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NOTE ON SPECTRA OF NON-SELFADJOINT OPERATORS OVER DYNAMICAL SYSTEMS

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Abstract We consider equivariant continuous families of discrete one-dimensional operators over arbitrary dynamical systems. We introduce the concept of a pseudo-ergodic element of a dynamical system. We then show that all operators associated to pseudo-ergodic elements have the same spectrum and that this spectrum agrees with their essential spectrum. As a consequence we obtain that the spectrum is constant and agrees with the essential spectrum for all elements in the dynamical system if minimality holds.

Keywords: minimal dynamical system; pseudo-ergodicity; spectrum

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1. Introduction

Selfadjoint random operators arise in the quantum mechanical treatment of disordered solids. Their study has been a key focus of mathematical physics in the last four decades. Indeed, in an impressive number of (classes of) specific examples explicit spectral features (such as pure point spectrum or purely singular continuous spectrum or Cantor spectra) could be proven, see e.g. the surveys and monographs of [**6**, **13**, **15**, **28**, **37**, **44**].

A particularly rich class of examples has been treated in one dimension. Corresponding models arise mostly by codings of topological dynamical systems via sample functions.

A very basic result in this context is constancy of the spectrum provided the underlying dynamical system is minimal and the selfadjoint operators satisfy a weak continuity condition. In fact, this constancy of the spectrum has been (re)proven in various works. For almost periodic operators it can be inferred from [**22**], see Chapter 10 of [**11**] as well. For special quasicrystal operators, a statement is contained in [**3**]. A rather general result

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for minimal systems is then discussed in [**29**]. In any case, the constancy is a rather direct consequence of a semicontinuity property of the spectrum of selfadjoint operators.

Now, recent years have seen quite some interest in non-selfadjoint random type operators, see e.g. [**4**, **5**, **12**, **16**, **18**–**20**, **35**, **36**, **42**] and references therein. In this context many spectral questions are wide open. In fact, even the most basic issue of constancy of the spectrum of operators associated to minimal dynamical systems cannot be inferred immediately as the basic argument from the selfadjoint case completely breaks down. The reason for this break down is that the spectra of non-selfadjoint operators do not have a semicontinuity property (as is well known, see e.g. [**23**, Example IV.3.8], compare § 4 below as well).

At the same time the concept of pseudo-ergodicity has been brought forward in [**16**] (and has been successfully employed since, see e.g. [**7**, **9**, **31**, **33**]) in the context of non-selfadjoint operators in order to deal with random examples without having to worry about a stochastic component. In this context, some version of constancy of the spectrum could be shown. However, this does not give constancy of the spectrum for all involved operators but only among those satisfying the pseudo-ergodicity condition.

The aim of the present note is to reconcile these different points of view. Specifically, we introduce the concept of a pseudo-ergodic element of an arbitrary dynamical system in § 3 as well as the setting of equivariant operator families over a dynamical system in § 4. We then combine these considerations to obtain our main abstract result in § 5. This result, Theorem 5.3, gives constancy of the spectrum among the pseudo-ergodic elements of the dynamical system. As discussed in § 6, this generalizes the result of [**16**] (in the case that the underlying group is $\mathbb{Z}$). If, on the other hand, the dynamical system is minimal then all elements turn out to be pseudo-ergodic and constancy of the spectrum for all involved operators follows, Corollary 5.5. This corollary extends to the non-selfadjoint case the results mentioned above. In § 7 we present some examples of minimal systems which are heavily studied in the selfadjoint case. We also indicate there some non-selfadjoint operators of interest to which the corollary can be applied.

As this discussion shows, Theorem 5.3 can be seen as a generalization of both the result mentioned above for selfadjoint operators in the minimal case and the result mentioned above for non-selfadjoint operators in the pseudo-ergodic case. Along the way we will also show that the spectrum agrees with the essential spectrum (which is also known in the selfadjoint case).

The considerations below are phrased in the setting of dynamical systems over $\mathbb{Z}$. This is for convenience mostly. Indeed, the underlying theory of dynamical systems is valid for substantially more general systems over a discrete countable group Γ. Thus, our main result can be carried over to such systems whenever a suitable version of Theorem 2.1 is at hand.

2. Background: classes of operators

Let $\mathbb{N}$ be the set of all positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\mathbb{Z}$ the set of all integers. Then, for $p \in [1, \infty)$, $\ell^p := \ell^p(\mathbb{Z})$ denotes the space of all two-sided infinite sequences $f : \mathbb{Z} \to \mathbb{C}$ such that $\sum_{j\in\mathbb{Z}} |f(j)|^p$ is finite. Moreover, $\ell^\infty := \ell^\infty(\mathbb{Z})$ is the set of all two-sided infinite, bounded sequences.

A matrix $A : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ is called a *band matrix* if the following two conditions hold.

(i) The map A is bounded, i.e. $\sup_{i,j\in\mathbb{Z}} |A_{i,j}| < \infty$.

(ii) There exists a *band width* $w \in \mathbb{N}$ such that $A_{i,j} = 0$ for all $i, j \in \mathbb{Z}$ satisfying $|i - j| > w$, i.e. A has finitely many non-zero diagonals only.

Any band matrix A generates a linear operator A on each ℓ^p, $p \in [1,\infty]$, by

$$(Af)(i) = \sum_{j\in\mathbb{Z}} A_{i,j} f(j), \quad i \in \mathbb{Z}, f \in \ell^p.$$

Since a band matrix has a finite band width, this sum is always finite. Thus, the operator is well defined. Moreover, we immediately deduce that A is a bounded operator on the space ℓ^p, $p \in [1,\infty]$. Such an operator is called a *band operator*. In the following we will not distinguish between the matrix and the operator. Furthermore, the set of all band operators is denoted by $\mathcal{BO}$.

In the literature the matrix of the operator A is often called the *matrix representation*. One could define a norm by

$$\|A\|_{\mathcal{W}} := \sum_{k\in\mathbb{Z}} \sup_{j\in\mathbb{Z}} |A_{j+k,j}|$$

on the set of band operators. The closure $\mathcal{W}$ of the band operators $\mathcal{BO}$ with respect to this norm $\|\cdot\|_{\mathcal{W}}$ is called the *Wiener algebra*.

Let $p \in [1,\infty]$ be given. Then $\mathcal{BO}$ is a subset of $L(\ell^p)$, the bounded linear operators on ℓ^p. Note that $\mathcal{BO}$ is not closed in $L(\ell^p)$. Let $\mathcal{BDO}(\ell^p)$ be the closure of $\mathcal{BO} \subseteq L(\ell^p)$. These operators are called *band-dominated operators*.

For $m \in \mathbb{N}_0$ define P_m to be the operator of multiplication by the characteristic function of $\{-m, \dots, m\}$ and $\mathcal{P} := \{P_m : m \in \mathbb{N}_0\}$. We then set (see e.g. [**39**, § 1.1])

$$L(\ell^p, \mathcal{P}) := \{A \in L(\ell^p) : \forall m \in \mathbb{N}_0 : \lim_{n\to\infty} \|P_m A(I - P_n)\| + \|(I - P_n)AP_m\| = 0\}.$$

Note that (see e.g. § 1.3.7 in [**31**])

$$\mathcal{BO} \subseteq \mathcal{W} \subseteq \mathcal{BDO}(\ell^p) \subseteq L(\ell^p, \mathcal{P}) \subseteq L(\ell^p)$$

for all $p \in [1,\infty]$. Moreover, $L(\ell^p, \mathcal{P}) = L(\ell^p)$ for $1 < p < \infty$ (but this equality fails for $p = 1$ or $p = \infty$). All three, $(\mathcal{W}, \|.\|_{\mathcal{W}})$, $(\mathcal{BDO}(\ell^p), \|.\|)$ and $(L(\ell^p, \mathcal{P}), \|.\|)$ are Banach algebras that are closed under passing to the inverse operator (see e.g. Theorems 1.1.9, 2.1.8 and 2.5.3 in [**39**]).

Denote by U the shift operator on the set of two-sided infinite sequences with values in $\mathbb{C}$, i.e. $(Uf)(k) := f(k-1)$ for all $k \in \mathbb{Z}$ and $f : \mathbb{Z} \to \mathbb{C}$. Its inverse is given by $(U^{-1}f)(k) := f(k+1)$ for all $k \in \mathbb{Z}$. Clearly, U descends to an isometric bijective operator on any ℓ^p.

For $A = (A_{i,j})_{i,j\in\mathbb{Z}} \in L(\ell^p, \mathcal{P})$ with $p \in [1,\infty]$, we will now look at partial limits (with respect to matrix-entrywise convergence) of the operator sequence $(U^{-n}AU^n)_{n\in\mathbb{Z}}$: A matrix $B : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ is called a *limit operator induced by the operator* A whenever there exists a sequence $(h_k)_{k\in\mathbb{N}}$ of integers such that $\lim_{k\to\infty} |h_k| = \infty$ and

$$B_{i,j} = \lim_{k\to\infty} A_{i+h_k,j+h_k}, \quad i, j \in \mathbb{Z}.$$

Clearly, if A is a band operator with band width $w \in \mathbb{N}$ then a limit operator B induced by A is a band operator as well. Furthermore, its band width is smaller than or equal to w. Similarly, B belongs, respectively, to $\mathcal{W}$, $\mathcal{BDO}(\ell^p)$ or $L(\ell^p, \mathcal{P})$ if A does.

We define by $\sigma^{\mathrm{op}}(A)$ the set of all limit operators induced by A, which is sometimes called the *operator spectrum of* A. By [31, Corollary 3.24], we have $\sigma^{\mathrm{op}}(A) \neq \varnothing$ for $A \in \mathcal{BDO}(\ell^p)$. An operator $A \in L(\ell^p, \mathcal{P})$ is called *self-similar* if $A \in \sigma^{\mathrm{op}}(A)$ holds.

For $p \in [1, \infty]$ and $A \in L(\ell^p, \mathcal{P})$ we write $\mathrm{spec}(A)$ for the spectrum of A and $\mathrm{spec}_{\mathrm{ess}}(A)$ for the essential spectrum of A, that is the spectrum of A modulo compact operators.

With this notation at hand the following theorem holds, as shown in [41].

Theorem 2.1 (see [41, Corollary 12 and Theorem 16]). *Let* $p \in [1, \infty]$ *and* $A \in L(\ell^p, \mathcal{P})$. *Then*

$$\mathrm{spec}_{\mathrm{ess}}(A) \supseteq \bigcup_{B\in\sigma^{\mathrm{op}}(A)} \mathrm{spec}(B).$$

In particular,

$$\mathrm{spec}(A) = \mathrm{spec}_{\mathrm{ess}}(A)$$

if A *is self-similar.*

For band-dominated operators, one can even obtain an equality, see [34], and also [9, 25, 30, 38, 39] for earlier versions.

Proposition 2.2 ([34, Corollary 12]). *Let* $p \in [1, \infty]$ *and* $A \in \mathcal{BDO}(\ell^p)$. *Then*

$$\mathrm{spec}_{\mathrm{ess}}(A) = \bigcup_{B\in\sigma^{\mathrm{op}}(A)} \mathrm{spec}(B).$$

As it turns out, the spectrum and the essential spectrum of $A \in \mathcal{W}$ are independent of $p \in [1, \infty]$, see [24, 32]. We will use the notation $\mathrm{spec}^{\infty}_{\mathrm{point}}(A)$ for the set of eigenvalues of the operator A on the space ℓ^∞.

Proposition 2.3 (see [8, Theorem 3.1]). *Let* $A \in \mathcal{W}$. *Then*

$$\mathrm{spec}_{\mathrm{ess}}(A) = \bigcup_{B\in\sigma^{\mathrm{op}}(A)} \mathrm{spec}(B) = \bigcup_{B\in\sigma^{\mathrm{op}}(A)} \mathrm{spec}^{\infty}_{\mathrm{point}}(B).$$

Remark 2.4. For special selfadjoint operators on ℓ^2 related results are contained in [27], see [6, 37] as well for related results in the context of random selfadjoint operators.

For the remaining part of the paper, we fix $p \in [1, \infty]$.

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3. Background: dynamical systems

Let (X, T) be a dynamical system, i.e. X is a compact metric space and $T : X \to X$ is a homeomorphism. Then, for $x \in X$ the limit sets $L^+(x)$ and $L^-(x)$ are defined by

$$L^{\pm}(x) := \{y \in X : \text{there exists } h_k \to \infty \text{ such that } \lim_{k\to\infty} T^{\pm h_k} x = y\}.$$

Moreover, the *orbit* of $x \in X$ is given as

$$\mathrm{Orb}(x) := \{T^n x : n \in \mathbb{Z}\}.$$

Remark 3.1. In the literature often the limit set $L^+(x)$ is called the ω-limit set and the limit set $L^-(x)$ is called the α-limit set of x.

The following proposition is well known. We include a proof for the convenience of the reader.

Proposition 3.2. *Let (X, T) be a dynamical system. Then, for all $x \in X$ the sets $L^{\pm}(x)$ are non-empty, compact and invariant under T and T^{-1}.*

Proof. Let $x \in X$. We will give a proof for the set $L^+(x)$ only. Analogously, the statement for $L^-(x)$ can be proven.

By compactness of X the sequence $(T^n x)_{n\in\mathbb{N}}$ has a convergent subsequence. Thus, the set $L^+(x)$ is non-empty. Let $y = \lim_{k\to\infty} T^{h_k} x \in L^+(x)$ be arbitrary. Since T is a homeomorphism the limits $\lim_{k\to\infty} T^{h_k \pm 1} x$ exist and they are equal to $T^{\pm 1} y$. Hence, Ty and $T^{-1}y$ are elements of $L^+(x)$ implying that $L^+(x)$ is T and T^{-1}-invariant.

We now turn to proving the compactness of $L^+(x)$. As X is a compact metric space it suffices to show closedness. Thus, let $(y_n)_{n\in\mathbb{N}}$ in $L^+(x)$ be convergent to y in X. For each $n \in \mathbb{N}$ there exists $(h_k^n)_{k\in\mathbb{N}}$ such that $y_n = \lim_{k\to\infty} T^{h_k^n} x$. Thus, there exists a subsequence $(k_n)_{n\in\mathbb{N}}$ such that for $(\tilde{h}_n)_{n\in\mathbb{N}} := (h_{k_n}^n)_{n\in\mathbb{N}}$ we have $\tilde{h}_n \to \infty$ and $T^{\tilde{h}_n} x \to y$. Consequently, $L^+(x) \subseteq X$ is closed. □

We will be interested in those elements of X for which the union of the limit sets agrees with X.

Definition 3.3 (pseudo-ergodic elements). Let (X, T) be a dynamical system. An $x \in X$ is called *pseudo-ergodic* if

$$X = L^+(x) \cup L^-(x).$$

The set of all pseudo-ergodic elements of X is denoted as $X_{\Psi\mathrm{E}}$.

We have the following characterization of pseudo-ergodicity of an $x \in X$, which is not isolated.

Proposition 3.4. *Let (X, T) be a dynamical system. For an $x \in X$, which is not isolated, the following assertions are equivalent.*

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(i) *The orbit of x is dense.*

(ii) *The element x is pseudo-ergodic.*

Proof. The implication (ii) $\Longrightarrow$ (i) is clear (and holds for any $x \in X$). We now show (i) $\Longrightarrow$ (ii). As $L^+(x) \cup L^-(x)$ is closed and invariant under T, it suffices to show that it contains x. As x is not isolated, there exists a sequence (y_n) in X converging to x such that the y_n, $n \in \mathbb{N}$, are pairwise different and none of them equals x. By (i) we can find for any y_n an index k_n with $d(T^{k_n}x, y_n) \leqslant (1/3)d(y_n, x)$, where we denote a metric on X by d. The assumption on the (y_n) gives that the index set $\{k_n : n \in \mathbb{N}\}$ is infinite. Moreover,

$$\lim_{n\to\infty} T^{k_n}x = \lim_{n\to\infty} y_n = x.$$

This shows $x \in L^+(x) \cup L^-(x)$. □

Remark 3.5. It is not hard to see that the closure of $\mathrm{Orb}(x)$ equals $L^+(x) \cup L^-(x)$ if and only if x belongs to $L^+(x) \cup L^-(x)$. Of course, one can easily give examples where x does not belong to $L^+(x) \cup L^-(x)$. Consider e.g. the space $\{0,1\}^{\mathbb{Z}}$ of sequences with values in $\{0,1\}$ over $\mathbb{Z}$ with the shift operation $Tx(n) = x(n+1)$. Let 1_0 be the characteristic function of $\{0\}$. Then, both $L^+(1_0)$ and $L^-(1_0)$ consist only of the function with value 0 everywhere. Thus, if we define X to be the closure of the orbit of 1_0, the orbit of 1_0 will be dense in X but 1_0 will not be pseudo-ergodic. This shows that the assumption that x is not isolated is necessary in the previous proposition.

A dynamical system (X, T) is called *minimal* if the orbit of x is dense in X for each $x \in X$. We are now going to study the relationship between minimality and pseudo-ergodicity (of all elements). We start with the following well-known result. We include a proof for the convenience of the reader.

Proposition 3.6. *Let (X, T) be a dynamical system. Then the following assertions are equivalent.*

(i) *The dynamical system (X, T) is minimal.*

(ii) *For all $x \in X$ the equation $L^+(x) = X$ holds.*

(iii) *For all $x \in X$ the equation $L^-(x) = X$ holds.*

Proof. We will prove that (i) and (ii) are equivalent. The equivalence of (i) and (iii) follows similarly. By the obvious inclusion $L^+(x) \subseteq \overline{\mathrm{Orb}(x)}$ for each $x \in X$ the implication (ii) $\Rightarrow$ (i) is clear.

Assume now (i) is true. Let $x \in X$ and choose an $y \in L^+(x)$. Such a choice is possible as $L^+(x)$ is not empty due to Proposition 3.2. By the same proposition the set $L^+(x)$ is closed and invariant under T and T^{-1}. Thus, we conclude $\overline{\mathrm{Orb}(y)} \subseteq L^+(x)$. By (i) this immediately implies

$$X = \overline{\mathrm{Orb}(y)} \subseteq L^+(x) \subseteq X$$

leading to assertion (ii). □

From the preceding considerations, we rather directly obtain the following characterization of minimality in terms of pseudo-ergodic elements.

Proposition 3.7. *Let (X, T) be a dynamical system. Then the following assertions are equivalent.*

(i) *The dynamical system (X, T) is minimal.*

(ii) *The equality $X = X_{\Psi\mathrm{E}}$ holds.*

(iii) *The set $X_{\Psi\mathrm{E}}$ is closed and non-empty.*

Proof. The implication (i) $\Longrightarrow$ (ii) follows directly from Proposition 3.6. The implication (ii) $\Longrightarrow$ (iii) is clear. It remains to show (iii) $\Longrightarrow$ (i). Now, the set $X_{\Psi\mathrm{E}}$ is clearly invariant under T. Thus, with any x it will contain $\mathrm{Orb}(x)$ and from (iii) and the definition of pseudo-ergodicity we then infer that $X_{\Psi\mathrm{E}} \supseteq \overline{\mathrm{Orb}(x)} \supseteq L^+(x) \cup L^-(x) = X$. Thus any element of x is pseudo-ergodic. In particular, the orbit of any element of x is dense and (i) follows. $\square$

4. Operators on dynamical systems

Given a dynamical system (X, T), a map $A : X \to L(\ell^p, \mathcal{P})$ is called a *family of operators over* (X, T) if the following conditions hold.

(i) $\sup_{x\in X} \sup_{i,j\in\mathbb{Z}} |A(x)_{i,j}| < \infty$. (Uniform boundedness)

(ii) $A(Tx) = U^{-1}A(x)U$ for all $x \in X$. (Equivariance)

(iii) The map $x \mapsto A(x)_{i,j}$ is continuous for each $i, j \in \mathbb{Z}$. (Continuity)

Recall that U is the shift operator, acting as an isometric bijection on our space ℓ^p. The boundedness assumption (i) follows, via the uniform boundedness principle, directly from weak continuity of the map A. More specifically, we have the following result.

Proposition 4.1. *Let a dynamical system (X, T), $A : X \to L(\ell^p, \mathcal{P})$ be given such that the following conditions hold:*

- *$A(Tx) = U^{-1}A(x)U$ for all $x \in X$.*
- *The map A is continuous with respect to the weak operator topology.*

Then, A is a family of operators over (X, T).

Proof. Condition (ii) of the preceding definition is satisfied by assumption. Condition (iii) can be inferred directly from the continuity in the weak operator topology (as the map $L(\ell^p) \longrightarrow \mathbb{C}, B \mapsto B_{i,j}$ is continuous with respect to the weak operator topology for each $i, j \in \mathbb{Z}$). As for (i), we note that the uniform boundedness principle together with compactness of X gives that the family $(A(x))_{x\in X}$ is bounded with respect to the norm of $L(\ell^p)$ (which is the usual operator norm). This directly gives (i). $\square$

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For later use, we also note the following simple consequence of the definition.

Proposition 4.2. *Let (X,T) be a dynamical system, $A : X \to L(\ell^p, \mathcal{P})$ a family of operators over (X,T). Then,*

$$A(T^n x)_{i,j} = A(x)_{i+n,j+n}, \quad i,j,n \in \mathbb{Z}, x \in X.$$

Proof. It suffices to consider the cases $n = 1$ and $n = -1$. (Then, the remaining statements follow easily by induction.) Let $e_j : \mathbb{Z} \to \mathbb{C}$ be defined by $e_j(i) := \delta_{j,i}$ where δ denotes the Kronecker delta. For $n = 1$ a short computation shows

$$A(Tx)_{i,j} = (A(Tx)e_j)(i) = \left(U^{-1}A(x)Ue_j\right)(i) = (A(x)e_{j+1})(i+1) = A(x)_{i+1,j+1}$$

for $i,j \in \mathbb{Z}$ and $x \in X$. The case $n = -1$ can be treated similarly. □

As a bounded operator A on ℓ^p always satisfies $\sup_{i,j\in\mathbb{Z}} |A_{i,j}| < \infty$, it is natural to ask whether condition (i) on the uniform boundedness of a family of operators can be relaxed. This is indeed possible as discussed in the following proposition.

Proposition 4.3. *Let (X,T) be a dynamical system with one dense orbit in X. Consider a map $A : X \to L(\ell^p, \mathcal{P})$ such that $A(Tx) = U^{-1}A(x)U$ holds for all $x \in X$ and the map $x \mapsto A(x)_{i,j}$ is continuous for each $i,j \in \mathbb{Z}$. Then, A is a family of operators over (X,T).*

Proof. We need to show that $\sup_{x\in X} \sup_{i,j\in\mathbb{Z}} |A(x)_{i,j}|$ is finite. Let $y \in X$ with dense orbit $\mathrm{Orb}(y)$ be given. Since $A(y) \in L(\ell^p)$ we know that $\sup_{i,j\in\mathbb{Z}} |A(y)_{i,j}| < \infty$. Furthermore, by Proposition 4.2, we get $\sup_{i,j\in\mathbb{Z}} |A(x)_{i,j}| = \sup_{i,j\in\mathbb{Z}} |A(y)_{i,j}|$ for all $x \in \mathrm{Orb}(y)$. As $\mathrm{Orb}(y) \subseteq X$ is dense and $A(\cdot)_{i,j} : X \to \mathbb{C}, i,j \in \mathbb{Z}$ is continuous it follows

$$\sup_{x\in X} \sup_{i,j\in\mathbb{Z}} |A(x)_{i,j}| = \sup_{x\in\mathrm{Orb}(y)} \sup_{i,j\in\mathbb{Z}} |A(x)_{i,j}| = \sup_{i,j\in\mathbb{Z}} |A(y)_{i,j}| < \infty,$$

which means that A also satisfies the uniform boundedness condition. □

5. Bringing it all together: the main result

In this section we combine the considerations and concepts of the previous sections to state and prove our main result.

Proposition 5.1. *Let (X,T) be a dynamical system, $A : X \to L(\ell^p, \mathcal{P})$ a family of operators over (X,T). Then, the equation*

$$\sigma^{\mathrm{op}}(A(x)) = \{A(y) : y \in L^+(x) \cup L^-(x)\}$$

holds for all $x \in X$.

Proof. We first show '$\subseteq$'. Let $x \in X$ and $B \in \sigma^{\mathrm{op}}(A(x))$ be given. Then, there exists a sequence $(h_k)_{k\in\mathbb{N}}$ of integers with $|h_k| \to \infty$ as $k \to \infty$, such that

$$B_{i,j} = \lim_{k\to\infty} A(x)_{i+h_k,j+h_k} = \lim_{k\to\infty} A(T^{h_k}x)_{i,j}, \quad i,j \in \mathbb{Z},$$

where the second equality follows from Proposition 4.2. Since $|h_k| \to \infty$, $k \to \infty$, and X is compact we can select a subsequence $(h_{k_j})_{j\in\mathbb{N}}$ such that $(T^{h_{k_j}}x)_{j\in\mathbb{N}}$ is convergent to

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$y \in X$ and $(h_{k_j})_{j\in\mathbb{N}}$ tends to ∞ or $-\infty$. Thus, $y \in L^+(x) \cup L^-(x)$. For $i, j \in \mathbb{Z}$, using the continuity of $A(\cdot)_{i,j} : X \to \mathbb{C}$, the equation $B_{i,j} = A(y)_{i,j}$ follows with $y \in L^+(x) \cup L^-(x)$. Hence, B is an element of the set $\{A(y) : y \in L^+(x) \cup L^-(x)\}$.

We now turn to proving '$\supseteq$'. For $x \in X$ and $y \in L^+(x) \cup L^-(x)$ there is a sequence $(h_k)_{k\in\mathbb{N}}$ tending to ∞ or $-\infty$ such that $\lim_{k\to\infty} T^{h_k}x = y$. Then, for $i, j \in \mathbb{Z}$, the continuity of $A(\cdot)_{i,j} : X \to \mathbb{C}$ and Proposition 4.2 imply

$$A(y)_{i,j} = \lim_{k\to\infty} A(T^{h_k}x)_{i,j} = \lim_{k\to\infty} A(x)_{i+h_k,j+h_k}$$

leading to $A(y) \in \sigma^{\mathrm{op}}(A(x))$. □

As a consequence we immediately deduce the following lemma. On the technical level this lemma is the crucial ingredient in the proof of our main result.

Lemma 5.2. *Let (X, T) be a dynamical system and $A : X \to L(\ell^p, \mathcal{P})$ a family of operators over (X, T). Then,*

$$\sigma^{\mathrm{op}}(A(x)) = \{A(y) : y \in X\}$$

holds for all $x \in X_{\Psi\mathrm{E}}$. In particular, for any $x \in X_{\Psi\mathrm{E}}$ the operator $A(x)$ is self-similar (i.e. $A(x) \in \sigma^{\mathrm{op}}(A(x))$).

Proof. This is a direct consequence of Proposition 5.1 and the definition of $X_{\Psi\mathrm{E}}$. □

Now we are able to prove that the spectrum is constant and agrees with the essential spectrum for any pseudo-ergodic element.

Theorem 5.3. *Let (X, T) be a dynamical system and $A : X \to L(\ell^p, \mathcal{P})$ a family of operators over (X, T). Set*

$$\Sigma := \bigcup_{x\in X} \mathrm{spec}(A(x)).$$

Then

$$\mathrm{spec}(A(x)) = \mathrm{spec}_{\mathrm{ess}}(A(x)) = \Sigma$$

holds for all $x \in X_{\Psi\mathrm{E}}$.

Remark 5.4. A similar result was proven for so-called pseudo-ergodic operators in [**16**]. In fact, the above result is a generalization of the result of [**16**] (in the case $\Gamma = \mathbb{Z}$). Details are discussed below in § 6. We note already here that every pseudo-ergodic operator is self-similar and for pseudo-ergodic operators the operator spectrum $\sigma^{\mathrm{op}}(A)$ is very large.

Proof of Theorem 5.3. Let $x \in X_{\Psi\mathrm{E}}$. By Theorem 2.1 and Lemma 5.2 it follows

$$\mathrm{spec}(A(x)) \supseteq \mathrm{spec}_{\mathrm{ess}}(A(x)) \supseteq \bigcup_{B\in\sigma^{\mathrm{op}}(A(x))} \mathrm{spec}(B) = \bigcup_{y\in X} \mathrm{spec}(A(y)) = \Sigma.$$

As $x \in X$ appears in the rightmost union, we obtain the assertion. □

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Corollary 5.5. *Let (X, T) be a minimal dynamical system, $A : X \to L(\ell^p, \mathcal{P})$ a family of operators over (X, T). Set*

$$\Sigma := \bigcup_{x \in X} \operatorname{spec}(A(x)).$$

Then

$$\operatorname{spec}(A(x)) = \operatorname{spec}_{\mathrm{ess}}(A(x)) = \Sigma$$

holds for all $x \in X$.

Proof. Due to minimality we have $X_{\Psi \mathrm{E}} = X$ by Proposition 3.7. Thus, the statement follows directly from the previous theorem. □

Remark 5.6. In particular, the corollary gives that the spectrum of $A(x)$, $x \in X$ is constant and agrees with the essential spectrum for all operators $A(x)$, $x \in X$. As discussed in the introduction, this is well known in the case $p = 2$ whenever $A(x)$, $x \in X$, is selfadjoint. For such operators the proof of constancy of the spectrum relies on a semi-continuity property of the spectrum found e.g. in [**40**, Theorem VIII.24 (a)]. This semi-continuity does not apply in our case. Indeed, we can consider the example given in [**23**, Example IV.3.8]: For $c \in \mathbb{R}$ let

$$A_c := \begin{pmatrix} \ddots & & & & & \\ \ddots & 0 & & & & \\ & 1 & 0 & & & \\ & & c & 0 & & \\ & & & 1 & 0 & \\ & & & & \ddots & \ddots \end{pmatrix}.$$

Then, for $c \neq 0$ we have $\operatorname{spec}(A_c) = \{z \in \mathbb{C} : |z| = 1\}$. However, $\operatorname{spec}(A_0) = \{z \in \mathbb{C} : |z| \leqslant 1\}$, although we have strong convergence of A_c to A_0 and in fact even norm convergence $\|A_c - A_0\| \to 0$ as $c \to 0$. Thus, the corollary gives a new result even for $p = 2$ whenever A is not selfadjoint.

Remark 5.7. For a minimal dynamical system (X, T) and a family of operators $A : X \to \mathcal{W}$, a combination of Proposition 2.3 and Theorem 5.3 states that for each $x \in X$ and any $\lambda \in \operatorname{spec}(A(x))$ there is $y \in X$ with a generalized bounded eigenfunction to the eigenvalue λ. So one might ask whether there exists a generalized bounded eigenfunction to $A(x)$ itself for $\lambda \in \operatorname{spec}(A(x))$. We consider this an interesting question.

6. Pseudo-ergodic operators over $\mathbb{Z}$

In this section we discuss shortly how a main result of [**16**] is a special case of Theorem 5.3.

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Consider a compact subset S of the complex plane with the induced topology. Let $X := S^{\mathbb{Z}}$ with the product topology and define T via

$$Tx(n) = x(n+1).$$

Then, (X, T) is a dynamical system, known as *shift over* S. In this situation Davies [**16**] calls an $x \in X$ pseudo-ergodic if for any $\varepsilon > 0$, any $n \in \mathbb{N}$ and any $(y_1, \ldots, y_n) \in S^n$ there exists a $k \in \mathbb{Z}$ with

$$\|(x(k+1), \ldots, x(k+n)) - (y_1, \ldots, y_n)\| \leqslant \varepsilon.$$

(Here, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{C}^n$.) Then, it is not hard to see (compare also [**31**]) that x is pseudo-ergodic in the sense of Davies if and only if $\mathrm{Orb}(x)$ is dense in X. This, in turn, is equivalent to x being pseudo-ergodic in the sense of our definition. Indeed, if S consists of at least two elements, then x can not be an isolated element of X and, hence, Proposition 3.4 gives the desired equivalence. If S consists of only one element then X consists of one element only and this element is clearly pseudo-ergodic both in the sense of our definition and the definition of Davies. Thus, our setting contains the setting of [**16**] and hence our main result, Theorem 5.3, generalizes the corresponding result of [**16**].

One further remark may be in order here: The setting of [**16**] is not restricted to shifts with respect to $\mathbb{Z}$. Instead rather general discrete groups are allowed for. In this sense, the results of Davies are still somewhat more general than ours.

7. Examples of minimal dynamical systems

In this section we discuss some examples where Corollary 5.5 can be applied. These are Sturmian models, quasiperiodic models, and almost periodic models. For all of these classes of models associated selfadjoint Schrödinger type operators attract considerable attention.

7.1. Sturmian models

We consider $\{0, 1\}$ with the discrete topology and equip $\{0, 1\}^{\mathbb{Z}}$ with the product topology and the 'shift' operation T given by $Tx(n) = x(n+1)$. In this way $(\{0, 1\}^{\mathbb{Z}}, T)$ becomes a dynamical system. Consider now $\alpha \in (0, 1)$ irrational and define

$$V_\alpha : \mathbb{Z} \longrightarrow \{0, 1\}, V_\alpha(n) := 1_{(1-\alpha, 1]}(n\alpha \bmod 1).$$

Let X_α be the closure of the orbit of V_α in $\{0, 1\}^{\mathbb{Z}}$. Then, X_α is invariant under T and (X_α, T) is a dynamical system. It is known as a *Sturmian dynamical system* or *Sturmian subshift (with rotation number* α*)*. Sturmian dynamical systems are minimal.

Sturmian dynamical systems play an important role in the investigations of a special type of solid discovered in 1982 and later called quasicrystals, see [**13**, **15**, **28**] for surveys on such operators and further references. In fact, the most prominent model in the investigation of quasicrystals is the Sturmian subshift with rotation number $\alpha =$ golden mean. This is known as the *Fibonacci model*.

Sturmian dynamical systems have the following complexity feature: For any natural number n the set

$$\{(\omega(k+1)\ldots\omega(k+n)) : \omega \in X_\alpha, k \in \mathbb{Z}\}$$

has exactly $n+1$ elements. In fact, this latter property even characterizes Sturmian dynamical systems (among the minimal subshifts over $\{0,1\}$). As we do not need this, we refrain from further discussion.

For the quantum mechanical treatment of conductance properties of quasicrystals (in one dimension) mostly Sturmian models are considered. There, one considers the function

$$\delta : X_\alpha \longrightarrow \{0,1\}, \delta(x) := x(0).$$

This is then used to define, for any $x \in X_\alpha$, the multiplication operator V_x on ℓ^2 satisfying

$$(V_x f)(n) := \delta(T^n x) f(n) = x(n) f(n).$$

The conductance properties are then encoded in the spectral theory of the family of operators

$$H_x := U + U^{-1} + \lambda V_x : \ell^2 \longrightarrow \ell^2$$

for $x \in X_\alpha$. Here, $\lambda \neq 0$ is arbitrary and U is the shift $Uf(n) = f(n-1)$ (which was already discussed above). This is a family of selfadjoint operators, see the mentioned references [**13**, **28**] for further discussion.

However, we could easily go over to a family of non-selfadjoint operators by considering e.g. the family $A : X_\alpha \longrightarrow \mathcal{BO}$ with

$$A(x) := U + \lambda V_x,$$

$x \in X_\alpha$, for $\lambda \neq 0$.

7.2. Quasiperiodic models

We consider for $n \in \mathbb{N}$ the set $\mathbb{T} := \mathbb{R}^n/\mathbb{Z}^n$. For $\beta \in \mathbb{R}^n$ we then define the 'rotation' action R on $\mathbb{T}$ by

$$R : \mathbb{T} \longrightarrow \mathbb{T},\ R(v + \mathbb{Z}^n) := v + \beta + \mathbb{Z}^n.$$

Then, $(\mathbb{T}, R)$ is a dynamical system. If the entries of β are rationally independent, then this dynamical system is minimal. Consider now a continuous function $\varphi : \mathbb{T} \longrightarrow \mathbb{C}$. Then, this function, for any $x \in \mathbb{T}$, gives rise to the multiplication operator V_x acting on ℓ^2 via $(V_x f)(n) = \varphi(R^n x) f(n)$. This then induces the family

$$H_x := U + U^{-1} + \lambda V_x : \ell^2 \longrightarrow \ell^2$$

for $x \in \mathbb{T}$. Here, $\lambda \neq 0$ is arbitrary and U is, again, the shift $Uf(n) = f(n-1)$. If φ is real-valued, this is a family of selfadjoint operators. They are known as *discrete quasiperiodic Schrödinger operators*.

The most prominent example is the case $n = 1$, $\varphi(x + \mathbb{Z}) = \cos(2\pi x)$, and β irrational. The associated operator is known as the *almost Mathieu operator*. It has been studied a great deal, see e.g. the surveys [**14**, **21**, **26**].

Again, we can easily go over to a family of non-selfadjoint operators by considering e.g. the family $A : X_\alpha \longrightarrow \mathcal{BO}$ with

$$A(x) := U + \lambda V_x,$$

$x \in X_\alpha$, for $\lambda \neq 0$.

7.3. Almost periodic models

Consider again the shift (X, T) over a compact subset $S \subseteq \mathbb{C}$ as in § 6 (i.e. $X := S^{\mathbb{Z}}$ with the product topology and $Tx(n) := x(n+1)$). Then, $x \in X$ is called *almost periodic* if $\mathrm{Orb}(x)$ is relatively compact in ℓ^∞. In this case, let X_x be the closure of $\mathrm{Orb}(x)$ in X, which due to the relative compactness in ℓ^∞ coincides with the closure in ℓ^∞ (sometimes called the *hull* of x). Clearly, (X_x, T) is a minimal dynamical system. For $y \in X_x$ let

$$H_y := U + U^{-1} + M_y : \ell^2 \longrightarrow \ell^2,$$

where U is again the shift $Uf(n) = f(n-1)$ and $M_y f(n) := y(n)f(n)$ is the multiplication operator induced by y. For $S \subseteq \mathbb{R}$, the operators H_y are selfadjoint and known as *discrete almost periodic Schrödinger operators.*

Starting with [**17**], almost periodic models (in arbitrary dimensions) were studied intensively, see e.g. [**10**, **37**, **43**]. The first systematic study of almost periodic Schrödinger operators was given in [**1**, **2**].

Note that almost periodic models generalize the above-mentioned quasiperiodic models, as can easily be inferred from the continuity of $\mathbb{T} \longrightarrow \ell^\infty$, $v \mapsto (\varphi(R^n v))_{n \in \mathbb{Z}}$.

8. Some further aspects

In this section we discuss how a family of operators can be seen as a dynamical system itself (under a weak continuity assumption) and how this dynamical system is related to the original dynamical system.

Let a dynamical system (X, T) and a family of operators $A : X \to L(\ell^p, \mathcal{P})$ be given. Assume that A is continuous with respect to the weak operator topology. (This is for example the case if A takes values in $\mathcal{BO}$ and there is a uniform upper bound for the band width.) Then, the set $X_A := \{A(x) : x \in X\}$ (with the weak operator topology) is compact and the map

$$X \longrightarrow X_A, x \mapsto A(x),$$

is continuous. Moreover, the map

$$Ad_U : B \mapsto U^{-1} B U$$

is a homeomorphism of this set. Thus, (X_A, Ad_U) is a dynamical system. In this way, any such family of operators comes with two dynamical systems viz the system (X, T) and the system (X_A, Ad_U). We will now study the relationship between these two dynamical systems.

Clearly, (X_A, Ad_U) is a factor of (X, T) via the map A, i.e. A gives a continuous surjective map from X to X_A intertwining the actions of T and Ad_U.

Moreover, if the family of operators $A : X \to L(\ell^p, \mathcal{P})$ is furthermore injective, both systems are conjugate (i.e. there exists a homeomorphism between them intertwining the actions of T and Ad_U). Thus, in that case one of the dynamical systems is minimal if and only if the other one is minimal as well. We lose this as soon as we do not have the injectivity of $A : X \to L(\ell^p, \mathcal{P})$. This can be seen by the next example.

Example 8.1. Let (X, T) be a an arbitrary non-minimal dynamical system and $B \in \mathcal{BO}$ be a band operator satisfying $U^{-1}BU = B$. Then, the constant family $A(x) := B$, $x \in X$ is a family of operators over (X, T) and continuous with respect to the weak operator topology. Then, (X_A, Ad_U) is minimal (as it only consists of one point). Clearly, $A : X \to \mathcal{BO}$ is not injective in this case.

As we have seen in Lemma 5.2, minimality of (X, T) implies that A is self-similar. One might ask if the converse holds as well. Clearly, the injectivity of $A : X \to L(\ell^p, \mathcal{P})$ is a necessary condition. But if A is injective and self-similar, it is also not necessarily true that (X, T) is minimal. This can be seen by the following example.

Example 8.2. Let (X_1, T) and (X_2, T) be two different minimal subshifts with alphabet $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathcal{A}_1 \cap \mathcal{A}_2 = \varnothing$. Consider two bijective maps $\Phi_1 : \mathcal{A}_1 \to \{1, \ldots, |\mathcal{A}_1|\}$ and $\Phi_2 : \mathcal{A}_2 \to \{|\mathcal{A}_1| + 1, \ldots, |\mathcal{A}_1| + |\mathcal{A}_2|\}$. Define two families of operators $A^{(1)} : X_1 \to \mathcal{BO}$ and $A^{(2)} : X_2 \to \mathcal{BO}$ by

$$A^{(k)}(x)_{i,i} := \Phi_k(x(i)) \quad \text{and} \quad A^{(k)}(x)_{i,j} := 0, \quad x \in X_k,\ i, j \in \mathbb{Z},\ i \neq j,\ k = 1, 2.$$

Then, $X := X_1 \cup X_2$ together with T forms a dynamical system. By $A(x) := A^{(k)}(x)$, $x \in X_k$ we define a map $A : X \to \mathcal{BO}$. By minimality of (X_k, T), $k = 1, 2$ it follows that $A : X \to \mathcal{BO}$ is a family of operators and each $A(x)$, $x \in X$ is self-similar. Moreover, A is injective. On the other hand, it can be seen immediately that (X, T) is not minimal.

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