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ON WEAKLY *S*-PERMUTABLY EMBEDDED SUBGROUPS OF FINITE GROUPS

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Abstract

In this paper, we obtain some criteria for p-nilpotency and p-supersolvability of a finite group and extend some known results concerning weakly S-permutably embedded subgroups. In particular, we generalise the main results of Zhang *et al.* ['Sylow normalizers and p-nilpotence of finite groups', *Comm. Algebra* **43**(3) (2015), 1354–1363].

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1. Introduction

Throughout the paper, we suppose that G is a finite group and p is a prime. Let $\pi(G)$ be the set of all the prime divisors of $|G|$. To state our results, we need to recall some notation. According to Kegel [10], a subgroup H of a finite group G is called an S-permutable subgroup of G if H permutes with every Sylow subgroup of G. According to Ballester-Bolinches and Pedraza-Aguilera [2], a subgroup H of a finite group G is said to be S-permutably embedded in G if, for each prime p dividing $|H|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of some S-permutable subgroup of G. According to Li *et al.* [16], a subgroup H of a finite group G is called a weakly S-permutably embedded subgroup of G if there exist $T \trianglelefteq\trianglelefteq G$ and $H_1 \leq G$ such that $G = HT$, $H \cap T \leq H_1 \leq H$ and H_1 is S-permutably embedded in G. Following Berkovich and Isaacs [3], if G is a finite group and p is a prime divisor of $|G|$, we denote by G_p^* the unique smallest normal subgroup of G for which the corresponding factor group is abelian of exponent dividing $p-1$. It is well known that G is p-supersolvable if and only if G_p^* is p-nilpotent (see [3]).

Let p be a prime dividing the order of a finite group G and $P \in \mathrm{Syl}_p(G)$. Let $\mathbb{D}(P)$ denote the set of subgroups $P_1 \leq P$ for which there exists $P_2 \leq G$ with $P_1 \cap O^p(G_p^*) \leq P_2 \leq P_1$ and P_2 is S-permutably embedded in G. It is not difficult to see that if $P_1 \leq P$ is weakly S-permutably embedded in G, then $P_1 \in \mathbb{D}(P)$. However, there exist a finite group G with p an odd prime divisor of $|G|$, and $P \in \mathrm{Syl}_p(G)$ with $|P| \geq p^{e+1}$, where e

is a positive integer, such that every subgroup P_1 of P of order p^e is in $\mathbb{D}(P)$, but P has a subgroup P_3 of order p^e which is not weakly S-permutably embedded in G. See the following example.

EXAMPLE 1.1. Let $T = \langle a, b \mid a^{p^2} = b^2 = 1, b^{-1}ab = a^{-1}\rangle \cong D_{2p^2}$, where p is an odd prime. There exists $c \in Aut(T)$ such that $a^c = a$ and $b^c = ba$. Consider $G = T \rtimes \langle c\rangle \cong \langle a, b, c \mid a^{p^2} = b^2 = c^{p^2} = 1, b^{-1}ab = a^{-1}, c^{-1}ac = a, c^{-1}bc = ba\rangle$. Let $P = \langle a\rangle \times \langle c\rangle$. Then P is the normal Sylow p-subgroup of G with order p^4. It is not difficult to see that G is p-supersolvable and thus $P \cap O^p(G^*_p) = 1$. Hence, for any subgroup P_1 of P with order p, $P_1 \in \mathbb{D}(P)$. Consider $\langle c^p\rangle$ and note that $|\langle c^p\rangle| = p$. It is not difficult to see that $\langle c^p\rangle$ is not weakly S-permutably embedded in G.

There are many criteria for p-nilpotency of a finite group in the literature. Recently, Huang *et al.* and Zhang *et al.* both proved the following theorem.

THEOREM 1.2 ([7, Theorem 1.8] and [20, Theorem 3.2]). *Let p be a prime dividing the order of a finite group G, e a positive integer and $P \in \mathrm{Syl}_p(G)$ with $|P| \geq p^{e+1}$. Suppose that $N_G(P)$ is p-nilpotent and, for any subgroup D of P with order p^e, D is weakly S-permutably embedded in G. In addition, suppose that all cyclic subgroups of P with order 4 are weakly S-permutably embedded in G if $p = 2$, $e = 1$ and P is nonabelian. Then G is p-nilpotent.*

In [18], the author investigated some necessary and sufficient conditions for p-supersolvability and p-nilpotency of a finite group. In this note, we continue the work of [18] and prove the following results, which generalise the main theorems of [7, 11–13, 19] and [20].

THEOREM 1.3. *Let p be an odd prime dividing the order of a finite group G, e a positive integer and $P \in \mathrm{Syl}_p(G)$ with $|P| \geq p^{e+1}$. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and, for any subgroup P_1 of P with order p^e, $P_1 \in \mathbb{D}(P)$.*

DEFINITION 1.4. Let p be a prime and P a nonidentity p-group with $|P| = p^n$. Fix an integer k with $1 \leq k \leq n$. We define the set $\mathbb{L}_k(P)$ as follows.

(i) Assume that $k = 1$. If $p = 2$ and P is nonabelian, $\mathbb{L}_1(P) = \{P_1 \mid P_1 \leq P, |P_1| = 2\} \cup \{P_2 \mid P_2 \leq P$ and P_2 is a cyclic subgroup of order $4\}$. Otherwise, set $\mathbb{L}_1(P) = \{P_1 \mid P_1 \leq P$ and $|P_1| = p\}$.

(ii) Assume that $n \geq 2$ and $2 \leq k \leq n$. Then $\mathbb{L}_k(P) = \{P_1 \mid P_1 \leq P$ and $|P_1| = p^k\}$.

THEOREM 1.5. *Let p be a prime dividing the order of a finite group G and suppose that $(|G|, p-1) = 1$. Let e be a positive integer and $P \in \mathrm{Syl}_p(G)$ with $|P| \geq p^{e+1}$. Note that $G^*_p = G$. Then the following statements are equivalent.*

(a) *G is p-nilpotent.*

(b) *For any $P_1 \in \mathbb{L}_e(P)$, $P_1 \in \mathbb{D}(P)$.*

(c) *For any $P_1 \in \mathbb{L}_e(P)$, $P_1 \cap O^p(G)$ is S-permutably embedded in G.*

Theorems 1.3 and 1.5 generalise [7, Theorems 1.3 and 1.8] and [20, Theorem 3.2].

THEOREM 1.6. *Let p be an odd prime dividing the order of a finite group G, e a positive integer and $P \in \mathrm{Syl}_p(G)$ with $|P| \geq p^{e+1}$. Then G is p-nilpotent if and only if for any subgroup $P_1 \leq P$ with $|P_1| = p^e$, $P_1 \in \mathbb{D}(P)$ and $N_G(P_1)$ is p-nilpotent.*

Theorems 1.5 and 1.6 generalise [11, Theorem 1.2], [13, Theorem 1.3] and [19, Theorem 3.2].

THEOREM 1.7. *Let p be a prime dividing the order of a finite group G, $P \in \mathrm{Syl}_p(G)$ and $P' \leq P_1 \leq \Phi(P)$. Assume that $K \leq G$ with $P_1 \in \mathrm{Syl}_p(K)$. Also assume that there exist $H, K_1 \leq G$ such that $G = KH$, $K \cap H \leq K_1 \leq K$ and, for any $L \in \mathrm{Syl}_p(K_1)$, L is S-permutably embedded in G. If $N_G(P)$ is p-nilpotent, then G is p-nilpotent.*

Here, as usual, $\Phi(P)$ denotes the Frattini subgroup of P. Theorem 1.7 generalises [20, Theorems 3.3 and 3.4]. Finally, we also give another criterion for p-nilpotency of a finite group.

THEOREM 1.8. *Let p be a prime dividing the order of a finite group G and $P \in \mathrm{Syl}_p(G)$. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and there exists $P_1 \in \mathbb{D}(P)$ such that $P' \leq P_1 \leq \Phi(P)$.*

The proofs of the main results of [7, 11, 12, 19] and [20] all require Thompson's normal p-complement criterion (see [9, Theorem 7.1]). However, our proof does not appeal to Thompson's normal p-complement criterion and is much more elementary.

2. Preliminaries

LEMMA 2.1. *Let p be a prime dividing the order of a finite group G and let $P_1 \leq G$ be a p-group, $X \leq G$ and $N \trianglelefteq G$.*

(a) *If P_1 is S-permutable in G, then $P_1 \leq O_p(G)$ and $O^p(G) \leq N_G(P_1)$* [18, Lemma 2.5(a)].
(b) *P_1 is S-permutable in G if and only if $O^p(G) \leq N_G(P_1)$* [18, Lemma 2.5(b)].
(c) *If P_1 is S-permutable in G, then $P_1 \cap N$ is S-permutable in G* [corollary of (b)].
(d) *If X is S-permutable in G, then $X \trianglelefteq\trianglelefteq G$* [10].
(e) *If X is S-permutable in G and $L \leq G$, then $X \cap L$ is S-permutable in L.*
(f) *If X is S-permutably embedded in G and $X \leq L$, then X is S-permutably embedded in L* [corollary of (e)].
(g) *If X is S-permutable (S-permutably embedded) in G, then XN/N is S-permutable (S-permutably embedded) in G/N.*
(h) *If X is S-permutable (S-permutably embedded) in G, then $X \cap O^p(G)$ is S-permutable (S-permutably embedded) in G.*

PROOF. (d) See [10]. Here we derive Lemma 2.1(d) from Wielandt's famous zipper lemma (see [9, Theorem 2.9]). Firstly, we claim that if $H \leq K \leq G$, then H is S-permutable in K. For any $p \in \pi(K)$ and for any $P_1 \in \mathrm{Syl}_p(K)$, there exists $P \in \mathrm{Syl}_p(G)$ with $P_1 = P \cap K$. Since $HP = PH$, by Dedekind's lemma, it follows that $HP_1 = P_1H$. Hence, H is S-permutable in K.

We claim that for any $p \in \pi(G)$ and for any $P \in \mathrm{Syl}_p(G)$, $H \trianglelefteq\trianglelefteq HP$. Note that H is S-permutable in HP. Since $[HP : H]$ is a power of p, it is not difficult to see that $O^p(HP) \leq H$ and thus $H \trianglelefteq\trianglelefteq HP$.

Suppose that G is a counterexample with minimal order; we work to obtain a contradiction. Note that $H < G$. For any proper subgroup K of G such that $H \leq K$, H is S-permutable in K. Hence, $H \trianglelefteq\trianglelefteq K$. Recall that we assumed that H is not subnormal in G. By Wielandt's zipper lemma, there is a unique maximal subgroup L of G that contains H. For any $p \in \pi(G)$ and for any $P \in \mathrm{Syl}_p(G)$, since $H \trianglelefteq\trianglelefteq HP$, it follows that $HP < G$ and thus $HP \leq L$. In particular, we have $P \leq L$. Since G is generalised by all of its Sylow subgroups, it follows that $G \leq L$. This is the desired contradiction.

(e) It is no loss to assume that $L > 1$. For any $p \in \pi(L)$ and for any $P_1 \in \mathrm{Syl}_p(L)$, there is a $P \in \mathrm{Syl}_p(G)$ such that $P_1 = P \cap L$. By (d), $X \trianglelefteq\trianglelefteq G$. Since $PX = XP$, we see that $O^p(PX) = O^p(X) \leq X$. Consider $PX \cap L$. Since $P_1 = P \cap L \in \mathrm{Syl}_p(L)$ and $P_1 = P \cap L \leq PX \cap L \leq L$, it follows that $P_1 \in \mathrm{Syl}_p(PX \cap L)$. Observe that $O^p(PX \cap L) \leq PX \cap L \cap O^p(PX) \leq PX \cap L \cap X = X \cap L$. Hence, $PX \cap L = P_1 O^p(PX \cap L) = P_1(X \cap L)$ and $P_1(X \cap L) = (X \cap L)P_1$. This completes the proof.

(g) The proof is not difficult.

(h) It suffices to show that if X is S-permutable in G, then $X \cap O^p(G)$ is S-permutable in G. If $\pi(G)\backslash\{p\} \neq \emptyset$, then for any $q \in \pi(G)\backslash\{p\}$ and any $Q \in \mathrm{Syl}_q(G)$, $XQ = QX$. By Dedekind's lemma, $(X \cap O^p(G))Q = Q(X \cap O^p(G))$. For any $P \in \mathrm{Syl}_p(G)$, we have $PX = XP$. By (d), $X \trianglelefteq\trianglelefteq G$ and $O^p(PX) = O^p(X) \leq X \cap O^p(G)$. Hence, $PX = PO^p(X) = P(X \cap O^p(G))$ and $P(X \cap O^p(G)) = (X \cap O^p(G))P$. This completes the proof. □

Lemma 2.2 [18, Lemma 2.1]. *Let p be a prime dividing the order of a finite group G, $P \in \mathrm{Syl}_p(G)$, $N \trianglelefteq G$ and e a positive integer. Write $P_1 = P \cap N$. Assume that $P_1 \trianglelefteq N$ and N is not p-nilpotent. Also assume that $|P_1| \leq p^e$ and $|P| \geq p^{e+1}$. Then P has a normal subgroup P_2 of order p^e with $[P_1 : P_1 \cap P_2] = p$.*

Lemma 2.3 [18, Lemma 2.2]. *Let p be a prime dividing the order of a finite group G and $P \in \mathrm{Syl}_p(G)$. Write $P_1 = P \cap O^p(G_p^*)$. Assume that $P_1 > 1$ and P_1 has a maximal subgroup T with $T \trianglelefteq G$. Then $P_1 \ntrianglelefteq G$.*

Lemma 2.4 [18, Theorem 1.3]. *Let p be a prime dividing the order of a finite group G, $e \geq 2$ an integer and $P \in \mathrm{Syl}_p(G)$ with $|P| \geq p^{e+1}$. Then G is p-supersolvable if and only if $P_1 \cap O^p(G_p^*)$ is S-permutable in G for all subgroups $P_1 \leq P$ with $|P_1| = p^e$.*

Lemma 2.5 [18, Theorem 1.4]. *Let p be a prime dividing the order of a finite group G and $P \in \mathrm{Syl}_p(G)$. Then G is p-supersolvable if and only if $P_1 \cap O^p(G_p^*)$ is S-permutable in G for all subgroups $P_1 \leq P$ with $|P_1| = p$ and, if $p = 2$ and P is nonabelian, $P_2 \cap O^p(G_p^*)$ is also S-permutable in G for all cyclic subgroups $P_2 \leq P$ with $|P_2| = 4$.*

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Lemma 2.6 [18, Lemma 2.8]. *Let p be a prime dividing the order of a finite group G and P_1 be a p-subgroup of G. Let $L \trianglelefteq G$ and N be a normal p'-subgroup of G. Then $P_1N/N \cap LN/N = (P_1 \cap L)N/N$.*

Lemma 2.7 [18, Lemma 2.9]. *Let p be a prime dividing the order of a finite group G and $N \trianglelefteq G$. Then $(G/N)^*_p = G^*_pN/N$, $O^p(G/N) = O^p(G)N/N$ and $O^p((G/N)^*_p) = O^p(G^*_p)N/N$.*

Lemma 2.8. *Let p be a prime dividing the order of a finite group G and $P \in \mathrm{Syl}_p(G)$. Let $X \leq G$ be S-permutable in G. Assume that $P \cap O^p(G) \nleq P \cap X$. Then $PX < G$.*

Proof. By Lemma 2.1(d), $X \trianglelefteq\trianglelefteq G$. If $PX = G$, then $O^p(G) = O^p(X) \leq X$ and thus $P \cap O^p(G) \leq P \cap X$, which is a contradiction. □

Lemma 2.9. *Let p be a prime dividing the order of a finite group G, $P \in \mathrm{Syl}_p(G)$ and $P_1 \leq P$. Assume that there exists $X \leq G$ such that $P_1 \cap O^p(G) \in \mathrm{Syl}_p(X)$ and X is S-permutable in G. If $P \cap O^p(G) \nleq P_1$, then $PX < G$.*

Proof. Note that $P \cap X = P_1 \cap O^p(G)$. By Lemma 2.8, it follows that $PX < G$. □

Lemma 2.10. *Let p be a prime dividing the order of a finite group G, $P \in \mathrm{Syl}_p(G)$, $P_1 \leq P$ and $P_1 \trianglelefteq G$.*

(a) *If $P_1 \leq \Phi(P)$ and G/P_1 is p-nilpotent, then G is p-nilpotent* [18, Lemma 2.13].

(b) *If $P' \leq P_1 \leq \Phi(P)$ and $N_G(P)$ is p-nilpotent, then G is p-nilpotent.*

Proof. (b) Consider G/P_1. Note that $P/P_1 \in \mathrm{Syl}_p(G/P_1)$ and P/P_1 is abelian. Since $N_{G/P_1}(P/P_1) = N_G(P)/P_1$ is p-nilpotent, by Burnside's theorem (see [9, Theorem 5.13]), G/P_1 is p-nilpotent. By (a), G is p-nilpotent. □

Lemma 2.11 (Tate; see [8, Satz IV.4.7] or [6, Theorem A]). *Let p be a prime dividing the order of a finite group G, $P \in \mathrm{Syl}_p(G)$ and $N \trianglelefteq G$. If $P \cap N \leq \Phi(P)$, then N is p-nilpotent.*

Lemma 2.12 [18, Corollary 3.8]. *Let p be a prime dividing the order of a finite group G and $P \in \mathrm{Syl}_p(G)$. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and there exists $P' \leq P_1 \leq \Phi(P)$ such that $P_1 \cap O^p(G^*_p) \trianglelefteq G^*_p$.*

3. Main results

Proof of Theorem 1.3. We only need to prove the sufficiency. Suppose that G is a counterexample with minimal order; we work in the following steps to obtain a contradiction.

Step 1. If $P \leq H < G$, then H is p-nilpotent. By Lemma 2.1(f), the hypotheses are inherited by H. Hence, H is p-nilpotent.

Step 2. $O_{p'}(G) = 1$. By Lemmas 2.1(g), 2.6 and 2.7, the hypotheses are inherited by $G/O_{p'}(G)$. If $O_{p'}(G) > 1$, then $G/O_{p'}(G)$ is p-nilpotent and thus G is p-nilpotent. Hence, $O_{p'}(G) = 1$.

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Step 3. G is not p-supersolvable, $G_p^* = G$ and, for any subgroup P_1 of P with order p^e, $P_1 \cap O^p(G)$ is S-permutably embedded in G. Assume that G is p-supersolvable and thus G is p-solvable with p-length 1. Since $O_{p'}(G) = 1$, it follows that $P \trianglelefteq G$ and thus $G = N_G(P)$ is p-nilpotent. Hence, G is not p-supersolvable. Assume that $G_p^* < G$. By Step 1, it follows that G_p^* is p-nilpotent and thus G is p-supersolvable. This is a contradiction. Hence, $G_p^* = G$. For any subgroup P_1 of P with order p^e, there exists $P_2 \leq G$ such that $P_1 \cap O^p(G) \leq P_2 \leq P_1$ and P_2 is S-permutably embedded in G. Since $P_1 \cap O^p(G) = P_2 \cap O^p(G)$, by Lemma 2.1(h), $P_1 \cap O^p(G)$ is S-permutably embedded in G.

Step 4. There exists a subgroup P_1 of P with order p^e such that $P_1 \cap O^p(G)$ is not S-permutable in G. Assume that for any subgroup P_1 of P with order p^e, $P_1 \cap O^p(G)$ is S-permutable in G. By Lemmas 2.4 and 2.5, G is p-supersolvable. But this contradicts Step 3.

Step 5. If $P_1 \leq P$ with $|P_1| = p^e$, $X \leq G$ is such that $P_1 \cap O^p(G) \in \mathrm{Syl}_p(X)$ and X is S-permutable in G and $PX < G$, then $P_1 \cap O^p(G)$ is S-permutable in G. By Step 1, PX is p-nilpotent and thus X is p-nilpotent. Since $X \trianglelefteq\trianglelefteq G$ (Lemma 2.1(d)) and $O_{p'}(G) = 1$, it follows that $X = P_1 \cap O^p(G)$. Hence, $P_1 \cap O^p(G)$ is S-permutable in G.

Step 6. If $P_1 \leq P$ is such that $|P_1| = p^e$ and $P \cap O^p(G) \nleq P_1$, then $P_1 \cap O^p(G)$ is S-permutable in G. There exists $X \leq G$ such that $P_1 \cap O^p(G) \in \mathrm{Syl}_p(X)$ and X is S-permutable in G. By Lemma 2.9, it follows that $PX < G$. By Step 5, $P_1 \cap O^p(G)$ is S-permutable in G.

Step 7. $|P \cap O^p(G)| \leq p^e$. By Steps 4 and 5, there exist $P_1 \leq P$ with $|P_1| = p^e$ and $X \leq G$ such that $P_1 \cap O^p(G) \in \mathrm{Syl}_p(X)$, X is S-permutable in G and $G = PX$. Since $X \trianglelefteq\trianglelefteq G$ (Lemma 2.1(d)), it follows that $O^p(G) = O^p(X) \leq X$ and thus $P \cap O^p(G) \leq P \cap X = P_1 \cap O^p(G) \leq P_1$.

Step 8. If $\widehat{P} = P \cap O^p(G)$, then $\widehat{P}$ has a maximal subgroup T such that $T \leq O_p(G)$. Since $|\widehat{P}| \leq p^e$ and $|P| \geq p^{e+1}$, there exists $\widetilde{P} \leq P$ such that $|\widetilde{P}| = p^{e+1}$ and $\widehat{P} \leq \widetilde{P}$. Assume that $\widehat{P} \leq \Phi(\widetilde{P})$. Observe that $\widetilde{P} \in \mathrm{Syl}_p(\widetilde{P}O^p(G))$. By Lemma 2.11, $O^p(G)$ is p-nilpotent, that is, G is p-nilpotent. This is a contradiction. Hence, $\widehat{P} \nleq \Phi(\widetilde{P})$ and $\widetilde{P}$ has a maximal subgroup P_1 with $\widehat{P} \nleq P_1$. Then $|P_1| = p^e$, $\widetilde{P} = \widehat{P}P_1$ and $[\widehat{P} : \widehat{P} \cap P_1] = p$. Let $T = \widehat{P} \cap P_1$. Since $\widehat{P} \nleq P_1$, by Step 6, $P_1 \cap O^p(G)$ is S-permutable in G. By Lemma 2.1(a), $T = \widehat{P} \cap P_1 = P_1 \cap O^p(G) \leq O_p(G)$.

Step 9. The final contradiction. Consider $N_G(\widehat{P})$. Assume that $N_G(\widehat{P}) < G$. Then $T = O_p(O^p(G))$. Note that $P \leq N_G(\widehat{P}) < G$. By Step 1, $N_G(\widehat{P})$ is p-nilpotent. Hence, $N_{O^p(G)/T}(\widehat{P}/T)$ is p-nilpotent. Note that $\widehat{P}/T \in \mathrm{Syl}_p(O^p(G)/T)$ and $|\widehat{P}/T| = p$. By Burnside's theorem, $O^p(G)/T$ is p-nilpotent. Then $O^p(O^p(G)) < O^p(G)$. This is a contradiction. Assume that $N_G(\widehat{P}) = G$, that is, $\widehat{P} \trianglelefteq G$. Since $O^p(G)$ is not p-nilpotent, by Lemma 2.2, P has a normal subgroup P_1 with order p^e such that $[\widehat{P} : \widehat{P} \cap P_1] = p$ and, in particular, $\widehat{P} \nleq P_1$. By Step 6, $P_1 \cap O^p(G)$ is S-permutable in G. By Lemma 2.1(b), $\widehat{P} \cap P_1 = P_1 \cap O^p(G) \trianglelefteq O^p(G)$. Since $\widehat{P}, P_1 \trianglelefteq P$, it follows that

$\widehat{P} \cap P_1 \trianglelefteq P$. Hence, $\widehat{P} \cap P_1 \trianglelefteq G$. Recall that $G_p^* = G$. By Lemma 2.3, $\widehat{P} \ntrianglelefteq G$. This contradicts $\widehat{P} \trianglelefteq G$. Hence, we obtain the final contradiction. □

Proof of Theorem 1.5. The implications (a) ⇒ (b) and (a) ⇒ (c) are not difficult to prove. By Lemma 2.1(h), (b) is equivalent to (c).

To prove (c) ⇒ (a), we modify the proof of Theorem 1.3. Since $(|G|, p-1) = 1$, G is p-nilpotent if and only if G is p-supersolvable. Suppose that G is a counterexample with minimal order; we work in the following steps to obtain a contradiction. Steps 1–6 use the same arguments as in the proof of Theorem 1.3.

Step 1. If $P \leq H < G$, then H is p-nilpotent.

Step 2. $O_{p'}(G) = 1$.

Step 3. G is not p-supersolvable.

Step 4. There exists a subgroup $P_1 \in \mathbb{L}_e(P)$ such that $P_1 \cap O^p(G)$ is not S-permutable in G.

Step 5. If $P_1 \in \mathbb{L}_e(P)$, $X \leq G$ is such that $P_1 \cap O^p(G) \in \mathrm{Syl}_p(X)$ and X is S-permutable in G and $PX < G$, then $P_1 \cap O^p(G)$ is S-permutable in G.

Step 6. If $P_1 \in \mathbb{L}_e(P)$ and $P \cap O^p(G) \nleq P_1$, then $P_1 \cap O^p(G)$ is S-permutable in G.

Step 7. $e \geq 2$ and $|P \cap O^p(G)| \leq p^e$. By Steps 4 and 5, there exist $P_1 \in \mathbb{L}_e(P)$ and $X \leq G$ such that $P_1 \cap O^p(G) \in \mathrm{Syl}_p(X)$, X is S-permutable in G and $G = PX$. By the proof of Step 7 of Theorem 1.3, $P \cap O^p(G) \leq P \cap X = P_1 \cap O^p(G) \leq P_1$. Assume that $e = 1$; we work to obtain a contradiction. Since $e = 1$, it follows that P_1 is cyclic and thus $P \cap O^p(G)$ is cyclic. Since $(|G|, p-1) = 1$, by Burnside's theorem, $O^p(G)$ is p-nilpotent, that is, G is p-nilpotent. This is the desired contradiction. Hence, $e \geq 2$, $|P_1| = p^e$ and $|P \cap O^p(G)| \leq p^e$.

Step 8. If $\widehat{P} = P \cap O^p(G)$, then $\widehat{P}$ has a maximal subgroup T such that $T \trianglelefteq O^p(G)$. The assertion follows by the same argument as in the proof of Step 8 of Theorem 1.3.

Step 9. The final contradiction. Consider $O^p(G)/T$. Note that $\widehat{P}/T \in \mathrm{Syl}_p(O^p(G)/T)$ and $|\widehat{P}/T| = p$. Since $(|G|, p-1) = 1$, by Burnside's theorem, it follows that $O^p(G)/T$ is p-nilpotent. Then $O^p(O^p(G)) < O^p(G)$. This is a contradiction. □

Proof of Theorem 1.6. We only need to prove the sufficiency. Suppose that G is a counterexample with minimal order; we work to obtain a contradiction. We mimic the proof of Theorem 1.3. In fact, we only need to modify Step 3 of Theorem 1.3.

Step 3. G is not p-supersolvable and $G_p^* = G$ and, for any subgroup P_1 of P with order p^e, $P_1 \cap O^p(G)$ is S-permutably embedded in G. Assume that G is p-supersolvable and thus G is p-solvable with p-length 1. Since $O_{p'}(G) = 1$ (Step 2), $P \trianglelefteq G$. Since G is p-supersolvable, there exists a subgroup $P_1 \leq P$ such that $|P_1| = p^e$ and $P_1 \trianglelefteq G$. Then $G = N_G(P_1)$ is p-nilpotent. Hence, G is not p-supersolvable. By the proof of Step 3 of Theorem 1.3, it follows that $G_p^* = G$ and, for any subgroup P_1 of P with order p^e, $P_1 \cap O^p(G)$ is S-permutably embedded in G. □

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Lemma 3.1. *Let p be a prime dividing the order of a finite group G, $P \in \mathrm{Syl}_p(G)$ and $P' \le P_1 \le \Phi(P)$. Suppose that P_1 is S-permutably embedded in G and $N_G(P)$ is p-nilpotent; then G is p-nilpotent.*

Proof. By the hypotheses, there exists $X \le G$ such that $P_1 \in \mathrm{Syl}_p(X)$ and X is S-permutable in G. We work by induction on $|G|$. By Lemma 2.1(g), the hypotheses are inherited by $G/O_{p'}(G)$. If $O_{p'}(G) > 1$, by induction, $G/O_{p'}(G)$ is p-nilpotent and thus G is p-nilpotent. Hence, we can assume that $O_{p'}(G) = 1$.

By Lemma 2.1(d), $X \trianglelefteq\trianglelefteq G$. Since $PX = XP$, we have $P \cap O^p(PX) = P \cap O^p(X) \le P \cap X = P_1 \le \Phi(P)$. In the subgroup PX, by Lemma 2.11, PX is p-nilpotent and thus X is p-nilpotent. Since $X \trianglelefteq\trianglelefteq G$ and $O_{p'}(G) = 1$, it follows that $X = P_1$. Then P_1 is S-permutable in G. By Lemma 2.1(a), $O^p(G) \le N_G(P_1)$. Since $P' \le P_1 \le P$, we see that $P_1 \trianglelefteq P$. Hence, $P_1 \trianglelefteq G$. By Lemma 2.10(b), G is p-nilpotent. □

Proof of Theorem 1.7. By Sylow's second theorem, there exists $k \in K$ such that $P^k \cap H \in \mathrm{Syl}_p(H)$. Since $G = KH$, it follows that $P^k = P_1^k(P^k \cap H)$. Since $P_1^k \le \Phi(P^k)$, it follows that $P^k = P^k \cap H$. Hence, $P^k \le H$ and thus $P_1^k \le K \cap H \le K_1 \le K$. Hence, $P_1^k \in \mathrm{Syl}_p(K_1)$ and thus P_1^k is S-permutably embedded in G. By Lemma 3.1, G is p-nilpotent. □

Proof of Theorem 1.8. It is no loss to assume that $O_{p'}(G) = 1$. Since $P_1 \in \mathbb{D}(P)$, there exist $P_2, X \le G$ such that $P_1 \cap O^p(G_p^*) \le P_2 \le P_1$, $P_2 \in \mathrm{Syl}_p(X)$ and X is S-permutable in G. By the proof of Lemma 3.1, it follows that $P_2 = X$ is S-permutable in G. By Lemma 2.1(a), $O^p(G) \le N_G(P_2)$. Hence, $P_1 \cap O^p(G_p^*) = P_2 \cap O^p(G_p^*) \trianglelefteq O^p(G)$. Since $P' \le P_1 \le P$, it follows that $P_1 \cap O^p(G_p^*) \trianglelefteq P$. Hence, $P_1 \cap O^p(G_p^*) \trianglelefteq G$. By Lemma 2.12, it follows that G is p-nilpotent. □

4. Applications

Recently, Ballester-Bolinches and Li proved the following theorem, which includes the main theorems of [17].

Theorem 4.1 [1, Theorem 3]. *Let p be a prime dividing the order of a finite group G, e a positive integer and $P \in \mathrm{Syl}_p(G)$ with $|P| \ge p^{e+1}$. Suppose that for any subgroup P_1 of P with order p^e, P_1 is S-permutably embedded in G. In addition, suppose that all cyclic subgroups of P of order 4 are S-permutably embedded in G if $p = 2$, $e = 1$ and P is nonabelian. Then G is p-supersolvable.*

Here we take a different approach to Theorem 4.1. We need the following lemma.

Lemma 4.2. *Let P be a finite nonidentity p-group, where p is a prime. Let $P_1 < P$ with $P_1 \not\le \Phi(P)$ and let $L = \{T \mid T$ is a maximal subgroup of P with $P_1 \not\le T\}$. We write $|L|$ to denote the cardinality of L. Then $|L| \ge p$.*

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Proof. Note that $P_1\Phi(P) < P$. Let $L_1 = \{T \mid T$ is a maximal subgroup of $P\}$ and $L_2 = \{T \mid T$ is a maximal subgroup of P with $P_1 \le T\}$, so $L = L_1 \backslash L_2$. Since $P_1 \nleq \Phi(P)$, it follows that $|L| \ge 1$. Now $L_2 = \{T \mid T$ is a maximal subgroup of P with $P_1\Phi(P) \le T\}$. There exist $n, m \in N_+$ such that $[P : \Phi(P)] = p^n$ and $[P : P_1\Phi(P)] = p^m$. It is not difficult to see that $|L_1| = (p^n - 1)/(p - 1)$ and $|L_2| = (p^m - 1)/(p - 1)$. Hence, $|L_1| \equiv |L_2| \equiv 1 \pmod p$ and $|L| \equiv 0 \pmod p$. Since $|L| \ge 1$, it follows that $|L| \ge p$. □

We modify the proof of Theorem 1.3 to prove Theorem 4.1.

Proof of Theorem 4.1. By Theorem 1.5, it suffices to prove the theorem for the case that p is an odd prime. Suppose that G is a counterexample with minimal order; we work in the following steps to obtain a contradiction.

Step 1. If $P \le H < G$, then H is p-supersolvable. By Lemma 2.1(f), the hypotheses are inherited by H. Hence, H is p-supersolvable.

Step 2. $O_{p'}(G) = 1$. By Lemma 2.1(g), the hypotheses are inherited by $G/O_{p'}(G)$. If $O_{p'}(G) > 1$, then $G/O_{p'}(G)$ is p-supersolvable and thus G is p-supersolvable. Hence, $O_{p'}(G) = 1$.

Step 3. There exists a subgroup P_1 of P with order p^e such that P_1 is not S-permutable in G. Assume that for any subgroup P_1 of P with order p^e, P_1 is S-permutable in G. By Lemmas 2.4 and 2.5, G is p-supersolvable. This is a contradiction.

Step 4. If $P_1 \le P$ with $|P_1| = p^e$, $X \le G$ is such that $P_1 \in \mathrm{Syl}_p(X)$ and X is S-permutable in G and $PX < G$, then P_1 is S-permutable in G. By Step 1, PX is p-supersolvable and thus X is p-supersolvable and, in particular, X is p-solvable with p-length 1. Since $X \trianglelefteq\trianglelefteq G$ (Lemma 2.1(d)) and $O_{p'}(G) = 1$, it follows that P_1 is the normal Sylow p-subgroup of X and thus $P_1 \trianglelefteq\trianglelefteq G$. For any $q \in \pi(G)\backslash\{p\}$ and any $Q \in \mathrm{Syl}_q(G)$, we have $XQ = QX$. Since $P_1 \trianglelefteq\trianglelefteq G$ and $P_1 \in \mathrm{Syl}_p(XQ)$, it follows that P_1 is the normal Sylow p-subgroup of XQ and thus $Q \le N_G(P_1)$. Hence, $O^p(G) \le N_G(P_1)$. By Lemma 2.1(b), P_1 is S-permutable in G.

Step 5. If $P_1 \le P$ is such that $|P_1| = p^e$ and $P \cap O^p(G^*_p) \nleq P_1$, then P_1 is S-permutable in G. There exists $X \le G$ such that $P_1 \in \mathrm{Syl}_p(X)$ and X is S-permutable in G. Assume that $G = PX$; we work to obtain a contradiction. Since $X \trianglelefteq\trianglelefteq G$ (Lemma 2.1(d)), it follows that $O^p(G) = O^p(X) \le X$ and thus $P \cap O^p(G^*_p) \le P \cap O^p(G) \le P \cap X = P_1$. This is the desired contradiction. Hence, $PX < G$. By Step 4, P_1 is S-permutable in G.

Step 6. $|P \cap O^p(G^*_p)| \le |P \cap O^p(G)| \le p^e$. By Steps 3 and 4, there exist $P_1 \le P$ with $|P_1| = p^e$ and $X \le G$ such that $P_1 \in \mathrm{Syl}_p(X)$, X is S-permutable in G and $G = PX$. By the proof of Step 5, $P \cap O^p(G^*_p) \le P \cap O^p(G) \le P \cap X = P_1$.

Step 7. The final contradiction. Let $\widehat{P} = P \cap O^p(G^*_p)$. Since $|\widehat{P}| \le p^e$ and $|P| \ge p^{e+1}$, there exists $\widetilde{P} \le P$ such that $|\widetilde{P}| = p^{e+1}$ and $\widehat{P} \le \widetilde{P}$. Assume that $\widehat{P} \le \Phi(\widetilde{P})$. Observe that $\widetilde{P} \in \mathrm{Syl}_p(\widetilde{P}O^p(G^*_p))$. By Lemma 2.11, $O^p(G^*_p)$ is p-nilpotent and thus G is

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p-supersolvable. This is a contradiction. Hence, $\widehat{P} \nleq \Phi(\widetilde{P})$. By Lemma 4.2, $\widetilde{P}$ has two different maximal subgroups P_1, P_2 with $\widehat{P} \nleq P_1$ and $\widehat{P} \nleq P_2$. By Step 5, P_1 and P_2 are both S-permutable in G. By Lemma 2.1(a), $P_1, P_2 \leq O_p(G)$ and thus $\widetilde{P} \leq O_p(G)$. Hence, $\widehat{P} \leq O_p(G)$ and thus $\widehat{P} = O_p(G) \cap O^p(G_p^*) \trianglelefteq G$. Since $O^p(G_p^*)$ is not p-nilpotent, by Lemma 2.2, P has a normal subgroup P_1 with order p^e such that $[\widehat{P} : \widehat{P} \cap P_1] = p$ and, in particular, $\widehat{P} \nleq P_1$. By Step 5, P_1 is S-permutable in G. By Lemma 2.1(a), $O^p(G) \leq N_G(P_1)$ and thus $\widehat{P} \cap P_1 = P_1 \cap O^p(G_p^*) \trianglelefteq O^p(G)$. Since $\widehat{P}, P_1 \trianglelefteq P$, it follows that $\widehat{P} \cap P_1 \trianglelefteq P$. Hence, $\widehat{P} \cap P_1 \trianglelefteq G$. By Lemma 2.3, $\widehat{P} \ntrianglelefteq G$. This contradicts $\widehat{P} \trianglelefteq G$. Hence, we obtain the final contradiction. □

Remark 4.3. There exists a finite non-p-supersolvable group G such that p is an odd prime divisor of $|G|$, e is a positive integer, $P \in \mathrm{Syl}_p(G)$ and $|P| \geq p^{e+1}$ such that for every subgroup P_1 of P with $|P_1| = p^e$, there are nonidentity subgroups P_2 and X of G with $P_1 \cap O^p(G_p^*) \leq P_2 \leq P_1$, $P_2 \in \mathrm{Syl}_p(X)$ and X is S-permutable in G. For example, let $p \geq 5$ be a prime and $e \geq 2$ be an integer and consider $G = SL(2, p) \times \mathbb{Z}_{p^e}$.

However, for p-solvable groups, we have the following result.

Theorem 4.4. *Let p be an odd prime dividing the order of a finite p-solvable group G, e a positive integer and $P \in \mathrm{Syl}_p(G)$ with $|P| \geq p^{e+1}$. Then G is p-supersolvable if and only if for any subgroup P_1 of P with order p^e, $P_1 \in \mathbb{D}(P)$.*

Proof. We only need to prove the sufficiency. Suppose that G is a counterexample with minimal order; we work in the following steps to obtain a contradiction. The arguments are the same as in Steps 1–8 of Theorem 1.3.

Step 1. If $P \leq H < G$, then H is p-supersolvable.

Step 2. $O_{p'}(G) = 1$.

Step 3. There exists a subgroup P_1 of P with order p^e such that $P_1 \cap O^p(G_p^*)$ is not S-permutable in G.

Step 4. If $P_2 \leq P_1 \leq P$, $X \leq G$ is such that $|P_1| = p^e$, $P_1 \cap O^p(G_p^*) \leq P_2 \leq P_1$, $P_2 \in \mathrm{Syl}_p(X)$ and X is S-permutable in G and if $PX < G$, then $P_1 \cap O^p(G_p^*)$ is S-permutable in G.

Step 5. If $P_1 \leq P$ is such that $|P_1| = p^e$ and $P \cap O^p(G_p^*) \nleq P_1$, then $P_1 \cap O^p(G_p^*)$ is S-permutable in G.

Step 6. $|P \cap O^p(G_p^*)| \leq p^e$.

Step 7. If $\widehat{P} = P \cap O^p(G_p^*)$, then $\widehat{P}$ has a maximal subgroup T such that $T \leq O_p(G)$.

Step 8. The final contradiction. Let $L = O_p(G) \cap O^p(G_p^*)$. By Step 7, $T \leq L$. Note that $L \leq \widehat{P}$. Assume that $L = T$. Consider G/T. Assume that $T > 1$. Since $|T| \leq p^{e-1}$, by induction, G/T is p-supersolvable. Hence, $O^p(G_p^*)/T$ is p-nilpotent. Then $O^p(O^p(G_p^*)) < O^p(G_p^*)$. This is a contradiction. Assume that $T = 1$. Since

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$\widehat{P} \trianglelefteq P$ and $\widehat{P} \cap O_p(G) = L = T = 1$, we have $[\widehat{P}, O_p(G)] = 1$. Since G is p-solvable and $O_{p'}(G) = 1$, by the Hall–Higman lemma (see [9, Theorem 3.21]), it follows that $\widehat{P} = 1$ and thus G_p^* is p-nilpotent. Hence, G is p-supersolvable. This is also a contradiction. Assume that $L = \widehat{P}$, that is, $\widehat{P} \trianglelefteq G$. Using the same arguments as in the proof of Step 9 of Theorem 1.3, we can obtain the final contradiction. □

Recently, Li *et al.* (see [12, 14, 15]) introduced the concept of E–S-supplemented subgroups. A subgroup H of a finite group G is said to be E–S-supplemented in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{eG}$, where H_{eG} is the subgroup of H generated by all those subgroups of H which are S-permutably embedded in G. They used E–S-supplemented subgroups to establish many results. One of their results is the following theorem.

Theorem 4.5 [12, Theorem 1.4]. *Let p be an odd prime dividing $|G|$ and P a Sylow p-subgroup of G. Suppose that there exists a subgroup D of P with $1 < D < P$ such that every subgroup H of P with order $|D|$ is E–S-supplemented in G and $N_G(P)$ is p-nilpotent. Then G is p-nilpotent.*

We point out that for a p-subgroup H of G, the concept of E–S-supplemented subgroups coincides with the concept of weakly S-permutably embedded subgroups. In order to show this, we appeal to the following significant theorem of Deskins.

Theorem 4.6 [5, Theorem 1]. *If a subgroup H of a finite group G is S-permutable in G, then $H/Core_G(H)$ is nilpotent and thus, for any $p \in \pi(G)$ and for any $P \in \mathrm{Syl}_p(G)$, $PCore_G(H)/Core_G(H)$ is S-permutable in $G/Core_G(H)$.*

Lemma 4.7. *Let p be a prime dividing the order of a finite group G, $P \in \mathrm{Syl}_p(G)$ and $P_1, P_2 \leq P$. If P_1 and P_2 are both S-permutably embedded in G, then $\langle P_1, P_2\rangle$ is S-permutably embedded in G.*

Proof. There exist $X_1, X_2 \leq G$ such that $P_1 \in \mathrm{Syl}_p(X_1)$, $P_2 \in \mathrm{Syl}_p(X_2)$ and X_1, X_2 are S-permutable in G. Let $H_1 = Core_G(X_1)$ and $H_2 = Core_G(X_2)$. By Deskins' theorem 4.6, P_1H_1/H_1 is S-permutable in G/H_1 and P_2H_2/H_2 is S-permutable in G/H_2. In G/H_1H_2, by Lemma 2.1(g), $\overline{P_1}$ and $\overline{P_2}$ are both S-permutable in $\overline{G}$. It is not difficult to see that $\overline{\langle P_1, P_2\rangle} = \langle \overline{P_1}, \overline{P_2}\rangle$ is S-permutable in $\overline{G}$. Hence, $\langle P_1, P_2\rangle H_1H_2$ is S-permutable in G. Let $B = \langle P_1, P_2\rangle H_1H_2$. Since P permutes with B, we have $P \cap B \in \mathrm{Syl}_p(B)$. It is not difficult to see that $P \cap H_1H_2 = (P \cap H_1)(P \cap H_2)$. Hence, by Dedekind's lemma, $P \cap B = \langle P_1, P_2\rangle(P \cap H_1)(P \cap H_2) = \langle P_1, P_2\rangle$. Hence, $\langle P_1, P_2\rangle$ is S-permutably embedded in G. □

Let p be a prime dividing the order of a finite group G and H a p-subgroup of G. By Lemma 4.7, H_{eG} is S-permutably embedded in G. Hence, for H, the concept of E–S-supplemented subgroups coincides with the concept of weakly S-permutably embedded subgroups. Hence, our Theorems 1.3, 1.5 and 1.6 also generalise the main theorems of [12].

Recently, in [4], Chen *et al.* introduced the concept of SE-quasinormal subgroups, which is the same as the concept of E–S-supplemented subgroups. Hence, our Theorem 1.5 also generalises [4, Theorem B].

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