mathcal{O}_n$ is the universal C^* algebra generated by n isometries $S_1, \ldots, S_n$ whose ranges sum to 1. There is a one-to-one correspondence $v \mapsto \lambda_v$ between unitaries in $\mathcal{O}_n$ and unital $*$-endomorphisms of $\mathcal{O}_n$ that associates to $v \in \mathcal{U}(\mathcal{O}_n)$ the endomorphism λ_v defined by $\lambda_v(S_i) = vS_i$ (where $i = 1, \ldots, n$). The canonical core UHF subalgebra $\mathcal{F}_n$ is the norm-closure of the union of the matrix subalgebras $\mathcal{F}_n^k \simeq M_n \otimes \cdots \otimes M_n$ (k factors), where

$$\mathcal{F}_n^k = \mathrm{span}\{S_{\alpha_1} \cdots S_{\alpha_k} S_{\beta_k}^* \cdots S_{\beta_1}^*,\ 1 \leq \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \leq n\}.$$

The UHF subalgebra $\mathcal{F}_n$ admits a unique normalized trace, denoted τ. We also consider $\mathcal{D}_n$, the diagonal subalgebra of $\mathcal{F}_n$, which is equal to the norm-closure of the union of the commutative subalgebras $\mathcal{D}_n^k = \mathcal{D}_n \cap \mathcal{F}_n^k \simeq \mathbb{C}^{n^k}$. We define the canonical endomorphism φ of $\mathcal{O}_n$ by $\varphi(x) = \sum_{i=1}^n S_i x S_i^*$ for all $x \in \mathcal{O}_n$. This shift endomorphism satisfies $S_i x = \varphi(x) S_i$ when $x \in \mathcal{O}_n$ and $i = 1, \ldots, n$. For a unital C^* algebra B, $\mathrm{Aut}(B)$ and $\mathrm{Inn}(B)$ are the groups of automorphisms and inner automorphisms of B, respectively. For unital C^* algebras $A \subseteq B$, $\mathrm{Aut}(B, A)$ and $\mathrm{Aut}_A(B)$ are the groups of automorphisms of B leaving A globally and pointwise invariant, respectively.

2. Main result

We refer to [8, Ch. VI] for generalities about UHF algebras and their automorphisms. To simplify the notation we will often represent elements of $\mathcal{F}_n$ by tensor products of matrices, through the canonical identification of $\mathcal{F}_n$ with $\bigotimes_{i=1}^{\infty} M_n$.

Given a sequence of unitaries $\underline{u} = (u_i)$ with $u_i \in \mathcal{U}(\mathcal{F}_n^1) \simeq U(n)$, we consider the associated automorphism $\alpha_{\underline{u}}$ of $\mathcal{F}_n$ such that

$$\alpha_{\underline{u}}(S_\alpha S_\beta^*) = u_1 S_{\alpha_1} u_2 S_{\alpha_2} \cdots u_k S_{\alpha_k} S_{\beta_k}^* u_k^* \cdots S_{\beta_2}^* u_2^* S_{\beta_1}^* u_1^*, \tag{2.1}$$

for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_k)$ of the same length k and for all $k \geq 1$. In the tensor product picture, $\alpha_{\underline{u}}$ is nothing more than the infinite tensor product automorphism $\bigotimes_{i=1}^{\infty} \mathrm{Ad}(u_i)$. It is also clear that if

$$\lim_{k\to\infty} u_1 \otimes u_2 \otimes \cdots \otimes u_k \otimes 1 \otimes 1 \otimes \cdots =: u$$

exists in $\bigotimes_{i=1}^{\infty} M_n$ then $\bigotimes_{i=1}^{\infty} \mathrm{Ad}(u_i) = \mathrm{Ad}(u)$ is inner, while the converse holds true whenever $1 \in \sigma(u_i)$ for all i (the latter assumption can always be satisfied by rotating the u_i if necessary); see [8, Theorem 6.3].

Of course, if $\alpha_{\underline{u}} = \mathrm{Ad}(u)$ is inner then it extends to an inner automorphism $\lambda_{u\varphi(u^*)}$ of $\mathcal{O}_n$. It is quite possible that even though $\alpha_{\underline{u}}$ is outer, it still extends to an automorphism of $\mathcal{O}_n$. A simple example of this situation arises when $\underline{u}$ is a nonscalar constant sequence: $u_1 = u_2 = u_3 = \cdots$. Then $\alpha_{\underline{u}} = \bigotimes_{i=1}^{\infty} \mathrm{Ad}(u_1)$ is an outer automorphism of $\mathcal{F}_n$ and extends to the (still outer) Bogolubov automorphism λ_{u_1} of $\mathcal{O}_n$. At this point one could suspect that the possibility of extending $\alpha_{\underline{u}}$ to $\mathcal{O}_n$ depends on whether

$$\lim_{k\to\infty} u_1 \otimes u_2 u_1^* \otimes u_3 u_2^* \otimes \cdots \otimes u_{k+1} u_k^* \otimes 1 \otimes 1 \cdots \tag{2.2}$$

exists in $\bigotimes_{i=1}^{\infty} M_n$. This is indeed the case, as we see from the following result.

THEOREM 2.1. *Let $u_1, u_2, u_3, \ldots$ be an infinite sequence of unitaries in $\mathcal{U}(\mathcal{F}_n^1)$. If the limit in Equation (2.2) exists and thus defines a unitary element $v \in \mathcal{F}_n$, then λ_v is an automorphism of $\mathcal{O}_n$ that, in restriction to $\mathcal{F}_n$, coincides with $\alpha_{\underline{u}} = \bigotimes_{i=1}^{\infty} \mathrm{Ad}(u_i)$. Conversely, suppose that $\alpha_{\underline{u}}$ extends to an endomorphism of $\mathcal{O}_n$. Then there are phases $(e^{i\theta_k})_k$ such that the limit in Equation (2.2) exists for the sequence (u_k'), where $u_k' = e^{i\theta_k} u_k$.*

PROOF. If v is defined by the limit as above, then one can easily check that λ_v coincides with $\alpha_{\underline{u}}$ in restriction to $\mathcal{F}_n$, and thus $\lambda_v \in \mathrm{Aut}(\mathcal{O}_n)$ by [3, Corollary 4.9].

Conversely, let λ_w be an endomorphism of $\mathcal{O}_n$ such that $\lambda_w(x) = \alpha_{\underline{u}}(x)$ for all $x \in \mathcal{F}_n$. Then $\lambda_w \in \mathrm{Aut}(\mathcal{O}_n)$ and $w \in \mathcal{F}_n$. Now,

$$w S_i S_j^* w^* = u_1 S_i S_j^* u_1^*$$

https://doi.org/10.1017/S1446788711001017 Published online by Cambridge University Press

for all $i, j = 1, \ldots, n$ and thus $w^* u_1 \in (\mathcal{F}_n^1)' \cap \mathcal{F}_n = \varphi(\mathcal{F}_n)$, that is, $w = u_1 \varphi(z_1)$ for some $z_1 \in \mathcal{U}(\mathcal{F}_n)$. But then

$$w\varphi(w) S_i S_j S_k^* S_h^* \varphi(w^*) w^* = u_1 \varphi(u_2) S_i S_j S_k^* S_h^* \varphi(u_2^*) u_1^*$$

for all $i, j, h, k = 1, \ldots, n$, that is,

$$\varphi(w^*) w^* u_1 \varphi(u_2) \in (\mathcal{F}_n^2)' \cap \mathcal{F}_n = \varphi^2(\mathcal{F}_n).$$

As argued above, this means that $w^* z_1^* u_2 = \varphi(z_1^*) u_1^* z_1^* u_2 \in \varphi(\mathcal{F}_n)$. Thus $u_1^* z_1^* u_2 = \varphi(z_2^*) \in \varphi(\mathcal{F}_n)$ for some unitary $z_2 \in \mathcal{U}(\mathcal{F}_n)$ from which $z_1^* = u_1 \varphi(z_2^*) u_2^* = \varphi(z_2^*) u_1 u_2^*$ and consequently

$$w = u_1 \varphi(u_2 u_1^*) \varphi^2(z_2).$$

Repeating this argument, one obtains that, for any positive integer k,

$$w = u_1 \varphi(u_2 u_1^*) \varphi^2(u_3 u_2^*) \cdots \varphi^k(u_{k+1} u_k^*) \varphi^{k+1}(z_{k+1})$$

for a suitable unitary $z_k \in \mathcal{F}_n$. Moreover, there is a sequence of phases $(e^{i\theta_k})_k$ such that, after replacing u_k with $e^{i\theta_k} u_k$, one can always assume that $\tau(z_k) \in \mathbb{R}_+$.

Consider now the τ-invariant conditional expectation $E_k : \mathcal{F}_n \to \mathcal{F}_n^k$ defined by $E_k = \mathrm{id}_k \otimes \tau$, where id_k is the identity on $\mathcal{F}_n^k$. For this, $x = \lim_k (E_k(x))$ for all $x \in \mathcal{F}_n$. As argued above, it is clear that $E_k(w) = u_1 \otimes \cdots \otimes u_{k+1} u_k^* \tau(z_{k+1})$ for all $k \geq 1$ and thus

$$1 = \|w\| = \lim_{k \to \infty} \|u_1 \otimes \cdots \otimes u_{k+1} u_k^*\| \tau(z_{k+1}) = \lim_{k \to \infty} \tau(z_{k+1}).$$

That is,

$$\lim_{k \to \infty} \|w - u_1 \otimes \cdots \otimes u_{k+1} u_k^*\| = \lim_{k \to \infty} \|w - u_1 \otimes \cdots \otimes u_{k+1} u_k^* \tau(z_{k+1})\| = 0,$$

as required. □

The automorphisms of $\mathcal{F}_n$ of the form $\alpha_{\underline{u}}$ satisfy $\alpha_{\underline{u}}(\mathcal{F}_n^k) = \mathcal{F}_n^k$ for all k. An endomorphism of the Cuntz algebra is said to be *localized* if it is induced by a unitary element of some finite matrix algebra, that is, it is in $\bigcup_{k=1}^{\infty} \mathcal{F}_n^k$, [2].

COROLLARY 2.2. *Suppose that $\alpha_{\underline{u}}$ extends to an automorphism of the Cuntz algebra. Then the following conditions are equivalent:*
(a) *one extension of $\alpha_{\underline{u}}$ is localized;*
(b) *all extensions of $\alpha_{\underline{u}}$ are localized;*
(c) *eventually $u_{k+1} u_k^* \in \mathbb{T}1$.*

PROOF. This result follows from the fact that any two such extensions must differ by a gauge automorphism. Moreover, a unitary map implementing any of them is obtained through Equation (2.2) up to phases, as explained above. □

https://doi.org/10.1017/S1446788711001017 Published online by Cambridge University Press

3. Outlook

We would like to mention a few related problems. On the one hand, one should investigate the detailed structure of $\mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n)$ and find an intrinsic characterization of automorphisms of $\mathcal{F}_n$ that extend to $\mathcal{O}_n$. On the other hand, one should study the analogous problems for the extension of automorphisms from the diagonal $\mathcal{D}_n$ to $\mathcal{F}_n$ and from $\mathcal{D}_n$ to $\mathcal{O}_n$.

In this respect, we can say something more about the extension of automorphisms from $\mathcal{D}_n$ to $\mathcal{O}_n$. In particular, we are now ready to show the existence of automorphisms of $\mathcal{D}_n$ that do not extend to (possibly proper) endomorphisms of $\mathcal{O}_n$. This will easily follow also from Theorem 3.3 below, but the following observation is more in line with the criterion given in Theorem 2.1.

PROPOSITION 3.1. *Let α be a product type automorphism of the diagonal, that is, $\alpha(\varphi^k(\mathcal{D}_n^1)) = \varphi^k(\mathcal{D}_n^1)$ for all $k \geq 0$. Then α extends to a (possibly proper) endomorphism of $\mathcal{O}_n$ if and only if the action on each $\varphi^k(\mathcal{D}_n^1)$ is eventually identical. In that case, α extends to a permutation automorphism of $\mathcal{O}_n$.*

PROOF. Suppose that α extends to an endomorphism λ_u of $\mathcal{O}_n$. Then it follows that $\lambda_u(\varphi^k(\mathcal{D}_n^1)) = \varphi^k(\mathcal{D}_n^1)$ for all k and an easy induction shows that $\mathrm{Ad}(u)$ preserves each $\varphi^k(\mathcal{D}_n^1)$. Thus, in particular, u is in the normalizer of $\mathcal{D}_n$. Consequently (see [10]), we may write $u = wv$ where $w \in \mathcal{S}_n$ and $v \in \mathcal{U}(\mathcal{D}_n)$, where $\mathcal{S}_n$ is the subgroup of $\mathcal{U}(\mathcal{O}_n)$ of unitaries that can be written as finite sums of words in S_i and S_i^*.

It now follows that the restriction α of λ_u to $\mathcal{D}_n$ coincides with the restriction of λ_w to $\mathcal{D}_n$. Thus $\mathrm{Ad}(w)$ preserves each $\varphi^k(\mathcal{D}_n^1)$ and so we may deduce that the restriction to $\mathcal{D}_n$ of the trace τ is $\mathrm{Ad}(w)$-invariant. However, this is only possible if w belongs to $\mathcal{F}_n$, that is, if w is a permutation matrix.

Since $\mathrm{Ad}(w)$ preserves each $\varphi^k(\mathcal{D}_n^1)$, it follows that

$$w = w_1 \varphi(w_2) \varphi^2(w_3) \cdots \varphi^r(w_{r+1})$$

for some positive integer r and permutation matrices w_j in $\mathcal{F}_n^1$ (where $j = 1, \ldots, r$). This implies that the restriction of λ_w to $\varphi^k(\mathcal{F}_n^1)$ coincides with the restriction of $\mathrm{Ad}(\varphi^k(w_{k+1} w_k \cdots w_2 w_1))$. This means that λ_w, in restriction to $\mathcal{F}_n$, is a product type automorphism of $\mathcal{F}_n$ for which the limit in Equation (2.2) is equal to 1 (actually the terms of the sequence eventually stabilize). Therefore, by Theorem 2.1, λ_w is (and thus α extends to) an automorphism of $\mathcal{O}_n$. □

We denote by $\mathrm{EndAut}(\mathcal{O}_n, \mathcal{D}_n)$ the subsemigroup of $\mathrm{End}(\mathcal{O}_n)$ of those endomorphisms that restrict to automorphisms of $\mathcal{D}_n$ and by $\mathrm{End}_{\mathcal{D}_n}(\mathcal{O}_n)$ the subsemigroup of $\mathrm{EndAut}(\mathcal{O}_n, \mathcal{D}_n)$ of those endomorphisms acting trivially on $\mathcal{D}_n$. Then

$$\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n) \subset \mathrm{EndAut}(\mathcal{O}_n, \mathcal{D}_n),$$

by the analysis in [4]. However, the following result is already implicit in [6].

https://doi.org/10.1017/S1446788711001017 Published online by Cambridge University Press

PROPOSITION 3.2. *With the notation above,*

$$\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n) = \mathrm{End}_{\mathcal{D}_n}(\mathcal{O}_n).$$

PROOF. Let u be a unitary element of $\mathcal{O}_n$. If $\lambda_u(x) = x$ for all $x \in \mathcal{D}_n$ then it is easy to see by induction on k that u commutes with $\varphi^k(\mathcal{D}_n^1)$ for all $k \geq 1$, and therefore

$$u \in \Big(\bigcup_{k\geq 0} \varphi^k(\mathcal{D}_n^1)\Big)' \cap \mathcal{O}_n = \mathcal{D}_n' \cap \mathcal{O}_n.$$

Since the diagonal $\mathcal{D}_n$ is a maximal abelian subalgebra of $\mathcal{O}_n$, it follows that $u \in \mathcal{D}_n$ and therefore $\lambda_u \in \mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n)$ by [6]. □

THEOREM 3.3. *The restriction map*

$$r : \mathrm{EndAut}(\mathcal{O}_n, \mathcal{D}_n) \to \mathrm{Aut}(\mathcal{D}_n) \tag{3.1}$$

is not surjective. Furthermore, $r(\mathrm{EndAut}(\mathcal{O}_n, \mathcal{D}_n))$ is not a subgroup of $\mathrm{Aut}(\mathcal{D}_n)$. Indeed, it is the disjoint union of the subgroup of those automorphisms that extend to automorphisms of $\mathcal{O}_n$, and the subsemigroup of those automorphisms that extend to proper endomorphisms of $\mathcal{O}_n$.

PROOF. The first statement follows from Proposition 3.1.

Let u be a unitary element of $\mathcal{O}_n$ such that λ_u is a proper endomorphism of $\mathcal{O}_n$ and $\lambda_u(\mathcal{D}_n) = \mathcal{D}_n$ (see [4]). We claim that $(\lambda_u|_{\mathcal{D}_n})^{-1} \in \mathrm{Aut}(\mathcal{D}_n)$ does not extend to an endomorphism of $\mathcal{O}_n$. For otherwise, let λ_v be such an extension. Then $\lambda_v\lambda_u(x) = x$ for all $x \in \mathcal{D}_n$ and thus $\lambda_v\lambda_u$ is an automorphism of $\mathcal{O}_n$, by Proposition 3.2, which is a contradiction. A similar argument shows that if two endomorphisms λ_{u_1} and λ_{u_2} of $\mathcal{O}_n$ restrict to the same automorphism of $\mathcal{D}_n$ then they are either both automorphisms or both proper endomorphisms. □

From this discussion, the automorphisms of $\mathcal{D}_n$ obtained by restriction of proper endomorphisms of $\mathcal{O}_n$ are necessarily not of product type.

Finally, we observe that the subgroup $r(\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n))$ is not normal in $\mathrm{Aut}(\mathcal{D}_n)$. Indeed, by Gelfand duality, $\mathrm{Aut}(\mathcal{D}_n) \simeq \mathrm{Homeo}(\mathcal{C})$, where $\mathcal{C}$ is the Cantor set, and the latter group is known to be simple.

References

[1] R. J. Archbold, 'On the "flip-flop" automorphism of $C^*(S_1, S_2)$', *Q. J. Math.* **30** (1979), 129–132.

[2] R. Conti and C. Pinzari, 'Remarks on endomorphisms of Cuntz algebras', *J. Funct. Anal.* **142** (1996), 369–405.

[3] R. Conti, M. Rørdam and W. Szymański, 'Endomorphisms of $\mathcal{O}_n$ which preserve the canonical UHF-subalgebra', *J. Funct. Anal.* **259** (2010), 602–617.

[4] R. Conti and W. Szymański, 'Labeled trees and localized automorphisms of the Cuntz algebras', *Trans. Amer. Math. Soc.,* arXiv:0805.4654, to appear.

[5] J. Cuntz, 'Simple C^*-algebras generated by isometries', *Comm. Math. Phys.* **57** (1977), 173–185.

[6] J. Cuntz, ‘Automorphisms of certain simple C^*-algebras’, in: *Quantum Fields—Algebras, Processes* (ed. L. Streit) (Springer, Vienna–New York, 1980).

[7] K. R. Davidson, *C^*-Algebras by Example*, Fields Institute Monographs, 6 (American Mathematical Society, Providence, RI, 1996).

[8] D. E. Evans and Y. Kawahigashi, *Quantum Symmetries on Operator Algebras*, Oxford Mathematical Monographs (Oxford University Press, Oxford, 1998).

[9] J. G. Glimm, ‘On a certain class of operator algebras’, *Trans. Amer. Math. Soc.* **95** (1960), 318–340.

[10] S. C. Power, ‘Homology for operator algebras, III. Partial isometry homotopy and triangular algebras’, *New York J. Math.* **4** (1998), 35–56.

ROBERTO CONTI, Dipartimento di Scienze,
Università di Chieti-Pescara ‘G. D’Annunzio’, Viale Pindaro 42,
I-65127 Pescara, Italy
e-mail: conti@sci.unich.it

https://doi.org/10.1017/S1446788711001017 Published online by Cambridge University Press