THE JOURNAL OF SYMBOLIC LOGIC
Volume 82, Number 4, December 2017

# UNIVERSALLY BAIRE SETS AND GENERIC ABSOLUTENESS

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**Abstract.** We prove several equivalences and relative consistency results regarding generic absoluteness beyond Woodin's $(\utilde{\Sigma}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness result for a limit of Woodin cardinals $\lambda$. In particular, we prove that two-step $\exists^{\mathbb{R}}(\utilde{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below a measurable limit of Woodin cardinals has high consistency strength and is equivalent, modulo small forcing, to the existence of trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas. The construction of these trees uses a general method for building an absolute complement for a given tree $T$ assuming many "failures of covering" for the models $L(T, V_\alpha)$ for $\alpha$ below a measurable cardinal.

**Introduction.** Generic absoluteness principles assert that certain properties of the set-theoretic universe cannot be changed by forcing. Some properties, such as the truth or falsity of the continuum hypothesis, can always be changed by forcing. Accordingly, one approach to formulating generic absoluteness principles is to consider properties of a limited complexity such as those corresponding to pointclasses in descriptive set theory: $\utilde{\Sigma}^1_2$, $\utilde{\Sigma}^1_3$, projective, and so on. (Another approach is to limit the class of allowed forcing notions. For a survey of results in this area, see Bagaria [1].)

By Shoenfield's absoluteness theorem, $\utilde{\Sigma}^1_2$ statements are always generically absolute. Generic absoluteness principles for larger pointclasses tend to be equiconsistent with strong axioms of infinity, and they may also relate to the extent of the universally Baire sets. For example, one-step $\utilde{\Sigma}^1_3$ generic absoluteness is equiconsistent with the existence of a $\Sigma_2$-reflecting cardinal and equivalent with the statement that every $\utilde{\Delta}^1_2$ set of reals is universally Baire (Feng, Magidor, and Woodin [3, Corollary 3.1 and Theorem 3.3]).

Another example is that two-step $\utilde{\Sigma}^1_3$ generic absoluteness, which is the statement that one-step $\utilde{\Sigma}^1_3$ generic absoluteness holds in every generic extension, is equivalent with the statement that every set has a sharp (Woodin [20, Lemma 1]) and also with the statement that every $\utilde{\Sigma}^1_2$ set of reals is universally Baire (Feng, Magidor, and Woodin [3, Theorem 3.4]).

A third example is that projective generic absoluteness (either one-step or two-step) is equiconsistent with the existence of infinitely many strong cardinals (Hauser [6], Woodin). Feng, Magidor, and Woodin asked whether projective

Received January 21, 2016.
2010 *Mathematics Subject Classification.* Primary 03E57, Secondary 03E15, 03E55.
*Key words and phrases*. generic absoluteness, universally Baire, Woodin cardinal.


0022-4812/17/8204-0003
DOI:10.1017/jsl.2017.35

generic absoluteness implies, or is implied by, the statement that every projective set is universally Baire. This question seems to remain open.

A bit higher in the complexity hierarchy we reach an obstacle: the continuum hypothesis is a $\Sigma^2_1$ statement that cannot be generically absolute, so generic absoluteness principles at this level of complexity must be limited in some way in order to be consistent. One approach is to consider only the generic extensions that satisfy the continuum hypothesis. Woodin showed that the absoluteness of $\Sigma^2_1$ statements to such generic extensions follows from large cardinals (see Larson [8, Theorem 3.2.1]).

The approach we take in this article is to replace $\Sigma^2_1$ with $(\Sigma^2_1)^\Gamma$ where $\Gamma$ is a pointclass and a statement is called $(\Sigma^2_1)^\Gamma$ if it has the form

$$\exists A \in \Gamma\, (\mathrm{HC}; \in, A) \models \varphi$$

for some statement $\varphi$ in the language of set theory expanded by a unary predicate symbol. We will consider pointclasses $\Gamma$ that are "well behaved" and in particular do not contain well orderings of $\mathbb{R}$. The pointclass uB of universally Baire sets of reals and its local version $\mathrm{uB}_\lambda$ are both examples of such pointclasses, and indeed Woodin has shown that if $\lambda$ is a limit of Woodin cardinals then generic absoluteness holds for $(\utilde{\Sigma}^2_1)^{\mathrm{uB}_\lambda}$ statements with respect to generic extensions by posets of cardinality less than $\lambda$ (see Steel [17, Theorem 6.1]).

In this article, we investigate generic absoluteness principles for pointclasses beyond $(\utilde{\Sigma}^2_1)^{\mathrm{uB}_\lambda}$ in relation to strong axioms of infinity and determinacy, and also in relation to the extent of the universally Baire sets. First, we consider generic absoluteness for $\exists^{\mathbb{R}}(\utilde{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ statements. This level of complexity is interesting because, whereas $(\utilde{\Sigma}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness follows from the modest large cardinal hypothesis that $\lambda$ is a limit of Woodin cardinals, generic absoluteness for the slightly larger pointclass $\exists^{\mathbb{R}}(\utilde{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ is not known to follow from any large cardinal assumption whatsoever (although it can be forced from large cardinals). Moreover, inner model theory suggests a possible reason that it might *not* follow from large cardinals; see Remarks 2.2 and 2.6 below.

In Section 2, we consider the principle of one-step $\exists^{\mathbb{R}}(\utilde{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below a limit of Woodin cardinals $\lambda$, obtain a consistency strength upper bound for it in terms of large cardinals, and obtain an equivalent characterization in terms of a closure property of the pointclass of $\lambda$-universally Baire sets. The problem of finding a consistency strength lower bound for this generic absoluteness principle remains open.

In Section 3, we prove some lemmas for constructing absolute complements of trees. These lemmas do not require any facts about generic absoluteness or pointclasses and may be read independently of the rest of the article.

In the remaining sections, we consider stronger principles of generic absoluteness and we derive consistency strength lower bounds for these principles in the form of strong axioms of determinacy. Given a limit $\lambda$ of Woodin cardinals, we show that certain generic absoluteness principles imply strong axioms of determinacy in the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ associated to a $V$-generic filter on the poset $\mathrm{Col}(\omega, {<}\lambda)$. It is convenient to express these strong axioms of determinacy in terms of the extent of the Suslin sets.

In Section 4, we consider the principle of two-step $\exists^{\mathbb{R}}(\utilde{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below a limit of Woodin cardinals $\lambda$. Woodin showed (see Remark 4.4 below) that

https://doi.org/10.1017/jsl.2017.35 Published online by Cambridge University Press

an upper bound for the consistency strength of this generic absoluteness principle is the existence of a limit $\lambda$ of Woodin cardinals and a cardinal less than $\lambda$ that is $<\lambda$-strong. Under the additional assumption that the limit $\lambda$ of Woodin cardinals is measurable, we obtain the following lower bound.

THEOREM 0.1. *If $\lambda$ is a measurable limit of Woodin cardinals and two-step $\exists^{\mathbb{R}}(\undertilde{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds with respect to generic extensions by posets of cardinality less than $\lambda$, then the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ satisfies $\mathsf{AD}$ + "every $\undertilde{\Pi}^2_1$ set of reals is Suslin."*

We almost have equiconsistency (and would have, if not for the assumption of measurability) because the theory $\mathsf{ZF} + \mathsf{AD}$ + "every $\undertilde{\Pi}^2_1$ set of reals is Suslin" is equiconsistent with $\mathsf{ZFC}$ + "there is a limit $\lambda$ of Woodin cardinals and a cardinal less than $\lambda$ that is $<\lambda$-strong" (Woodin; see Steel [16, Section 15]).

It is worth mentioning, although we will not use this fact, that the statement "every $\undertilde{\Pi}^2_1$ set of reals is Suslin" is equivalent under $\mathsf{ZF} + \mathsf{AD}^+$ to the statement "$\theta_0 < \Theta$" about the length of the Solovay sequence.[1] The forward direction of this equivalence uses the uniformization property of Suslin sets and is essentially given by Solovay [14, Lemma 2.2]. The reverse direction is due to Woodin, building on results of Martin and Woodin [9]. Woodin's argument is unpublished; see Wilson [19] for a slightly sharper version.

In Section 5, we consider generic absoluteness of the theory of $L(\mathrm{uB}_\lambda, \mathbb{R})$ below a limit of Woodin cardinals $\lambda$. Woodin showed (see Theorem 5.1 below) that an upper bound for the consistency strength of this generic absoluteness principle is the existence of a limit $\lambda$ of Woodin cardinals and a cardinal less than $\lambda$ that is $<\lambda$-supercompact. Under the additional assumption that the limit $\lambda$ of Woodin cardinals is measurable, we obtain the following lower bound.

THEOREM 0.2. *If $\lambda$ is a measurable limit of Woodin cardinals and $L(\mathrm{uB}_\lambda, \mathbb{R}) \equiv L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]}$ for every generic extension $V[g]$ by a poset of cardinality less than $\lambda$, then the models $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ and $L(\mathrm{uB}_\lambda, \mathbb{R})$ both satisfy $\mathsf{AD} + \mathsf{DC}$ + "every set of reals is Suslin."*

The theory $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$ + "every set of reals is Suslin" is equiconsistent with the theory $\mathsf{ZFC}$ + "there is a limit $\lambda$ of Woodin cardinals such that $\lambda$ many cardinals less than $\lambda$ are $<\lambda$-strong" (Woodin and Steel, unpublished). Because this large cardinal hypothesis is much weaker than the existence of a limit $\lambda$ of Woodin cardinals and a cardinal less than $\lambda$ that is $<\lambda$-supercompact, Theorem 0.2 is far from equiconsistency.

It is worth mentioning that the statement $\mathsf{AD}$ + "every set of reals is Suslin" is equivalent under $\mathsf{ZF} + \mathsf{DC}$ to $\mathsf{AD}_{\mathbb{R}}$, the axiom of determinacy for real games, by unpublished work of Martin and Woodin (see Steel [17, Theorems 9.1, 9.2]).

**§1. $\lambda$-universally Baire sets and $(\undertilde{\Sigma}^2_1)^{\mathrm{uB}_\lambda}$ sets.** For our purposes, a *tree* is a tree on $\omega^k \times \mathrm{Ord}$ for some natural number $k$, meaning a collection of finite sequences of elements of $\omega^k \times \mathrm{Ord}$ that is closed under initial segments and ordered by reverse inclusion. Usually we assume $k = 1$ for simplicity and leave obvious generalizations

[1] The axiom $\mathsf{AD}^+$ is a technical strengthening of $\mathsf{AD}$ due to Woodin that holds in all known models of $\mathsf{AD}$ and in particular in models of the form $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$.

https://doi.org/10.1017/jsl.2017.35 Published online by Cambridge University Press

to the reader. We may abuse notation by treating a finite sequence of elements of $\omega^k \times \mathrm{Ord}$ as a $(k+1)$-tuple of finite sequences. According to the usual convention, elements of the Baire space $\omega^\omega$ are called *reals* and the Baire space itself may be denoted by $\mathbb{R}$ when appropriate. Given a tree $T$ on $\omega \times \mathrm{Ord}$ and a real $x$, we define a tree $T_x$ on Ord, called a *section* of $T$, by

$$T_x = \{s \in \mathrm{Ord}^{<\omega} : (x \restriction |s|, s) \in T\}.$$

The *projection* $\mathrm{p}[T]$ of $T$ is the set of reals defined by

$$\mathrm{p}[T] = \{x \in \mathbb{R} : T_x \text{ is ill founded}\}.$$

An equivalent definition of the projection is to let $[T]$ be the set of infinite branches of $T$, which is a closed subset of $\omega^\omega \times \mathrm{Ord}^\omega$, and to let $\mathrm{p}[T]$ be its projection onto the first coordinate.

For any given real $x$ the statement $x \in \mathrm{p}[T]$ is generically absolute by the absoluteness of well foundedness. New reals appearing in generic extensions may or may not be in $\mathrm{p}[T]$.

A set of reals $A$ is *Suslin* if it is the projection of a tree. That is, $A = \mathrm{p}[T]$ for some tree $T$ on $\omega \times \mathrm{Ord}$. The pointclass of Suslin sets is a natural generalization of the pointclass of $\utilde{\Sigma}^1_1$ (analytic) sets, which have the form $\mathrm{p}[T]$ for trees $T$ on $\omega \times \omega$. The Suslin sets play an important role under the axiom of determinacy. However, under the axiom of choice every set of reals is trivially Suslin by a tree $T$ on $\omega \times 2^{\aleph_0}$, so in this context one has to impose some condition on the tree in order to get an interesting definition.

A pair of trees $(T, \tilde{T})$ on $\omega \times \mathrm{Ord}$ is *$\lambda$-absolutely complementing*, where $\lambda$ is a cardinal, if in every generic extension by a poset of cardinality less than $\lambda$ the trees project to complements:

$$\mathrm{p}[T] = \mathbb{R} \setminus \mathrm{p}[\tilde{T}].$$

Note that for any pair of trees $(T, \tilde{T})$ on $\omega \times \mathrm{Ord}$ the statement $\mathrm{p}[T] \cap \mathrm{p}[\tilde{T}] = \emptyset$ is generically absolute by a standard argument using the absoluteness of well foundedness. Therefore if the statement $\mathrm{p}[T] = \mathbb{R} \setminus \mathrm{p}[\tilde{T}]$ holds in $V$ and the statement $\mathrm{p}[T] \cup \mathrm{p}[\tilde{T}] = \mathbb{R}$ holds in every generic extension by a poset of cardinality less than $\lambda$, then the pair $(T, \tilde{T})$ is $\lambda$-absolutely complementing.

A set of reals $A$ is *$\lambda$-universally Baire* if there is a $\lambda$-absolutely complementing pair of trees $(T, \tilde{T})$ such that $A = \mathrm{p}[T]$.[2] The pointclass $\mathrm{uB}_\lambda$ is defined to consist of the sets of reals that are $\lambda$-universally Baire. Note that if $\lambda$ is a limit cardinal and a set of reals is $\kappa$-universally Baire for all cardinals $\kappa < \lambda$ then it is $\lambda$-universally Baire; this can be seen by "amalgamating" a transfinite sequence of trees into a single tree whose projection is the union of their projections. A set of reals is *universally Baire* (uB) if it is $\lambda$-universally Baire for all cardinals $\lambda$.

For a set of reals $A \in \mathrm{uB}_\lambda$ and a generic extension $V[g]$ by a poset of cardinality less than $\lambda$, there is a *canonical extension* $A^{V[g]} \subset \mathbb{R}^{V[g]}$ of $A$ defined by

$$A^{V[g]} = \mathrm{p}[T]^{V[g]}$$

[2]In this article, we use the definition of $\lambda$-universal Baireness from Larson [8] and Steel [17], not the one from Feng, Magidor, and Woodin [3] that involves posets of cardinality equal to $\lambda$.

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for any $\lambda$-absolutely complementing pair of trees $(T, \tilde{T}) \in V$ with $A = \mathrm{p}[T]^V$. This extension of $A$ does not depend on the choice of the $\lambda$-absolutely complementing pair of trees.

Universal Baireness was introduced by Feng, Magidor, and Woodin [3] as a generalization of the property of Baire that implies some other classical regularity properties such as Lebesgue measurability. Although universally Baire sets can fail to be determined, as in $L$, if there is a Woodin cardinal less than $\lambda$ then every $\lambda$-universally Baire set of reals is determined by Neeman [11, Theorem 6.17].[3]

If $\lambda$ is a limit of Woodin cardinals then we obtain not only the determinacy of the $\mathrm{uB}_\lambda$ sets, but also a proper class model of $\mathsf{ZF} + \mathsf{AD}$ via Woodin's "derived model" construction, which will be useful to us as a way of bringing together the $\mathrm{uB}_\lambda$ sets existing in various generic extensions under one umbrella. We define the following standard notation:

Definition 1.1. Let $\lambda$ be a limit of Woodin cardinals and let $G \subset \mathrm{Col}(\omega, {<}\lambda)$ be a $V$-generic filter. We define

$$\mathbb{R}^*_G = \bigcup_{\alpha<\lambda} \mathbb{R}^{V[G\restriction\alpha]} \quad \text{and} \quad \mathrm{HC}^*_G = \bigcup_{\alpha<\lambda} \mathrm{HC}^{V[G\restriction\alpha]}.$$

For a set of reals $A \in \mathrm{uB}_\lambda$, we define

$$A^*_G = \bigcup_{\alpha<\lambda} A^{V[G\restriction\alpha]} \subset \mathbb{R}^*_G.$$

Similarly, for a set of reals $A \in \mathrm{uB}_\lambda^{V[G\restriction\xi]}$ where $\xi < \lambda$ we define

$$A^*_G = \bigcup_{\xi\leq\alpha<\lambda} A^{V[G\restriction\alpha]} \subset \mathbb{R}^*_G.$$

Finally, we define the pointclass

$$\mathrm{Hom}^*_G = \{A^*_G : A \in \mathrm{uB}_\lambda^{V[G\restriction\xi]} \text{ for some } \xi < \lambda\},$$

which might just as aptly have been called $\mathrm{uB}^*_G$.[4]

The following theorem is a special case of Woodin's derived model theorem. For a proof of this special case, see Steel [17]. The axiom $\mathsf{AD}^+$ in the theorem is a strengthening of $\mathsf{AD}$ due to Woodin that holds in all known models of $\mathsf{AD}$.[5]

Theorem 1.2 (Woodin). *Let $\lambda$ be a limit of Woodin cardinals and let $G \subset \mathrm{Col}(\omega, {<}\lambda)$ be a $V$-generic filter. Then the model $L(\mathrm{Hom}^*_G, \mathbb{R}^*_G)$ satisfies* $\mathsf{AD}^+$.

The theory of the model $L(\mathrm{Hom}^*_G, \mathbb{R}^*_G)$ does not depend on the choice of generic filter $G$ because the Levy collapse poset $\mathrm{Col}(\omega, {<}\lambda)$ is homogeneous, so we will sometimes omit $G$ from the notation and refer to "the" model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$.

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[3] In the case that there are two Woodin cardinals less than $\lambda$, the determinacy of the $\lambda$-universally Baire sets of reals follows from earlier work of Martin, Steel, and Woodin.

[4] The name $\mathrm{Hom}^*_G$ comes from the notion of *${<}\lambda$-homogeneity*, which in the case that $\lambda$ is a limit of Woodin cardinals is equivalent to $\lambda$-universal Baireness by work of Martin, Solovay, Steel, and Woodin (see Steel [17, Sections 2–4] or Larson [8, Theorem 3.3.13]).

[5] For the definition of $\mathsf{AD}^+$, see Woodin [21, Definition 22]. Instead of using $\mathsf{AD}^+$ directly, we will use some consequences due to Woodin, namely Theorems 1.4 and 1.6 below and the $\Sigma_1$-reflection theorem. For a proof of the $\Sigma_1$-reflection theorem, see Steel and Trang [15].

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In the past, Theorem 1.2 has been called the *derived model theorem* and the model $L(\mathrm{Hom}^*_G, \mathbb{R}^*_G)$ has been called a *derived model*. More recently, the term "derived model" refers to a model that may properly contain $L(\mathrm{Hom}^*_G, \mathbb{R}^*_G)$ and the term "derived model theorem" refers to the fact that this larger model satisfies $\mathsf{AD}^+$ (Woodin [21, Theorem 31]; see Zhu [22] for a proof). The model $L(\mathrm{Hom}^*_G, \mathbb{R}^*_G)$ will suffice for this article.

The following generic absoluteness theorem will be used many times throughout the article. For a proof, see Steel [17, Theorem 6.1; Lemmas 7.3 and 7.4].

THEOREM 1.3 (Woodin). *Let $\lambda$ be a limit of Woodin cardinals. For every real $x$, every set of reals $A \in \mathrm{uB}_\lambda$, every formula $\varphi(v)$ in the language of set theory expanded by two new predicate symbols, every generic extension $V[g]$ by a poset of cardinality less than $\lambda$, and every generic extension $V[G]$ by* $\mathrm{Col}(\omega, {<}\lambda)$ *containing $V[g]$, the following equivalences hold*:

$$\begin{aligned} &\exists B \in \mathrm{uB}^V_\lambda\ (\mathrm{HC}^V; \in, A, B) \models \varphi[x] \\ \iff &\exists B \in \mathrm{uB}^{V[g]}_\lambda\ (\mathrm{HC}^{V[g]}; \in, A^{V[g]}, B) \models \varphi[x] \\ \iff &\exists B \in \mathrm{Hom}^*_G\ (\mathrm{HC}^*_G; \in, A^*_G, B) \models \varphi[x] \\ \iff &\exists B \in L(\mathrm{Hom}^*_G, \mathbb{R}^*_G)\ (\mathrm{HC}^*_G; \in, A^*_G, B) \models \varphi[x]. \end{aligned}$$

Two special cases of Theorem 1.3 will be used often. First, in the case $A = \emptyset$, we get generic absoluteness for various restricted notions of $\Sigma^2_1(x)$; in particular we get $(\Sigma^2_1(x))^{\mathrm{uB}_\lambda}$ generic absoluteness between $V$ and $V[g]$. Second, in the case that $\varphi$ does not mention $B$, we get generic absoluteness for statements that are projective in $A$ and its expansions $A^{V[g]}$ and $A^*$, respectively.

The main consequence of $\mathsf{AD}^+$ in the derived model that we will need is given by the following theorem.

THEOREM 1.4 (Woodin). $\mathsf{AD}^+$ *implies that every $\Sigma^2_1$ set of reals is Suslin and is the projection of a tree $T$ on* $\omega \times \mathrm{Ord}$ *that is definable without parameters.*

The tree in Theorem 1.4 comes from the scale property of $\Sigma^2_1$ (see Steel [17, Section 8] for the case of derived models). Note that if $\lambda$ is a limit of Woodin cardinals and $T$ is a tree obtained by applying Theorem 1.4 in the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$, then we have $T \in V$ by the homogeneity of the poset $\mathrm{Col}(\omega, {<}\lambda)$. We will often use the following immediate corollary of Theorems 1.3 and 1.4.

COROLLARY 1.5 (Trees for $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ formulas). *Let $\lambda$ be a limit of Woodin cardinals. For every formula $\varphi(v)$ in the language of set theory expanded by a new predicate symbol, there is a tree $T_\varphi \in V$ such that for every generic extension $V[g]$ of $V$ by a poset of cardinality less than $\lambda$ and every real $x \in V[g]$ we have*

$$x \in \mathrm{p}[T_\varphi] \iff \exists B \in \mathrm{uB}^{V[g]}_\lambda\ (\mathrm{HC}^{V[g]}; \in, B) \models \varphi[x].$$

We remark that these trees can be used to get $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness between $V$ and $V[g]$ by a standard argument using the absoluteness of well foundedness, just as Shoenfield trees can be used to get $\Sigma^1_2$ generic absoluteness. In the case of $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness it is simpler to prove the absoluteness directly, using the stationary tower, than to build the trees $T_\varphi$ of Corollary 1.5. However, these trees $T_\varphi$ will still be quite useful for other purposes.

The following theorem can be considered as a basis theorem for the pointclass $\Sigma^2_1$. For a proof in the case of derived models, see Steel [17, Section 8].

THEOREM 1.6 (Woodin). $\mathsf{AD}^+$ *implies that every true* $\Sigma^2_1$ *statement has a witness that is a* $\Delta^2_1$ *set of reals.*

Woodin's basis theorem easily generalizes to say that for every real $x$, every true $\Sigma^2_1(x)$ statement has a $\Delta^2_1(x)$ witness, uniformly in $x$ in the following sense. (The author is probably not the first to take note of this generalization, but does not know of a reference for it.)

LEMMA 1.7. *Assume* $\mathsf{AD}^+$ *and let* $\varphi(v)$ *be a formula in the language of set theory expanded by a unary predicate symbol. Consider the* $\Sigma^2_1$ *set* $S$ *defined by*

$$y \in S \iff \exists B \subset \mathbb{R}\ (\mathrm{HC}; \in, B) \models \varphi[y].$$

*Then to each real* $y \in S$, *we can assign* $B(y) \subset \mathbb{R}$ *satisfying* $(\mathrm{HC}; \in, B(y)) \models \varphi[y]$ *in such a way that the following binary relations are both* $\Sigma^2_1$:

$$\{(y,z) \in \mathbb{R}\times\mathbb{R} : y \in S \ \ \&\ \ z \in B(y)\},$$
$$\{(y,z) \in \mathbb{R}\times\mathbb{R} : y \in S \ \ \&\ \ z \notin B(y)\}.$$

*In other words* $B(y) \in \Delta^2_1(y)$ *for each real* $y \in S$, *uniformly in* $y$.

PROOF. This follows easily from the proof of Woodin's basis theorem (see Steel [17, Section 8]) and in particular from the fact that for every real $y$, if there is a set of reals $B$ such that $(\mathrm{HC}; \in, B) \models \varphi[y]$, then there is such a set $B$ with the additional property that $B \in \mathrm{OD}^{L(A,\mathbb{R})}_y$ for some set of reals $A$. Note that the model $L(A,\mathbb{R})$ depends only on the Wadge rank of $A$ and not on $A$ itself. Minimizing this Wadge rank and then minimizing the witness $B$ in the canonical well ordering of $\mathrm{OD}^{L(A,\mathbb{R})}_y$ sets, a straightforward computation shows that the witness $B(y)$ we obtain is $\Delta^2_1(y)$ uniformly in $y$. ⊣

In particular the above lemma applies to the pointclass $\Sigma^2_1$ of the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ where $\lambda$ is a limit of Woodin cardinals. Next we will obtain a version of the lemma in terms of the pointclass $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$.

LEMMA 1.8. *Let* $\lambda$ *be a limit of Woodin cardinals and let* $\varphi(v)$ *be a formula in the language of set theory expanded by a unary predicate symbol. Consider the* $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ *set* $S$ *defined by*

$$y \in S \iff \exists B \in \mathrm{uB}_\lambda\ (\mathrm{HC}; \in, B) \models \varphi[y].$$

*Then to each real* $y \in S$, *we can assign* $B(y) \in \mathrm{uB}_\lambda$ *satisfying* $(\mathrm{HC}; \in, B(y)) \models \varphi[y]$ *in such a way that the following binary relations are both* $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$:

$$\{(y,z) \in \mathbb{R}\times\mathbb{R} : y \in S \ \ \&\ \ z \in B(y)\},$$
$$\{(y,z) \in \mathbb{R}\times\mathbb{R} : y \in S \ \ \&\ \ z \notin B(y)\}.$$

*In other words* $B(y) \in (\Delta^2_1(y))^{\mathrm{uB}_\lambda}$ *for each real* $y \in S$, *uniformly in* $y$.

PROOF. Working in $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$, consider the $\Sigma^2_1$ set of reals $S^*$ defined by

$$y \in S^* \iff \exists B^* \subset \mathbb{R}^*_\lambda\ (\mathrm{HC}^*_\lambda; \in, B^*) \models \varphi[y].$$

By Lemma 1.7, to each real $y \in S^*$, we can assign a witness $B^*(y) \subset \mathbb{R}^*_\lambda$ satisfying $(\mathrm{HC}^*_\lambda; \in, B^*(y)) \models \varphi[y]$ such that $B^*(y) \in \Delta^2_1(y)$ for each real $y \in S^*$, uniformly in $y$. Note that in the case $y \in V$, we have $B^*(y) \cap V \in V$ by the homogeneity of the Levy collapse poset.

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By Woodin's generic absoluteness (Theorem 1.3), we have $S^* \cap V = S$ and, defining $B(y) = B^*(y) \cap V$, we have $B(y) \in (\Delta^2_1(y))^{\mathrm{uB}_\lambda}$ for each real $y \in S$, uniformly in $y$. Also by Theorem 1.3, the structure $(\mathrm{HC}; \in, B(y))$ is an elementary substructure of $(\mathrm{HC}^*_\lambda; \in, B^*(y))$, so it satisfies $\varphi[y]$. For each real $y \in S$, the trees for the set $B^*(y)$ and its complement given by applying Theorem 1.4 in the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ are in $V$ by homogeneity, where they form an $\lambda$-absolutely complementing pair for the set $B(y)$. Therefore $B(y) \in \mathrm{uB}_\lambda$. ⊣

## §2. One-step $\exists^{\mathbb{R}}(\underset{\sim}{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness.

Next we will consider principles of generic absoluteness for pointclasses beyond $(\underset{\sim}{\Sigma}^2_1)^{\mathrm{uB}_\lambda}$. First we define a "lightface" (effective) version.

Definition 2.1. Let $\lambda$ be a limit of Woodin cardinals. Then $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ *generic absoluteness below* $\lambda$ is the statement that for every formula $\varphi(v)$ in the language of set theory expanded by a unary predicate symbol, and for every extension $V[g]$ of $V$ by a poset of cardinality less than $\lambda$, we have

$$\exists y \in \mathbb{R}^V\ \forall B \in \mathrm{uB}^V_\lambda\ (\mathrm{HC}^V; \in, B) \models \varphi[y]$$
$$\iff \exists y \in \mathbb{R}^{V[g]}\ \forall B \in \mathrm{uB}^{V[g]}_\lambda\ (\mathrm{HC}^{V[g]}; \in, B) \models \varphi[y].$$

Remark 2.2. The canonical inner model $M_\omega$ for $\omega$ many Woodin cardinals does not satisfy $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below its limit of Woodin cardinals $\lambda$, as mentioned in Steel [17, Remark 6.2]. This is because it satisfies the $\forall^{\mathbb{R}}(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ statement "every real is in a mouse with a $\mathrm{uB}_\lambda$ iteration strategy," which fails in small forcing extensions that add reals.

This remark applies not only to $M_\omega$ but also to some other mice satisfying stronger large cardinal axioms, as shown in Steel [16]. It is an open question whether $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below $\lambda$ is implied by any large cardinal hypothesis on $\lambda$.

Remark 2.3. The upward direction of $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness, from $V$ to $V[g]$, is automatic from $(\underset{\sim}{\Sigma}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness. More specifically, it follows from $(\Sigma^2_1(y))^{\mathrm{uB}_\lambda}$ generic absoluteness where $y$ is a real witnessing that the $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ statement holds in $V$.

In the terminology of Hamkins and Löwe [4], the existence of a real $y$ witnessing an $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ statement is a *button* from the point of view of $V_\lambda$. That is, once "pushed" (made true) by forcing, it cannot be "unpushed" (made false) by any further forcing. Therefore $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness is a special case of the maximality principle MP defined by Hamkins [5]: if $\lambda$ is a limit of Woodin cardinals and $V_\lambda \models \mathsf{MP}$ then $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds below $\lambda$.

The consistency of $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness can be established by a compactness argument based on the one given by Hamkins [5] for the consistency of the maximality principle MP. We just have to localize the argument to $V_\lambda$ and check that the hypothesis "$\lambda$ is a limit of Woodin cardinals" is preserved, which we do below for the convenience of the reader.

Proposition 2.4. *If the theory* ZFC+ *"there are infinitely many Woodin cardinals" is consistent, then so is* ZFC + *"$\lambda$ is a limit of Woodin cardinals and $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds below $\lambda$."*

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Proof. Let $M$ be a model of ZFC + "there are infinitely many Woodin cardinals" and let $\lambda^M \in M$ be a limit of Woodin cardinals of $M$. Let $T$ be the theory in the language of set theory expanded by a constant symbol $\lambda$ consisting of the ZFC axioms, the assertion that $\lambda$ is a limit of Woodin cardinals, and for each formula $\varphi(v)$, the assertion "if the statement $\exists y \in \mathbb{R}\, \forall B \in \mathrm{uB}_\lambda\ (\mathrm{HC}; \in, B) \models \varphi[y]$ holds in some generic extension of $V$ by a poset of cardinality less than $\lambda$, then it holds in $V$." The theory $T$ implies $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below $\lambda$. We will show that $T$ is consistent.

Indeed, given any finite subset $T_0 \subset T$ let $\varphi_0(v), \dots, \varphi_{n-1}(v)$ enumerate all the formulas $\varphi(v)$ that are mentioned in $T_0$ and have the property that the $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ statement "$\exists y \in \mathbb{R}\, \forall B \in \mathrm{uB}_\lambda\ (\mathrm{HC}; \in, B) \models \varphi[y]$" holds in some generic extension of $M$ by a poset of cardinality less than $\lambda^M$. For each $i < n$ take a poset $\mathbb{P}_i \in (V_\lambda)^M$ whose top condition forces this $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ statement. Then any generic extension of $(M, \lambda^M)$ by the product forcing $\mathbb{P}_0 \times \cdots \times \mathbb{P}_{n-1}$ satisfies the statements "$\exists y \in \mathbb{R}\, \forall B \in \mathrm{uB}_\lambda\ (\mathrm{HC}; \in, B) \models \varphi_i[y]$" for all $i < n$ by Remark 2.3, and also it still satisfies ZFC + "$\lambda$ is a limit of Woodin cardinals" because Woodin cardinals are preserved by small forcing, so it satisfies $T_0$. ⊣

The next result is an equivalent condition for $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness to hold below $\lambda$ in terms of a closure property of the pointclass $\mathrm{uB}_\lambda$ of $\lambda$-universally Baire sets of reals.

Proposition 2.5. *For any limit of Woodin cardinals $\lambda$, the following statements are equivalent.*

1. $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ *generic absoluteness holds below $\lambda$.*
2. *Every $(\Delta^2_1)^{\mathrm{uB}_\lambda}$ set of reals is $\lambda$-universally Baire.*

Proof. (1) $\Longrightarrow$ (2): The proof of this direction is analogous to that of Feng, Magidor, and Woodin [3, Theorem 3.1] with the pointclass $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ in place of the pointclass $\Sigma^1_2$. Let $A$ be a $(\Delta^2_1)^{\mathrm{uB}_\lambda}$ set of reals and take formulas $\varphi(v)$ and $\psi(v)$ such that for all reals $y$ we have

$$y \in A \iff \exists B \in \mathrm{uB}_\lambda\, (\mathrm{HC}; \in, B) \models \varphi[y] \text{ and}$$
$$y \notin A \iff \exists B \in \mathrm{uB}_\lambda\, (\mathrm{HC}; \in, B) \models \psi[y].$$

Let $T_\varphi$ and $T_\psi$ be trees such that in every generic extension $V[g]$ of $V$ by a poset of cardinality less than $\lambda$ we have, for every real $y \in V[g]$,

$$y \in \mathrm{p}[T_\varphi] \iff \exists B \in \mathrm{uB}_\lambda\, (\mathrm{HC}; \in, B) \models \varphi[y] \text{ and}$$
$$y \in \mathrm{p}[T_\psi] \iff \exists B \in \mathrm{uB}_\lambda\, (\mathrm{HC}; \in, B) \models \psi[y].$$

In particular, in $V$ we have $A = \mathrm{p}[T_\varphi] = \mathbb{R} \setminus \mathrm{p}[T_\psi]$. We claim that the trees $T_\varphi$ and $T_\psi$ are $\lambda$-absolutely complementing. Let $V[g]$ be a generic extension of $V$ by a poset of cardinality less than $\lambda$. As usual, the absoluteness of well foundedness gives $\mathrm{p}[T_\varphi] \cap \mathrm{p}[T_\psi] = \emptyset$ in $V[g]$. On the other hand, the $\forall^{\mathbb{R}}(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ statement

$$\forall y \in \mathbb{R}\, \exists B \in \mathrm{uB}_\lambda\, (\mathrm{HC}; \in, B) \models \varphi[y] \vee \psi[y]$$

holds in $V$, so by our hypothesis (1) it holds in $V[g]$ and we have

$$V[g] \models \mathrm{p}[T_\varphi] \cup \mathrm{p}[T_\psi] = \mathbb{R}.$$

Therefore the trees $T_\varphi$ and $T_\psi$ project to complements in $V[g]$.

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(2) $\implies$ (1): Suppose that the $\forall^{\mathbb{R}}(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ statement

$$\forall y \in \mathbb{R}\, \exists B \in \mathrm{uB}_\lambda\ (\mathrm{HC}; \in, B) \models \varphi[y]$$

holds in $V$. We want to show that it continues to hold in generic extensions by posets of cardinality less than $\lambda$. (The other direction of generic absoluteness is automatic; see Remark 2.3.) In this case Lemma 1.8 gives a total function $y \mapsto B(y)$ uniformly choosing $(\Delta^2_1(y))^{\mathrm{uB}_\lambda}$ sets of reals to witness our true $(\Sigma^2_1(y))^{\mathrm{uB}_\lambda}$ statements. By hypothesis (2), the $(\Delta^2_1)^{\mathrm{uB}_\lambda}$ relation

$$W = \{(y, z) \in \mathbb{R} \times \mathbb{R} : z \in B(y)\}$$

is $\lambda$-universally Baire. The true statement

$$\forall y \in \mathbb{R}\ (\mathrm{HC}; \in, W_y) \models \varphi[y]$$

can be expressed as a first-order property of the structure $(\mathrm{HC}; \in, W)$. Letting $V[g]$ be any generic extension of $V$ by a poset of cardinality less than $\lambda$ and letting $W^{V[g]} \subset \mathbb{R}^{V[g]} \times \mathbb{R}^{V[g]}$ denote the canonical extension of $W$ to $V[g]$, a special case of Theorem 1.3 (Woodin's generic absoluteness theorem) gives

$$(\mathrm{HC}; \in, W) \prec (\mathrm{HC}^{V[g]}; \in, W^{V[g]}).$$

Therefore we have

$$\forall y \in \mathbb{R}^{V[g]}\ \left(\mathrm{HC}^{V[g]}; \in, (W^{V[g]})_y\right) \models \varphi[y].$$

This shows that the sections $(W^{V[g]})_y$ for reals $y \in V[g]$ witness that our $\forall^{\mathbb{R}}(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ statement holds in $V[g]$, as desired. $\dashv$

Remark 2.6. Recall that the canonical inner model $M_\omega$ for $\omega$ many Woodin cardinals does not satisfy $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below its limit of Woodin cardinals $\lambda$. This can now be seen to follow from the fact that $M_\omega$ has a $(\Delta^2_1)^{\mathrm{uB}_\lambda}$ well ordering of its reals, and that no well ordering of the reals can have the Baire property (which is implied by universal Baireness).

In general, the set of reals appearing in mice with $\mathrm{uB}_\lambda$ iteration strategies is a $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ set with a $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ well ordering given by the comparison theorem for mice. Therefore if every real is in such a mouse (as it is in $M_\omega$; see Steel [17, Remark 6.2]) then this well ordering is a $(\Delta^2_1)^{\mathrm{uB}_\lambda}$ well ordering of the reals.

This remark applies not only to $M_\omega$ but also to some other mice satisfying stronger large cardinal axioms. It is an open question whether every large cardinal hypothesis on $\lambda$ is consistent with the existence of a $(\Delta^2_1)^{\mathrm{uB}_\lambda}$ well ordering of the reals.

Next we consider a boldface generic absoluteness principle that allows all reals in $V$ as parameters. It is called one-step generic absoluteness to distinguish it from the two-step generic absoluteness principle defined in Section 4.

Definition 2.7. *One-step* $\exists^{\mathbb{R}}(\utilde{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ *generic absoluteness below* $\lambda$, where $\lambda$ is a limit of Woodin cardinals, is the statement that for every formula $\varphi(v, v')$ in the language of set theory expanded by a unary predicate symbol, every real parameter $x \in V$, and every generic extension $V[g]$ of $V$ by a poset of cardinality less than $\lambda$, we have

$$\exists y \in \mathbb{R}^V\, \forall B \in \mathrm{uB}^V_\lambda\ (\mathrm{HC}^V; \in, B) \models \varphi[x, y]$$
$$\iff \exists y \in \mathbb{R}^{V[g]}\, \forall B \in \mathrm{uB}^{V[g]}_\lambda\ (\mathrm{HC}^{V[g]}; \in, B) \models \varphi[x, y].$$

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By a straightforward relativization of Proposition 2.5, this principle is equivalent to a closure property of the pointclass of $\lambda$-universally Baire sets:

PROPOSITION 2.8. *For any limit of Woodin cardinals $\lambda$, the following statements are equivalent.*

1. *One-step $\exists^{\mathbb{R}}(\undertilde{\boldsymbol{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds below $\lambda$.*
2. *Every $(\undertilde{\boldsymbol{\Delta}}^2_1)^{\mathrm{uB}_\lambda}$ set of reals is $\lambda$-universally Baire.*

Obtaining one-step $\exists^{\mathbb{R}}(\undertilde{\boldsymbol{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness in the first place is not simply a matter of relativization, however. Even if we can force below $\lambda$ to make an $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formula hold for all the real parameters in $V$ for which it can be forced to hold, doing so might add more reals that we must consider as parameters, so we might need to force again, *etc.* Fortunately, a mild large cardinal assumption is sufficient to show that this process eventually reaches a stopping point. Namely, the assumption that some cardinal $\delta < \lambda$ is $\Sigma_2$-*reflecting in* $V_\lambda$, meaning that it is inaccessible and $V_\delta \prec_{\Sigma_2} V_\lambda$. (Feng, Magidor, and Woodin [3, Theorem 3.3] similarly used the Levy collapse of a $\Sigma_2$-reflecting cardinal to obtain one-step $\undertilde{\boldsymbol{\Sigma}}^1_3$ generic absoluteness.)

REMARK 2.9. One can also obtain one-step $\exists^{\mathbb{R}}(\undertilde{\boldsymbol{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below a limit $\lambda$ of Woodin cardinals as an application (in $V_\lambda$) of a "boldface maximality principle" that is shown in Hamkins [5] to hold after the Levy collapse of a (fully-) reflecting cardinal.

We note a useful reformulation: $V_\delta \prec_{\Sigma_2} V_\lambda$ if and only if for every set $x \in V_\delta$ and every formula $\psi$, if there is an ordinal $\beta < \lambda$ such that $V_\beta \models \varphi[x]$, then there is an ordinal $\bar{\beta} < \delta$ such that $V_{\bar{\beta}} \models \varphi[x]$. This reformulation is usually proved for $\lambda = \mathrm{Ord}$, but when $\lambda$ is a limit of Woodin cardinals the model $V_\lambda$ satisfies enough of ZFC for the proof.

It is convenient to split the consistency proof of one-step $\exists^{\mathbb{R}}(\undertilde{\boldsymbol{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness into two parts. First we prove a lemma using only the hypothesis $V_\delta \prec_{\Sigma_2} V_\lambda$ and then we use the inaccessibility of $\delta$ to get the full result.

LEMMA 2.10. *Let $\lambda$ be a limit of Woodin cardinals and let $\delta < \lambda$ be a cardinal such that $V_\delta \prec_{\Sigma_2} V_\lambda$. Let $\varphi(v, v')$ be a formula in the language of set theory expanded by a unary predicate symbol and let $x$ be a real parameter. Suppose there is a poset $\mathbb{P} \in V_\lambda$ such that*

$$1 \Vdash_{\mathbb{P}} \exists y \in \mathbb{R}\, \forall B \in \mathrm{uB}_\lambda\ (\mathrm{HC}; \in, B) \models \varphi[x, y].$$

*Then there is a poset $\bar{\mathbb{P}} \in V_\delta$ with the same property*:

$$1 \Vdash_{\bar{\mathbb{P}}} \exists y \in \mathbb{R}\, \forall B \in \mathrm{uB}_\lambda\ (\mathrm{HC}; \in, B) \models \varphi[x, y].$$

PROOF. Take a cardinal $\kappa < \lambda$ large enough that $\mathbb{P} \in V_\kappa$. We may assume that $\kappa$ is inaccessible, which implies $(\mathrm{uB})^{V_\kappa} = \mathrm{uB}_\kappa$. After forcing with $\mathbb{P}$, we still have $(\mathrm{uB})^{V_\kappa} = \mathrm{uB}_\kappa$ because $\kappa$ remains inaccessible. Also, by taking $\kappa < \lambda$ to be sufficiently large we may ensure that $\mathrm{uB}_\kappa = \mathrm{uB}_\lambda$ after forcing with $\mathbb{P}$ by an observation of Steel and Woodin; see Larson [8, Theorem 3.3.5].[6] Therefore our assumption on $\mathbb{P}$ yields

$$V_\kappa \models 1 \Vdash_{\mathbb{P}} \exists y \in \mathbb{R}\, \forall B \in \mathrm{uB}\ (\mathrm{HC}; \in, B) \models \varphi[x, y].$$

[6]This observation is usually stated in terms of homogeneously Suslin sets: $\mathrm{Hom}_\eta = \mathrm{Hom}_{<\lambda}$ for all sufficiently large $\eta < \lambda$. The present version is equivalent; one has only to let $\kappa$ be greater than the second Woodin cardinal above $\eta$.

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Now because $V_\delta \prec_{\Sigma_2} V_\lambda$ and the inaccessibility of $\kappa$ is a first-order property of $V_{\kappa+1}$, we can take an inaccessible cardinal $\bar{\kappa} < \delta$ and a poset $\bar{\mathbb{P}}$ such that

$$V_{\bar{\kappa}} \models 1 \Vdash_{\bar{\mathbb{P}}} \exists y \in \mathbb{R}\, \forall B \in \mathrm{uB}\ (\mathrm{HC}; \in, B) \models \varphi[x, y].$$

After forcing with $\bar{\mathbb{P}}$ we have $(\mathrm{uB})^{V_{\bar{\kappa}}} = \mathrm{uB}_{\bar{\kappa}}$ because $\bar{\kappa}$ is inaccessible, and we trivially have $\mathrm{uB}_\lambda \subset \mathrm{uB}_{\bar{\kappa}}$ because $\bar{\kappa} < \lambda$, so the desired conclusion follows. ⊣

Proposition 2.11. *Let $\lambda$ be a limit of Woodin cardinals and let $\delta < \lambda$ be an inaccessible cardinal such that $V_\delta \prec_{\Sigma_2} V_\lambda$. Let $G \subset \mathrm{Col}(\omega, {<}\delta)$ be a $V$-generic filter. Then $V[G]$ satisfies one-step $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below $\lambda$.*

Proof. Let $x \in V[G]$ be a real parameter. We will show that $V[G]$ satisfies $\exists^{\mathbb{R}}(\Pi^2_1(x))^{\mathrm{uB}_\lambda}$ generic absoluteness below $\lambda$. Because $\delta$ is inaccessible, the real $x$ is contained in the generic extension of $V$ by some proper initial segment of the generic filter $G$. Because our large cardinal hypotheses on $\delta$ and $\lambda$ are preserved by forcing with posets of cardinality less than $\delta$, we may assume $x \in V$.

By Lemma 2.10, every $\exists^{\mathbb{R}}(\Pi^2_1(x))^{\mathrm{uB}_\lambda}$ statement that can be forced by a poset in $V_\lambda$ (over $V[G]$, or equivalently over $V$) can also be forced by a poset in $V_\delta$. Such a poset in $V_\delta$ is absorbed into $\mathrm{Col}(\omega, {<}\delta)$ by universality, so the desired statement holds in $V[G]$ by the upward direction of $\exists^{\mathbb{R}}(\Pi^2_1(x))^{\mathrm{uB}_\lambda}$ generic absoluteness, which is automatic. ⊣

We remark that strong cardinals are $\Sigma_2$-reflecting, so the hypothesis of Proposition 2.11 follows from $\lambda$ being a limit of Woodin cardinals and $\delta < \lambda$ being ${<}\lambda$-strong (the $\mathsf{AD} + \theta_0 < \Theta$ hypothesis). However, it is much weaker than this because if $\lambda$ is a Mahlo cardinal then there are many inaccessible cardinals $\delta < \lambda$ such that $V_\delta$ is a fully elementary substructure of $V_\lambda$. We have not proved any consistency strength lower bound, leading to the obvious question.

Question 2.12. *What is the consistency strength of the theory* $\mathsf{ZFC}$ + *"there is a limit $\lambda$ of Woodin cardinals such that one-step $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds below $\lambda$"*?

## §3. Building absolute complements for trees.

In this section, which may be read independently of the rest of the article, we introduce a method for building an absolute complement to a given tree (Lemma 3.1 below). Two related theorems are that if $\kappa$ is supercompact then every tree becomes weakly homogeneous in some small forcing extension (Martin and Woodin [9, Theorem 3.2]) and if $\kappa$ is a Woodin cardinal then every tree becomes ${<}\kappa$-weakly homogeneous in some small forcing extension (Woodin; see Larson [8, Theorem 1.5.12]).

Unlike these two theorems, our method only gives $\kappa$-absolute complementation, not ${<}\kappa$-weak homogeneity. Although we will not have a ${<}\kappa$-weak homogeneity system to work with, something like a Martin–Solovay construction (see Steel [17, Section 2]) will still produce an absolutely complementing tree. A similar argument is used in another theorem of Woodin (see Steel [17, Theorem 4.5]).

Another difference from the results mentioned above is that because our large cardinal hypothesis of measurability is so weak, we will need to augment it with a smallness assumption about the tree, namely that not too many sets are constructible from it; this is made precise in the statement of the lemma. A related argument appears in Wilson [18, Lemma 9.4], where the large cardinal hypothesis is weak compactness and the smallness assumption is about a pointclass called the envelope.

The absolutely complementing tree constructed in the proof of the lemma will be the tree of a semiscale, so we will briefly review the essential properties of semiscales and their trees. For more information, see Kechris and Moschovakis [7] or Moschovakis [10].

First, a *norm* on a set of reals $A$ is a function $\psi : A \to \mathrm{Ord}$. A sequence of norms $\vec{\psi} = (\psi_i : i < \omega)$ on $A$ can be used to define a strong notion of convergence as follows. For a sequence of reals $(x_n : n < \omega)$ in $A$ and a real $y$, we write $x_n \to y \pmod{\vec{\psi}}$ to mean that $(x_n : n < \omega)$ converges to $y$ in the usual topology of the Baire space $\omega^\omega$ and for every $i < \omega$ the sequence of ordinals $(\psi_i(x_n) : n < \omega)$ is eventually constant.

A *semiscale* on $A$ is a sequence of norms $\vec{\psi}$ on $A$ such that for every sequence of reals $(x_n : n < \omega)$ in $A$ and every real $y$, if $x_n \to y \pmod{\vec{\psi}}$, then $y \in A$. If a set of reals admits a semiscale, then it is Suslin. More specifically, given a semiscale $\vec{\psi}$ on a set of reals $A$ we have $\mathrm{p}[T_{\vec{\psi}}] = A$ where

$$T_{\vec{\psi}} = \left\{ \left(x \upharpoonright i, \left(\psi_0(x), \dots, \psi_{i-1}(x)\right)\right) : x \in A \text{ and } i < \omega \right\}.$$

The tree $T_{\vec{\psi}}$ defined in this manner is called the *tree of the semiscale* $\vec{\psi}$.

Lemma 3.1. *Let $T$ be a tree on $\omega \times \gamma$ for some ordinal $\gamma$. Let $\kappa$ be a measurable cardinal and suppose there is a normal measure on $\kappa$ concentrating on the set of $\alpha < \kappa$ such that*

$$|\wp(V_\alpha) \cap L(T, V_\alpha)| = \alpha.$$

*Then there is a generic extension $V[g]$ of $V$ by a poset of cardinality less than $\kappa$ in which $T$ is $\kappa$-absolutely complemented.*

Proof. Consider an elementary embedding $j : V \to M$ that is the ultrapower map by such a normal measure on $\kappa$, so by Łoś's theorem, we have $|\wp(V_\kappa) \cap L(j(T), V_\kappa)| = \kappa$ in $M$ and equivalently in $V$. Take a $V$-generic filter $G \subset \mathrm{Col}(\omega, {<}\kappa)$ and extend $j$ to an elementary embedding

$$\hat{\jmath} : V[G] \to M[H],$$

where $H \subset \mathrm{Col}(\omega, {<}j(\kappa))$ is an $M$-generic filter. Note that in $V[G]$ the trees $T$ and $j(T)$ have the same projection by the elementarity of $\hat{\jmath}$ and the absoluteness of well foundedness. In $V[G]$, define the set of reals

$$A = \omega^\omega \setminus \mathrm{p}[T] = \omega^\omega \setminus \mathrm{p}[j(T)].$$

For every finite sequence of ordinals $t \in j(\gamma)^{<\omega}$, we define a norm $\varphi_t$ on $A$ by

$$\varphi_t(x) = \begin{cases} \mathrm{rank}_{j(T)_x}(t) & \text{if } t \in j(T)_x, \\ 0 & \text{if } t \notin j(T)_x. \end{cases}$$

Note that in $M[H]$, we have

$$\hat{\jmath}(A) = \omega^\omega \setminus \mathrm{p}[j(T)] = \omega^\omega \setminus \mathrm{p}[j(j(T))]$$

by the elementarity of $\hat{\jmath}$. We can define a collection of norms on $\hat{\jmath}(A)$ by

$$\mathcal{C} = \{\hat{\jmath}(\varphi_t) : t \in j(\gamma)^{<\omega}\}.$$

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Note that $\mathcal{C}$ is in $M[H]$ and is countable there: every norm $\varphi_t$ has a $\mathrm{Col}(\omega, {<}\kappa)$-name in $L(j(T), V_\kappa)$, and the number of such names is at most $\kappa$ by our hypothesis, so the pointwise image of the set of names is in $M$ and has cardinality at most $\kappa$ there.[7] In $M[H]$, take an enumeration $\vec{\psi}$ of $\mathcal{C}$ in order type $\omega$.

Claim. *$\vec{\psi}$ is a semiscale on the set of reals $\hat{\jmath}(A)$.*

Assuming for now that the claim holds, working in $M[H]$, we define $\tilde{T} = T_{\vec{\psi}}$, the tree of the semiscale $\vec{\psi}$. Then $\tilde{T}$ is definable from $\vec{\psi}$ and projects to $\hat{\jmath}(A)$, which by definition is the complement of the projection of $j(T)$.

Because each norm in the collection $\mathcal{C}$ is definable in the model $M[H]$ from the tree $j(j(T)) \in M$ and a finite sequence of ordinal parameters of the form $j(t)$ where $t \in j(\gamma)^{<\omega}$, the semiscale $\vec{\psi}$ and its tree $\tilde{T}$ are definable in $M[H]$ from $j(j(T))$ and a countable sequence of ordinal parameters. Therefore by a standard argument using the inaccessability of $j(\kappa)$ in $M$ and the homogeneity of the Levy collapse forcing, we have $\tilde{T} \in M[H \restriction \alpha]$ for some $\alpha < j(\kappa)$.

In this intermediate model $M[H \restriction \alpha]$ the tree $\tilde{T}$ is a $j(\kappa)$-absolute complement for $j(T)$. By the elementarity of $\hat{\jmath}$ it follows that the tree $T$ itself is $\kappa$-absolutely complemented in some small forcing extension of $V$, as desired.

It remains to prove the claim. Assume toward a contradiction that in $M[H]$ there is a sequence of reals $(x_n : n < \omega)$ in $\hat{\jmath}(A)$ such that $x_n \to y \pmod{\vec{\psi}}$ for some real $y$, but $y \notin \hat{\jmath}(A)$. Because $y \notin \hat{\jmath}(A)$, we have $y \in \mathrm{p}[j(T)]$ as witnessed by a sequence of ordinals $f \in j(\gamma)^\omega$ such that $(y, f) \in [j(T)]$. We will obtain a contradiction by considering, for each $i < \omega$, the norm $\hat{\jmath}(\varphi_{f\restriction i}) \in \mathcal{C}$ and the eventual value of the corresponding sequence of ordinals:

$$\lambda_i = \lim_{n<\omega} \hat{\jmath}(\varphi_{f\restriction i})(x_n).$$

By the definition of the norm $\varphi_{f\restriction i}$ and the elementarity of $\hat{\jmath}$, we can characterize the norm $\hat{\jmath}(\varphi_{f\restriction i})$ by

$$\hat{\jmath}(\varphi_{f\restriction i})(x) = \begin{cases} \mathrm{rank}_{j(j(T))_x}(j(f \restriction i)) & \text{if } j(f \restriction i) \in j(j(T))_x, \\ 0 & \text{if } j(f \restriction i) \notin j(j(T))_x. \end{cases}$$

Consider any fixed $i < \omega$. By our choice of the sequence of ordinals $f$, we have $(y \restriction i, f \restriction i) \in j(T)$. Because of the convergence $x_n \to y$, for all sufficiently large $n < \omega$ we have $x_n \restriction i = y \restriction i$ and therefore $(x_n \restriction i, f \restriction i) \in j(T)$, and so by the elementarity of $j$, we have $(x_n \restriction i, j(f \restriction i)) \in j(j(T))$, or in other words $j(f \restriction i) \in j(j(T))_{x_n}$. Therefore for all sufficiently large $n < \omega$, we are in the first case of the above characterization of the norm $\hat{\jmath}(\varphi_{f\restriction i})$ as applied to the real $x_n$, not the trivial second case where the norm is defined to be zero, so we have

$$\lambda_i = \lim_{n<\omega} \mathrm{rank}_{j(j(T))_{x_n}}(j(f \restriction i)).$$

Considering this characterization of $\lambda_i$ and $\lambda_{i+1}$ for any sufficiently large $n$, we see that $\lambda_{i+1}$ is less than $\lambda_i$ because they are the ranks of the node $j(f \restriction (i+1))$ and

[7] If $\kappa$ were supercompact, then we could take the set $\{j(t) : t \in j(\gamma)^{<\omega}\}$ itself to be in $M$ and we would not need the hypothesis that $j(T)$ does not construct too many sets.

its predecessor $j(f \restriction i)$, respectively, in the well founded tree $j(j(T))_{x_n}$. Therefore we obtain an infinite decreasing sequence of ordinals

$$\lambda_0 > \lambda_1 > \lambda_2 > \cdots .$$

This contradiction completes the proof of the claim and the lemma. ⊣

The consequences of Lemma 3.1 that we will use in our applications to generic absoluteness are stated below.

LEMMA 3.2. *Let $T$ be a tree on $\omega \times \gamma$ for some ordinal $\gamma$. Let $\kappa$ be a measurable cardinal. Assume that every generic extension of $V$ by a poset of cardinality less than $\kappa$ satisfies "$\mathbb{R} \cap L[T, x]$ is countable for every real $x$." Then some generic extension of $V$ by a poset of cardinality less than $\kappa$ satisfies "$T$ is $\kappa$-absolutely complemented."*

PROOF. Let $\alpha < \kappa$ be a cardinal such that $|V_\alpha| = \alpha$. After forcing with $\mathrm{Col}(\omega, \alpha)$ we get a real $x$ coding $V_\alpha$. By our hypothesis, the set $\mathbb{R} \cap L[T, x]$ is countable in this generic extension, so we have $|\wp(V_\alpha) \cap L(T, V_\alpha)| = \alpha$ in the ground model. Because every normal measure concentrates on cardinals $\alpha$ such that $|V_\alpha| = \alpha$, the desired conclusion follows from Lemma 3.1. ⊣

We can weaken the hypothesis of Lemma 3.2 by a standard argument using Solovay's almost disjoint coding method. (This gives a stronger lemma, but it is not essential to prove our main generic absoluteness results.)

LEMMA 3.3. *Let $T$ be a tree on $\omega \times \gamma$ for some ordinal $\gamma$. Let $\kappa$ be a measurable cardinal. Assume that every generic extension of $V$ by a poset of cardinality less than $\kappa$ satisfies "for every real $x$ there is a real $y$ such that $y \notin L[T, x]$." Then some generic extension of $V$ by a poset of cardinality less than $\kappa$ satisfies "$T$ is $\kappa$-absolutely complemented."*

PROOF. By Lemma 3.2 it suffices to show that if some generic extension $V[g_0]$ of $V$ by a poset of cardinality less than $\kappa$ satisfies "$\mathbb{R} \cap L[T, x_0]$ is uncountable for some real $x_0$," then some further generic extension $V[g]$ of $V[g_0]$ by a poset of cardinality less than $\kappa$ satisfies "$\mathbb{R} \subset L[T, x]$ for some real $x$."

Take a small forcing extension $V[g_0]$ containing a real $x_0$ and satisfying "$\mathbb{R} \cap L[T, x_0]$ is uncountable." By forcing with $\mathrm{Col}(\omega_1, \mathbb{R})$ if necessary to ensure CH, we may assume that

$$V[g_0] \models |\mathbb{R}| = |\mathbb{R} \cap L[T, x_0]| = \omega_1.$$

In $V[g_0]$, we have a subset $X_1$ of $\omega_1$ coding HC. Forcing over $V[g_0]$, we will use Solovay's almost disjoint coding to code our subset $X_1$ of $\omega_1$ by a real $x_1$. This is a standard argument, which we include for the reader's convenience. We let

$$\vec{a} = (a_\xi : \xi < \omega_1^{V[g_0]}) \in L[T, x_0]$$

be a family of almost disjoint subsets of $\omega$ and let $\mathbb{P}_{\vec{a}, X_1}$ denote the forcing notion consisting of partial functions $p : \omega \to 2$ such that $p^{-1}(\{1\})$ is finite and $\mathrm{dom}(p) \cap a_\xi$ is finite for every $\xi \in X_1$.

Let $g_1$ be a $V[g_0]$-generic filter for $\mathbb{P}_{\vec{a}, X_1}$ and let $x_1 \subset \omega$ be the corresponding generic real, meaning that for every $n < \omega$, we have $n \in x_1$ if and only if $\left(\bigcup g_1\right)(n) = 1$. Then $x_1$ codes $X_1$ relative to $\vec{a}$ in the sense that for every $\xi < \omega_1$, we have $\xi \in X_1$ if and only if $a_\xi \cap x_1$ is infinite. We have $x_1, \vec{a} \in L[T, x_0, x_1]$ so we have $X_1, \mathbb{P}_{\vec{a}, X_1} \in L[T, x_0, x_1]$ as well.

In the model $V[g_0]$ the forcing $\mathbb{P}_{\vec{a},X_1}$ is a subset of HC and has the countable chain condition, so every real $y \in V[g_0][g_1]$ is the interpretation of a hereditarily countable $\mathbb{P}_{\vec{a},X_1}$-name $\dot{y} \in V[g_0]$ by the generic filter $g_1$. The set $X_1$ codes the name $\dot{y}$, among other elements of $\mathrm{HC}^{V[g_0]}$, so we have $\dot{y} \in L[T, x_0, x_1]$. But the model $L[T, x_0, x_1]$ contains the generic filter $g_1$, so we have $y \in L[T, x_0, x_1]$. This shows that, letting $V[g] = V[g_0][g_1]$ and $x = \langle x_0, x_1\rangle$, we have $V[g] \models \mathbb{R} \subset L[T, x]$ as desired. ⊣

**§4. Two-step $\exists^{\mathbb{R}}(\boldsymbol{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness.** In this section we consider the following generic absoluteness principle, which is a strengthening of one-step $\exists^{\mathbb{R}}(\boldsymbol{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness.

Definition 4.1. *Two-step $\exists^{\mathbb{R}}(\boldsymbol{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below $\lambda$*, where $\lambda$ is a limit of Woodin cardinals, is the statement that every generic extension $V[g]$ of $V$ by a poset of cardinality less than $\lambda$ satisfies one-step $\exists^{\mathbb{R}}(\boldsymbol{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below $\lambda$.

The essential difference between one-step and two-step generic absoluteness is that in the definition of two-step generic absoluteness we allow real parameters $x$ from $V[g]$ and not just from $V$. For the lightface pointclass $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ or its relativization to any particular real $x \in V$, two-step generic absoluteness follows automatically from one-step generic absoluteness. The distinction between one-step and two-step generic absoluteness only exists for the boldface versions.

Applying Proposition 2.8 in generic extensions, we get a characterization of two-step $\exists^{\mathbb{R}}(\boldsymbol{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness in terms of a closure property of $\mathrm{uB}_\lambda$ in generic extensions.

Proposition 4.2. *For any limit of Woodin cardinals $\lambda$, the following statements are equivalent.*

1. *Two-step $\exists^{\mathbb{R}}(\boldsymbol{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds below $\lambda$.*
2. *In every generic extension $V[g]$ of $V$ by a poset of cardinality less than $\lambda$, every $(\boldsymbol{\Delta}^2_1)^{\mathrm{uB}_\lambda}$ set of reals is $\lambda$-universally Baire.*

One can obtain two-step $\exists^{\mathbb{R}}(\boldsymbol{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness from trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas by a standard argument using the absoluteness of well foundedness. By "trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas," we mean trees $\tilde{T}_\varphi$ that are analogous to the trees $T_\varphi$ for $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ formulas given by Corollary 1.5. To be precise:

Definition 4.3. We say *there are trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas* if for every formula $\varphi(v)$ there is a tree $\tilde{T}_\varphi$ such that for every generic extension $V[g]$ of $V$ by a poset of cardinality less than $\lambda$ and every real $x \in V[g]$ we have

$$x \in \mathrm{p}[\tilde{T}_\varphi] \iff \forall B \in \mathrm{uB}_\lambda^{V[g]}\,(\mathrm{HC}^{V[g]}; \in, B) \models \neg\varphi[x].$$

(We negate the formula $\varphi$ so that the tree $\tilde{T}_\varphi$ will be a $\lambda$-absolute complement of the tree $T_\varphi$ from Corollary 1.5.)

Remark 4.4. If $\lambda$ is a limit of Woodin cardinals and $\delta < \lambda$ is a $<\lambda$-strong cardinal, then trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas appear after forcing with $\mathrm{Col}(\omega, 2^{2^\delta})$. This is implicit in Woodin's proof from the same large cardinal hypothesis that $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ satisfies AD+"every $\boldsymbol{\Pi}^2_1$ set of reals is Suslin" (see Steel [17, Section 9]).

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The existence of trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas can be shown to be equivalent to some of its obvious consequences:

PROPOSITION 4.5. *For any limit of Woodin cardinals $\lambda$, the following statements are equivalent.*

1. *There are trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas.*
2. *In every generic extension $V[g]$ of $V$ by a poset of cardinality less than $\lambda$, every $(\utilde{\Sigma}^2_1)^{\mathrm{uB}_\lambda}$ set of reals is $\lambda$-universally Baire.*
3. *Every $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ set of reals is $\lambda$-universally Baire.*

PROOF. (1) $\implies$ (2): The trees $\tilde{T}_\varphi$ for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas are $\lambda$-absolute complements of the trees $T_\varphi$ for $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ formulas given by Corollary 1.5. Therefore in $V$ and in every generic extension by a poset of cardinality less than $\lambda$, the pair $(T_\varphi, \tilde{T}_\varphi)$ witnesses that the corresponding $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ set of reals is $\lambda$-universally Baire. Every $(\utilde{\Sigma}^2_1)^{\mathrm{uB}_\lambda}$ set of reals is a section of a $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ set of reals, making it $\lambda$-universally Baire as well. The implication (2) $\implies$ (3) is trivial.

(3) $\implies$ (1): Let $\varphi(v)$ be a formula in the language of set theory expanded by a unary predicate symbol. By Corollary 1.5 there is a tree $T_\varphi \in V$ that projects to the $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ set of reals defined by $\varphi$ in every generic extension by a poset of cardinality less than $\lambda$. We will show that the tree $T_\varphi$ is $\lambda$-absolutely complemented. The $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ set

$$A = \mathrm{p}[T_\varphi] = \{y \in \mathbb{R} : \exists B \in \mathrm{uB}_\lambda \ (\mathrm{HC}; \in, B) \models \varphi[y]\}$$

is $\lambda$-universally Baire by our hypothesis, so there is some $\lambda$-absolutely complementing pair of trees $(T, \tilde{T})$ such that $\mathrm{p}[T] = \mathrm{p}[T_\varphi]$.

We claim that the pair $(T_\varphi, \tilde{T})$ is also $\lambda$-absolutely complementing, or equivalently that for every generic extension $V[g]$ of $V$ by a poset of cardinality less than $\lambda$ we have

$$V[g] \models \mathrm{p}[T_\varphi] = \mathrm{p}[T].$$

We have $\mathrm{p}[T_\varphi] \cap \mathrm{p}[\tilde{T}] = \emptyset$ in $V$, so by the usual argument this holds in $V[g]$ as well, giving $V[g] \models \mathrm{p}[T_\varphi] \subset \mathrm{p}[T]$. For the reverse inclusion let $y \mapsto B(y)$ be the partial function given by Lemma 1.8 that chooses $(\Delta^2_1(y))^{\mathrm{uB}_\lambda}$ witnesses $B(y)$ uniformly for reals $y \in A$. Then the relation

$$W = \{(y, z) \in \mathbb{R} \times \mathbb{R} : y \in A \ \ \& \ \ z \in B(y)\}$$

is $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$, and we have

$$\forall y \in A \ (\mathrm{HC}; \in, W_y) \models \varphi[y],$$

where $W_y = B(y)$ is the corresponding section of $W$. By our hypothesis, the relation $W$ is $\lambda$-universally Baire. Let $A^{V[g]}$ and $W^{V[g]}$ denote the canonical extensions of $A$ and $W$, respectively, to $V[g]$. Then we have

$$(\mathrm{HC}; \in, A, W) \prec (\mathrm{HC}^{V[g]}; \in, A^{V[g]}, W^{V[g]}),$$

and it follows that

$$\forall y \in A^{V[g]} \left(\mathrm{HC}^{V[g]}; \in, (W_y)^{V[g]}\right) \models \varphi[y],$$

which shows that $V[g] \models \mathrm{p}[T] \subset \mathrm{p}[T_\varphi]$. This completes the proof that $\tilde{T}$ is a $\lambda$-absolute complement of the tree $T_\varphi$. Accordingly, we write $\tilde{T}_\varphi = \tilde{T}$. ⊣

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A natural question is whether two-step $\exists^{\mathbb{R}}(\undertilde{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below $\lambda$ can be added to the list of equivalences in 4.5, rather than being strictly weaker. A positive answer to this question could be seen as an explanation of this generic absoluteness principle in terms of a continuous reduction to the absoluteness of well foundedness.

QUESTION 4.6. *Let $\lambda$ be a limit of Woodin cardinals and assume two-step $\exists^{\mathbb{R}}(\undertilde{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below $\lambda$. Must trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas exist?*

If there are trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas then the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ satisfies (in addition to AD) the statement "every $\undertilde{\Pi}^2_1$ set of reals is Suslin," as noted in Steel [17, Section 9]. This conclusion follows even if the trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas do not appear in $V$ but only in small forcing extensions of $V$, because small forcing does not affect $\mathrm{Hom}^*_\lambda$ or $\mathbb{R}^*_\lambda$. Accordingly, one might ask the weaker question:

QUESTION 4.7. *Let $\lambda$ be a limit of Woodin cardinals and assume two-step $\exists^{\mathbb{R}}(\undertilde{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below $\lambda$. Must $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ satisfy "every $\undertilde{\Pi}^2_1$ set of reals is Suslin"?*

In the case where $\lambda$ is measurable we may apply Lemma 3.2 to get a positive answer, yielding a proof of Theorem 0.1. The proof is similar to that of "every set has a sharp" from two-step $\undertilde{\Sigma}^1_3$ generic absoluteness (Woodin [20, Lemma 1]) except that it uses Lemma 3.2 in place of Jensen's covering lemma for $L$.

PROOF OF THEOREM 0.1. Assume that $\lambda$ is a measurable limit of Woodin cardinals and two-step $\exists^{\mathbb{R}}(\undertilde{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds below $\lambda$. We will show that the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ satisfies "every $\undertilde{\Pi}^2_1$ set of reals is Suslin."

Fix a formula $\varphi(v)$ and let $T_\varphi$ be the tree for the corresponding $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ formula as given by Corollary 1.5. We will use Lemma 3.2 to obtain an absolute complement for $T_\varphi$. Given any generic extension $V[g]$ of $V$ by a poset of cardinality less than $\lambda$ and any real $x \in V[g]$, we want to show $\mathbb{R} \cap L[T_\varphi, x]$ is countable in $V[g]$. Because the tree $T_\varphi$ (as it is obtained from Theorem 1.4) is definable without parameters in the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$, we have $\mathbb{R} \cap L[T_\varphi, x] \subset C_x$ where

$$C_x = (\mathbb{R} \cap \mathrm{OD}_x)^{L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)}.$$

Therefore it suffices to show that $C_x$ is countable in $V[g]$. (Note that $C_x$ is in $V[g]$ by the homogeneity of the Levy collapse forcing.)

In the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$, the set of reals $C_x$ can be shown to be $\Sigma^2_1(x)$ using Woodin's $\Sigma_1$-reflection theorem. (See Steel and Trang [15] for a proof of the reflection theorem. Alternatively, we could use a more local version of ordinal-definability as in Steel [17, Remark 8.4].) Therefore by Woodin's generic absoluteness (Theorem 1.3) the set of reals $C_x$ is $(\Sigma^2_1(x))^{\mathrm{uB}_\lambda}$ uniformly in all generic extensions of $V[g]$ by posets of cardinality less than $\lambda$.

The statement "$C_x$ is countable" means there is a single real coding all reals in $C_x$, so it is an $\exists^{\mathbb{R}}(\Pi^2_1(x))^{\mathrm{uB}_\lambda}$ statement uniformly in all generic extensions of $V[g]$ by posets of cardinality less than $\lambda$. This $\exists^{\mathbb{R}}(\Pi^2_1(x))^{\mathrm{uB}_\lambda}$ statement becomes true after forcing over $V[g]$ to collapse $C_x$ (or simply $\mathbb{R}^{V[g]}$ itself) to be countable, so by our generic absoluteness hypothesis, $C_x$ is already countable in $V[g]$.

Therefore the set $\mathbb{R} \cap L[T_\varphi, x]$, which is contained in $C_x$, is also countable, and by Lemma 3.2 the tree $T_\varphi$ has a $\lambda$-absolute complement in some small forcing extension

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of $V$. As noted in Steel [17, Section 9], the existence of such absolute complements implies that the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ satisfies "every $\utilde{\Pi}^2_1$ set of reals is Suslin." ⊣

We can weaken the hypothesis of Theorem 0.1 by using Lemma 3.3 instead of Lemma 3.2. A similar argument is used by Caicedo and Schindler [2] to construct a projective well ordering of the reals from an anti-large-cardinal hypothesis. Recall that two-step $\exists^{\mathbb{R}}(\utilde{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below $\lambda$ is equivalent to the statement that, in every generic extension by a poset of cardinality less than $\lambda$, every $(\utilde{\Delta}^2_1)^{\mathrm{uB}_\lambda}$ set of reals is $\lambda$-universally Baire. Note that in particular this statement rules out the existence of a $(\utilde{\Delta}^2_1)^{\mathrm{uB}_\lambda}$ well ordering of the reals in a small forcing extension.

Proposition 4.8. *If $\lambda$ is a measurable limit of Woodin cardinals and every generic extension by a poset of cardinality less than $\lambda$ satisfies "there is no $(\utilde{\Delta}^2_1)^{\mathrm{uB}_\lambda}$ well ordering of the reals," then the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ satisfies* AD + *"every $\utilde{\Pi}^2_1$ set of reals is Suslin."*

Proof. Fix a formula $\varphi(v)$ and let $T_\varphi$ be the tree for the corresponding $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ formula as given by Corollary 1.5. We will use Lemma 3.3 to obtain an absolute complement for $T_\varphi$. Given any generic extension $V[g]$ of $V$ by a poset of cardinality less than $\lambda$ and any real $x \in V[g]$, we want to show that there is a real $y \in V[g]$ such that $y \notin L[T_\varphi, x]$.

Because the tree $T_\varphi$ (as it is obtained from Theorem 1.4) is definable without parameters in the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$, we have $\mathbb{R} \cap L[T_\varphi, x] \subset C_x$ where

$$C_x = (\mathbb{R} \cap \mathrm{OD}_x)^{L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)}.$$

Therefore it suffices to show that there is a real $y \in V[g]$ such that $y \notin C_x$. (Note that $C_x$ is in $V[g]$ by the homogeneity of the Levy collapse forcing.)

In the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$, the set of reals $C_x$ is $\Sigma^2_1(x)$ and moreover it has a $\Sigma^2_1(x)$ well ordering (both facts can be shown using Woodin's $\Sigma_1$-reflection theorem). So by absoluteness, in $V[g]$ the set $C_x$ is $(\Sigma^2_1(x))^{\mathrm{uB}_\lambda}$ and has a $(\Sigma^2_1(x))^{\mathrm{uB}_\lambda}$ well ordering. By our assumption, $V[g]$ has no $(\Delta^2_1(x))^{\mathrm{uB}_\lambda}$ well ordering of its reals, so there must be some real $y \in V[g]$ that is not in the domain of this partial well ordering. In other words $y \notin C_x$, so $y \notin L[T_\varphi, x]$ as desired.

Therefore by Lemma 3.3 the tree $T_\varphi$ has a $\lambda$-absolute complement in some small forcing extension of $V$. As noted in Steel [17, Section 9], the existence of such absolute complements implies that the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ satisfies "every $\utilde{\Pi}^2_1$ set of reals is Suslin." ⊣

§5. **The theory of $L(\mathrm{uB}_\lambda, \mathbb{R})$.** In this section, we consider the principle of generic absoluteness for the theory of $L(\mathrm{uB}_\lambda, \mathbb{R})$. The following theorem of Woodin (see Larson [8, Theorems 3.4.18,19] for a proof of a slightly more general version) gives an upper bound in terms of large cardinals for the consistency strength of this generic absoluteness principle. It also says something about what the generically absolute theory is in this situation.

Theorem 5.1 (Woodin). *If $\lambda$ is a limit of Woodin cardinals and $\delta < \lambda$ is $<\lambda$-supercompact, then letting $g \subset \mathrm{Col}(\omega, 2^\delta)$ be a $V$-generic filter we have*

$$L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]} \equiv L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g][h]}$$

*for every generic extension $V[g][h]$ of $V[g]$ by a poset of cardinality less than $\lambda$, and the model $L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]}$ satisfies* AD + DC + *"every set of reals is Suslin."*

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In order to establish Theorem 0.2, which gives a lower bound for the consistency strength of generic absoluteness of the theory of $L(\mathrm{uB}_\lambda, \mathbb{R})$ in the case that $\lambda$ is measurable as well as being a limit of Woodin cardinals, we first apply a relativized version of Proposition 4.8 to all $\lambda$-universally Baire sets appearing in all small forcing extensions of $V$, obtaining the following result.

PROPOSITION 5.2. *If $\lambda$ is a measurable limit of Woodin cardinals and for every generic extension $V[g]$ by a poset of cardinality less than $\lambda$ we have*

$$L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]} \models \text{“there is no well ordering of the reals,”}$$

*then the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ satisfies* $\mathsf{AD} + \mathsf{DC} +$ *“every set of reals is Suslin.”*

PROOF. To show that $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ satisfies “every set of reals is Suslin,” it suffices by the argument of Steel [17, Section 9] to show that every $\underset{\sim}{\Pi}^2_1(A^*)$ set of reals is Suslin for every set of reals $A^* \in \mathrm{Hom}^*_\lambda$.

Let $A^* \in \mathrm{Hom}^*_\lambda$. By increasing the complexity of $A^*$, we may assume that there is a homogeneity system for $A^*$ that is definable from $A^*$ in the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$. (Because every set in $\mathrm{Hom}^*_\lambda$ has a homogeneity system coded by a set of reals in $\mathrm{Hom}^*_\lambda$ and the pointclass $\mathrm{Hom}^*_\lambda$ is closed under countable joins, we can obtain a set with the desired property in $\omega$ many steps.) This property allows Woodin's results on the scale and basis properties for the pointclass $\Sigma^2_1$ to be relativized to the pointclass $\Sigma^2_1(A^*)$; see Steel [17, Lemma 8.2].[8]

By passing to a small forcing extension, we may assume that $A^*$ is the canonical extension of some set of reals $A \in \mathrm{uB}_\lambda$. For every generic extension $V[g]$ by a poset of cardinality less than $\lambda$, every $(\underset{\sim}{\Delta}^2_1(A^{V[g]}))^{\mathrm{uB}_\lambda}$ set of reals is an element of the model $L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]}$, so by our hypothesis there cannot be a $(\underset{\sim}{\Delta}^2_1(A^{V[g]}))^{\mathrm{uB}_\lambda}$ well ordering of the reals. Then by a straightforward relativization of Proposition 4.8, every $\underset{\sim}{\Pi}^2_1(A^*)$ set of reals is Suslin in $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ as desired.

Now we get DC by a standard argument using the inaccessibility of $\lambda$. Work in $V[G]$ where $G \subset \mathrm{Col}(\omega, {<}\lambda)$ is a generic filter realizing $\mathbb{R}^*_\lambda$ and $\mathrm{Hom}^*_\lambda$. Because $\lambda$ is inaccessible, every countable sequence of sets in $\mathrm{Hom}^*_\lambda$ is coded by a set in $\mathrm{Hom}^*_\lambda$. Therefore the fragment of DC for binary relations on $\mathrm{Hom}^*_\lambda$ is downward absolute from $V[G]$ to $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$. In the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$, every set is ordinal-definable from a set in $\mathrm{Hom}^*_\lambda$, so this fragment of DC implies full DC. ⊣

The other ingredient that we will need to prove Theorem 0.2 is the following lemma, which is obtained by a straightforward reflection argument using the measurability of $\lambda$. (The Woodin cardinals in the hypothesis of the lemma are only needed because our definition of $\mathrm{Hom}^*_G$ required them.)

LEMMA 5.3. *Let $\lambda$ be a measurable limit of Woodin cardinals and let $\varphi$ be a formula in the language of set theory. If $L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]} \models \varphi$ for every generic extension $V[g]$ of $V$ by a poset of cardinality less than $\lambda$, then $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda) \models \varphi$.*

[8] The author thanks Nam Trang for pointing out to him that some condition is necessary for this relativization because if $A^*$ is a complete $\Pi^2_1$ set then $\Sigma^2_1(A^*)$ cannot have the scale property. The author also thanks Martin Zeman for a helpful conversation about conditions on $A^*$ sufficient for relativization.

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Proof. Assume that $L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]} \models \varphi$ for every generic extension $V[g]$ of $V$ by a poset of cardinality less than $\lambda$. Let $j : V \to M$ be an elementary embedding witnessing the measurability of $\lambda$ and let $G \subset \mathrm{Col}(\omega, {<}\lambda)$ be a filter that is $V$-generic, hence also $M$-generic. By our assumption and the elementarity of $j$, we have $L(\mathrm{uB}_{j(\lambda)}, \mathbb{R})^{M[G]} \models \varphi$. We want to show that $L(\mathrm{Hom}^*_G, \mathbb{R}^*_G) \models \varphi$. In fact, we will show that

$$L(\mathrm{Hom}^*_G, \mathbb{R}^*_G) = L(\mathrm{uB}_{j(\lambda)}, \mathbb{R})^{M[G]}.$$

We have $\mathbb{R}^*_G = \mathbb{R}^{V[G]} = \mathbb{R}^{M[G]}$, so it remains to show that

$$\mathrm{Hom}^*_G = \mathrm{uB}^{M[G]}_{j(\lambda)}.$$

Let $A \in \mathrm{Hom}^*_G$. By passing from $V$ to a small forcing extension and extending $j$ to this small forcing extension, we may assume that $A = \mathrm{p}[T]$ for some $\lambda$-absolutely complementing pair of trees $(T, \tilde{T}) \in V$. Then $(j(T), j(\tilde{T}))$ is a $j(\lambda)$-absolutely complementing pair of trees in $M$ by elementarity of $j$, and mapping branches pointwise by $j$ gives $\mathrm{p}[T] \subset \mathrm{p}[j(T)]$ and $\mathrm{p}[\tilde{T}] \subset \mathrm{p}[j(\tilde{T})]$ in any model containing these trees.

With respect to the common set of reals $\mathbb{R}^*_G = \mathbb{R}^{V[G]} = \mathbb{R}^{M[G]}$, both pairs of trees $(T, \tilde{T})$ and $(j(T), j(\tilde{T}))$ project to complements and so the inclusions imply equality: $\mathrm{p}[T] = \mathrm{p}[j(T)]$ and $\mathrm{p}[\tilde{T}] = \mathrm{p}[j(\tilde{T})]$. Therefore $A = \mathrm{p}[j(T)]$ and the $j(\lambda)$-absolutely complementing pair of trees $(j(T), j(\tilde{T}))$ witnesses that $A$ is $j(\lambda)$-universally Baire in $M[G]$, as desired.

Conversely, let $A$ be a $j(\lambda)$-universally Baire set of reals in $M[G]$ and take a $j(\lambda)$-absolutely complementing pair of trees $(T, \tilde{T}) \in M[G]$ with $\mathrm{p}[T] = A$. Extend $j$ to an elementary embedding $\hat{\jmath} : V[G] \to M[H]$ where $H \subset \mathrm{Col}(\omega, {<}j(\lambda))$ is a $V$-generic filter. Mapping branches pointwise by $j$, we have $\mathrm{p}[T] \subset \mathrm{p}[\hat{\jmath}(T)]$ and $\mathrm{p}[\tilde{T}] \subset \mathrm{p}[\hat{\jmath}(\tilde{T})]$ in any model containing these trees.

With respect to the set of reals $\hat{\jmath}(\mathbb{R}^*_G) = \mathbb{R}^*_H$, both pairs of trees $(T, \tilde{T})$ and $(\hat{\jmath}(T), \hat{\jmath}(\tilde{T}))$ project to complements and so the inclusions imply equality: $\mathrm{p}[T] = \mathrm{p}[\hat{\jmath}(T)]$ and $\mathrm{p}[\tilde{T}] = \mathrm{p}[\hat{\jmath}(\tilde{T})]$. Therefore $\mathrm{p}[T] = \hat{\jmath}(A)$ and the $j(\lambda)$-absolutely complementing pair of trees $(T, \tilde{T}) \in M[G]$ witnesses that $\hat{\jmath}(A) \in \mathrm{Hom}^*_H$. By the elementarity of $\hat{\jmath}$, it follows that $A \in \mathrm{Hom}^*_G$ as desired. ⊣

Now it is a simple matter to prove the theorem.

Proof of Theorem 0.2. Assume that $\lambda$ is a measurable limit of Woodin cardinals and $L(\mathrm{uB}_\lambda, \mathbb{R}) \equiv L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]}$ for every small forcing extension $V[g]$. We want to show that the models $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ and $L(\mathrm{uB}_\lambda, \mathbb{R})$ both satisfy $\mathsf{AD} + \mathsf{DC}$ + "every set of reals is Suslin."

The model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ has no well ordering of its reals by $\mathsf{AD}$, or more simply by the argument of Solovay [13]. By Lemma 5.3, it follows that there is some small forcing extension $V[g]$ such that $L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]}$ has no well ordering of its reals, and by our generic absoluteness hypothesis, for every small forcing extension $V[g]$ the model $L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]}$ has no well ordering of its reals.

By Proposition 5.2, it follows that $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ satisfies $\mathsf{AD} + \mathsf{DC}$ + "every set of reals is Suslin." By Lemma 5.3 again, there is some small forcing extension $V[g]$ such that $L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]}$ satisfies $\mathsf{AD} + \mathsf{DC}$ + "every set of reals is Suslin," and by our generic absoluteness hypothesis again, $L(\mathrm{uB}_\lambda, \mathbb{R})$ itself satisfies $\mathsf{AD} + \mathsf{DC}$ + "every set of reals is Suslin." ⊣

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One might hope to improve the consistency strength lower bound given by Theorem 0.2. The next natural target would be the theory $\mathsf{AD}_{\mathbb{R}}$ + "$\Theta$ is regular." (By convention, we write $\mathsf{AD}_{\mathbb{R}}$ here instead of the equivalent theory $\mathsf{AD}$ + "every set of reals is Suslin.")

QUESTION 5.4. *Assume that $\lambda$ is a measurable limit of Woodin cardinals and $L(\mathrm{uB}_\lambda, \mathbb{R}) \equiv L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]}$ for every generic extension $V[g]$ by a poset of cardinality less than $\lambda$.*

1. *Does $L(\mathrm{uB}_\lambda, \mathbb{R})$ satisfy $\mathsf{AD}_{\mathbb{R}}$ + "$\Theta$ is regular"?*
2. *Does $L(\Gamma, \mathbb{R})$ satisfy $\mathsf{AD}_{\mathbb{R}}$ + "$\Theta$ is regular" for some pointclass $\Gamma \subset \mathrm{uB}_\lambda$?*

In unpublished work, Woodin strengthened the conclusion of Theorem 5.1 to get $\mathsf{AD}_{\mathbb{R}}$ + "$\Theta$ is regular" in the model $L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]}$, making a positive answer to 5.4(1) plausible. However, 5.4(2) might be a more reasonable target for current inner-model-theoretic techniques such as those developed by Sargsyan [12]. One might also hope to do without the hypothesis that $\lambda$ is measurable.

§**6. Note.** Ralf Schindler pointed out an error in the proof of Proposition 2.4. We proved the consistency of a schema, but the principle of $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness was defined as a single statement quantifying over formulas. We give a correct proof using forcing instead of compactness. Let $G \subset \mathrm{Col}(\omega, {<}\lambda)$ be a $V$-generic filter. In $L(\mathrm{Hom}^*_G, \mathbb{R}^*_G)$, use countable choice (which follows from $\mathsf{AD}$) to choose reals witnessing all true $\exists^{\mathbb{R}}\Pi^2_1$ statements. The real coding the sequence of witnesses appears in $V[G \restriction \alpha]$ for some $\alpha < \lambda$, and $V[G \restriction \alpha]$ satisfies $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below $\lambda$.

§**7. Acknowledgments.** The author would like to thank John Steel, Nam Trang, Martin Zeman, and the referees for their many helpful comments on the article. The author gratefully acknowledges support from NSF grant DMS-1044150.

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https://doi.org/10.1017/jsl.2017.35 Published online by Cambridge University Press