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ON THE CONTINUATION OF SOLUTIONS OF NON-AUTONOMOUS SEMILINEAR PARABOLIC PROBLEMS

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Abstract We study non-autonomous parabolic equations with critical exponents in a scale of Banach spaces E_σ, $\sigma \in [0, 1 + \mu)$. We consider a suitable $E_{1+\varepsilon}$-solution and describe continuation properties of the solution. This concerns both a situation when the solution can be continued as an $E_{1+\varepsilon}$-solution and a situation when the $E_{1+\varepsilon}$-norm of the solution blows up, in which case a piecewise $E_{1+\varepsilon}$-solution is constructed.

Keywords: abstract parabolic equations; continuation of solutions; critical exponents

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1. Introduction

In this paper, given a family of unbounded linear operators in the Banach space E_0, $A(t)\colon D_{E_0} \subset E_0 \to E_0$, $t \in \mathbb{R}$, we focus on the well-posedness of a Cauchy problem of the form

$$\dot{u}(t) + A(t)u(t) = F(t, u(t)), \quad t > \tau,\ u(\tau) = u_\tau, \tag{1.1}$$

where the linear main part operator depends on the time variable.

Following the pioneering work of Sobolevskiĭ [**26**], such problems have been considered by many authors in wide generality and many results have been obtained (see, for example, the monographs [**7**, **20**, **22**, **23**, **27**, **30**] and references therein). Here, our main concern will be *critically growing nonlinearities*, that is, roughly speaking we will allow $F(t, u(t))$ to exhibit the same order of magnitude as the linear main part operator $A(t)$ (see [**9**, **13**, **15**, **29**]).

For subcritical nonlinearities, continuation of solutions is satisfactorily described for both autonomous and non-autonomous problems (see, for example, [**4**]). As observed in [**29**], this is no longer the case for nonlinearities satisfying a critical growth condition (see also [**13**]). On the other hand, some previous results concerning continuation properties of solutions of autonomous problems [**13**] cannot be directly applied to (1.1) and require essential modifications. This is our main goal in the present paper.

To describe our results, we start from the following two general assumptions. Conditions sufficient for them in terms of the operators $A(t)$ will be discussed in detail in § 3.

Assumption 1.1. *Given a family of Banach spaces* $\{E_\alpha,\ \alpha \in [0, 1+\mu)\}$*, there exists a continuous process* $\{U(t,\tau)\colon t \geqslant \tau \in \mathbb{R}\} \subset L(E_0)$ *in* E_0 *such that, given* $\tau \in \mathbb{R}$ *and* $u_\tau \in E_0$*, the map* $[\tau,\infty) \ni t \to u(t) = U(t,\tau)u_\tau \in E_0$ *is a classical solution of the linear problem*

$$\dot{u}(t) + A(t)u(t) = 0, \quad t > \tau,\ u(\tau) = u_\tau.$$

Furthermore, given any point $t_0 \in \mathbb{R}$*, there is a time interval* $I \subset \mathbb{R}$ *centred at* t_0 *such that for any* $1+\mu > \sigma \geqslant \zeta \geqslant 0$ *a constant* $M > 0$ *exists for which*

$$\|U(t,\tau)\|_{L(E_\zeta,E_\sigma)} \leqslant M(t-\tau)^{\zeta-\sigma}, \quad t,\tau \in I,\ t > \tau. \tag{1.2}$$

Assumption 1.2. *Given* $t_0 \in \mathbb{R}$*, there is also a time interval* $I \subset \mathbb{R}$ *centred at* t_0 *such that whenever* $1+\mu > \zeta > \sigma \geqslant 0$, $1 \geqslant \zeta - \sigma > 0$*, a constant* $M > 0$ *exists for which*

$$\|U(t,\tau) - \mathrm{Id}\|_{L(E_\zeta,E_\sigma)} \leqslant M(t-\tau)^{\zeta-\sigma}, \quad t,\tau \in I,\ t > \tau. \tag{1.3}$$

Concerning the right-hand side in (1.1), we will assume that F belongs to a class of maps $\mathcal{L}(\varepsilon,\rho,\gamma(\varepsilon),\eta,C_\eta)$ satisfying a suitable Lipschitz condition relative to $\{E_\alpha,\ \alpha \in [0,1+\mu)\}$. Note that any such F falls, in particular, into the class of ε-regular maps considered in [**9**].

Definition 1.3. We say that a continuous function $F\colon \mathbb{R} \times E_{1+\varepsilon} \to E_{\gamma(\varepsilon)}$ is of the class $\mathcal{L}(\varepsilon,\rho,\gamma(\varepsilon),\eta,C_\eta)$ of Lipschitz maps relative to $\{E_\alpha,\ \alpha \in [0,1+\mu)\}$, with constants $\rho > 1$, $0 < \varepsilon < \min\{1/\rho,\mu\}$, $\gamma(\varepsilon) \in [\rho\varepsilon, 1)$, $\eta > 0$ and $C_\eta > 0$, if and only if for any bounded time interval $I \subset \mathbb{R}$ there exists $c > 0$ such that for each $v, w \in E_{1+\varepsilon}$ and $t \in I$ we have

$$\|F(t,v) - F(t,w)\|_{E_{\gamma(\varepsilon)}} \leqslant c\|v-w\|_{E_{1+\varepsilon}}(\eta\|v\|^{\rho-1}_{E_{1+\varepsilon}} + \eta\|w\|^{\rho-1}_{E_{1+\varepsilon}} + C_\eta) \tag{1.4}$$

and

$$\|F(t,v)\|_{E_{\gamma(\varepsilon)}} \leqslant c(\eta\|v\|^{\rho}_{E_{1+\varepsilon}} + C_\eta). \tag{1.5}$$

We single out for special attention the case when in (1.4) and (1.5) one has $\gamma(\varepsilon) = \rho\varepsilon$ and not $\gamma(\varepsilon) \in (\rho\varepsilon, 1)$ as it exhibits criticality of F relative to (E_1, E_0) (see [**9**]).

Definition 1.4. For the case in which, for a certain $\eta > 0$, (1.4) and (1.5) hold with $\gamma(\varepsilon) \in (\rho\varepsilon, 1)$, we say that F is *subcritical*. When, for a certain $\eta > 0$, (1.4) and (1.5) hold with $\gamma(\varepsilon) = \rho\varepsilon$ but not with $\gamma(\varepsilon) \in (\rho\varepsilon, 1)$, F is called *critical* and ρ is called a *critical exponent*. For the case in which F is critical and (1.4) and (1.5) hold with any $\eta > 0$, we say that F is an *almost critical* map.

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We will consider the following solution of (1.1) as in Definition 1.5 (see [**15**] and also [**9**, **13**]).

Definition 1.5. Given F of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha,\ \alpha \in [0, 1+\mu)\}$, $\tau > 0$ and $u_\tau \in E_0$, we say that $u\colon [\tau, T] \to E_0 \cup E_{1+\varepsilon}$ is a mild $E_{1+\varepsilon}$-solution (an $E_{1+\varepsilon}$-solution for short) of (1.1) on the interval $[\tau, T]$ if and only if $u \in L^\infty_{\mathrm{loc}}((\tau, T], E_{1+\varepsilon})$, there exists the limit $\lim_{t\to\tau^+}(t-\tau)^\varepsilon \|u(t)\|_{E_{1+\varepsilon}} = 0$, $u(\tau) = u_\tau$ and for $t \in (\tau, T]$ we have

$$u(t) = U(t, \tau)u_\tau + \int_\tau^t U(t, s)F(s, u(s))\,\mathrm{d}s. \tag{1.6}$$

If, given $a \in (\tau, \infty]$, u is an $E_{1+\varepsilon}$-solution of (1.1) on $[\tau, T]$ for any $T \in (\tau, a)$, then we say that u is an $E_{1+\varepsilon}$-solution on the interval $[\tau, a)$.

With these assumptions, the $E_{1+\varepsilon}$-solution will be unique and Hölder continuous away from τ.

Theorem 1.6. *Suppose that Assumptions 1.1 and 1.2 hold, F is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha,\ \alpha \in [0, 1+\mu)\}$, $\tau \in \mathbb{R}$ and $u_\tau \in E_0$.*

There then exists at most one $E_{1+\varepsilon}$-solution $u = u(\cdot, \tau, u_\tau)$ of (1.1) on $[\tau, T]$ and $u \in C^\nu_{\mathrm{loc}}((\tau, T], E_{1+\theta})$ for any $0 < \theta < \min\{\gamma(\varepsilon), \mu\}$, $0 < \nu < \nu^* = \min\{\gamma(\varepsilon), \mu\} - \theta$.

To describe a set of initial data for which (1.1) has a unique $E_{1+\varepsilon}$-solution, we will consider a linear subspace $\mathfrak{E}^\tau_\varepsilon$ of E_0,

$$\mathfrak{E}^\tau_\varepsilon = \Big\{\varphi \in E_0 \colon \text{there exists} \lim_{t\to\tau^+}(t-\tau)^\varepsilon \|U(t, \tau)\varphi\|_{E_{1+\varepsilon}} = 0\Big\}.$$

We also define, for some $\delta > 0$,

$$\|\varphi\|^{\mathfrak{E}^\tau_\varepsilon}_\delta = \sup_{t\in(\tau,\tau+\delta]} (t-\tau)^\varepsilon \|U(t, \tau)\varphi\|_{E_{1+\varepsilon}}, \quad \varphi \in \mathfrak{E}^\tau_\varepsilon,$$

and

$$B^\delta_{\mathfrak{E}^\tau_\varepsilon}(w_0, r) = \{\varphi \in \mathfrak{E}^\tau_\varepsilon \colon \|\varphi - w_0\|^{\mathfrak{E}^\tau_\varepsilon}_\delta < r\}, \quad w_0 \in \mathfrak{E}^\tau_\varepsilon.$$

With the above set-up, we first state the local well-posedness result, which complements the earlier considerations of [**9**, Theorem 1], [**13**, Theorem 2.1] and [**15**, Theorem 3.1]. In what follows, $B(a, b) = \int_0^1 s^{a-1}(1-s)^{b-1}\,\mathrm{d}s$, $a, b > 0$, denotes Euler's beta function and

$$B_{\varepsilon,\rho} := \max\{B(1-\rho\varepsilon, \gamma(\varepsilon)-\varepsilon), B(\gamma(\varepsilon)-\varepsilon, 1-\varepsilon)\} = B(1-\rho\varepsilon, \gamma(\varepsilon)-\varepsilon). \tag{1.7}$$

Theorem 1.7. *Suppose that Assumptions 1.1 and 1.2 are satisfied and that F is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha,\ \alpha \in [0, 1+\mu)\}$.*

The following then hold.

(i) *Given $t_0 \in \mathbb{R}$, $w_0 \in \mathfrak{E}^\tau_\varepsilon$ and given τ in a certain interval $\mathcal{J} \subset \mathbb{R}$ centred at t_0, there exist $\bar\delta_0 \in (0, 1]$ and $\bar r_0 = 1/4(8c\eta M B_{\varepsilon,\rho})^{1/(\rho-1)}$, where $M = M(1+\varepsilon, \gamma(\varepsilon), \mathcal{J})$ and $B_{\varepsilon,\rho}$ are as in (1.2) and (1.7), such that for any initial condition u_τ satisfying*

$$u_\tau \in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(w_0, r) \tag{1.8}$$

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with

$$\delta_0 \in (0, \bar{\delta}_0] \quad \text{and} \quad r \in (0, \bar{r}_0], \tag{1.9}$$

there exists a unique $E_{1+\varepsilon}$*-solution* $u = u(\cdot, \tau, u_\tau)$ *of (1.1) on* $[\tau, \tau + \delta_0]$.

(ii) *When* F *is subcritical or* F *is almost critical, the time of existence* δ_0 *can be chosen uniformly with respect to the initial condition* $u_\tau \in B_{\mathfrak{E}^\tau_\varepsilon}(w_0, r)$ *for arbitrarily large* r.

(iii) *For any* $0 \leqslant \theta < \min\{\gamma(\varepsilon), \mu\}$ *we have*

$$\lim_{t \to \tau^+} (t - \tau)^\theta \|u(t, \tau, u_\tau)\|_{E_{1+\theta}} = 0, \quad u_\tau \in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(w_0, r) \cap \mathfrak{E}^\tau_\theta, \tag{1.10}$$

and

$$\sup_{t \in [\tau, \tau + \delta_0]} (t - \tau)^\theta \|u(t, \tau, u^1_\tau) - u(t, \tau, u^2_\tau)\|_{E_{1+\theta}} \leqslant C(\theta)(\|u^1_\tau - u^2_\tau\|^{\mathfrak{E}^\tau_\theta}_{\delta_0} + \|u^1_\tau - u^2_\tau\|^{\mathfrak{E}^\tau_\epsilon}_{\delta_0}), \quad u^1_\tau, u^2_\tau \in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(w_0, r) \cap \mathfrak{E}^\tau_\theta. \tag{1.11}$$

(iv) *Whenever* $0 \leqslant \theta < \min\{\gamma(\varepsilon), \mu\}$ *and* $u_\tau \in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(w_0, r)$, *we have*

$$\lim_{t \to \tau^+} (t - \tau)^\theta \|U(t, \tau) u_\tau - u_\tau\|_{E_{1+\theta}} = 0 \implies \lim_{t \to \tau^+} (t - \tau)^\theta \|u(t, \tau, u_\tau) - u_\tau\|_{E_{1+\theta}} = 0. \tag{1.12}$$

Although so far we have not used embedding properties, it is typical for applications that

$$E_\beta \text{ is densely embedded in } E_\alpha \text{ whenever } 0 \leqslant \alpha \leqslant \beta < 1 + \mu. \tag{1.13}$$

Remark 1.8.

(i) Under Assumption 1.1 and (1.13), $E_1 \subset \mathfrak{E}^\tau_\varepsilon$, $\|\cdot\|^{\mathfrak{E}^\tau_\varepsilon}_\delta$ is the norm in $\mathfrak{E}^\tau_\varepsilon$ and $B^\delta_{\mathfrak{E}^\tau_\varepsilon}(w_0, r)$ contains a ball in E_1 centred at w_0 and of radius r/M.

(ii) If (1.13) holds, then Theorem 1.7 can be applied with $w_0 \in E_1$ and the time of existence δ_0 can then be chosen uniformly with respect to $\tau \in \mathcal{J}$.

Remark 1.9. With (1.13) and the assumptions of Theorem 1.6, if $u = u(\cdot, \tau, u_\tau)$ is an $E_{1+\varepsilon}$-solution of (1.1) as in this theorem, then the following hold.

(i) $u \in C([\tau, \tau + \delta_0], E_\alpha) \cap C^\nu_{\text{loc}}((\tau, \tau + \delta_0], E_{1+\theta})$ whenever $u_\tau \in E_\alpha$, $\alpha \in [0, 1]$, $0 < \theta < \min\{\gamma(\varepsilon), \mu\}$ and $0 < \nu < \min\{\gamma(\varepsilon), \mu\} - \theta$.

(ii) $u \in C([\tau, \tau + \delta_0], E_{1+\varepsilon}) \cap C^\nu_{\text{loc}}((\tau, \tau + \delta_0], E_{1+\theta})$ whenever $u_\tau \in E_{1+\varepsilon}$.

(iii) $u(t, \tau, u_\tau)$ is continuous in E_1 with respect to $(t, u_\tau) \in [\tau, \tau + \delta_0] \times E_1$.

Given $\tau \in \mathbb{R}$ and $u_\tau \in \mathfrak{E}^\tau_\varepsilon$, we next define

$$I(u_\tau) := \{T \in (\tau, \infty) \colon \text{there exists a unique } E_{1+\varepsilon}\text{-solution of (1.1) on } [\tau, T]\}.$$

Under the assumptions of Theorem 1.7, $I(u_\tau)$ is non-empty, in which case we define

$$T_{u_\tau} := \sup I(u_\tau) \tag{1.14}$$

and call $[\tau, T_{u_\tau})$ the *maximal interval of existence* of the $E_{1+\varepsilon}$-solution.

Since in applications E_1 often plays the role of a space in which (1.1) is expected to define a continuous process, we now state the theorem that involves characterization of the maximal time of existence of the $E_{1+\varepsilon}$-solution in terms of the E_1-norm even in a certain critical case. This is significant for applications as any 'better' estimate may often be impossible to find.

Theorem 1.10. *Suppose that Assumptions 1.1 and 1.2 hold, F is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha,\ \alpha \in [0, 1+\mu)\}$, $\tau \in \mathbb{R}$, $u_\tau \in \mathfrak{E}^\tau_\varepsilon$ and $u = u(\cdot, \tau, u_\tau)$ is the $E_{1+\varepsilon}$-solution of (1.1) on a maximal interval of existence $[\tau, T_{u_\tau})$. Assume also that (1.13) holds.*

(i) *If F is subcritical or F is almost critical, then*

$$T_{u_\tau} < \infty \quad \textit{implies that} \quad \limsup_{t \to T^-_{u_\tau}} \|u(t, \tau, u_\tau)\|_{E_1} = \infty.$$

(ii) *In either case, when F is subcritical, almost critical or critical, $T_{u_\tau} < \infty$ implies that there does not exist even one sequence $t_n \to T^-_{u_\tau}$ for which $\{u(t_n, \tau, u_\tau)\}$ converges in E_1; in particular, the map $[\tau, T_{u_\tau}) \ni t \to u(t) \in E_1$ cannot be uniformly continuous.*

Note that in Theorem 1.10 for F subcritical, almost critical or critical, we have that

$$T_{u_\tau} < \infty \quad \text{implies that} \quad \limsup_{t \to T^-_{u_\tau}} \|u(t)\|_{E_{1+\varepsilon}} = \infty. \tag{1.15}$$

However, the $E_{1+\varepsilon}$-estimate may not be easy to find in applications.

It is next reasonable to generalize the notion of an $E_{1+\varepsilon}$-solution and investigate the possibility of a continuing $E_{1+\varepsilon}$-solution even though its $E_{1+\varepsilon}$-norm may blow up.

Definition 1.11. Suppose that F is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha,\ \alpha \in [0, 1+\mu)\}$, $\tau > 0$, $v_0 \in \mathfrak{E}^\tau_\varepsilon$ and $I_\tau \subset \mathbb{R}$ is an interval of the form $[\tau, a)$.

We say that $\mathcal{U} \colon I_\tau \to E_0 \cup E_{1+\varepsilon}$ is a *piecewise $E_{1+\varepsilon}$-solution* of (1.1) on I_τ if and only if $\mathcal{U}(\tau) = u_\tau$ and, for each $T \in I_\tau \setminus \{\tau\}$, there exist a number $N_T \in \mathbb{N}$ and a partition

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$\tau = \tau_0 < \tau_1 < \cdots < \tau_{N_T} < T = \tau_{N_T+1}$ of $[\tau, T]$ such that

$$\|\mathcal{U}(t) - \mathcal{U}(\tau_{i-1})\|_{E_0} \xrightarrow{t\to\tau_{i-1}^-} 0, \quad i = 2, \ldots, N_T + 1, \tag{1.16}$$

$$\limsup_{t\to\tau_{i-1}^-} \|\mathcal{U}(t)\|_{E_{1+\varepsilon}} = \infty, \quad i = 2, \ldots, N_T + 1, \tag{1.17}$$

$$\mathcal{U} \in L^\infty_{\text{loc}}((\tau_{i-1}, \tau_i), E_{1+\varepsilon}), \quad i = 1, \ldots, N_T + 1, \tag{1.18}$$

$$(t - \tau_{i-1})^\varepsilon \|\mathcal{U}(t)\|_{E_{1+\varepsilon}} \xrightarrow{t\to\tau_{i-1}^+} 0, \quad i = 1, \ldots, N_T + 1, \tag{1.19}$$

$$\mathcal{U}(\tau_{i-1}) = u_{\tau_{i-1}}, \quad i = 1, \ldots, N_T + 1, \tag{1.20}$$

$$\mathcal{U}(t) = U(t, \tau_{i-1})u_{\tau_{i-1}} + \int_{\tau_{i-1}}^t U(t, s)F(s, \mathcal{U}(s))\, \mathrm{d}s, \quad t \in (\tau_{i-1}, \tau_i),\ i = 1, \ldots, N_T + 1. \tag{1.21}$$

If the interval $I_\tau = [\tau, a)$ is finite, $\mathcal{U}\colon [\tau, a) \to E_0 \cup E_{1+\varepsilon}$ is a piecewise $E_{1+\varepsilon}$-solution of (1.1) on $I_\tau = [\tau, a)$ and a is the limit of a strictly increasing sequence $\{\tau_i, i \in \mathbb{N}\}$ of times such that $\limsup_{t\to\tau_i^-} \|\mathcal{U}(t)\|_{E_{1+\varepsilon}} = \infty$, then a is called an accumulation time of singular times.

In the theorem below, the $E_{1+\varepsilon}$-solution will be continued as a piecewise $E_{1+\varepsilon}$-solution.

Theorem 1.12. *Suppose that Assumptions 1.1 and 1.2 are satisfied and F is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha,\ \alpha \in [0, 1 + \mu)\}$. Suppose additionally that (1.13) holds, E_1 is reflexive and, given any $\tau \in \mathbb{R}$, $u_\tau \in \mathfrak{E}_\varepsilon^\tau$,*

$$\sup_{t\in[\tau,T)} \|u(t)\|_{E_1} < \infty \tag{1.22}$$

whenever $T \in (\tau, \infty)$ and an $E_{1+\varepsilon}$-solution $u = u(\cdot, \tau, u_\tau)$ of (4.13) exists for all $t \in [\tau, T)$. Finally, suppose that

when $\tau \in \mathbb{R}$, $u_\tau \in E_1$ and $T_{u_\tau} < \infty$, the map $[\tau, T_{u_\tau}) \ni t \to u(t) \in E_0$, where $u = u(\cdot, \tau, u_\tau)$ is a $E_{1+\varepsilon}$-solution of (1.1), is uniformly continuous. (1.23)

Under these assumptions, given $\tau \in \mathbb{R}$, $u_\tau \in E_1$ and having a unique $E_{1+\varepsilon}$-solution $u = u(\cdot, \tau, T_{u_\tau})$ of (1.1) for which $T_{u_\tau} < \infty$, there exist $a \in (T_{u_\tau}, \infty]$ and the unique extension $\mathcal{U}\colon [\tau, a) \to E_1$ of u such that $\mathcal{U}$ is a piecewise $E_{1+\varepsilon}$-solution of (1.1) on $[\tau, a)$ and either $a = \infty$ or a is an accumulation time of singular times.

To enhance the accessibility of the above-mentioned abstract results and assumptions, let us consider an illustrative problem of the form

$$\begin{cases} u_t + \theta(t)(-\Delta)^m u = f(t, x, u), & t > \tau,\ x \in \Omega, \\ \dfrac{\partial^j u}{\partial \nu^j}\Big|_{\partial\Omega} = 0, & j = 0, \ldots, m - 1,\ u(\tau, \cdot) = u_\tau \in L^2(\Omega) = E_1, \end{cases} \tag{1.24}$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded C^{2m}-domain, $\theta \in C^{\mu}_{\mathrm{loc}}(\mathbb{R},(0,\infty))$ for some $\mu \in (0,1]$ and the $\theta(t)(-\Delta)^m =: A(t)$ are self-adjoint positive definite operators in $L^2(\Omega)$.

Dealing with (1.24), on the one hand, one can relate E_α to the complex interpolation space $([L^2(\Omega), H^{2m}(\Omega) \cap H_0^m(\Omega)]_{1-\alpha})'$ when $\alpha \in (0,1)$ and to $[L^2(\Omega), H^{2m}(\Omega) \cap H_0^m(\Omega)]_{\alpha-1}$ when $\alpha \in (1,2)$. On the other hand, E_α can be viewed as the extrapolated fractional scale generated by $(L^2(\Omega), A(t))$.

Since $\sup_{\|\phi\|_{E_1}=1} \|(A(t)-A(s))\phi\|_{E_0}$ can be bounded from above by $|\theta(t)-\theta(s)|$ and thus, for $t, s \in I$, by a multiple of $|t-s|^\mu$, we have

$$A(\cdot) \in C^{\mu}_{\mathrm{loc}}(\mathbb{R}, L(E_1, E_0)) \quad \text{with } E_1 = L^2(\Omega),\ E_0 = (H^{2m}(\Omega) \cap H_0^m(\Omega))'.$$

There then exists a continuous process $\{U(t,\tau)\colon (t,\tau) \in \mathbb{R}^2,\ t \geqslant \tau \in \mathbb{R}\} \subset L(E_0)$ associated with the Dirichlet initial boundary-value problem for

$$u_t + \theta(t)(-\Delta)^m u = 0, \quad t > \tau,\ x \in \Omega,$$

which possesses smoothing properties as in (1.2) and (1.3).

The above-mentioned properties of the linear process enable a satisfactory treatment of (1.24) in $L^2(\Omega)$ including the case of critical exponent, which for $m = 1$ is

$$\rho_{\mathrm{c}} = \frac{N+4}{N}. \tag{1.25}$$

A crucial ingredient in this consideration is that the nonlinear right-hand side in (1.24) will then belong to the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ of Lipschitz maps relative to $\{E_\alpha,\ \alpha \in [0, 1+\mu)\}$.

Coming back to (1.24) and following Remark 1.8, we can now apply Theorem 1.7 with $w_0 \in E_1 = L^2(\Omega)$. Consequently, (1.24) is locally well posed in $L^2(\Omega)$ up to the critical case, which for $m = 1$ involves the exponent ρ_{c} as in (1.25).

It does happen that we often have an *a priori*-like $L^2(\Omega)$-bound on the solution of (1.24);

$$\|u(t)\|^2_{L^2(\mathbb{R}^N)} \leqslant g(\tau, \|u_\tau\|_{L^2(\Omega)}, T), \quad t \in [\tau, T). \tag{1.26}$$

The estimate in (1.26) can be derived by taking the $L^2(\Omega)$-product of both sides of the equation with u if f in (1.24) is a logistic-type map of the form $u - u|u|^{\rho_{\mathrm{c}}-1}$ or, more generally, when f satisfies a structure condition $uf(t,x,u) \leqslant C(t,x)u^2 + D(t,x)$, for some $C \in L^\infty_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and $D \in L^1_{\mathrm{loc}}(\mathbb{R}, L^1(\Omega))$, that does not involve any monotonicity condition.

If f is almost critical, that is, if $|f|$ is estimated by a power function $c(C_\eta + \eta|u|^{\rho_{\mathrm{c}}})$ with η suitably small, then, due to Theorem 1.10, (1.26) implies that the solution exists globally in time, which, however, may not be true in the critical case.

For continuation of the solution in the critical case, besides (1.26) some additional information becomes needed. This could be an *a priori*-like bound in $L^\infty(\Omega)$ or in $H^1(\Omega)$, which, however, can rarely be found for (1.24) without more specific assumptions or techniques.

Even though no such 'better' estimate on the solutions of (1.24) can be provided, we can obtain the information that

$$u \in W^{1,1}((\tau, T), (H^{2m}(\Omega) \cap H_0^m(\Omega))'). \tag{1.27}$$

Then viewing u as a uniformly continuous map in $E_0 = (H^{2m}(\Omega) \cap H_0^m(\Omega))'$, we can construct continuation of the solution as in Theorem 1.12.

Since in applications characterization of the scale in terms of suitable function spaces plays an important role, we finally mention that a situation does arise when the length of a scale interval for which such an effective characterization applies shrinks to 1. An example of the operator $\mathcal{A}$ can be found in [**18**] such that the associated scale can be characterized with the aid of Bessel potentials spaces $H_p^{4\sigma}(\mathbb{R}^N)$ as long as $\sigma \in [-1 - N/4p' + N/4r, 1 + N/4p - N/4r]$. Consequently, although the associated process $U(t, s) = \mathrm{e}^{-(t-s)\mathcal{A}}$, $\infty > t \geqslant s > -\infty$, satisfies, for some $\omega \in \mathbb{R}$, estimates of the form

$$\|U(t, s)\|_{\mathcal{L}(H_p^{4\sigma}(\mathbb{R}^N), H_p^{4\xi}(\mathbb{R}^N))} \leqslant M\mathrm{e}^{-\omega(t-s)}/(t - s)^{\xi-\sigma}, \quad t > s,$$

this can be ensured only for $\sigma \in [-1 - N/4p' + N/4r, 1 + N/4p - N/4r]$. Furthermore, the length of the latter interval is equal to $2 + N/4 - N/2r$ and it actually tends to one as $r \searrow N/4$ and $N = 4$.

The proofs of the above-mentioned abstract results will be given in § 2. In § 3 we discuss sufficient conditions for Assumptions 1.1 and 1.2 in terms of $A(t)$. In § 4 we show from a broader perspective how the abstract results work in applications.

2. Proofs of abstract results

2.1. Uniqueness and Hölder continuity of the $E_{1+\varepsilon}$-solution: proof of Theorem 1.6

Given $-\infty < \tau < T < \infty$ we define

$$\mathfrak{M}_\tau^T := \Big\{\psi \in L^\infty_{\mathrm{loc}}((\tau, T], E_{1+\varepsilon})\colon \lim_{t\to\tau^+} (t - \tau)^\varepsilon \|\psi(t)\|_{E_{1+\varepsilon}} = 0\Big\}. \tag{2.1}$$

Theorem 1.6 is a consequence of the following two lemmas.

Lemma 2.1. *Suppose that Assumptions 1.1 and 1.2 hold, F is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha,\ \alpha \in [0, 1+\mu)\}$, $u \in \mathfrak{M}_\tau^T$ and (1.6) is valid for $t \in (\tau, T]$ with some $u_\tau \in E_0$.*

Then $u \in C^\nu([\delta, T], E_{1+\theta})$ for any $\delta \in (\tau, T)$, $\nu \in (0, \nu_*)$ and $\nu_* = \min\{\gamma(\varepsilon), \mu\} - \theta > 0$.

Proof. Due to Assumptions 1.1 and 1.2, given a bounded time interval $[-T, T] \subset \mathbb{R}$ and any $0 \leqslant \zeta \leqslant \sigma < 1 + \mu$, one can choose a positive constant M for which we have

$$\|U(t, \tau)\|_{L(E_\zeta, E_\sigma)} \leqslant M(t - \tau)^{-(\sigma-\zeta)}, \quad T \geqslant t > \tau \geqslant -T, \tag{2.2}$$

and, if $1 \geqslant \sigma - \zeta \geqslant 0$,

$$\|U(t, \tau) - \mathrm{Id}\|_{L(E_\sigma, E_\zeta)} \leqslant M(t - \tau)^{\sigma-\zeta}, \quad T \geqslant t > \tau \geqslant -T. \tag{2.3}$$

On the other hand, since $u \in \mathfrak{M}_\tau^T$, for any $\delta > \tau$ close enough to τ we have that $\|u(t)\|_{E_{1+\varepsilon}} \leqslant (t-\tau)^{-\varepsilon}$ for $t \in (\tau, \delta)$, and letting $\tilde{c} = c(\eta + C_\eta)$ we deduce from (1.5) that

$$\|F(s, u(s))\|_{E_{\gamma(\varepsilon)}} \leqslant \tilde{c}((s-\tau)^{-\varepsilon\rho} + 1), \quad t \in (\tau, \delta). \tag{2.4}$$

Without loss of generality, we assume that $\delta > \tau$ is close enough to τ and (2.4) holds. Since $u \in \mathfrak{M}_\tau^T$ and $\delta > \tau$, it follows that $\|u\|_{L^\infty((\delta,T),E_{1+\varepsilon})} \leqslant c_\delta$ and by (1.5) we conclude that

$$m_\delta := \|F(t, u(t))\|_{L^\infty((\delta,T),E_{\gamma(\varepsilon)})} < \infty. \tag{2.5}$$

For $\tau < \delta \leqslant t \leqslant t+h \leqslant T$, from the variation of constants formula we infer that

$$\begin{aligned}
\|u(t+h) - u(t)\|_{E_{1+\theta}} &\leqslant \|(U(t+h,\tau) - U(t,\tau))u_\tau\|_{E_{1+\theta}} \\
&\quad + \int_t^{t+h} \|U(t+h,s)F(s,u(s))\|_{E_{1+\theta}} \,\mathrm{d}s \\
&\quad + \int_\delta^t \|(U(t+h,s) - U(t,s))F(s,u(s))\|_{E_{1+\theta}} \,\mathrm{d}s \\
&\quad + \int_\tau^\delta \|(U(t+h,s) - U(t,s))F(s,u(s))\|_{E_{1+\theta}} \,\mathrm{d}s \\
&=: J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

Choosing arbitrary

$$\hat{\varepsilon} \in (\theta, \mu) \cap (\theta, \gamma(\varepsilon)) \tag{2.6}$$

and applying (2.2) and (2.3), we obtain for $J_1 = \|(U(t+h,t) - \mathrm{Id})U(t,\tau)u_\tau\|_{E_{1+\theta}}$ that

$$\begin{aligned}
J_1 &\leqslant \|U(t+h,t) - \mathrm{Id}\|_{L(E_{1+\hat{\varepsilon}},E_{1+\theta})} \|U(t,\tau)\|_{L(E_0,E_{1+\hat{\varepsilon}})} \|u_\tau\|_{E_0} \\
&\leqslant M^2 h^{\hat{\varepsilon}-\theta}(t-\tau)^{-1-\hat{\varepsilon}} \|u_\tau\|_{E_0} \\
&\leqslant M^2 h^{\hat{\varepsilon}-\theta}(\delta-\tau)^{-1-\hat{\varepsilon}} \|u_\tau\|_{E_0}.
\end{aligned}$$

Using (2.2), (2.5) and (2.6) we get $J_2 \leqslant \int_t^{t+h} \|U(t+h,s)\|_{L(E_{\gamma(\varepsilon)},E_{1+\theta})} \|F(s,u(s))\|_{E_{\gamma(\varepsilon)}} \,\mathrm{d}s$ and

$$J_2 \leqslant M m_\delta \int_t^{t+h} (t+h-s)^{\gamma(\varepsilon)-\theta-1} \,\mathrm{d}s \leqslant M m_\delta (\gamma(\varepsilon)-\theta)^{-1} (T-\tau)^{\gamma(\varepsilon)-\hat{\varepsilon}} h^{\hat{\varepsilon}-\theta}.$$

On the other hand, by (2.3), (2.5) and (2.6), we have

$$\begin{aligned}
J_3 &\leqslant \int_\delta^t \|U(t+h,t) - \mathrm{Id}\|_{L(E_{1+\hat{\varepsilon}},E_{1+\theta})} \|U(t,s)\|_{L(E_{\gamma(\varepsilon)},E_{1+\hat{\varepsilon}})} \|F(s,u(s))\|_{E_{\gamma(\varepsilon)}} \,\mathrm{d}s \\
&\leqslant M^2 m_\delta \int_\delta^t h^{\hat{\varepsilon}-\theta}(t-s)^{\gamma(\varepsilon)-1-\hat{\varepsilon}} \,\mathrm{d}s \\
&\leqslant M^2 (\gamma(\varepsilon)-\hat{\varepsilon})^{-1} m_\delta h^{\hat{\varepsilon}-\theta} (T-\tau)^{\gamma(\varepsilon)-\hat{\varepsilon}},
\end{aligned}$$

whereas due to (2.4) we get

$$\begin{aligned} J_4 &\leqslant \int_\tau^\delta \|U(t+h,t)-\mathrm{Id}\|_{L(E_{1+\hat\varepsilon},E_{1+\theta})}\|U(t,s)\|_{L(E_{\gamma(\varepsilon)},E_{1+\hat\varepsilon})}\|F(s,u(s))\|_{E_{\gamma(\varepsilon)}}\,\mathrm{d}s \\ &\leqslant \tilde c M^2\int_\tau^\delta h^{\hat\varepsilon-\theta}(t-s)^{\gamma(\varepsilon)-1-\hat\varepsilon}((s-\tau)^{-\varepsilon\rho}+1)\,\mathrm{d}s \\ &\leqslant \tilde c M^2 h^{\hat\varepsilon-\theta}\bigg(B(\gamma(\varepsilon)-\hat\varepsilon,1-\varepsilon\rho)\frac{(T_{u_\tau}-\tau)^{\gamma(\varepsilon)-\hat\varepsilon}}{(\delta-\tau)^{\varepsilon\rho}}+(\gamma(\varepsilon)-\hat\varepsilon)^{-1}(T_{u_\tau}-\tau)^{\gamma(\varepsilon)-\hat\varepsilon}\bigg). \end{aligned}$$

Hence, for any $\delta>\tau$ close enough to τ there is a $\bar c>0$ such that $\|u(t+h)-u(t)\|_{E_{1+\theta}}\leqslant \bar c h^{\hat\varepsilon-\theta}$ for each $\delta\leqslant t\leqslant t+h\leqslant T$. Since $\hat\varepsilon$ could be any number satisfying (2.6), we obtain the result. □

Lemma 2.2. *If $\varphi,\tilde\varphi\in\mathfrak{M}_\tau^T$, $u_\tau\in E_0$ and (1.6) is valid in $(\tau,T]$ both for $u=\varphi$ and $u=\tilde\varphi$, then φ and $\tilde\varphi$ are identical on $(\tau,T]$.*

Proof. By assumption, we have

$$\begin{aligned} &\|\varphi(t)-\tilde\varphi(t)\|_{E_{1+\varepsilon}} \\ &\quad\leqslant cC_\eta M\int_\tau^t (t-s)^{\gamma(\varepsilon)-1-\varepsilon}\|\varphi(s)-\tilde\varphi(s)\|_{E_{1+\varepsilon}}\,\mathrm{d}s \\ &\qquad + c\eta M\int_\tau^t (t-s)^{\gamma(\varepsilon)-1-\varepsilon}\|\varphi(s)-\tilde\varphi(s)\|_{E_{1+\varepsilon}}(\|\varphi(s)\|_{E_{1+\varepsilon}}^{\rho-1}+\|\tilde\varphi(s)\|_{E_{1+\varepsilon}}^{\rho-1})\,\mathrm{d}s, \\ &\qquad\qquad t\in(\tau,T]. \end{aligned}$$

Since $\varphi,\tilde\varphi\in\mathfrak{M}_\tau^T$, given $\xi\in(0,1)$, there is a certain $h\in(0,\xi)$ such that

$$(t-\tau)^\varepsilon\|\varphi(t)\|_{E_{1+\varepsilon}}+(t-\tau)^\varepsilon\|\tilde\varphi(t)\|_{E_{1+\varepsilon}}\leqslant\xi,\quad t\in(\tau,\tau+h).$$

Using this and restricting t to the interval $(\tau,\tau+h)$, where $h\in(0,\xi)$, we obtain

$$\begin{aligned} &(t-\tau)^\varepsilon\|\varphi(t)-\tilde\varphi(t)\|_{E_{1+\varepsilon}} \\ &\quad\leqslant cC_\eta MB(\gamma(\varepsilon)-\varepsilon,1-\varepsilon)\xi^{\gamma(\varepsilon)-\varepsilon}\sup_{s\in(\tau,\tau+h)}(s-\tau)^\varepsilon\|\varphi(s)-\tilde\varphi(s)\|_{E_{1+\varepsilon}} \\ &\qquad+\xi^{\rho-1+\gamma(\varepsilon)-\varepsilon\rho}2c\eta MB(1-\varepsilon\rho,\gamma(\varepsilon)-\varepsilon)\sup_{s\in(\tau,\tau+h)}(s-\tau)^\varepsilon\|\varphi(s)-\tilde\varphi(s)\|_{E_{1+\varepsilon}}. \end{aligned}$$

We remark that the inequality above will hold true if we replace its left-hand side by $\sup_{s\in(\tau,\tau+h)}(s-\tau)^\varepsilon\|\varphi(s)-\tilde\varphi(s)\|_{E_{1+\varepsilon}}$. On the other hand, recalling that $\rho>1$, $\gamma(\varepsilon)\geqslant\rho\varepsilon$ and choosing $\xi>0$ small enough, we can ensure that the right-hand side above is less than $\frac12\sup_{s\in(\tau,\tau+h)}(s-\tau)^\varepsilon\|\varphi(s)-\tilde\varphi(s)\|_{E_{1+\varepsilon}}$. Consequently, $\sup_{s\in(\tau,\tau+h)}(s-\tau)^\varepsilon\|\varphi(s)-\tilde\varphi(s)\|_{E_{1+\varepsilon}}=0$, and thus $\varphi=\tilde\varphi$ in $[\tau,\tau+h]$ for some $h>0$.

Now, if $\tau^*\in(\tau,\tau+h]$ is such that $\varphi(\tau^*)=\tilde\varphi(\tau^*)$, then, applying the variation of constants formula with the initial time τ^* and with the initial value $\varphi(\tau^*)=\tilde\varphi(\tau^*)$, we obtain

$$\varphi(t)-\tilde\varphi(t)=\int_{\tau^*}^t U(t,s)(F(s,\varphi(s))-F(s,\tilde\varphi(s)))\,\mathrm{d}s\quad\text{in }[\tau^*,T].$$

Hence, letting $c^* = \sup_{s\in[\tau^*,T]}(\|\varphi(s)\|^{\rho-1}_{E_{1+\varepsilon}} + \|\tilde{\varphi}(s)\|^{\rho-1}_{E_{1+\varepsilon}})$,

$$\|\varphi(t) - \tilde{\varphi}(t)\|_{E_{1+\varepsilon}} \leqslant cM(C_\eta + \eta c^*) \int_{\tau^*}^{t} (t-s)^{\gamma(\varepsilon)-1-\varepsilon} \|\varphi(s) - \tilde{\varphi}(s)\|_{E_{1+\varepsilon}} \,\mathrm{d}s \quad \text{in } [\tau^*, T],$$

and by the singular version of Gronwall's inequality we conclude that φ and $\tilde{\varphi}$ coincide. □

2.2. Proof of Theorem 1.7

Let us fix an interval $I = (t_0 - \xi, t_0 + \xi)$ around t_0 such that (1.2) holds with $\zeta = \gamma(\varepsilon)$ and $\sigma = 1+\varepsilon$. Let us also choose an interval $\mathcal{J}$ centred at t_0 such that $I \setminus \mathcal{J}$ is the union of two intervals of length $l > 0$.

We first note that if $\delta^* \in (0, l)$, $\tau \in \mathcal{J}$, $\delta \in (0,1] \cap (0, \delta^*)$, $v \in C((\tau, \tau+\delta], E_{1+\varepsilon})$, $\lambda(v,t) := \sup_{s\in(\tau,t]}\{(s-\tau)^\varepsilon \|v(s)\|_{E_{1+\varepsilon}}\}$, $R > 0$, $t \in (\tau, \tau+\delta]$ and $\lambda(v,t) \leqslant R$, then, by Assumption 1.1 and (1.5), we have

$$\|U(t,s)F(s,v(s))\|_{E_{1+\varepsilon}} \leqslant \|U(t,s)\|_{L(E_{\gamma(\varepsilon)},E_{1+\varepsilon})} \|F(s,v(s))\|_{E_{\gamma(\varepsilon)}}$$

and

$$\|U(t,s)F(s,v(s))\|_{E_{1+\varepsilon}} \leqslant M(t-s)^{-1+\gamma(\varepsilon)-\varepsilon} c(\eta\|v(s)\|^\rho_{E_{1+\varepsilon}} + C_\eta).$$

Consequently,

$$\begin{aligned}
(t-\tau)^\varepsilon &\left\| \int_\tau^t U(t,s)F(s,v(s))\,\mathrm{d}s \right\|_{E_{1+\varepsilon}} \\
&\leqslant cC_\eta M(t-\tau)^\varepsilon \int_\tau^t (t-s)^{-1+\gamma(\varepsilon)-\varepsilon} \,\mathrm{d}s \\
&\qquad + c\eta M(t-\tau)^\varepsilon \int_\tau^t (t-s)^{-1+\gamma(\varepsilon)-\varepsilon} (s-\tau)^{-\rho\varepsilon} [(s-\tau)^\varepsilon \|v(s)\|_{E_{1+\varepsilon}}]^\rho \,\mathrm{d}s \\
&\leqslant cMB(1-\rho\varepsilon, \gamma(\varepsilon)-\varepsilon)[C_\eta(t-\tau)^{\gamma(\varepsilon)} + \eta\lambda^\rho(v,t)] \\
&\leqslant cMB_{\varepsilon,\rho}[C_\eta(t-\tau)^{\gamma(\varepsilon)} + \eta R^\rho]. \qquad (2.7)
\end{aligned}$$

Also, if $v, \tilde{v} \in C((\tau, \tau+\delta], E_{1+\varepsilon})$, $t \in (\tau, \tau+\delta]$, $\lambda(v,t) \leqslant R$ and $\lambda(\tilde{v},t) \leqslant R$, then, with a similar usage of Assumption 1.1 and (1.4), we obtain

$$\begin{aligned}
(t-\tau)^\varepsilon &\left\| \int_\tau^t U(t,s)[F(s,v(s)) - F(s,\tilde{v}(s))]\,\mathrm{d}s \right\|_{E_{1+\varepsilon}} \\
&\leqslant cC_\eta M(t-\tau)^\varepsilon \int_\tau^t (t-s)^{-1+\gamma(\varepsilon)-\varepsilon} (s-\tau)^{-\varepsilon} (s-\tau)^\varepsilon \|v(s) - \tilde{v}(s)\|_{E_{1+\varepsilon}} \,\mathrm{d}s \\
&\quad + c\eta M(t-\tau)^\varepsilon \int_\tau^t (t-s)^{-1+\gamma(\varepsilon)-\varepsilon} (s-\tau)^{-\rho\varepsilon} \\
&\qquad\qquad \times (((s-\tau)^\varepsilon \|v(s)\|_{E_{1+\varepsilon}})^{\rho-1} + ((s-\tau)^\varepsilon \|\tilde{v}(s)\|_{E_{1+\varepsilon}})^{\rho-1}) \\
&\qquad\qquad\qquad \times (s-\tau)^\varepsilon \|v(s) - \tilde{v}(s)\|_{E_{1+\varepsilon}} \,\mathrm{d}s.
\end{aligned}$$

Hence, letting

$$\Gamma_\varepsilon(t) := cMB_{\varepsilon,\rho}[C_\eta(t-\tau)^{\gamma(\varepsilon)-\varepsilon} + 2\eta R^{\rho-1}], \tag{2.8}$$

we conclude that

$$(t-\tau)^\varepsilon \left\| \int_\tau^t U(t,s)[F(s,v(s)) - F(s,\tilde{v}(s))]\,\mathrm{d}s \right\|_{E_{1+\varepsilon}} \leqslant \Gamma_\varepsilon(t) \sup_{s\in(\tau,t]} \{(s-\tau)^\varepsilon \|v(s)-\tilde{v}(s)\|_{E_{1+\varepsilon}}\}. \tag{2.9}$$

We now choose $R_0 \geqslant R > 0$ and $\delta \in (0,1] \cap (0,\delta^*)$ such that

$$c\eta MB_{\varepsilon,\rho}R_0^{\rho-1} = \tfrac{1}{8} \quad \text{and} \quad cC_\eta MB_{\varepsilon,\rho}\delta^{\gamma(\varepsilon)-\varepsilon} = \min\{\tfrac{1}{8}R, \tfrac{1}{4}\}. \tag{2.10}$$

We also set

$$r := \tfrac{1}{4}R \leqslant \tfrac{1}{4}R_0 = \frac{1}{4(8c\eta MB_{\varepsilon,\rho})^{1/(\rho-1)}}$$

and, since $\lim_{t\to\tau^+} \|(t-\tau)^\varepsilon U(t,\tau)w_0\|_{E_{1+\varepsilon}} = 0$, we choose $\bar{\delta}_0 \in (0,\delta]$ such that

$$\|(t-\tau)^\varepsilon U(t,\tau)w_0\|_{E_{1+\varepsilon}} \leqslant \tfrac{1}{2}R, \quad \tau < t \leqslant \tau + \bar{\delta}_0. \tag{2.11}$$

For any fixed $\delta_0 \in (0,\bar{\delta}_0]$, $u_\tau \in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(w_0,r)$, we then define

$$K(R,\tau) = \left\{ v \in C((\tau,\tau+\delta_0],E_{1+\varepsilon}),\ \sup_{t\in(\tau,\tau+\delta_0]} \{(t-\tau)^\varepsilon \|v(t)\|_{E_{1+\varepsilon}}\} \leqslant R \right\}$$

and let $d(v,\tilde{v}) = \sup_{t\in(\tau,\tau+\delta_0]}\{(t-\tau)^\varepsilon\|v(t)-\tilde{v}(t)\|_{E_{1+\varepsilon}}\}$ for $v,\tilde{v} \in K(R,\tau)$.

$(K(R,\tau),d)$ is a complete metric space and we next consider the map

$$(\mathcal{T}v)(t) = U(t,\tau)u_\tau + \int_\tau^t U(t,s)F(s,v(s))\,\mathrm{d}s, \quad v \in K(R,\tau),\ t \in (\tau,\tau+\delta_0].$$

Adapting Lemma 2.1, one can see that $\mathcal{T}v \in C((\tau,\tau+\delta_0],E_{1+\varepsilon})$ for $v \in K(R,\tau)$.

It then follows from (1.8), (2.7), (2.10) and (2.11) that

$$\begin{aligned}
\|(t-\tau)^\varepsilon(\mathcal{T}v)(t)\|_{E_{1+\varepsilon}}
&\leqslant (t-\tau)^\varepsilon \left\| U(t,\tau)u_\tau + \int_\tau^t U(t,s)F(s,v(s))\,\mathrm{d}s \right\|_{E_{1+\varepsilon}} \\
&\leqslant \|(t-\tau)^\varepsilon U(t,\tau)u_\tau\|_{E_{1+\varepsilon}} + cM(t-\tau)^\varepsilon \int_\tau^t (t-s)^{-1+\gamma(\varepsilon)-\varepsilon}(\eta\|v(s)\|^\rho_{E_{1+\varepsilon}} + C_\eta)\,\mathrm{d}s \\
&\leqslant \|(t-\tau)^\varepsilon U(t,\tau)(u_\tau-\omega_0)\|_{E_{1+\varepsilon}} + \|(t-\tau)^\varepsilon U(t,\tau)\omega_0\|_{E_{1+\varepsilon}} + cM\eta B_{\varepsilon,\rho}R^\rho \\
&\qquad + cMC_\eta B_{\varepsilon,\rho}\delta_0^{\gamma(\varepsilon)} \\
&\leqslant r + \|(t-\tau)^\varepsilon U(t,\tau)\omega_0\|_{E_{1+\varepsilon}} + cM\eta B_{\varepsilon,\rho}R^\rho + cMC_\eta B_{\varepsilon,\rho}\delta_0^{\gamma(\varepsilon)-\varepsilon}\delta_0^\varepsilon \\
&\leqslant R,
\end{aligned}$$

which yields that $\mathcal{T}$ takes $K(R,\tau)$ into $K(R,\tau)$. On the other hand, applying (2.8)–(2.10), we get $d(\mathcal{T}v_1,\mathcal{T}v_2)\leqslant \frac{1}{2}d(v_1,v_2)$.

Consequently, due to the Banach fixed-point theorem, we infer that $\mathcal{T}$ a unique fixed point $u=u(\cdot,\tau,u_\tau)$ in $K(R,\tau)$ and we now show that $\lim_{t\to\tau^+}\|(t-\tau)^\varepsilon u(t)\|_{E_{1+\varepsilon}}=0$.

Adapting (2.7), we have for each $t\in(\tau,\tau+\delta_0]$ and the above fixed point u,

$$(t-\tau)^\varepsilon\|u(t)\|_{E_{1+\varepsilon}}\leqslant (t-\tau)^\varepsilon\|U(t,\tau)u_\tau\|_{E_{1+\varepsilon}}+cMB_{\varepsilon,\rho}[C_\eta(t-\tau)^{\gamma(\varepsilon)}+\eta R^{\rho-1}\lambda(u,t)],$$

where, by assumption, given any $\xi>0$ we can choose $h\in(0,\xi)$ such that for $t\in(\tau,\tau+h)$ we have $(t-\tau)^\varepsilon\|U(t,\tau)u_\tau\|_{E_{1+\varepsilon}}<\xi$. Hence, we obtain

$$(t-\tau)^\varepsilon\|u(t)\|_{E_{1+\varepsilon}}\leqslant \xi+cMB_{\varepsilon,\rho}[C_\eta\xi^{\gamma(\varepsilon)}+\eta R^{\rho-1}\lambda(u,t)],\quad t\in(\tau,\tau+h).$$

Since the right-hand side above is a non-decreasing function of t, we obtain

$$\lambda(u,t)\leqslant \xi+cMB_{\varepsilon,\rho}[C_\eta\xi^{\gamma(\varepsilon)}+\eta R^{\rho-1}\lambda(u,t)],\quad t\in(\tau,\tau+h),$$

and, via (2.10), $\frac{7}{8}\lambda(u,t)\leqslant\xi+cMB_{\varepsilon,\rho}C_\eta\xi^{\gamma(\varepsilon)}$, $t\in(\tau,\tau+h)$. This yields

$$\lambda(u,t)=\sup_{s\in(\tau,t]}\{(s-\tau)^\varepsilon\|u(s)\|_{E_{1+\varepsilon}}\}\to 0\quad\text{as } t\to\tau^+,\tag{2.12}$$

which ensures that $(t-\tau)^\varepsilon u(t)\to 0$ in $E_{1+\varepsilon}$ as $t\to\tau^+$.

Finally, letting $u(\tau)=u_\tau$, we extend the fixed point $u=u(\cdot,\tau,u_\tau)$ constructed above to the interval $[\tau,\tau+\delta_0]$ and obtain the $E_{1+\varepsilon}$-solution of (1.1). Since the uniqueness follows from Theorem 2.2, part (i) of Theorem 1.7 is proved.

Part (ii) now follows from Corollary 2.3 (see also Remark 2.4).

Corollary 2.3. *Suppose that Assumptions 1.1 and 1.2 hold,* $F\colon\mathbb{R}\times E_{1+\varepsilon}\to E_{\gamma(\varepsilon)}$ *is continuous and constants* $\rho>1$, $0<\varepsilon<\min\{1/\rho,\mu\}$, $\gamma(\varepsilon)\in[\rho\varepsilon,1)$ *exist such that for each* $\eta>0$ *there exists* $C_\eta>0$ *and, moreover, for any bounded time interval* I *there exists* $c>0$ *for which*

$$\|F(t,v)-F(t,w)\|_{E_{\gamma(\varepsilon)}}\leqslant c\|v-w\|_{E_{1+\varepsilon}}(\eta\|v\|^{\rho-1}_{E_{1+\varepsilon}}+\eta\|w\|^{\rho-1}_{E_{1+\varepsilon}}+C_\eta),\quad v,w\in E_{1+\varepsilon},\ t\in I,$$

and

$$\|F(t,v)\|_{E_{\gamma(\varepsilon)}}\leqslant c(\eta\|v\|^{\rho}_{E_{1+\varepsilon}}+C_\eta),\quad v,w\in E_{1+\varepsilon},\ t\in I.$$

Then, given any $t_0\in\mathbb{R}$, τ *in a certain interval* $\mathcal{J}\subset\mathbb{R}$ *centred at* t_0 *and given any* $r_0>0$, *there exists* $\delta_0>0$ *such that for any initial condition* $u_\tau\in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(0,r_0)$ *there exists a unique* $E_{1+\varepsilon}$*-solution* $u=u(\cdot,\tau,u_\tau)$ *of (1.1) on* $[\tau,\tau+\delta_0]$.

Proof. Letting $w_0=0$, observe via Theorem 1.7 that given $r_0>0$ one can now choose $\eta>0$ such that r in (1.9) satisfies $r>r_0$. Proceeding as in the proof of Theorem 1.7, we obtain for any $u_\tau\in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(0,r)$ the existence of an $E_{1+\varepsilon}$-solution $u=u(\cdot,\tau,u_\tau)$ of (1.1) on $[\tau,\tau+\delta_0]$. □

Remark 2.4. For subcritical F, without loss of generality one can assume that $\eta > 0$ in (1.4) and (1.5) can be chosen arbitrarily small. Indeed, given $\rho > 1$, $\varepsilon \in (0, 1/\rho)$, $\varepsilon < \mu$, $\gamma(\varepsilon) \in (\rho\varepsilon, 1)$, we can choose $\tilde{\rho} > \rho$ close enough to ρ and have $\varepsilon \in (0, 1/\tilde{\rho})$ and $\gamma(\varepsilon) \in (\tilde{\rho}\varepsilon, 1)$. Then $\eta\|w\|^{\rho-1}_{E_{1+\varepsilon}}$ can be estimated by $\tilde{\eta}\|w\|^{\tilde{\rho}-1}_{E_{1+\varepsilon}} + c_{\tilde{\eta},\eta}$, which yields (1.4) and (1.5) with ε and $\gamma(\varepsilon)$ as before, ρ replaced by $\tilde{\rho}$ suitably close to ρ and η replaced by $\tilde{\eta}$, which we can fix as small as we wish. Parameters ε, $\gamma(\varepsilon)$ and c in (1.4) and (1.5) will remain the same and the only difference will come from the replacement of C_η by $C_\eta + c_{\tilde{\eta},\eta}$, which will not influence the heart of our consideration.

Proof. We now prove conditions (1.10)–(1.12). As in (2.7), for $\theta \in (0, \gamma(\varepsilon)) \cap (0, \mu)$ and for the unique $E_{1+\varepsilon}$-solution $u = u(\cdot, \tau, u_\tau)$ of (1.1) we obtain

$$
\begin{aligned}
(t-\tau)^\theta \|u(t)\|_{E_{1+\theta}} &\leqslant (t-\tau)^\theta \|U(t,\tau)u_\tau\|_{E_{1+\theta}} + (t-\tau)^\theta \int_\tau^t \|U(t,s)F(s,u(s))\|_{E_{1+\theta}}\,\mathrm{d}s \\
&\leqslant (t-\tau)^\theta \|U(t,\tau)u_\tau\|_{E_{1+\theta}} + cMC_\eta(\gamma(\varepsilon)-\theta)^{-1}(t-\tau)^{\gamma(\varepsilon)} \\
&\quad + \eta cMB(1-\varepsilon\rho, \gamma(\varepsilon)-\theta)\Big(\sup_{\tau<s\leqslant t}\{(s-\tau)^\varepsilon \|u(s)\|_{E_{1+\varepsilon}}\}\Big)^\rho.
\end{aligned}
$$

Recalling that $u_\tau \in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(w_0, r) \cap \mathfrak{E}^\tau_\theta$ and by using (2.12), we conclude that $(t-\tau)^\theta \|u(t)\|_{E_{1+\theta}} \to 0$ as $t \to \tau^+$, which proves (1.10).

For $\theta \in (0, \gamma(\varepsilon)) \cap (0, \mu)$, as in (2.9) we next have

$$
\begin{aligned}
&(t-\tau)^\theta \|u(t,\tau,u^1_\tau) - u(t,\tau,u^2_\tau)\|_{E_{1+\theta}} \\
&\quad\leqslant (t-\tau)^\theta \|U(t,\tau)(u^1_\tau - u^2_\tau)\|_{E_{1+\theta}} \\
&\qquad + (t-\tau)^\theta \int_\tau^t \|U(t,s)[F(s,u(s,\tau,u^1_\tau)) - F(s,u(s,\tau,u^2_\tau))]\|_{E_{1+\theta}}\,\mathrm{d}s \\
&\quad\leqslant (t-\tau)^\theta \|U(t,\tau)(u^1_\tau - u^2_\tau)\|_{E_{1+\theta}} \\
&\qquad + \mathit{\Gamma}_\theta(t) \sup_{\tau<s\leqslant\tau+\delta_0}\{(s-\tau)^\varepsilon \|u(s,\tau,u^1_\tau) - u(s,\tau,u^2_\tau)\|_{E_{1+\varepsilon}}\}, \qquad (2.13)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathit{\Gamma}_\theta(t) = cM(1+\theta,\gamma(\varepsilon),T)\max\{B(\gamma(\varepsilon)-\theta, 1-\varepsilon), B(1-\rho\varepsilon, \gamma(\varepsilon)-\theta)\} \\
\times [C_\eta(t-\tau)^{\gamma(\varepsilon)-\theta} + 2\eta R^{\rho-1}].
\end{aligned}
$$

Taking $\theta = \varepsilon$ we have

$$
\begin{aligned}
&(t-\tau)^\varepsilon \|u(t,\tau,u^1_\tau) - u(t,\tau,u^2_\tau)\|_{E_{1+\varepsilon}} \\
&\quad\leqslant (t-\tau)^\varepsilon \|U(t,\tau)(u^1_\tau - u^2_\tau)\|_{E_{1+\varepsilon}} \\
&\qquad + \mathit{\Gamma}_\varepsilon(t) \sup_{\tau<s\leqslant\tau+\delta_0}\{(s-\tau)^\varepsilon \|u(s,\tau,u^1_\tau) - u(s,\tau,u^2_\tau)\|_{E_{1+\varepsilon}}\}.
\end{aligned}
$$

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Since, by (2.10), $\Gamma_\varepsilon(\tau+\delta_0) \leqslant \frac{1}{2}$ and $\Gamma_\varepsilon(t)$ is increasing with respect to t, we conclude that

$$\sup_{\tau<s\leqslant\tau+\delta_0} \{(s-\tau)^\varepsilon \|u(s,\tau,u_\tau^1) - u(s,\tau,u_\tau^2)\|_{E_{1+\varepsilon}}\} \leqslant 2 \sup_{\tau<s\leqslant\tau+\delta_0} (s-\tau)^\varepsilon \|U(s,\tau)(u_\tau^1 - u_\tau^2)\|_{E_{1+\varepsilon}}.$$

Consequently, using the above inequality and (2.13), we obtain (1.11).

Assuming that $0 \leqslant \theta < \min\{\gamma(\varepsilon), \mu\}$ and $\lim_{t\to\tau^+}(t-\tau)^\theta \|U(t,\tau)u_\tau - u_\tau\|_{E_{1+\theta}} = 0$, we now show that $\lim_{t\to\tau^+}(t-\tau)^\theta \|u(t,\tau,u_\tau) - u_\tau\|_{E_{1+\theta}} = 0$, for which we first use the variation of constants formula, (1.2) and (1.5) to obtain for each $t \in (\tau, \tau+\delta_0]$,

$$\begin{aligned}
&(t-\tau)^\theta \|u(t) - u_\tau\|_{E_{1+\theta}} \\
&\quad\leqslant (t-\tau)^\theta \|U(t,\tau)u_\tau - u_\tau\|_{E_{1+\theta}} + (t-\tau)^\theta \int_\tau^t \|U(t,s)F(s,u(s))\|_{E_{1+\theta}}\,\mathrm{d}s \\
&\quad\leqslant (t-\tau)^\theta \|U(t,\tau)u_\tau - u_\tau\|_{E_{1+\theta}} \\
&\qquad + cM(t-\tau)^\theta \int_\tau^t (t-s)^{\gamma(\varepsilon)-1-\theta}(C_\eta + \eta(s-\tau)^{-\varepsilon\rho}\|(s-\tau)^\varepsilon u(s)\|_{E_{1+\varepsilon}}^\rho)\,\mathrm{d}s \\
&\quad\leqslant (t-\tau)^\theta \|U(t,\tau)u_\tau - u_\tau\|_{E_{1+\theta}} + cC_\eta MB(1-\rho\varepsilon, \gamma(\varepsilon)-\theta)(t-\tau)^{\gamma(\varepsilon)} \\
&\qquad + cM\eta B(1-\rho\varepsilon, \gamma(\varepsilon)-\theta)\lambda^\rho(u,t). \qquad (2.14)
\end{aligned}$$

Thus, (1.12) is a consequence of (2.12) and (2.14). The proof of Theorem 1.7 is complete. □

2.3. Proof of Remark 1.8

Note first that $E_{1+\varepsilon} \subset \mathfrak{E}_\varepsilon^\tau$ because whenever $\varphi \in E_{1+\varepsilon}$, by Assumption 1.1, we have

$$(t-\tau)^\varepsilon \|U(t,\tau)\varphi\|_{E_{1+\varepsilon}} \leqslant M(t-\tau)^\varepsilon \|\varphi\|_{E_{1+\varepsilon}} \to 0 \quad \text{as} \quad t \to \tau^+.$$

Now, if $\psi \in E_1$ and $E_{1+\varepsilon} \ni \varphi_n \xrightarrow{E_1} \psi$, then, again using Assumption 1.1, we obtain

$$(t-\tau)^\varepsilon \|U(t,\tau)\psi\|_{E_{1+\varepsilon}} \leqslant M\|\psi - \varphi_n\|_{E_1} + (t-\tau)^\varepsilon \|U(t,\tau)\varphi_n\|_{E_{1+\varepsilon}}$$

and for each $\zeta > 0$ we can choose $n \in \mathbb{N}$ and $h_\zeta > 0$ such that the right-hand side of the above inequality becomes less than ζ uniformly for $t \in (\tau, \tau+h_\zeta)$. This proves that $E_1 \subset \mathfrak{E}_\varepsilon^\tau$.

By assumption, $\|\varphi\|_\delta^{\mathfrak{E}_\varepsilon^\tau} = 0$ implies that $0 = \|U(t,\tau)\varphi\|_{E_0} \to \|\varphi\|_{E_0}$ as $t \to \tau^+$ and we get $\varphi = 0$. It then follows easily that $\|\cdot\|_\delta^{\mathfrak{E}_\varepsilon^\tau}$ is the norm in $\mathfrak{E}_\varepsilon^\tau$.

Finally, if $\psi \in \{\phi \in E_1 \colon \|\phi - w_0\|_{E_1} \leqslant r/M\}$, then $\sup_{s\in(\tau,\tau+\delta]}(s-\tau)^\varepsilon \|U(s,\tau)(\psi - w_0)\|_{E_{1+\varepsilon}} \leqslant M\|\psi - w_0\|_{E_1} \leqslant r$ and, evidently, $\psi \in B_{\mathfrak{E}_\varepsilon^\tau}^\delta(w_0, r)$, which completes the proof of part (i).

We can now apply Theorem 1.7 with $w_0 \in E_1$ and ensure that the time of existence δ_0 can be chosen uniformly in a certain neighbourhood of a given point $t_0 \in \mathbb{R}$. Actually, following the proof of the existential part of Theorem 1.7, it suffices to ensure that the number $\bar{\delta}_0(R)$ in (2.11) can be chosen uniformly with respect to $\tau \in \mathcal{J}$.

Choose $w_0 \in E_1$ and recall that $E_{1+\varepsilon}$ is dense in E_1. Using (1.2) for any $\phi \in E_{1+\varepsilon}$ we have

$$\begin{aligned}\sup_{\tau<t\leqslant\tau+\bar{\delta}_0} \|(t-\tau)^{\varepsilon}U(t,\tau)w_0\|_{E_{1+\varepsilon}} &\leqslant \sup_{\tau<t\leqslant\tau+\bar{\delta}_0} \|(t-\tau)^{\varepsilon}U(t,\tau)(w_0-\phi)\|_{E_{1+\varepsilon}} \\ &\quad + \sup_{\tau<t\leqslant\tau+\bar{\delta}_0} \|(t-\tau)^{\varepsilon}U(t,\tau)\phi\|_{E_{1+\varepsilon}} \\ &\leqslant M\|w_0-\phi\|_{E_1} + \bar{\delta}_0^{\varepsilon} M\|\phi\|_{E_{1+\varepsilon}}, \quad \tau \in \mathcal{J}.\end{aligned}$$

Note that ϕ can be chosen such that $M\|w_0-\phi\|_{E_1} \leqslant R/4$ and $\bar{\delta}_0$ such that $\bar{\delta}_0^{\varepsilon} M\|\phi\|_{E_{1+\varepsilon}} \leqslant R/4$, in which case $\sup_{\tau<t\leqslant\tau+\bar{\delta}_0} \|(t-\tau)^{\varepsilon}U(t,\tau)w_0\|_{E_{1+\varepsilon}} \leqslant R/2$ for any $\tau \in \mathcal{J}$. □

Proof of Remark 1.9. We first prove that

$$\lim_{t\to\tau^+} \|(U(t,\tau)-I)\psi\|_{E_\alpha} = 0 \quad \text{for } \psi \in E_\alpha,\ \alpha \in [0, 1+\mu). \tag{2.15}$$

For this, observe that

$$\|(U(t,\tau)-I)\phi\|_{E_\alpha} \leqslant M(t-\tau)^{\beta-\alpha}\|\phi\|_{E_\beta} \to 0 \quad \text{as} \quad t\to\tau^+$$

whenever $\phi \in E_\beta$, $0 \leqslant \alpha < \beta < 1+\mu$. On the other hand, if $E_\beta \ni \phi_n \xrightarrow{E_\alpha} \psi \in E_\alpha$, then

$$\|(U(t,\tau)-I)\psi\|_{E_\alpha} \leqslant (M+1)\|\psi-\phi_n\|_{E_\alpha} + \|(U(t,\tau)-I)\phi_n\|_{E_\alpha}.$$

For $\zeta > 0$ one can thus choose $n \in \mathbb{N}$ and $h_\zeta > 0$ such that the right-hand side of the above inequality will be less than ζ uniformly for $t \in (\tau, \tau+h_\zeta)$, which proves (2.15).

We next infer that

$$\lim_{t\to\tau^+} \|u(t)-U(t,\tau)u_\tau\|_{E_\alpha} = 0 \quad \text{for } \alpha \in [0,1),\ u_\tau \in E_\alpha. \tag{2.16}$$

Indeed, since u is an $E_{1+\varepsilon}$-solution, for $\delta > \tau$ close enough to τ and $t \in (\tau,\delta)$ we have $\|u(t)\|_{E_{1+\varepsilon}} \leqslant (t-\tau)^{-\varepsilon}$. Via (1.5), $\|F(s,u(s))\|_{E_{\gamma(\varepsilon)}} \leqslant \tilde{c}((s-\tau)^{-\varepsilon\rho}+1)$ for $t \in (\tau,\delta)$. Whenever $\gamma(\varepsilon) \leqslant \alpha < 1$ and $t \in (\tau,\delta)$, we can thus estimate $\|u(t)-U(t,\tau)u_\tau\|_{E_\alpha}$ by $\int_\tau^t \|U(t,s)\|_{L(E_{\gamma(\varepsilon)},E_\alpha)}\|F(s,u(s))\|_{E_{\gamma(\varepsilon)}}\,\mathrm{d}s$ and obtain

$$\begin{aligned}\|u(t)-U(t,\tau)u_\tau\|_{E_\alpha} &\leqslant \int_\tau^t M(t-s)^{\gamma(\varepsilon)-\alpha}\tilde{c}((s-\tau)^{-\varepsilon\rho}+1)\,\mathrm{d}s \\ &\leqslant \tilde{c}M((t-\tau)^{1+\gamma(\varepsilon)-\alpha-\varepsilon\rho}B(1+\gamma(\varepsilon)-\alpha, 1-\varepsilon\rho) \\ &\qquad + (1+\gamma(\varepsilon)-\alpha)^{-1}(t-\tau)^{1+\gamma(\varepsilon)-\alpha}),\end{aligned}$$

where the right-hand side tends to 0 as $t\to\tau^+$. Connecting (2.15) and (2.16), we obtain that $\lim_{t\to\tau^+}\|u(t)-u_\tau\|_{E_\alpha} = 0$ whenever $u_\tau \in E_\alpha$ and $\alpha \in [0,1)$. By (1.12) and (2.15), the latter is also true for $\alpha = 1$ and using Theorem 1.6 we obtain (i).

For the proof of (ii), note that given $u_\tau \in E_{1+\varepsilon}$ one can actually find a fixed point of $(\mathcal{T}v)(t) = U(t,\tau)u_\tau + \int_\tau^t U(t,s)F(s,v(s))\,\mathrm{d}s$ in a complete metric space

$$\mathcal{K}_\xi(R,\tau) = \{v \in C([\tau,\tau+\xi], E_{1+\varepsilon})\colon |||v - u_\tau||| \leqslant R\}$$

with some $R > 0$, $\xi > 0$ and $|||v||| = \sup_{t\in[\tau,\tau+\xi]} \|v(t)\|_{E_{1+\varepsilon}}$. Indeed, given $v \in \mathcal{K}_\xi(R,\tau)$, we have by (2.15), (1.2) and (1.5) that for a suitably small $\xi > 0$,

$$\begin{aligned}
&\|(Tv)(t) - u_\tau\|_{E_{1+\varepsilon}} \\
&\qquad \leqslant \|U(t,\tau)u_\tau - u_\tau\|_{E_{1+\varepsilon}} + \int_\tau^t \|U(t,s)\|_{L(E_{\gamma(\varepsilon)},E_{1+\varepsilon})} \|F(s,v(s))\|_{E_{\gamma(\varepsilon)}}\,\mathrm{d}s \\
&\qquad \leqslant \|U(t,\tau)u_\tau - u_\tau\|_{E_{1+\varepsilon}} + M\int_\tau^t (t-s)^{\gamma(\varepsilon)-1-\varepsilon} c(\eta\|v(s)\|^\rho_{E_{1+\varepsilon}} + C_\eta)\,\mathrm{d}s \\
&\qquad \leqslant \|U(t,\tau)u_\tau - u_\tau\|_{E_{1+\varepsilon}} + c(\eta R^\rho + C_\eta)M(\gamma(\varepsilon)-\varepsilon)^{-1}\xi^{\gamma(\varepsilon)-\varepsilon} \\
&\qquad \leqslant R, \quad t \in [\tau,\tau+\xi].
\end{aligned}$$

Hence, $\mathcal{T}$ takes $\mathcal{K}_\xi(R,\tau)$ into itself. On the other hand, (1.2) and (1.4) imply that, for $v, \tilde{v} \in \mathcal{K}_\xi(R,\tau)$,

$$\begin{aligned}
&\|(Tv)(t) - (T\tilde{v})(t)\|_{E_{1+\varepsilon}} \\
&\quad \leqslant \int_\tau^t \|U(t,s)\|_{L(E_{\gamma(\varepsilon)},E_{1+\varepsilon})} \|F(s,v(s)) - F(s,\tilde{v}(s))\|_{E_{\gamma(\varepsilon)}}\,\mathrm{d}s \\
&\quad \leqslant c(2\eta R^{\rho-1} + C_\eta) \sup_{t\in[\tau,\tau+\xi]} \|v(t) - \tilde{v}(t)\|_{E_{1+\varepsilon}} M(\gamma(\varepsilon)-\varepsilon)^{-1}\xi^{\gamma(\varepsilon)-\varepsilon}, \quad t \in [\tau,\tau+\xi],
\end{aligned}$$

so that for $\xi > 0$ small enough, $\mathcal{T}\colon \mathcal{K}_\xi(R,\tau) \to \mathcal{K}_\xi(R,\tau)$ is a contraction. By uniqueness, this ensures that an $E_{1+\varepsilon}$-solution of (1.1) can be viewed as a fixed point of $\mathcal{T}$ in $\mathcal{K}_\xi(R,\tau)$, and hence it is right-continuous in $E_{1+\varepsilon}$ at τ. Combining this with Theorem 1.6, we get (ii).

Finally, applying (ii) and (1.11) with $\theta = 0$, we obtain (iii). □

2.4. Proofs of continuation results

In what follows we prove Theorems 1.10 and 1.12.

Proof of Theorem 1.10 (i). Recalling Remark 1.8, we assume that $T_{u_\tau} < \infty$, $\limsup_{t\to T^-_{u_\tau}} \|u(t,\tau,u_\tau)\|_{E_1} < r^*$ for some $r^* > 0$ and for any $n \in \mathbb{N}$ large enough we define $\tau_n := T_{u_\tau} - 1/n$, $u_{\tau_n} := u(T_{u_\tau} - 1/n, \tau, u_\tau)$. We then consider the Cauchy problem

$$\dot{u}(t) + A(t)u(t) = F(t,u(t)), \quad t > \tau_n,\ u(\tau_n) = u_{\tau_n}, \tag{2.17}$$

where the initial conditions u_{τ_n} belong both to $E_{1+\varepsilon}$ and to a ball $B_{E_1}(0,r^*)$ in E_1 of radius r^*. Also, the initial times τ_n converge to T_{u_τ}.

We then have $\sup_{s\in(\tau_n,\tau_n+\delta]}(s-\tau_n)^\varepsilon \|U(s,\tau_n)u_{\tau_n}\|_{E_{1+\varepsilon}} \leqslant M\|u_{\tau_n}\|_{E_1} \leqslant Mr^*$, and hence $u_{\tau_n} \in B^\delta_{\mathfrak{E}^{\tau_n}_\varepsilon}(0, Mr^*)$.

Due to Theorem 1.7 (i) and (ii), there is a unique $E_{1+\varepsilon}$-solution of (2.17) on $[\tau_n, \tau_n+\delta_0]$, where δ_0 does not depend on n (see Remark 1.8 (ii)). By uniqueness, the solution coincides with $u(\cdot,\tau,u_\tau)$ on $[\tau_n, T_{u_\tau}]$ for all sufficiently large n. By concatenation, $u = u(\cdot,\tau,u_\tau)$ can thus be continued as an $E_{1+\varepsilon}$-solution onto $[\tau, T_{u_\tau}+\delta_0)$, which contradicts the definition of T_{u_τ}. □

Proof of Theorem 1.10 (ii). Assume that $T_{u_\tau} < \infty$ and let $\tau_n \to T_{u_\tau}^-$ be such that $u(\tau_n,\tau,u_\tau) \to w_0$ in E_1 as $n \to \infty$. Then $\sup_{s\in(\tau_n,\tau_n+\delta]}(s-\tau_n)^\varepsilon \|U(s,\tau_n)(u_{\tau_n} - w_0)\|_{E_{1+\varepsilon}} \leqslant M\|u_{\tau_n} - w_0\|_{E_1}$. Hence, if r is chosen as in Theorem 1.7 relative to w_0 and $N \in \mathbb{N}$ is such that $\|u_{\tau_n} - w_0\|_{E_1} \leqslant r/M$ for $n \geqslant N$, then $u_{\tau_n} \in B^\delta_{\mathfrak{E}_\varepsilon^{\tau_n}}(\omega_0, r)$ for $n \geqslant N$ and $\delta > 0$ close to zero.

Due to Theorem 1.7 (see Remark 1.8 (ii)) there is a unique $E_{1+\varepsilon}$-solution of (2.17) on $[\tau_n, \tau_n+\delta_0]$, where δ_0 does not depend on n. Again, by uniqueness, this solution coincides with $u(\cdot,\tau,u_\tau)$ on $[\tau_n, T_{u_\tau}]$ for each n sufficiently large, and thus $u = u(\cdot,\tau,u_\tau)$ can be continued as an $E_{1+\varepsilon}$-solution of (1.1) onto $[\tau, T_{u_\tau}+\delta_0)$, which contradicts the definition of T_{u_τ}. □

Proof of (1.15). Assume that $T_{u_\tau} < \infty$ and let $\limsup_{t\to T_{u_\tau}^+} \|u(t,\tau,u_\tau)\|_{E_{1+\varepsilon}} < r^*$ for some $r^* > 0$. For any $n \in \mathbb{N}$ large enough, define $\tau_n := T_{u_\tau} - 1/n$, $u_{\tau_n} := u(T_{u_\tau} - 1/n, \tau, u_\tau)$ and consider the Cauchy problem (2.17).

Since u_{τ_n} belongs to a ball $B_{E_{1+\varepsilon}}(0, r^*)$ in $E_{1+\varepsilon}$ of radius $r^* > 0$ around zero, for any $\delta > 0$ small enough, we have $\sup_{s\in(\tau_n,\tau_n+\delta]}(s-\tau_n)^\varepsilon \|U(s,\tau_n)u_{\tau_n}\|_{E_{1+\varepsilon}} \leqslant \delta^\varepsilon M\|u_{\tau_n}\|_{E_{1+\varepsilon}} \leqslant \delta^\varepsilon M r^*$. Hence, if $r > 0$ is chosen relatively to $w_0 = 0$ as in Theorem 1.7 and $\delta^\varepsilon \in (0, r/r^*M)$, we observe that u_{τ_n} belongs to $B^\delta_{\mathfrak{E}_\varepsilon^{\tau_n}}(0, r)$.

As a consequence of Theorem 1.7 (see Remark 1.8 (ii)), problem (2.17) has a unique $E_{1+\varepsilon}$-solution on $[\tau_n, \tau_n+\delta_0]$, where δ_0 does not depend on n. By uniqueness, the solution coincides with $u(\cdot,\tau,u_\tau)$ on $[\tau_n, T_{u_\tau}]$ for each n large enough and $u = u(\cdot,\tau,u_\tau)$ can be continued as an $E_{1+\varepsilon}$-solution of (1.1) onto $[\tau, T_{u_\tau}+\delta_0)$, which leads to a contradiction. □

Proof of Theorem 1.12. By assumption, given $\tau \in \mathbb{R}$ and $u_\tau \in \mathfrak{E}_\varepsilon^\tau$, we obtain from Theorem 1.7 that there exists a unique $E_{1+\varepsilon}$-solution u of (1.1) on the maximal interval of existence $[\tau, T_{u_\tau})$ and we define $u_0 := u_\tau$, $T_{u_0} := T_{u_\tau}$.

If $T_{u_0} < \infty$, then, using (1.22), (1.23) and the reflexivity of E_1, we conclude that there exists a certain $u_1 \in E_1$ such that

$$\lim_{t\to T_{u_0}^-} \|u(t,\tau,u_\tau) - u_1\|_{E_0} = 0 \quad \text{and} \quad u(t,\tau,u_\tau) \xrightarrow{t\to T_{u_0}^-} u_1$$

weakly in E_1. Thus, $u(t,\tau,u_\tau)$ can be extended to a function $\mathcal{U}_0$ defined on $[\tau, T_{u_0}]$ and satisfying the conditions $\mathcal{U}_0 \in L^\infty_{\mathrm{loc}}((\tau, T_{u_0}), E_{1+\varepsilon})$, $\mathcal{U}_0(\tau) = u_\tau = u_0$, $\mathcal{U}_0(t) \xrightarrow{E_0} \mathcal{U}_0(T_{u_0}) = u_1 \in \mathfrak{E}_\varepsilon^{T_{u_0}}$ as $t \to T_{u_0}^-$ and

$$\mathcal{U}_0(t) = U(t,\tau)u_0 + \int_\tau^t U(t,s)F(s,\mathcal{U}_0(s))\,\mathrm{d}s \quad \text{for } t \in (\tau, T_{u_0}).$$

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By Theorem 1.7, there exists a unique $E_{1+\varepsilon}$-solution $u(\cdot, T_{u_0}, u_1)$ of the Cauchy problem

$$\dot{u}(t) + A(t)u(t) = F(t, u(t)), \quad t > T_{u_0},\ u(T_{u_0}) = u_1,$$

which can be continued on the maximal interval of existence $[T_{u_0}, T_{u_1})$. Now, if $T_{u_1} < \infty$, repeating the above argument we find $u_2 \in E_1$ such that

$$\lim_{t \to T_{u_1}^-} \|u(t, T_{u_0}, u_1) - u_2\|_{E_0} = 0 \quad \text{and} \quad u(t, T_{u_0}, u_1) \xrightarrow{t \to T_{u_1}^-} u_2$$

weakly in E_1. Thus, $u(t, T_{u_0}, u_1)$ can be extended to a function $\mathcal{U}_1$ defined on $[T_{u_0}, T_{u_1}]$ and satisfying $\mathcal{U}_1 \in L^\infty_{\text{loc}}((T_{u_0}, T_{u_1}), E_{1+\varepsilon})$, $\mathcal{U}_1(T_{u_0}) = u_1$, $\mathcal{U}_1(t) \xrightarrow{E_0} \mathcal{U}_1(T_{u_1}) = u_2 \in \mathfrak{E}_\varepsilon^{T_{u_1}}$ as $t \to T_{u_1}^-$ and

$$\mathcal{U}_1(t) = U(t, T_{u_0})u_1 + \int_{T_{u_0}}^t U(t, s)F(s, \mathcal{U}_1(s))\,\mathrm{d}s \quad \text{for } t \in (T_{u_0}, T_{u_1}).$$

If in the $(k + 1)$th step we have $T_{u_k} = \infty$, then function $\mathcal{U}$ defined on $[\tau, \infty)$ by concatenations of $\mathcal{U}_j$, $j = 0, \dots, k + 1$, is an extension of u to a piecewise $E_{1+\varepsilon}$-solution on $[\tau, \infty)$.

Otherwise, proceeding inductively we obtain a sequence of maps $\mathcal{U}_j$ on $[\tau, T_{u_j}]$, $j = 0, 1, \dots,$ and by concatenations we define a piecewise $E_{1+\varepsilon}$-solution on $[\tau, a)$ with $a := \sum_{j=0}^{\infty} T_{u_j}$. Now either $a = \infty$ or, if $a < \infty$, a is an accumulation time of singular times $T_j := \sum_{l=0}^{j} T_{u_l}$, $j \in \mathbb{N}$.

The above construction ensures that the extension of a $E_{1+\varepsilon}$-solution to a piecewise $E_{1+\varepsilon}$-solution is uniquely defined, and hence the proof is complete. □

3. Linear non-autonomous parabolic problems

In what follows we discuss sufficient conditions for Assumptions 1.1 and 1.2 in terms of $A(t)$.

Definition 3.1. The family $\{A(t) \colon t \in \mathbb{R}\}$ of closed operators $A(t) \colon D_X \subset X \to X$, which are defined on the same dense subset D_X of the Banach space X, is *locally uniformly sectorial* (of the class $\mathcal{LUS}(D_X, X)$ for short) if and only if for each $t \in \mathbb{R}$ the complex half-plane $\{\lambda \in \mathbb{C} \colon \operatorname{Re}\lambda \leqslant 0\}$ is contained in the resolvent set $\rho(A(t))$ of $A(t)$ and for any bounded time interval $I \subset \mathbb{R}$ there exists a certain $M > 0$ such that

$$\|(\lambda I - A(t))^{-1}\|_{L(X)} \leqslant \frac{M}{1 + |\lambda|}, \quad \operatorname{Re}\lambda \leqslant 0,\ t \in I.$$

If $\{A(t) \colon t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X)$, then, for each $s \in \mathbb{R}$, $-A(s)$ generates the asymptotically decaying C^0-analytic semigroup $\{\mathrm{e}^{-A(s)t} \colon t \geqslant 0\}$ in X. Actually, for a family $\{A(t) \colon t \in \mathbb{R}\}$ of the class $\mathcal{LUS}(D_X, X)$, we have that $\operatorname{Re}\sigma(A(s)) > a > 0$ and

$$\|\mathrm{e}^{-A(t)s}\|_{L(X)} \leqslant C\mathrm{e}^{-as}, \quad s \geqslant 0, \qquad \|A(t)\mathrm{e}^{-A(t)s}\|_{L(X)} \leqslant \frac{C_1}{s}\mathrm{e}^{-as}, \quad s > 0,$$

where $a, C, C_1 > 0$ are independent of $s > 0$ and t in bounded time intervals (see [**26**, § 1.1]). Consequently, fractional powers $A^{\alpha}(t)$ can be defined as the inverse of $A^{-\alpha}(t)\colon X \to R(X)$,

$$A^{-\alpha}(t) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} s^{\alpha-1} \mathrm{e}^{-A(t)s}\,\mathrm{d}s, \quad \alpha > 0. \tag{3.1}$$

Also, one can consider the associated fractional power spaces $X^{\alpha}(t)$, $\alpha \geqslant 0$,

$$X^{\alpha}(t) := D(A^{\alpha}(t)) \quad \text{with the norm } \|\phi\|_{X^{\alpha}(t)} = \|A^{\alpha}(t)\phi\|_X \text{ for } \phi \in X^{\alpha},\ \alpha > 0,$$

where for $\alpha = 0$ we set $A^0(t) := \mathrm{Id}$, $X^0(t) := X$. As in [**26**, § 1.9, (1.56)], we then have

$$\|A^{\alpha}(t)\mathrm{e}^{-A(t)s}\|_{L(X)} \leqslant c_{\alpha}\mathrm{e}^{-as}s^{-\alpha}, \quad s > 0, \tag{3.2}$$

where c_{α} neither depends on $s > 0$ nor on t varying on bounded time intervals.

Since $A(t)$ coincides with the inverse of $A^{-1}(t)$, it follows that $X^1(t)$ coincides as a set with D_X for every $t \in \mathbb{R}$. Concerning topologies we have the following result.

Proposition 3.2. *If $\{A(t)\colon t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X)$ and $I \subset \mathbb{R}$ is such that*

$$\sup_{t,s \in I} \|A(t)A^{-1}(s)\|_{L(X)} < \infty, \tag{3.3}$$

then the $X^1(t)$ are independent of t except for norms, which are uniformly equivalent on I.

Proof. We have $\|\phi\|_{X^1(t)} = \|A(t)A^{-1}(s)A(s)\phi\|_X \leqslant c\|A(s)\phi\|_X = c\|\phi\|_{X^1(s)}$, $t, s \in I$. □

If $A(t)$ is a positive operator satisfying

$$\exists_{\epsilon>0} \sup_{s \in [-\epsilon,\epsilon]} \|A^{is}(t)\|_{L(X)} < \infty, \tag{3.4}$$

then fractional power spaces can be characterized as (see [**28**], also [**5**])

$$X^{(1-\theta)\alpha+\theta\beta}(t) = [X^{\alpha}(t), X^{\beta}(t)]_{\theta}, \quad 0 < \theta < 1,\ 0 \leqslant \alpha < \beta < \infty.$$

Remark 3.3. It is known that (3.4) holds in many applications (see [**8**, **11**, **19**, **24**, **25**, **28**]).

Definition 3.4. We will say that the family of positive operators $\{A(t)\colon t \in \mathbb{R}\}$ is of the class $\mathcal{BIP}(X)$, that is, it consists of operators possessing bounded imaginary powers, if and only if, given any $t \in \mathbb{R}$, $A(t)$ has the property (3.4).

Corollary 3.5. *If $\{A(t)\colon t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X) \cap \mathcal{BIP}(X)$ and $I \subset \mathbb{R}$ is such that (3.3) holds, then the $X^{\theta}(t)$, $\theta \in [0, 1]$, are independent of $t \in I$ except for norms, which are uniformly equivalent on I.*

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Following [**7**], given $\{A(t)\colon t \in \mathbb{R}\}$ of the class $\mathcal{LUS}(D_X, X)$, we consider the extrapolated space $X^{-1}(t)$ generated by $(X, A(t))$, where $X^{-1}(t)$ is the completion of $(X, \|A^{-1}(t)\cdot\|_X)$. We extend $A(t)$ to a closed operator in $X^{-1}(t)$ (with the same notation).

Whenever $t, s \in \mathbb{R}$ are such that $A^{-1}(s)A(t), A^{-1}(t)A(s)\colon D_X \subset X \to X$ are bounded operators (which happens, in particular, when the domains of the adjoint operators $A'(t)$ and $A'(s)$ are the same), $X^{-1}(t)$ coincides with $X^{-1}(s)$ as, for some $c_1, c_2 > 0$, we have

$$c_1\|A^{-1}(s)x\|_X \leqslant \|A^{-1}(t)x\|_X \leqslant c_2\|A^{-1}(s)x\|_X, \quad x \in X$$

(see [**6**]). This leads to the following counterpart of Proposition 3.2 for extrapolated spaces.

Proposition 3.6. *If* $\{A(t)\colon t \in \mathbb{R}\}$ *is of the class* $\mathcal{LUS}(D_X, X)$ *and*

$$\sup_{t,s\in I} \|\overline{A^{-1}(t)A(s)}\|_{L(X)} < \infty, \tag{3.5}$$

then the $X^{-1}(t)$ *are independent of* $t \in I$ *except for norms, which are uniformly equivalent on* I.

Due to [**7**, Proposition V.1.3.1], if $\{A(t)\colon t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X)$, then (the closed extension of) $A(t)$ belongs to a class $\operatorname{Lis}(X, X^{-1}(t))$ of linear isomorphisms from X into $X^{-1}(t)$. Furthermore, $\{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \leqslant 0\} \subset \rho(A(t))$ and, given any bounded time interval I,

$$\|(\lambda I - A(t))^{-1}\|_{L(X^{-1}(t))} \leqslant \frac{M}{1+|\lambda|}, \quad \operatorname{Re}\lambda \leqslant 0,\ t \in I, \tag{3.6}$$

for some $M > 0$. Letting $Y(t) = X^{-1}(t)$ and applying (3.1), one can associate with $(Y(t), A(t))$ the fractional power scale $\{Y^\alpha(t)\colon \alpha \geqslant 0\}$ and consider, as in [**7**, p. 266],

$$X^\alpha(t) := Y^{\alpha+1}(t), \quad \alpha \in [-1, \infty),$$

which is the extrapolated fractional power scale of order 1 generated by $(X, A(t))$.

Corollary 3.7. *If* $\{A(t)\colon t \in \mathbb{R}\}$ *is of the class* $\mathcal{LUS}(D_X, X) \cap \mathcal{BIP}(X)$ *and* $I \subset \mathbb{R}$ *is such that (3.3) and (3.5) hold, then for each* $\theta \in [-1, 1]$ *the spaces* $Y^{\theta+1}(t) = X^\theta(t)$ *are independent of* $t \in I$*, except for norms which are uniformly equivalent on* I*; that is, for every* $\theta \in [-1, 1]$,

$$\|\phi\|_{X^\theta(t)} \leqslant c\|\phi\|_{X^\theta(s)}, \quad s, t \in I,$$

for some $c > 0$ *and every* ϕ *from the set* $X^\theta(t) = X^\theta(s)$.

Given $t_0 \in \mathbb{R}$, $\alpha_0 \in [0, 1)$ and letting $\mu_0 := 1 - \alpha_0$, we next define

$$E_\alpha := Y^{\alpha+\alpha_0}(t_0), \quad \|\cdot\|_{E_\alpha} = \|A^{\alpha+\alpha_0}(t_0)\cdot\|_{Y(t_0)},\ \alpha \in [0, 1+\mu_0]. \tag{3.7}$$

Lemma 3.8. *Suppose that $\{A(t)\colon t\in\mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X,X)\cap\mathcal{BIP}(X)$, (3.3) and (3.5) hold on each bounded time interval I and $\{E_\alpha,\ \alpha\in[0,1+\mu_0]\}$ is as in (3.7), where $\mu_0>0$. Then*

(i) *$\{A(t)\colon t\in\mathbb{R}\}$ is of the class $\mathcal{LUS}(D_{E_0},E_0)$ with $D_{E_0}=E_1$;*

(ii) *for any bounded time interval $I\subset\mathbb{R}$ and $\sigma\in[0,1+\mu_0]$, there exist constants $c,c',c''>0$ such that for each $t,s\in I$ we have*

$$\|\phi\|_{E_\sigma}\leqslant c''\|A^\sigma(t)\phi\|_{E_0}\leqslant c\|A^\sigma(s)\phi\|_{E_0}\leqslant c'\|\phi\|_{E_\sigma},\quad \phi\in E_\sigma. \tag{3.8}$$

Proof. Recall that $\{Y^\alpha(t)\colon \alpha\geqslant 0\}$ is the fractional power scale generated by $(Y(t),A(t))$. Hence, $A(t)$ can be viewed as a closed densely defined operator in $Y^{\alpha_0}(t)$ with the domain $Y^{\alpha_0+1}(t)$. The resolvent set of $A(t)$ in this setting will still contain $\{\lambda\in\mathbb{C}\colon \operatorname{Re}\lambda\leqslant 0\}$, and for each bounded time interval I there will be a constant $M>0$ such that

$$\|(\lambda I-A(t))^{-1}\phi\|_{Y^{\alpha_0}(t)}\leqslant\frac{M}{1+|\lambda|}\|\phi\|_{Y^{\alpha_0}(t)},\quad \operatorname{Re}\lambda\leqslant 0,\ t\in I,\ \phi\in Y^{\alpha_0}(t).$$

Part (i) is thus a consequence of Corollary 3.7 and (3.6).

Concerning part (ii), we first observe that, due to Corollary 3.7, if $\phi\in E^\sigma$, then ϕ belongs to both of the sets $Y^{\sigma+\alpha_0}(t)$ and $Y^{\sigma+\alpha_0}(s)$ as these sets coincide for $t,s\in\mathbb{R}$ and $A^\sigma(t)\phi$ and $A^\sigma(s)\phi$ are the elements of $E^0=Y^{\alpha_0}(t_0)$. Actually, $A^\sigma(t)$ and $A^\sigma(s)$ are one-to-one from E_σ onto E_0.

Given a bounded time interval $I\subset\mathbb{R}$, we can thus use the equivalence of norms stated in Corollary 3.7 to obtain, for some constants $\bar c$, $\tilde c$ and $\hat c$ depending on I but not on $t,s\in I$, that

$$\begin{aligned}\|A^\sigma(t)\phi\|_{E_0}=\|A^\sigma(t)\phi\|_{Y^{\alpha_0}(t_0)}&\leqslant\bar c\|A^\sigma(t)\phi\|_{Y^{\alpha_0}(t)}=\bar c\|\phi\|_{Y^{\sigma+\alpha_0}(t)}\\&\leqslant\tilde c\|\phi\|_{Y^{\sigma+\alpha_0}(s)}=\tilde c\|A^\sigma(s)\phi\|_{Y^{\alpha_0}(s)}\\&\leqslant\hat c\|A^\sigma(s)\phi\|_{Y^{\alpha_0}(t_0)}=\hat c\|A^\sigma(s)\phi\|_{E_0}\end{aligned}$$

whenever $t,s\in I$. Similarly, using again the equivalence of norms, we also have

$$\begin{aligned}\|\phi\|_{E_\sigma}=\|\phi\|_{Y^{\sigma+\alpha_0}(t_0)}&\leqslant\tilde c\|\phi\|_{Y^{\sigma+\alpha_0}(t)}=\tilde c\|A^\sigma(t)\phi\|_{Y^{\alpha_0}(t)}\\&\leqslant\hat c\|A^\sigma(t)\phi\|_{Y^{\alpha_0}(t_0)}=\hat c\|A^\sigma(t)\phi\|_{E_0},\\\|A^\sigma(s)\phi\|_{E_0}=\|A^\sigma(s)\phi\|_{Y^{\alpha_0}(t_0)}&\leqslant\tilde c\|A^\sigma(s)\phi\|_{Y^{\alpha_0}(s)}=\tilde c\|\phi\|_{Y^{\sigma+\alpha_0}(s)}\\&\leqslant\hat c\|\phi\|_{Y^{\sigma+\alpha_0}(t_0)}=\hat c\|\phi\|_{E_\sigma},\end{aligned}$$

which proves (ii). □

Corollary 3.9. *Under the assumptions of Lemma 3.8, we have that for any bounded time interval $I\subset\mathbb{R}$ and $\sigma\in[0,1+\mu_0]$ there exists a constant $c>0$ such that*

$$\|A^\sigma(t)A^{-\sigma}(s)\|_{L(E_0)}\leqslant c\quad\textit{for each } t,s\in I.$$

Proof. It suffices to note that $A^\sigma(t)$ and $A^\sigma(s)$ are one-to-one from E_σ onto E_0 and use (3.8). □

We will next assume that $\{A(t)\colon t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X)$ and, in addition,

$$\exists_{\mu\in(0,1]}\forall_{T>0}\exists_{C>0}\forall_{t,\tau,s\in[-T,T]}\|(A(t)-A(\tau))A^{-1}(s)\|_{L(X)} \leqslant C|t-\tau|^\mu. \tag{3.9}$$

Following [**7**, **15**, **20**, **23**, **26**], we consider in X a non-autonomous linear problem

$$\dot{u}(t) + A(t)u(t) = 0, \quad t > \tau,\ u(\tau) = u_\tau. \tag{3.10}$$

Recall that a continuous function $[\tau,\infty) \ni t \to u(t) \in X$ is a *classical solution* of (3.10) if it is continuously differentiable in (τ,∞), $u(t) \in D_X$ for each $t > \tau$ and u satisfies (3.10). Recall also that a two parameter family $\{U(t,\tau)\colon (t,\tau) \in \mathbb{R}^2,\ t \geqslant \tau\}$ of maps $U(t,\tau)\colon X \to X$ is a continuous process in X provided that $U(\tau,\tau) = \mathrm{Id}$, $U(t,\sigma)U(\sigma,\tau) = U(t,\tau)$ for $t \geqslant \sigma \geqslant \tau \in \mathbb{R}$ and $\{(t,s) \in \mathbb{R}^2\colon t \geqslant s\} \times X \ni (t,\tau,v) \mapsto U(t,\tau)v \in X$ is a continuous map.

The following result is known (see [**15**, § 2] for the proof).

Proposition 3.10. *If $\{A(t)\colon t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X)$ and (3.9) holds, then there exists a continuous process $\{U(t,\tau)\colon t \geqslant \tau \in \mathbb{R}\} \subset L(X)$ in X such that, given $\tau \in \mathbb{R}$ and $u_\tau \in X$, the map $[\tau,\infty) \ni t \to u(t) = U(t,\tau)u_\tau \in X$ is a classical solution of (3.10).*

To describe smoothing properties of the process we state the following result.

Proposition 3.11. *Under the assumptions of Proposition 3.10, for each bounded time interval $I = [-T,T]$ there is a positive constant N such that, with μ as in (3.9), we have*

$$\|A^\sigma(t)U(t,\tau)A^{-\zeta}(\tau)\|_{L(X)} \leqslant N(t-\tau)^{\zeta-\sigma}, \quad 0 \leqslant \zeta \leqslant \sigma < 1+\mu,\ -T \leqslant \tau < t \leqslant T. \tag{3.11}$$

For the proof of (3.11) we refer the reader to [**26**] (see also [**15**, Theorem 2.2]). To obtain another smoothing property we will need the *additional assumption*

$$\forall_{1+\mu>\xi>0}\forall_{T>0}\exists_{c>0}\forall_{t,\tau\in[-T,T]}\|A^\xi(t)A^{-\xi}(\tau)\|_{L(X)} \leqslant c. \tag{3.12}$$

Proposition 3.12. *If $\{A(t)\colon t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X)$ and (3.9) holds, then*

$$\forall_{T>0}\forall_{\substack{1\geqslant\zeta>\sigma\geqslant0,\\ 1>\zeta-\sigma>\delta>0}}\exists_{N>0}\forall_{-T\leqslant\tau\leqslant t\leqslant T}\|A^\sigma(t)[U(t,\tau)-\mathrm{Id}]A^{-\zeta}(\tau)\|_{L(X)} \leqslant N(t-\tau)^\delta.$$

If (3.12) is also satisfied, then

$$\forall_{T>0}\forall_{\substack{1+\mu>\zeta>\sigma\geqslant0,\\ 1>\zeta-\sigma}}\exists_{N>0}\forall_{-T\leqslant\tau\leqslant t\leqslant T}\|A^\sigma(t)[U(t,\tau)-\mathrm{Id}]A^{-\zeta}(\tau)\|_{L(X)} \leqslant N(t-\tau)^{\zeta-\sigma}. \tag{3.13}$$

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Proof. From [**26**, (1.53)] we infer that

$$U(t,\tau)A^{-\zeta}(\tau) = \mathrm{e}^{(t-\tau)A(t)}A^{-\zeta}(\tau) + \int_\tau^t \mathrm{e}^{(t-s)A(t)}[A(s)-A(t)]U(s,\tau)A^{-\zeta}(\tau)\,\mathrm{d}s.$$

We next rewrite $A^\sigma(t)[U(t,\tau)-\mathrm{Id}]A^{-\zeta}(\tau)$ as a sum J_1+J_2, where

$$J_1 = A^\sigma(t)[\mathrm{e}^{(t-\tau)A(t)} - \mathrm{Id}]A^{-\zeta}(\tau)$$

and

$$J_2 = \int_\tau^t A^\sigma(t)\mathrm{e}^{(t-s)A(t)}[A(s)-A(t)]U(s,\tau)A^{-\zeta}(\tau)\,\mathrm{d}s.$$

Due to (3.9), (3.3) holds on any bounded time interval I and, assuming that $\{A(t)\colon t\in\mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X,X)$ and (3.9) holds, one obtains as in [**26**, § 1.9, (1.59)] that

$$\forall_{1\geqslant\zeta>\xi\geqslant 0}\forall_{T>0}\exists_{c>0}\forall_{t,\tau\in[-T,T]}\|A^\xi(t)A^{-\zeta}(\tau)\|_{L(X)}\leqslant c. \tag{3.14}$$

If $1\geqslant\zeta>\sigma\geqslant 0$ and $0<\delta<\zeta-\sigma<1$, then with [**21**, Theorem 1.4.3] we can estimate $\|J_1v\|_X$ by $(1/\delta)c_{1-\delta}(t-\tau)^\delta\|A^{\delta+\sigma}(t)A^{-\zeta}(\tau)v\|_X$, which, via (3.14), can be bounded on $[-T,T]$ by $(1/\delta)cc_{1-\delta}(t-\tau)^\delta\|v\|_X$.

If $1+\mu>\zeta>\sigma\geqslant 0$ and (3.12) holds, then, choosing $\tilde\delta=\zeta-\sigma$ and using [**21**, Theorem 1.4.3], we estimate $\|J_1v\|_X$ by

$$\frac{1}{\tilde\delta}c_{1-\tilde\delta}(t-\tau)^{\tilde\delta}\|A^{\tilde\delta+\sigma}(t)A^{-\zeta}(\tau)v\|_X = \frac{1}{\zeta-\sigma}c_{1-\zeta+\sigma}(t-\tau)^{\zeta-\sigma}\|A^\zeta(t)A^{-\zeta}(\tau)v\|_X,$$

which, via (3.12), is bounded on $[-T,T]$ by $(1/(\zeta-\sigma))cc_{1-\zeta+\sigma}(t-\tau)^{\zeta-\sigma}\|v\|_X$.

Consequently, by not assuming (3.12) we obtain that $\|J_1\|_{L(X)}\leqslant(1/\delta)cc_{1-\delta}(t-\tau)^\delta$, $0<\delta<\zeta-\sigma$, whereas by assuming (3.12) we obtain $\|J_1\|_{L(X)}\leqslant(1/(\zeta-\sigma))cc_{1-\zeta+\sigma}(t-\tau)^{\zeta-\sigma}$.

The integral J_2 is equal to

$$\int_\tau^t A^\sigma(t)\mathrm{e}^{(t-s)A(t)}[(A(s)-A(t))A^{-1}(s)]A(s)U(s,\tau)A^{-\zeta}(\tau)\,\mathrm{d}s,$$

where by (3.2) and (3.9) we have

$$\|A^\sigma(t)\mathrm{e}^{(t-s)A(t)}[(A(s)-A(t))A^{-1}(s)]\|_{L(X)}\leqslant c(t-s)^{-\sigma}(t-s)^\mu.$$

Note that if $0\leqslant\zeta\leqslant 1$, we obtain from (3.11) that $\|A(s)U(s,\tau)A^{-\zeta}(\tau)\|_{L(X)}\leqslant c(s-\tau)^{\zeta-1}$, whereas if $1+\mu>\zeta>1$, then $A(s)U(s,\tau)A^{-\zeta}(\tau)=A(s)U(s,\tau)A^{-1}(\tau)A^{1-\zeta}(\tau)$ and

$$\|A(s)U(s,\tau)A^{-\zeta}(\tau)\|_{L(X)}\leqslant\|A(s)U(s,\tau)A^{-1}(\tau)\|_{L(X)}\|A^{1-\zeta}(\tau)\|_{L(X)}\leqslant c,$$

as in this case

$$A^{1-\zeta}(\tau)=A^{-(\zeta-1)}(\tau)=\frac{1}{\Gamma(\zeta-1)}\int_0^\infty s^{\zeta-2}\mathrm{e}^{-A(\tau)s}\,\mathrm{d}s$$

is a bounded operator and

$$\|A^{1-\zeta}(\tau)\|_{L(X)} \leqslant \frac{C}{\Gamma(\zeta-1)} \int_0^\infty s^{\zeta-2} \mathrm{e}^{-as} \, \mathrm{d}s = Ca^{1-\zeta}.$$

Since t and τ vary in a bounded interval, we thus infer that for $0 \leqslant \zeta \leqslant 1$,

$$\begin{aligned} \|J_2\|_{L(X)} &\leqslant \tilde{c} \int_\tau^t (t-s)^{\mu-\sigma}(s-\tau)^{\zeta-1} \, \mathrm{d}s \\ &\leqslant \tilde{c}(t-\tau)^{\mu-\sigma+\zeta} B(1+\mu-\sigma, \zeta) \\ &= \tilde{c} B(1+\mu-\sigma,\zeta)(t-\tau)^{\mu}(t-\tau)^{\zeta-\sigma} \\ &\leqslant \bar{c}(t-\tau)^{\zeta-\sigma}, \end{aligned}$$

whereas for $1+\mu > \zeta > 1$,

$$\|J_2\|_{L(X^\alpha)} \leqslant \tilde{c} \int_\tau^t (t-s)^{\mu-\sigma} \, \mathrm{d}s = \hat{c}(t-\tau)^{1+\mu-\sigma} = \hat{c}(t-\tau)^{1+\mu-\zeta}(t-\tau)^{\zeta-\sigma} \leqslant \bar{c}(t-\tau)^{\zeta-\sigma}.$$

Combining the above estimates, we obtain the result. □

Theorem 3.13. *Suppose that $\{A(t)\colon t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X) \cap \mathcal{BIP}(X)$, conditions (3.3) and (3.5) hold on each bounded time interval $I \subset \mathbb{R}$, and $\{E_\alpha,\ \alpha \in [0, 1+\mu_0]\}$ is defined as in (3.7). Suppose furthermore that*

$$\exists_{\mu\in(0,\mu_0]} \forall_{T>0} \exists_{C>0} \forall_{t,\tau,s\in[-T,T]} \|(A(t)-A(\tau))A^{-1}(s)\|_{L(E^0)} \leqslant C|t-\tau|^{\mu}. \tag{3.15}$$

Under these assumptions the following hold.

(i) *There exists a continuous process $\{U(t,\tau)\colon t \geqslant \tau \in \mathbb{R}\} \subset L(E_0)$ defined by (3.10) in E_0 such that, given $\tau \in \mathbb{R}$ and $u_\tau \in E_0$, the map $[\tau,\infty) \ni t \to u(t) = U(t,\tau)u_\tau \in E^0$ is a classical solution of (3.10).*

(ii) $\|U(t,\tau)\|_{L(E_\zeta,E_\sigma)} \leqslant M(t-\tau)^{\zeta-\sigma}$, $0 \leqslant \zeta \leqslant \sigma < 1+\mu$, $-T \leqslant \tau < t \leqslant T$.

(iii) $\|U(t,\tau) - \mathrm{Id}\|_{L(E_\zeta,E_\sigma)} \leqslant M(t-\tau)^{\zeta-\sigma}$, $1+\mu > \zeta > \sigma \geqslant 0$, $1 \geqslant \zeta-\sigma > 0$, $-T \leqslant \tau < t \leqslant T$,

where the constant M in (ii) and (iii) can depend on ζ, σ and T but does not depend on $t, \tau \in [-T,T]$.

Proof. From Lemma 3.8 we obtain that $\{A(t)\colon t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_{E_0}, E_0)$ with $D_{E_0} = E_1$. From this and (3.15) we obtain (i) by applying Proposition 3.10 with $X = E_0$.

Due to Proposition 3.11, for each bounded time interval I there exists $N > 0$ such that

$$\|A^\sigma(t)U(t,\tau)A^{-\zeta}(\tau)\|_{L(E_0)} \leqslant N(t-\tau)^{\zeta-\sigma}, \quad 0 \leqslant \zeta \leqslant \sigma < 1+\mu,\ t \in I. \tag{3.16}$$

Since $A^\zeta(\tau)$ is one-to-one from E_σ onto E_0, (3.16) can be rewritten equivalently as

$$\|A^\sigma(t)U(t,\tau)\phi\|_{E_0} \leqslant N(t-\tau)^{\zeta-\sigma}\|A^\zeta(\tau)\phi\|_{E_0}, \quad \phi \in E_\zeta,$$

and by (3.8) also as $\|U(t,\tau)\phi\|_{E_\sigma} \leqslant M(t-\tau)^{\zeta-\sigma}\|\phi\|_{E_\zeta}$, $\phi \in E_\zeta$, which gives (ii).

Finally, by Corollary 3.9 we can use Proposition 3.12 with $X = E_0$ and obtain from (3.13),

$$\forall_{T>0}\forall_{\substack{1+\mu>\zeta>\sigma\geqslant 0,\\ 1>\zeta-\sigma}}\exists_{N>0}\forall_{-T\leqslant\tau\leqslant t\leqslant T}\|A^{\sigma}(t)[U(t,\tau)-\mathrm{Id}]A^{-\zeta}(\tau)\|_{L(E_0)}\leqslant N(t-\tau)^{\zeta-\sigma}. \tag{3.17}$$

Inequality (3.17) can be rewritten equivalently as

$$\|A^{\sigma}(t)[U(t,\tau)-\mathrm{Id}]\phi\|_{E_0}\leqslant N(t-\tau)^{\zeta-\sigma}\|A^{\zeta}(\tau)\phi\|_{E_0},\quad \phi\in E_{\zeta},$$

and by (3.8) also as $\|[U(t,\tau)-\mathrm{Id}]\phi\|_{E_\sigma}\leqslant M(t-\tau)^{\zeta-\sigma}\|\phi\|_{E_\zeta}$, $\phi\in E_\zeta$, which gives (iii). □

An equivalent form of (3.15) is expressed in the following proposition.

Proposition 3.14. *Suppose that $\{A(t)\colon t\in\mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X,X)\cap\mathcal{BIP}(X)$, (3.3) and (3.5) hold on each bounded time interval $I\subset\mathbb{R}$ and $\{E_\alpha,\ \alpha\in[0,1+\mu_0]\}$ is as in (3.7).*

Then (3.15) is equivalent to

$$A(\cdot)\in C^{\mu}_{\mathrm{loc}}(\mathbb{R},L(E_1,E_0)).$$

Proof. For any bounded time interval $I\subset\mathbb{R}$, (3.15) implies that

$$\|(A(t)-A(\tau))\phi\|_{E_0}\leqslant C|t-\tau|^{\mu}\|A(s)\phi\|_{E_0}\quad\text{and}\quad\|A(t)\phi\|_{E_0}\leqslant c\|A(s)\phi\|_{E_0}$$

whenever $t,\tau,s\in I$ and $\phi\in D_{E_0}$. Due to (3.8), we then have $\|(A(t)-A(\tau))\phi\|_{E_0}\leqslant\tilde{C}|t-\tau|^{\mu}\|\phi\|_{E_1}$ for $t,\tau\in I$, which proves that $A(\cdot)\in C^{\mu}(I,L(E_1,E_0))$. On the other hand, if $A(\cdot)\in C^{\mu}_{\mathrm{loc}}(I,L(E_1,E_0))$, then, given a bounded time interval I, we have that $A(\cdot)\in C^{\mu}(I,L(E_1,E_0))$. Combining this with (3.8), we obtain $\|(A(t)-A(\tau))\psi\|_{E_0}\leqslant C|t-\tau|^{\mu}\|\psi\|_{E_1}\leqslant\tilde{C}|t-\tau|^{\mu}\|A(s)\psi\|_{E_0}$, $t,\tau,s\in I$, $\psi\in E_1$. Letting $\phi=A^{-1}(s)\psi$ we obtain (3.15). □

Under the assumptions of Theorem 3.13, both Theorems 1.6 and 1.7 apply provided that the required assumption on F holds. In applications we often have some $\nu_0\in(0,1)$ such that for each bounded time interval $I\subset\mathbb{R}$ and B bounded in $E_{1+\varepsilon}$ there is a $c>0$ such that

$$\|F(t,v)-F(s,w)\|_{E_0}\leqslant c(|t-s|^{\nu_0}+\|v-w\|_{E_{1+\varepsilon}}),\quad t,s\in I,\ v,w\in B. \tag{3.18}$$

Then the $E_{1+\varepsilon}$-solution will have the properties of a classical solution; see Proposition 3.15.

Proposition 3.15. *Suppose that $\{A(t)\colon t\in\mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X,X)\cap\mathcal{BIP}(X)$ and that (3.3) and (3.5) hold on any bounded time interval I. Assume also that (3.15) and (3.18) hold and that F is of the class $\mathcal{L}(\varepsilon,\rho,\gamma(\varepsilon),\eta,C_\eta)$ relative to $\{E_\alpha,\ \alpha\in[0,1+\mu)\}$, where the E_α are as in (3.7).*

Then $\{A(t)\colon t\in\mathbb{R}\}$ is of the class $\mathcal{LUS}(D_{E_0},E_0)$ with $D_{E_0}=E_1$ and Theorems 1.6 and 1.7 apply. The unique $E_{1+\varepsilon}$-solution, $u=u(\cdot,\tau,u_\tau)$, is of the class $C^1((\tau,\tau+\delta_0],E_0)$, $u(t)\in D_{E_0}$ for $t\in(\tau,\tau+\delta_0]$ and $\dot{u}(t)+A(t)u(t)=F(t,u(t))$ for each $t\in(\tau,\tau+\delta_0]$.

Proof. By Theorem 3.13, we know that Theorems 1.7 and 1.6 apply. Hence, there is a unique $E_{1+\varepsilon}$-solution of (1.1), $u = u(\cdot, \tau, u_\tau)$ and $u \in C^{\nu}_{\mathrm{loc}}((0, T_{u_\tau}), E_{1+\varepsilon})$ for some $\nu \in (0,1)$. The last property and (3.18) yield that $F(\cdot, u(\cdot)) \in C^{\sigma}_{\mathrm{loc}}((0, T_{u_\tau}), E_0)$ for $\sigma = \min\{\nu_0, \nu\}$. The result now follows as in [**23**, § 5.7, Theorem 7.1] and [**15**, § 2.3]. □

Remark 3.16. Under the assumptions of Proposition 3.15, following [**30**, Theorem 3.10] and letting $\mathcal{F}^{1,\sigma}((\tau, \tau + \delta_0], E_0)$ as in [**30**, p. 5], we have for the $E_{1+\varepsilon}$-solution $u = u(\cdot, \tau, u_\tau)$ of (1.1),

$$A(\cdot)u(\cdot) \in C((\tau, \tau + \delta_0], E_0), \qquad \frac{\mathrm{d}}{\mathrm{d}t} u(\cdot) \in \mathcal{F}^{1,\sigma}((\tau, \tau + \delta_0], E_0).$$

4. Applications

In what follows we show how the abstract results apply in sample problems.

4.1. Non-autonomous wave equation with structural damping

In this example, following [**11–13**, **16**], we consider the initial boundary-value problem of the form:

$$\left.\begin{aligned} &u_{tt} + \eta(t)(-\Delta)^{1/2} u_t + \nu u_t + (-\Delta)u = f(t,u), \quad t > \tau,\ x \in \Omega, \\ &u(\tau, x) = u_\tau(x),\ x \in \Omega, \quad u_t(\tau, x) = v_\tau(x),\ x \in \Omega, \quad u_{|\partial\Omega} = 0, \end{aligned}\right\} \tag{4.1}$$

where $(u_\tau, v_\tau) \in H^1_0(\Omega) \times L^2(\Omega)$ and $-\Delta$ is the negative Dirichlet Laplacian in $L^2(\mathbb{R}^N)$.

Assumption 4.1. *Suppose that Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 3$, $\nu \geqslant 0$ and*

$$\eta \in C^{\mu}_{\mathrm{loc}}(\mathbb{R}, (0, \infty)) \quad \textit{for some } \mu \in (0, 1]. \tag{4.2}$$

We remark that due to (4.2), given any bounded time interval $I \subset \mathbb{R}$, there are constants $\kappa_1, \kappa_2 > 0$ such that $\eta(t) \in [\kappa_1, \kappa_2]$ for each $t \in I$. Letting $v = \dot{u}$, we rewrite (4.1) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ v \end{bmatrix} + A(t)\begin{bmatrix} u \\ v \end{bmatrix} = F\left(t, \begin{bmatrix} u \\ v \end{bmatrix}\right), \quad t > \tau, \quad \begin{bmatrix} u \\ v \end{bmatrix}_{t=\tau} = \begin{bmatrix} u_\tau \\ v_\tau \end{bmatrix}, \tag{4.3}$$

where $A(t)$ and $F(t, \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right])$ can be viewed in matrix form as

$$A(t) = \begin{bmatrix} 0 & -I \\ -\Delta & \eta(t)(-\Delta)^{1/2} + \nu I \end{bmatrix}, \qquad F\left(t, \begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} 0 \\ f^e(t, u) \end{bmatrix} \tag{4.4}$$

and f^e denotes a Nemitskiĭ operator associated with f.

In this example we set $X = H^1_0(\Omega) \times L^2(\Omega)$, $D_X = (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)$ and, referring to [**14**, proof of Lemma 1 (iii)] and [**11**, Proposition 1], we conclude that

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$\{A(t)\colon t\in\mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X,X)\cap\mathcal{BIP}(X)$. Furthermore, for any $t,s\in\mathbb{R}$ we obtain

$$A(t)A^{-1}(s)=\begin{bmatrix} I & 0\\ (\eta(s)-\eta(t))(-\Delta)^{1/2} & I\end{bmatrix},$$
$$A^{-1}(s)A(t)=\begin{bmatrix} I & (\eta(s)-\eta(t))(-\Delta)^{-1/2}\\ 0 & I\end{bmatrix}.$$

Hence, for any bounded time interval I, we obtain the counterparts of (3.3) and (3.5):

$$\sup_{t,s\in I}\|A(t)A^{-1}(s)\|_{L(X)}=\sup_{t,s\in I}\sup_{\|\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_X=1}\left\|\begin{bmatrix}\phi\\ (\eta(s)-\eta(t))(-\Delta)^{1/2}\phi+\psi\end{bmatrix}\right\|_X\leqslant(1+2\kappa_2),$$
$$\sup_{t,s\in I}\|\overline{A^{-1}(s)A(t)}\|_{L(X)}=\sup_{t,s\in I}\sup_{\|\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_X=1}\left\|\begin{bmatrix}\phi+(\eta(s)-\eta(t))(-\Delta)^{-1/2}\psi\\ \psi\end{bmatrix}\right\|_X\leqslant(1+2\kappa_2).$$

Let $\{Z^\alpha,\ \alpha\geqslant-1\}$ be the extrapolated fractional power scale generated by $(L^2(\Omega),-\Delta)$. As in (3.7), choosing $\alpha_0=0$ we define the spaces E_α, $\alpha\in[0,2]$. Due to [**11**, Theorem 2]:

$$E_\alpha:=Y^{\alpha+\alpha_0}(t_0)=Z^{\alpha/2}\times Z^{(\alpha-1)/2},\quad \alpha\in[0,2].$$

By [**7**], $Z^{-\alpha}(t)$, $\alpha\in(0,1)$, is viewed as the completion of $(L^2(\Omega),\|(-\Delta)^{-\alpha}\cdot\|_{L^2(\Omega)})$.

With this set-up, we observe that

$$A(\cdot)\in C^\mu_{\mathrm{loc}}(\mathbb{R},L(E_1,E_0))\quad\text{with } E_1=H^1_0(\Omega)\times L^2(\Omega)\text{ and }E_0=L^2(\Omega)\times H^{-1}(\Omega),\tag{4.5}$$

where μ is as in (4.2) because, given $t,s\in[-T,T]$, we have

$$\sup_{\|\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_{E_1}=1}\left\|[A(t)-A(s)]\begin{bmatrix}\phi\\ \psi\end{bmatrix}\right\|_{E_0}=|\eta(t)-\eta(s)|\sup_{\|\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_{E_1}=1}\|(-\Delta)^{1/2}\psi\|_{Z^{-1/2}}\leqslant c|t-s|^\mu.$$

Due to Proposition 3.14, (4.5) is equivalent to (3.15) and we can apply Theorem 3.13.

Proposition 4.2. *Suppose that Assumption 4.1 holds and let $E_\alpha=Z^{\alpha/2}\times Z^{(\alpha-1)/2}$ for $\alpha\in[0,1+\mu)$, where μ is as in (4.2).*

There then exists a continuous process $\{U(t,\tau)\colon (t,\tau)\in\mathbb{R}^2,\ t\geqslant\tau\in\mathbb{R}\}\subset L(E_0)$ in $E_0=L^2(\Omega)\times H^{-1}_0(\Omega)$ associated with

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}u\\ v\end{bmatrix}+\begin{bmatrix}0 & -I\\ -\Delta & \eta(t)(-\Delta)^{1/2}+\nu I\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix}=0,\quad t>0,\quad \begin{bmatrix}u\\ v\end{bmatrix}_{t=\tau}=\begin{bmatrix}u_\tau\\ v_\tau\end{bmatrix},$$

and $\{U(t,\tau)\colon (t,\tau)\in\mathbb{R}^2,\ t\geqslant\tau\in\mathbb{R}\}$ enjoys the smoothing properties (1.2) and (1.3).

Remark 4.3. Besides (4.5) we also have that $A(\cdot) \in C^{\mu}_{\mathrm{loc}}(\mathbb{R}, L(E_2, E_1))$ with $E_2 = H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$ and $E_1 = H^1_0(\Omega) \times L^2(\Omega)$ as whenever $t, s \in [-T, T]$, (4.2) yields

$$\sup_{\left\|\left[\begin{smallmatrix}\phi\\\psi\end{smallmatrix}\right]\right\|_{E_2}=1} \left\| [A(t) - A(s)] \begin{bmatrix}\phi\\\psi\end{bmatrix} \right\|_{E_1} = |\eta(t) - \eta(s)| \sup_{\left\|\left[\begin{smallmatrix}\phi\\\psi\end{smallmatrix}\right]\right\|_{E_2}=1} \|(-\Delta)^{1/2}\psi\|_{Z^0} \leqslant c|t-s|^{\mu}.$$

Assuming that $N \geqslant 3$ we now define

$$\rho_{\mathrm{c}} = \frac{N+2}{N-2},$$

which in this example plays the role of a critical exponent for initial data in $H^1_0(\Omega) \times L^2(\Omega)$.

Remark 4.4. To keep the notation short, we adapt the Landau symbols $O(\varphi)$ and $o(\varphi)$. We will write $h(t,x,s) = O(\varphi(s))$ if, given a bounded time interval I, $|h(t,x,s)| \leqslant c|\varphi(s)|$ for some $c > 0$ independent of $s \in \mathbb{R}$, $x \in \Omega$ and $t \in I$. We will write $h(t,x,s) = o(\varphi(s))$ if, given a bounded time interval I, $\lim_{|s|\to\infty} |h(t,x,s)|/\varphi(s) = 0$ uniformly with respect to $x \in \Omega$ and $t \in I$.

Proposition 4.5. *Suppose that $N \geqslant 3$, $f \in C(\mathbb{R}^2, \mathbb{R})$ has partial derivative $f'_u \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $E_\alpha = Z^{\alpha/2} \times Z^{(\alpha-1)/2}$ for $\alpha \in [0, 1+\mu)$, where μ is as in (4.2).*

(i) *If $f'_s(t,s) = O(c_\eta + \eta|s|^{\rho-1})$ for some $\eta > 0$ and $\rho \in (1, \rho_{\mathrm{c}})$, then the map $F(t, \left[\begin{smallmatrix}u\\v\end{smallmatrix}\right])$ in (4.4) is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha,\ \alpha \in [0, 1+\mu)\}$ and is subcritical.*

(ii) *If $f'_s(t,s) = O(c_\eta + \eta|s|^{\rho_{\mathrm{c}}-1})$ for some $\eta > 0$ and (i) does not apply, then the map $F(t, \left[\begin{smallmatrix}u\\v\end{smallmatrix}\right])$ in (4.4) is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha,\ \alpha \in [0, 1+\mu)\}$ and is critical.*

(iii) *If $f'_s(t,s) = o(|s|^{\rho_{\mathrm{c}}-1})$ and (i) does not apply, then $F(t, \left[\begin{smallmatrix}u\\v\end{smallmatrix}\right])$ in (4.4) is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha,\ \alpha \in [0, 1+\mu)\}$ and is almost critical.*

Proof. Parts (i) and (ii) follow in a similar manner to that found in [**11**, Lemma 3 and Corollary 2]. Also, (iii) can be proved analogously to [**13**, Lemma 3.1 and Corollary 3.1]. We thus omit the details. □

Corollary 4.6. *Suppose that Assumption 4.1 holds and let $E_\alpha = Z^{\alpha/2} \times Z^{(\alpha-1)/2}$ for $\alpha \in [0, 1+\mu)$, where μ is as in (4.2). Suppose also that the assumptions of Proposition 4.5 are satisfied; in particular, $f'_s(t,s) = O(c_\eta + \eta|s|^{\rho_{\mathrm{c}}-1})$ for some $\eta > 0$.*

Then Theorem 1.7 applies and, given $\tau \in \mathbb{R}$, $\left[\begin{smallmatrix}u_\tau\\v_\tau\end{smallmatrix}\right] \in H^1_0(\Omega) \times L^2(\Omega)$, (4.1) has a unique $E_{1+\varepsilon}$-solution $\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right] = \left[\begin{smallmatrix}u\\v\end{smallmatrix}\right](\cdot, \tau, \left[\begin{smallmatrix}u_\tau\\v_\tau\end{smallmatrix}\right])$ defined on the maximal interval of existence $[\tau, T_{u_\tau, v_\tau})$.

With an assumption on f as in Lemma 4.7, there will be a functional decreasing along $\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right](t, \tau, \left[\begin{smallmatrix}u_\tau\\v_\tau\end{smallmatrix}\right])$,

$$\mathcal{L}\left(\begin{bmatrix}w_1\\w_2\end{bmatrix}\right) = \tfrac{1}{2}\|w_2\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|(-\Delta)^{1/2}w_1\|^2_{L^2(\Omega)} - \int_\Omega \int_0^{w_1} f(s)\,\mathrm{d}s\,\mathrm{d}x, \quad \begin{bmatrix}w_1\\w_2\end{bmatrix} \in E_1. \tag{4.6}$$

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Lemma 4.7. *If f does not depend on t, that is, $f = f(u)$, then $\mathcal{L}$ in (4.6) takes bounded subsets of E_1 into bounded subsets of $\mathbb{R}$ and, given the $E_{1+\varepsilon}$-solution $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ of (4.3) on the interval I_τ, $\mathcal{L}(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right])$ is non-increasing for $t \in I_\tau$.*

If λ_1 denotes the first positive eigenvalue of the negative Dirichlet Laplacian in $L^2(\Omega)$ and

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < \lambda_1, \tag{4.7}$$

then $\mathcal{L}$ is also bounded from below and, for some constants $d_1, d_2 > 0$,

$$\left\| \begin{bmatrix} u \\ v \end{bmatrix} \left(t, \tau, \begin{bmatrix} u_\tau \\ v_\tau \end{bmatrix} \right) \right\|_{E_1} \leqslant d_1 \mathcal{L}\left(\begin{bmatrix} u_\tau \\ v_\tau \end{bmatrix} \right) + d_2. \tag{4.8}$$

Proof. Multiplying the first equation in (4.1) by $v = u_t$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{L}\left(\begin{bmatrix} u \\ v \end{bmatrix} \right) \right) = -\eta(t)\|(-\Delta)^{1/4} v\|^2_{L^2(\Omega)} - \nu \|v\|^2_{L^2(\Omega)} \leqslant 0,$$

which yields that $\mathcal{L}(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]) \leqslant \mathcal{L}(\left[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}\right])$ as long as the solution exists. On the other hand, using (4.7) we obtain that $-\int_\Omega \int_0^{w_1} f(s)\,\mathrm{d}s\,\mathrm{d}x$ is bounded from below by $-((\lambda_1 - \delta)/2)\|w_1\|^2_{L^2(\Omega)} - N_\delta |\Omega|$ for some $N_\delta > 0$ and $\delta > 0$ small enough. Consequently,

$$\mathcal{L}\left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right) \geqslant \frac{\delta}{2\lambda_1} \|(-\Delta)^{1/2} w_1\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|w_2\|^2_{L^2(\Omega)} - N_\delta |\Omega|, \quad \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in E_1,$$

and the result follows easily. This proves (4.8) for smooth solutions, that is, for solutions with smooth initial data that can be obtained with [**26**, Theorem 7] due to Remark 4.3. With $(1.11)_{\theta=0}$ (see Remark 1.9 (iii)) it then extends to $E_{1+\varepsilon}$-solutions and the proof is complete. □

Theorem 1.10 now leads to the following conclusion.

Corollary 4.8. *Suppose that Assumption 4.1 holds and assume that $f \in C^1(\mathbb{R}, \mathbb{R})$ does not depend on a time variable, (4.7) is satisfied and $f_s'(s) = o(|s|^{\rho_c - 1})$. Then, given $\tau \in \mathbb{R}$ and $\left[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}\right] \in E_1 = H^1_0(\Omega) \times L^2(\Omega)$, there exists a unique global $E_{1+\varepsilon}$-solution of (4.1).*

Suppose finally that we have $f_s'(s) = O(1 + |s|^{\rho_c - 1})$ but not $f_s'(s) = o(|s|^{\rho_c - 1})$. Note that (1.15) is rather difficult to verify, as an $E_{1+\varepsilon}$-estimate remains unknown. Nonetheless, since we know (4.8) and, in addition,

$$(-\Delta)^{-1/2}\dot{v} + \eta(t) v + \nu(-\Delta)^{-1/2} v + (-\Delta)^{1/2} u = (-\Delta)^{-1/2} f(u),$$

we infer that $u \in W^{1,1}((0, T_{u_\tau, v_\tau}), L^2(\Omega))$, $\dot{u} \in W^{1,1}((0, T_{u_\tau, v_\tau}), H^{-1}(\Omega))$ whenever $T_{u_\tau, v_\tau} < \infty$ and the map

$$[0, T_{u_\tau, v_\tau}) \ni t \to \begin{bmatrix} u(t, u_\tau, v_\tau) \\ v(t, u_\tau, v_\tau) \end{bmatrix} \in E_0 = L^2(\Omega) \times H^{-1}(\Omega)$$

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is uniformly continuous (see [**10**, Theorem I.2.2]). Thus, (1.22) and (1.23) hold and Theorem 1.12 applies.

Corollary 4.9. *Suppose that Assumption 4.1 holds, $f \in C^1(\mathbb{R},\mathbb{R})$ does not depend on a time variable, $f'_s(s) = O(1 + |s|^{\rho_c - 1})$ and (4.7) is satisfied.*

Whenever $\tau \in \mathbb{R}$ and $\left[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}\right] \in E_1$ are such that T_{u_τ,v_τ} is finite, there exist an $a \in (T_{u_\tau,v_\tau}, \infty]$ and an extension $\mathcal{U}\colon [\tau, a) \to E_1$ of a maximally defined $E_{1+\varepsilon}$-solution of (4.1) such that $\mathcal{U}$ is a piecewise $E_{1+\varepsilon}$-solution on $[\tau, a)$ and $a = \infty$ or a is an accumulation time of singular times.

4.2. Non-autonomous parabolic problems

Given

$$\mathcal{A}(t) = (-1)^m \sum_{|\sigma| \leqslant 2m} a_\sigma(t,x) D^\sigma, \quad t \in \mathbb{R},\ x \in \Omega, \tag{4.9}$$

$$B_j = \sum_{|\sigma| \leqslant m_j} b^j_\sigma(x) D^\sigma, \quad \text{where } j = 1,\dots,m, m_j \in \{0,1,\dots,2m-1\},\ x \in \partial\Omega,$$

and, adapting the notion of a regular parabolic initial boundary-value problem, we say that $\{(\mathcal{A}(t), \{B_j\}, \Omega, \partial\Omega),\ t \in \mathbb{R}\}$ is of the class $\mathcal{RPIBVP}$ of regular parabolic initial boundary-value problems of order $2m$ if $(\mathcal{A}(t), \{B_j\}, \Omega, \partial\Omega, \alpha)$ is a strongly α-regular elliptic boundary-value problem of class C^0 and order $2m$ for every $t \in \mathbb{R}$ as in [**3**, p. 655] and, in addition,

there exists $\mu \in (0,1]$ such that for each bounded time interval $I \subset \mathbb{R}$ and for any $|\sigma| \leqslant 2m$ the map $I \ni t \to a_\sigma(t,\cdot) \in C(\bar{\Omega},\mathbb{R})$ is of the class $C^\mu(I, C(\bar{\Omega},\mathbb{R}))$; in addition, whenever $|\sigma| = 2m$, the modulus of continuity of the maps $\bar{\Omega} \ni x \to a_\sigma(t,x) \in \mathbb{R}$ can be chosen uniformly for $t \in I$. (4.10)

We will next consider spaces $H^s_p(\Omega)$ as in [**28**]. For $p = 2$ they are Hilbert spaces and will be denoted by $H^s(\Omega)$. Following [**28**] we also define

$$H^s_{p,\{B_j\}}(\Omega) = \{\phi \in H^s_p(\Omega)\colon \forall_{i \in \{j\colon m_j < s - 1/p\}} B_i \phi_{|\partial\Omega} = 0\}.$$

Assuming that $\{(\mathcal{A}(t), \{B_j\}, \Omega, \partial\Omega),\ t \in \mathbb{R}\}$ is of the class $\mathcal{RPIBVP}$, we have the estimate

$$\|\varphi\|_{H^{2m}_p(\Omega)} \leqslant c^*(\|\mathcal{A}(t)\varphi\|_{L^p(\Omega)} + \|\varphi\|_{L^p(\Omega)}), \quad \varphi \in H^{2m}_{p,\{B_j\}}(\Omega),\ t \in I, \tag{4.11}$$

where $I \subset \mathbb{R}$ is an arbitrarily chosen bounded time interval. We emphasize that $c^* > 0$ actually depends on Ω, m, N, p, α, the moduli of continuity of the top order coefficients of $\mathcal{A}(t)$ with $t \in I$, the coefficients of boundary operators B_j and certain constants related to the notion of α-regular elliptic boundary-value problems, which are specified in [**3**, Theorems 12.1] (see also [**1**, **2**]). Thus, the constant c^* in (4.11) is independent of t in a bounded time interval $I \subset \mathbb{R}$.

We remark that, due to (4.9), (4.10) and properties of the $H_p^{2m}(\Omega)$-norm, we also have

$$\|\mathcal{A}(t)\varphi\|_{L^p(\Omega)} \leqslant c_*\|\varphi\|_{H_p^{2m}(\Omega)}, \quad \varphi \in H_p^{2m}(\Omega),\ t \in I, \tag{4.12}$$

where c_* depends on Ω, m and $L^\infty(I, C(\bar{\Omega}, \mathbb{R}))$-norms of coefficients of $A(t)$.

With the above set-up, we consider the $2m$th-order problem

$$\left.\begin{aligned} u_t + (-1)^m \sum_{|\xi|,|\zeta|\leqslant m} D^\zeta(a_{\xi,\zeta}(t,x)D^\xi u) = f(t,x,u), \quad t > \tau,\ x \in \Omega \subset \mathbb{R}^N,\\ B_0 u = \cdots = B_{m-1}u = 0 \quad \text{on } \partial\Omega, \quad u(\tau, x) = u_\tau(x),\ x \in \Omega. \end{aligned}\right\} \tag{4.13}$$

Letting

$$A(t)u = (-1)^m \sum_{|\xi|,|\zeta|\leqslant m} D^\zeta(a_{\xi,\zeta}(t,x)D^\xi u),$$

we summarize the conditions on (4.13).

Assumption 4.10. *$N > 2m$, $\Omega \subset \mathbb{R}^N$ is a bounded C^{2m}-domain, the coefficients $a_{\xi,\zeta}(t,\cdot) \in C^m(\bar{\Omega}, \mathbb{R})$ $(|\xi| \leqslant m,\ |\zeta| \leqslant m)$ of $A(t)$ are such that the maps $I \ni t \to D^\beta a_{\xi,\zeta}(t,\cdot) \in C(\bar{\Omega}, \mathbb{R})$ $(|\beta| \leqslant m)$ belong to the class $C^\mu(I, C(\bar{\Omega}, \mathbb{R}))$, and after rewriting $A(t)$ in the form (4.9), we have that $\{(\mathcal{A}(t), \{B_j\}, \Omega, \partial\Omega),\ t \in \mathbb{R}\}$ is of the class $\mathcal{RPIBVP}$.*

Assumption 4.11. *All $A(t)$ are self-adjoint in $L^2(\Omega)$ and, given a bounded time interval I,*

$$\langle A(t)\phi, \phi\rangle_{L^2(\Omega)} \geqslant s_*\|\phi\|^2_{L^2(\Omega)}, \quad t \in I, \tag{4.14}$$

where $s_* > 0$ can depend on I but not on $t \in I$.

Proposition 4.12. *Suppose that Assumptions 4.10 and 4.11 hold and*

$$E_\alpha = \begin{cases} (H^{2m}_{\{B_j\}}(\Omega))', & \alpha = 0,\\ ([L^2(\Omega), H^{2m}_{\{B_j\}}(\Omega)]_{1-\alpha})', & \alpha \in (0,1),\\ L^2(\Omega), & \alpha = 1,\\ [L^2(\Omega), H^{2m}_{\{B_j\}}(\Omega)]_{\alpha-1}, & \alpha \in (1, 1+\mu). \end{cases} \tag{4.15}$$

There then exists a continuous process in $E_0 = (H^{2m}_{\{B_j\}}(\Omega))'$ associated with

$$\begin{gathered} u_t + A(t)u = 0, \quad t > \tau,\ x \in \Omega \subset \mathbb{R}^N,\\ B_0 u = \cdots = B_{m-1}u = 0 \quad \text{on } \partial\Omega, \quad u(\tau,\cdot) = u_\tau \in L^2(\Omega), \end{gathered}$$

and possessing smoothing properties (1.2) and (1.3).

Proof. We will ensure that Theorem 3.13 applies with $X = L^2(\Omega)$ and $E_0 = (H^{2m}_{\{B_j\}}(\Omega))'$.

As in [**17**, Proposition 1.3.3] we get $\|(\lambda I - A(t))\phi\|_X \geqslant 2^{-1/2}|\lambda - s_*|$ whenever $\phi \in H^2_{\{B_j\}}(\Omega)$, $t \in I$, $\mathrm{Re}(\lambda) \leqslant s_*$. From this we conclude that $\{A(t) \colon t \in \mathbb{R}\}$ is of the class

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$\mathcal{LUS}(D_X, X)$. Since purely imaginary powers are unitary operators, $\{A(t)\colon t \in \mathbb{R}\}$ is of the class $\mathcal{BIP}(X)$.

We now fix a bounded time interval $I \subset \mathbb{R}$ and concentrate on points $t \in I$. Using (4.14), Schwartz's inequality and (4.11) we obtain, with s_* as in (4.14),

$$\|\varphi\|_{H^{2m}(\Omega)} \leqslant c^*(1 + s_*^{-1})\|A(t)\varphi\|_{L^2(\Omega)}, \quad \varphi \in H^{2m}_{\{B_j\}}(\Omega),\ t \in I. \tag{4.16}$$

Next, to obtain (3.3) we apply (4.12) with $p = 2$, $\varphi = A^{-1}(s)\psi$, $\psi \in L^2(\Omega)$, and use (4.16) with $t = s$ and $\varphi = A^{-1}(s)\psi$ to conclude that

$$\begin{aligned}\|A(t)A^{-1}(s)\psi\|_{L^2(\Omega)} &\leqslant c_*\|A^{-1}(s)\psi\|_{H^{2m}(\Omega)} \\ &\leqslant c_*c^*(1 + s_*^{-1})\|\psi\|_{L^2(\Omega)}, \quad \psi \in L^2(\Omega),\ t, s \in I.\end{aligned}$$

In the proof of (3.5) we adapt the idea of [**6**, Remark 6.6 (c)]. Since (from above) we have $\sup_{t,s\in I}\|A(s)A^{-1}(t)\|_{L(L^2(\Omega))} \leqslant N$, using this and self-adjointness of the operators we obtain

$$|\langle\phi, A^{-1}(t)A(s)\psi\rangle_{L^2(\Omega)}| \leqslant N\|\phi\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)}, \quad \phi \in L^2(\Omega),\ \psi \in H^{2m}_{\{B_j\}}(\Omega),\ t, s \in I.$$

This ensures that the set $\{A^{-1}(t)A(s)\psi\colon t, s \in I,\ \psi \in H^{2m}_{\{B_j\}}(\Omega),\ \|\psi\|_{L^2(\mathbb{R}^N)} \leqslant 1\}$ is bounded in $L^2(\Omega)$, and hence $\|\overline{A^{-1}(t)A(s)}\|_{L(L^2(\Omega))} \leqslant \bar{c}$, where $\bar{c} > 0$ does not depend on $t, s \in I$.

Letting $\alpha_0 = 0$, we define next spaces E_α, $\alpha \in [0, 1 + \mu_0] = [0, 2]$ as in (3.7), that are characterized as in (4.15). To ensure that

$$A(\cdot) \in C^\mu_{\mathrm{loc}}(\mathbb{R}, L(E_1, E_0)) \quad \text{with } E_1 = L^2(\Omega) \text{ and } E_0 = (H^{2m}_{\{B_j\}}(\Omega))', \tag{4.17}$$

observe that

$$\|(A(t) - A(s))\phi\|_{(H^{2m}_{\{B_j\}}(\Omega))'} = \sup_{\|\psi\|_{H^{2m}_{\{B_j\}}(\Omega)}=1}\left|\int_\Omega \phi(A(t) - A(s))\psi\right|$$

for $t, s \in \mathbb{R}$, $\phi \in L^2(\mathbb{R}^N)$. Hence, given t and s in a bounded time interval I and using (4.9) and (4.10), we obtain

$$\begin{aligned}&\sup_{\|\phi\|_{E_1}=1}\|[A(t) - A(s)]\phi\|_{E_0} \\ &\quad = \sup_{\|\phi\|_{L^2(\Omega)}=1}\ \sup_{\|\psi\|_{H^{2m}_{\{B_j\}}(\Omega)}=1}\left|\int_\Omega \phi(A(t) - A(s))\psi\right| \\ &\quad \leqslant \sup_{\|\phi\|_{L^2(\Omega)}=1}\ \sup_{\|\psi\|_{H^{2m}_{\{B_j\}}(\Omega)}=1}\ \sum_{|\sigma|\leqslant 2m}\|a_\sigma(t,\cdot) - a_\sigma(s,\cdot)\|_{C(\bar{\Omega},\mathbb{R})}\|\phi\|_{L^2(\Omega)}\|D^\sigma\psi\|_{L^2(\Omega)} \\ &\quad \leqslant c|t - s|^\mu.\end{aligned}$$

We can now apply Theorem 3.13 to get the result. □

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Remark 4.13. Besides (4.17), we also have $A(\cdot) \in C^{\mu}_{\mathrm{loc}}(\mathbb{R}, L(E_2, E_1))$ with $E_2 = H^{2m}_{\{B_j\}}(\Omega)$ and $E_1 = L^2(\Omega)$ as, by (4.10), whenever $\phi \in E_2$, s and t vary in a bounded time interval I,

$$\|(A(t) - A(s))\phi\|_{E_1} \leqslant \|a_\sigma(t, \cdot) - a_\sigma(s, \cdot)\|_{C(\bar{\Omega})} \sum_{|\sigma| \leqslant 2m} \|D^\sigma \phi\|_{E_1} \leqslant |t - s|^\mu \|\phi\|_{E_2}.$$

We now consider a nonlinear term, where we use the Landau symbols $O(\varphi)$ and $o(\varphi)$ as in Remark 4.4. For (4.13) with initial data in $L^2(\Omega)$, the role of a critical exponent is played by

$$\rho_{\mathrm{c}} = \frac{N + 4m}{N}.$$

Proposition 4.14. *Assume that $f, f'_u \in C(\mathbb{R}^{N+2}, \mathbb{R})$, let E_α, $\alpha \in [0, 1 + \mu)$, be as in (4.15) and let*

$$N > 4m. \tag{4.18}$$

(i) *If $f'_s(t, x, s) = O(c_\eta + \eta |s|^{\rho - 1})$ for some $\eta > 0$ and $\rho \in (1, \rho_{\mathrm{c}})$, then the map $F(t, u)$ in (4.13) is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha,\ \alpha \in [0, 1 + \mu)\}$ and is subcritical.*

(ii) *If $f'_s(t, x, s) = O(c_\eta + \eta |s|^{\rho_{\mathrm{c}} - 1})$ for some $\eta > 0$ and (i) does not apply, then the map $F(t, u)$ in (4.13) is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha,\ \alpha \in [0, 1 + \mu)\}$ and is critical.*

(iii) *If $f'_s(t, x, s) = o(|s|^{\rho_{\mathrm{c}} - 1})$ and (i) does not apply, then $F(t, u)$ in (4.13) is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha,\ \alpha \in [0, 1 + \mu)\}$ and is almost critical. Furthermore,*

(iv) *parts (i)–(iii) hold with $\varepsilon > 0$ as small as we wish. Actually, whenever t varies in a bounded time interval $I \subset \mathbb{R}$, there exists a certain $c > 0$ such that*

$$\|F(t, \phi)\|_{E_0} \leqslant c(1 + \|\phi\|^{\rho_{\mathrm{c}}}_{E_1}), \quad \phi \in E_1. \tag{4.19}$$

Proof. Note that when restricting the time variable t to a bounded time interval I, one needs to show that there are constants $c > 0$, $C_\eta > 0$ and $\varepsilon \in (0, 1/\rho)$, $\varepsilon < \mu$, $\rho\varepsilon \leqslant \gamma(\varepsilon) < 1$ such that

$$\|F(t, v) - F(t, w)\|_{E_{\gamma(\varepsilon)}} \leqslant c\|v - w\|_{E_{1+\varepsilon}} (C_\eta + \eta \|v\|^{\rho - 1}_{E_{1+\varepsilon}} + \eta \|w\|^{\rho - 1}_{E_{1+\varepsilon}}), \quad v, w \in E_{1+\varepsilon}. \tag{4.20}$$

We now describe admissible triples $(\rho, \varepsilon, \gamma(\varepsilon))$ for which (4.20) holds and prove that the map F is indeed critical for $\rho = \rho_{\mathrm{c}}$, while it is subcritical for $\rho \in (1, \rho_{\mathrm{c}})$.

Observe that due to (4.15) we have

$$\left.\begin{aligned} &E_{1+\varepsilon} \hookrightarrow L^s(\Omega), \qquad \varepsilon \in [0, \mu),\ 2m\varepsilon - \frac{N}{2} \geqslant -\frac{N}{s},\ s \geqslant 2, \\ &E_{\gamma(\varepsilon)} \hookleftarrow L^\sigma(\Omega), \quad \gamma(\varepsilon) \in [0, 1),\ \frac{2N}{N + 4m(1 - \gamma(\varepsilon))} \leqslant \sigma \leqslant 2,\ \sigma > 1, \end{aligned}\right\} \tag{4.21}$$

where $2N/(N + 4m(1 - \gamma(\varepsilon))) > 1$ provided that $\gamma(\varepsilon) > (4m - N)/4m =: \tilde{\gamma}$.

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By (4.21), $\|F(t,v)-F(t,w)\|_{E_{\gamma(\varepsilon)}}$ is bounded by $\hat{c}\|F(t,v)-F(t,w)\|_{L^{2N/(N+4m(1-\gamma(\varepsilon)))}(\Omega)}$ and whenever $f'_s(t,x,s) = O(c_\eta + \eta|s|^{\rho-1})$ we have

$$\|F(t,v) - F(t,w)\|_{E_{\gamma(\varepsilon)}} \leqslant \tilde{c}\||v-w|(c_\eta + \eta|v|^{\rho-1} + \eta|w|^{\rho-1})\|_{L^{2N/(N+4m(1-\gamma(\varepsilon)))}(\Omega)}. \quad (4.22)$$

Applying next Hölder's inequality with $q = (N+4m(1-\gamma(\varepsilon)))/(N-4m\varepsilon)$ and $q' = (N+4m(1-\gamma(\varepsilon)))/(4m(1-\gamma(\varepsilon)+\varepsilon))$, recalling the embedding $H^{2m\varepsilon}(\Omega) \hookrightarrow L^{2N/(N-4m\varepsilon)}(\Omega)$, and assuming that

$$H^{2m\varepsilon}(\Omega) \hookrightarrow L^{N(\rho-1)/(2m(1-\gamma(\varepsilon)+\varepsilon))}(\Omega), \quad (4.23)$$

we obtain

$$\begin{aligned}\|F(t,v) - F(t,w)\|_{E_{\gamma(\varepsilon)}} &\leqslant \tilde{c}\|v-w\|_{L^{2N/(N-4m\varepsilon)}(\Omega)}\|c_\eta + \eta|v|^{\rho-1} + \eta|w|^{\rho-1}\|_{L^{N/2m(1-\gamma(\varepsilon)+\varepsilon)}}(\Omega) \\ &\leqslant c\|v-w\|_{E_{1+\varepsilon}}(C_\eta + \eta\|v\|^{\rho-1}_{E_{1+\varepsilon}} + \eta\|w\|^{\rho-1}_{E_{1+\varepsilon}}), \quad v,w \in E_{1+\varepsilon},\end{aligned} \quad (4.24)$$

where (4.23) requires that

$$\bar{\gamma} := \frac{(4m\varepsilon - N)(\rho-1) + 4m(1+\varepsilon)}{4m} \geqslant \gamma(\varepsilon) \geqslant \frac{2m(1+\varepsilon) - N(\rho-1)}{2m} =: \underline{\gamma}. \quad (4.25)$$

We remark that $\bar{\gamma} > \tilde{\gamma}$ and that for $\rho \in (1, 1+4m/N]$ and $\varepsilon > 0$ we have $\bar{\gamma} > \underline{\gamma}$ and $\bar{\gamma} \geqslant \varepsilon\rho$. We also have $1 > \bar{\gamma}$ if $\varepsilon \in (0, N(\rho-1)/4m\rho)$.

The above ensures that any triple $(\rho, \varepsilon, \gamma(\varepsilon))$, where $\rho \in (1, 1+4m/N]$, $\varepsilon \in (0, \min\{\mu, N(\rho-1)/4m\rho\})$ and $\gamma(\varepsilon) \in [\rho\varepsilon, \bar{\gamma}] \cap [\max\{0, \underline{\gamma}\}, \bar{\gamma}] \cap (\tilde{\gamma}, \bar{\gamma}] =: \mathcal{I}(\varepsilon)$ is admissible.

For any admissible triple $(\rho, \varepsilon, \gamma(\varepsilon))$, (4.25) implies that $\rho \leqslant (N+4m-4m\gamma(\varepsilon))/(N-4m\varepsilon)$, and since $\gamma(\varepsilon) \geqslant \rho\varepsilon$, we have $\rho \leqslant (N+4m-4m\rho\varepsilon)/(N-4m\varepsilon)$, which holds if and only if $\rho \leqslant (N+4m)/N = \rho_{\mathrm{c}}$. Thus, $\rho = \rho_{\mathrm{c}}$ cannot be attained for $\gamma(\varepsilon) > \rho_{\mathrm{c}}\varepsilon$, and therefore $\rho = \rho_{\mathrm{c}}$ necessitates that $\gamma(\varepsilon) = \varepsilon\rho_{\mathrm{c}}$. Note that $\bar{\gamma}|_{\rho=\rho_{\mathrm{c}}} = \varepsilon\rho_{\mathrm{c}}$, that is, for $\rho = \rho_{\mathrm{c}}$ we have $\mathcal{I}(\varepsilon) = \{\varepsilon\rho_{\mathrm{c}}\}$. This completes the proof of (i) and (ii).

Note that having $|f'(t,x,s)| \leqslant O(c_\eta + \eta|s|^{\rho_{\mathrm{c}}-1})$ for each $\eta > 0$, we obtain (4.20) for any $\eta > 0$, which leads to (iii).

In describing admissible triples, we have already ensured that $\varepsilon > 0$ can be chosen arbitrarily close to zero. Actually, we also have

$$\begin{aligned}\|F(t,v) - F(t,0)\|_{E_0} &\leqslant \hat{c}\|F(t,v) - F(t,0)\|_{L^{2N/(N+4m)}(\Omega)} \\ &\leqslant \tilde{c}\||v|(c_\eta + \eta|v|^{\rho-1})\|_{L^{2N/(N+4m)}(\Omega)},\end{aligned}$$

which leads to (4.19) as $L^2(\Omega) \hookrightarrow L^{2N\rho/(N+4m)}(\Omega)$ for $\rho \in (1, \rho_{\mathrm{c}}]$. □

Remark 4.15. Note that by not assuming (4.18) in Proposition 4.14, we may not have (i)–(iii) satisfied for $\varepsilon > 0$ arbitrarily small, as stated in (iv) (see [**18**, § 3.1] for a similar proof).

Corollary 4.16. *Suppose that Assumptions 4.10 and 4.11 hold and the spaces E_α, $\alpha \in [0, 1+\mu)$, are as in (4.15). Suppose also that the assumptions of Proposition 4.14 are*

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satisfied; in particular, $f'_s(t,x,s) = O(c_\eta + \eta|s|^{\rho_c - 1})$ *for some* $\eta > 0$. *Then Theorem 1.7 applies and, given any* $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$, *the initial boundary-value problem (4.13) has a unique* $E_{1+\varepsilon}$*-solution* $u = u(\cdot, \tau, u_\tau)$ *defined on the maximal interval of existence* $[\tau, T_{u_\tau})$.

We now derive an $L^2(\Omega)$-estimate of the solutions.

Lemma 4.17. *Suppose that*

$$sf(t,x,s) \leqslant C(t,x)s^2 + D(t,x), \quad t \in \mathbb{R},\ x \in \Omega, \tag{4.26}$$

for some $C \in L^\infty_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ *and* $D \in L^1_{\mathrm{loc}}(\mathbb{R}, L^1(\Omega))$.

If $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$, $T \in (\tau, \infty)$ *and an* $E_{1+\varepsilon}$*-solution* u *of (4.13) exists for* $t \in [\tau, T)$, *then*

$$\|u(t,\tau,u_\tau)\|^2_{L^2(\mathbb{R}^N)} \leqslant g(\tau, \|u_\tau\|_{L^2(\Omega)}, T), \quad t \in [\tau, T), \tag{4.27}$$

where $g\colon \mathbb{R} \times [0,\infty) \times (0,\infty) \to [0,\infty)$ *is a certain continuous function.*

Proof. We restrict the time variable to $[\tau, T)$ here, which allows us to choose the constant s_* such that (4.14) holds uniformly for $t \in [\tau, T)$. We also define $C^* := \sup_{(t,x)\in[\tau,T)\times\Omega} 2|C(t,x)|$.

From (4.13), (4.14) and (4.26) we obtain for any $\lambda \in (0, s_*)$ an estimate of the form

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\Omega)} + (s_* - C^*)\|u(t)\|^2_{L^2(\Omega)} \leqslant \|D(t,\cdot)\|_{L^1(\Omega)}, \quad t \in [\tau, T).$$

Solving the above inequality we get

$$\|u(t)\|^2_{L^2(\Omega)} \leqslant \|u_\tau\|^2_{L^2(\Omega)}\mathrm{e}^{-2t(s_*-C^*)} + 2\int_\tau^t \|D(s,\cdot)\|_{L^1(\Omega)}\mathrm{e}^{-2(t-s)(s_*-C^*)}\,\mathrm{d}s, \quad t \in [\tau, T).$$

This proves (4.27) for smooth solutions, that is, for solutions with smooth initial data, which can be obtained with [**26**, Theorem 7] due to Remark 4.13. With $(1.11)_{\theta=0}$ (see Remark 1.9 (iii)) it then extends to $E_{1+\varepsilon}$-solutions, which completes the proof. □

Theorem 1.10 now implies the following result.

Corollary 4.18. *Suppose that the assumptions of Corollary 4.16 and Lemma 4.17 hold.*

If $f'_s(t,x,s) = o(|s|^{\rho_c - 1})$, *then, given any* $\tau \in \mathbb{R}$ *and* $u_\tau \in L^2(\Omega)$, *a unique* $E_{1+\varepsilon}$*-solution of (4.13) exists globally in time.*

Proof. With a finite maximal time of existence (i.e. $T_{u_\tau} < \infty$) we would have $\sup_{[\tau, T_{u_\tau})}\|u(t,\tau,u_\tau)\|_{L^2(\mathbb{R}^N)} < \infty$ (see Lemma 4.17) and Theorem 1.10 (i) with $E_1 = L^2(\Omega)$ would lead to a contradiction. □

In the critical case $\rho = \rho_c$, some better estimate of the solutions can be sometimes obtained if additional conditions are imposed on (4.13). For example, in the autonomous case, an $H^1(\Omega)$-estimate can be found as in [**29**]. Also, if $m = 1$ and the maximum

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principle applies, then an $L^\infty(\Omega)$-estimate may be known. However, without any such specific assumption, the estimate of the solutions of (4.13) in the $E_{1+\varepsilon}$-norm, needed to apply (1.15), remains unknown. On the other hand, Theorem 1.12 will yield the existence of a piecewise $E_{1+\varepsilon}$-solution on some larger time interval than the maximal interval of existence of the $E_{1+\varepsilon}$-solution.

Lemma 4.19. *Suppose that the assumptions of Corollary 4.16 and Lemma 4.17 are satisfied.*

If $\tau \in \mathbb{R}$, $u_\tau \in E_1 = L^2(\Omega)$ and $T_{u_\tau} < \infty$, the map $[\tau, T_{u_\tau}) \ni t \to u(t) \in E_0 = (H^{2m}_{\{B_j\}}(\Omega))'$, where u is a $E_{1+\varepsilon}$-solution of (4.13), is uniformly continuous.

Proof. From (4.13) we infer that

$$\|u_t(t)\|_{(H^{2m}_{\{B_j\}}(\Omega))'} \leqslant \|A(t)u(t)\|_{(H^{2m}_{\{B_j\}}(\Omega))'} + \|f(t,\cdot,u)\|_{(H^{2m}_{\{B_j\}}(\Omega))'}, \quad t \in (\tau, T_{u_\tau}).$$

Since

$$\|A(t)u\|_{(H^{2m}_{\{B_j\}}(\Omega)')} = \sup_{\|\psi\|_{H^{2m}_{\{B_j\}}(\Omega)}=1} \left| \int_\Omega uA(t)\psi \right| \leqslant \|u(t)\|_{L^2(\Omega)} \max_{|\sigma|\leqslant 2m} \|a_\sigma(t,\cdot)\|_{C(\bar{\Omega},\mathbb{R})},$$

by (4.10) and (4.27) we get

$$\|A(t)u\|_{L^\infty((\tau,T_{u_\tau}),(H^{2m}_{\{B_j\}}(\Omega))')} \leqslant cg(\tau, \|u_\tau\|_{L^2(\Omega)}, T_{u_\tau}).$$

From (4.19), $\|f(t,\cdot,u)\|_{(H^{2m}_{\{B_j\}}(\Omega))'}$ is bounded by a multiple of $(1 + \|u(t)\|^{\rho_c}_{L^2(\Omega)})$, and hence, by (4.27),

$$\|f(t,\cdot,u)\|_{L^\infty((\tau,T_{u_\tau}),(H^{2m}_{\{B_j\}}(\Omega))')} \leqslant c(1 + [g(\tau, \|u_\tau\|_{L^2(\Omega)}, T_{u_\tau})]^{\rho_c}).$$

The above estimates ensure that $u(\cdot,\tau,u_\tau) \in W^{1,1}((\tau,T),(H^{2m}_{\{B_j\}}(\Omega))')$ and the proof is complete (see [**10**, Theorem I.2.2]). □

Theorem 1.12 and Lemmas 4.17 and 4.19 now lead to the following conclusion.

Corollary 4.20. *Suppose that the assumptions of Corollary 4.16 and Lemma 4.17 hold.*

Whenever $\tau \in \mathbb{R}$ and $u_\tau \in L^2(\Omega)$ are such that $T_{u_\tau} < \infty$, there exist an $a \in (T_{u_\tau}, \infty]$ and an extension $\mathcal{U}$ of the maximally defined $E_{1+\varepsilon}$-solution of (4.13) such that $\mathcal{U}$ is a piecewise $E_{1+\varepsilon}$-solution on $[\tau, a)$ and either $a = \infty$ or a is an accumulation time of singular times.

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